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ALGORITHMIC ASPECTS OF ROMAN {3}-DOMINATION IN GRAPHS

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Abstract. Let G be a simple, undirected graph. A function $g : V(G) \to \{0, 1, 2, 3\}$ having the property that $\sum_{v \in N_G(u)} g(v) \geq 3$, if $g(u) = 0$, and $\sum_{v \in N_G(u)} g(v) \geq 2$, if $g(u) = 1$ for any vertex $u \in G$, where $N_G(u)$ is the set of vertices adjacent to u in G, is called a *Roman {3}-dominating function* (R3DF) of G. The weight of a R3DF g is the sum $g(V) = \sum_{v \in V} g(v)$. The minimum weight of a R3DF is called the *Roman {3}-domination number* and is denoted by $\gamma_{\{R3\}}(G)$. Given a graph G and a positive integer k, the Roman {3}-domination problem (R3DP) is to check whether G has a R3DF of weight at most k. In this paper, first we show that the R3DP is NP-complete for chordal graphs, planar graphs and for two subclasses of bipartite graphs namely, star convex bipartite graphs and comb convex bipartite graphs. The minimum Roman {3}-domination problem (MR3DP) is to find a R3DF of minimum weight in the input graph. We show that MR3DP is linear time solvable for bounded tree-width graphs, chain graphs and threshold graphs, a subclass of split graphs. We propose a $3(1+\ln(\Delta-1))$-approximation algorithm for the MR3DP, where Δ is the maximum degree of G and show that the MR3DP problem cannot be approximated within $(1-\epsilon)\ln|V|$ for any $\epsilon > 0$ unless NP $\subseteq$ DTIME($|V|^{O(\log\log|V|)}$). Next, we show that the MR3DP problem is APX-complete for graphs with maximum degree 4. We also show that the domination and Roman {3}-domination problems are not equivalent in computational complexity aspects. Finally, an ILP formulation for MR3DP is proposed.

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1. Introduction

Consider $G = (V, E)$ be a simple, undirected and connected graph. For a vertex $v \in V$, the *open neighborhood* of v in G is $N_G(v) = \{u \in V \mid (u, v) \in E\}$ and the *closed neighborhood* of v is defined as $N_G[v] = N_G(v) \cup \{v\}$. The *degree* $deg(v)$ of a vertex v is $|N_G(v)|$ and $\Delta(G)$ (or simply Δ) denotes the maximum degree of G. An *induced subgraph* is a graph formed from a subset D of vertices of G and all of the edges in G connecting pairs of vertices in that subset, denoted by $\langle D \rangle$. A *clique* is a subset of vertices of G such that every two distinct vertices in the subset are adjacent. An *independent set* is a set of vertices in which no two vertices are adjacent. A *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set. A vertex v of G is said to be a *pendant vertex* if $deg(v) = 1$ and is called *isolated vertex* if $deg(v) = 0$. An edge of G is said to be a *pendant edge* if one of its vertices is a pendant vertex. A *star* is a tree on n vertices with one

Keywords. Roman dominating function, Roman {3}-domination, NP-complete, APX-complete, Integer Linear Programming-domination, NP-complete, APX-complete, integer linear programming.

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vertex having degree $n-1$, called *central vertex*, and the other $n-1$ vertices having degree 1. A *comb* is a tree obtained by joining a single pendant edge to each vertex of a path. In comb, the path is called *backbone* and the pendant vertices are called *teeth*. A bipartite graph $G=(X,Y,E)$ is called *tree convex* if there exists a tree $T=(X,F)$ such that, for each y in Y, the neighbors of y induce a subtree in T. When T is a star (comb), G is called *star (comb) convex bipartite* graph [15]. A vertex u is *simplicial* if its neighborhood $N_G(u)$ induces a complete subgraph of G. An ordering of vertices $\sigma=(u_1,u_2,\dots,u_n)$ is called *Perfect Elimination Ordering* (PEO), if each u_i is simplicial in the subgraph induced by the vertices $\{u_1,\dots,u_i\}$. A graph G is *chordal graph* if and only if G admits a PEO. For undefined terminology and notations we refer to [26].

A vertex v in G dominates the vertices of its closed neighborhood. A set of vertices $S\subseteq V$ is a *dominating set* (DS) in G if for every vertex $u\in V\setminus S$, there exists at least one vertex $v\in S$ such that $(u,v)\in E$, *i.e.*, $N_G[S]=V(G)$. The *domination number* is the minimum cardinality of a dominating set in G and is denoted by $\gamma(G)$ [11].

The concept of Roman domination was introduced in 2004 by Cockayne *et al.* [7]. A function $g:V\to\{0,1,2\}$ is a *Roman Dominating Function* (RDF) on G if every vertex $u\in V$ for which $g(u)=0$ is adjacent to at least one vertex v for which $g(v)=2$. The literature on Roman domination in graphs has been surveyed in [9,22].

Roman $\{2\}$-domination was introduced in 2016 by Chellali *et al.* [5]. A Roman $\{2\}$-dominating function (R2DF) $g:V\to\{0,1,2\}$ has the property that for every vertex $v\in V$ with $g(v)=0$, either there exists a vertex $u\in N_G(v)$, with $g(u)=2$, or at least two vertices $x,y\in N_G(v)$ with $g(x)=g(y)=1$. The concept of Roman $\{2\}$-domination has been studied in [3,27].

Double Roman domination was initiated in 2016 by Robert *et al.* [24]. A *Double Roman Dominating Function* (DRDF) on G is a function $g:V\to\{0,1,2,3\}$ such that for every vertex $v\in V$ if $g(v)=0$, then v has at least two neighbors x,y with $g(x)=g(y)=2$ or one neighbor w with $g(w)=3$, and if $g(v)=1$, then v must have at least one neighbor w with $g(w)\geq 2$. The concept of double Roman domination has been surveyed in [1,4].

Recently, Mojdeh *et al.* [17] initiated the study of Roman $\{3\}$-domination and listed out its applications. A function $g:V\to\{0,1,2,3\}$ having the property that $\sum_{v\in N_G(u)} g(v)\geq 3$, if $g(u)=0$, and $\sum_{v\in N_G(u)} g(v)\geq 2$, if $g(u)=1$ for any vertex $u\in G$ is called a *Roman $\{3\}$-Dominating Function* (R3DF) of G.

The weight of a RDF (R2DF, DRDF, R3DF) g is the sum $g(V)=\sum_{v\in V} g(v)$. The minimum weight of a RDF, R2DF, DRDF and R3DF, respectively, is called the *Roman domination number, Roman $\{2\}$-domination number, double Roman domination number* and *Roman $\{3\}$-domination number*, respectively, denoted by $\gamma_R(G)$, $\gamma_{\{R2\}}(G)$, $\gamma_{dR}(G)$ and $\gamma_{\{R3\}}(G)$. The minimum Roman $\{3\}$-domination problem (MR3DP) is to find a R3DF of minimum weight in the input graph. The decision version of Roman $\{3\}$-domination problem is defined as follows.

ROMAN $\{3\}$-DOMINATION PROBLEM (R3DP)
INSTANCE: Graph $G=(V,E)$ and a positive integer k.
QUESTION: Does G have a R3DF of weight at most k?

Mojdeh *et al.* [17] have shown the defense strategy of Roman $\{3\}$ – Empire and proved that the R3DP is NP-complete for bipartite graphs. Motivated by their work [17], we investigate the complexity of R3DP in planar graphs, subclasses of bipartite graphs and chordal graphs. Ivanović [13] and ReVelle and Rosing [23] have proposed integer linear programming (ILP) formulations for the Roman domination problem. Motivated by this, we propose an ILP formulation for the MWCRDP.

2. Complexity results

In this section, we show that R3DP is NP-complete for star convex bipartite graphs, comb convex bipartite graphs and chordal graphs by giving a polynomial time reduction from Exact-3-Cover (X3C) [10], which is a famous NP-complete problem and is defined as follows.

EXACT-3-COVER (X3C)
INSTANCE: A finite set X with $|X|=3q$ and a collection C of 3-element subsets of X.

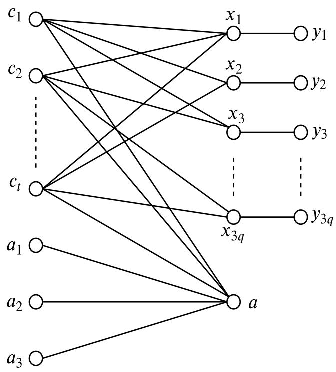

FIGURE 1. Construction of a star convex bipartite graph from an instance of X3C.

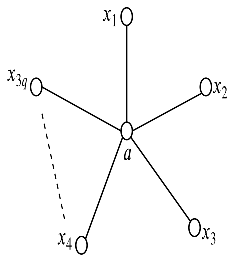

FIGURE 2. Star graph.

QUESTION: Is there a subcollection C' of C such that every element of X appears in exactly one member of C'?

Theorem 2.1. *R3DP is NP-complete for star convex bipartite graphs.*

Proof. Given a graph G and a function f, whether f is a R3DF of size at most k can be checked in polynomial time. Hence R3DP is a member of NP. Now we show that R3DP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3C, where $X = \{x_1, x_2, . . ., x_{3q}\}$ and $C = \{c_1, c_2, . . ., c_t\}$, to an instance $\langle G, k\rangle$ of R3DP as follows.

Create vertices x_i, y_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a_1, a_2 and a_3. Add edges (x_i, y_i) for each $x_i \in X$, (a_i, a) for each a_i and (c_i, a) for each c_i. Also add edges (c_j, x_i) if $x_i \in c_j$. The graph constructed is shown in the Figure 1. Let $A = \{a\} \cup \{x_i : 1 \le i \le 3q\}$ and $B = \{y_i : 1 \le i \le 3q\} \cup \{c_i : 1 \le i \le t\} \cup \{a_1, a_2, a_3\}$. Assume the set A induces a star with vertex a as central vertex, as shown in the Figure 2, and the neighbors of each element in B induce a subtree of star. Therefore G is a star convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3C in polynomial time.

Next we show that, X3C has a solution if and only if G has a R3DF with weight at most $7q+3$. Let $k = 7q+3$. Suppose C' is a solution for X3C with $|C'| = q$. We define a function $f : V \to \{0, 1, 2, 3\}$ as follows.

$$f(v) = \begin{cases} 3, & \text{if } v = a \\ 2, & \text{if } v \in \{y_i : 1 \le i \le 3q\} \\ 1, & \text{if } v \in C' \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

It can be easily verified that f is a R3DF of G and $f(V) = 7q + 3 = k$.

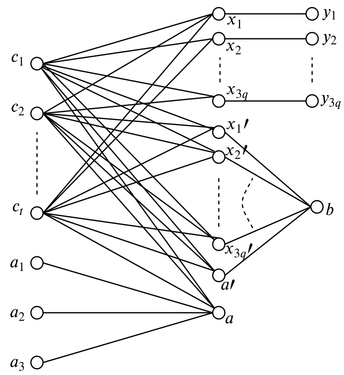

FIGURE 3. Construction of a comb convex bipartite graph from an instance of X3C.

Conversely, suppose that G has a R3DF g with weight k. Let $M = \{a, a_1, a_2, a_3\}$. Clearly, $\sum_{u\in M} g(u) \geq 3$. The following claim holds.

Claim 2.2. If $g(V) = k$ then for each pair of vertices $\{x_i, y_i\}$, $g(x_i) = 0$ and $g(y_i) = 2$.

Proof by contradiction. Assume $g(V) = k$ and there exist some pairs $\{x_i, y_i\}$ such that $g(x_i) + g(y_i) > 2$. Let m (≥ 1) be the number of pairs of $\{x_i, y_i\}$ with $g(x_i) + g(y_i) \geq 3$. The number of pairs of $\{x_i, y_i\}$ with $g(x_i) = 0$ and $g(y_i) = 2$ is $3q - m$. Since g is a R3DF of G, each x_i with $g(x_i) = 0$, where $g(y_i) = 2$, should have a neighbor c_j with $g(c_j) = 1$. Then minimum number of c_j's required with $g(c_j) = 1$ is $\lceil \frac{3q-m}{3} \rceil$. Also, $g(a) + g(a_1) + g(a_2) + g(a_3) \geq 3$. Hence $g(V) \geq 3 + 6q + m + \lceil \frac{3q-m}{3} \rceil$, which is greater than k. Our assumption leads to a contradiction. Therefore for each pair $\{x_i, y_i\}$, $g(x_i) = 0$ and $g(y_i) = 2$. Hence the claim. □

Since each c_i has exactly three neighbors in X, clearly, there exist at least q number of c_i's with weight exactly 1 such that $\left(\bigcup_{g(c_i)\geq 1} N_G(c_i)\right) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 1\}$ is an exact cover for C. □

Theorem 2.3. *R3DP is NP-complete for comb convex bipartite graphs.*

Proof. Clearly, R3DP for comb convex bipartite graphs is a member of NP. We transform an instance $\langle X, C\rangle$ of X3C, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{c_1, c_2, \ldots, c_t\}$, to an instance $\langle G, k\rangle$ of R3DP as follows.

Create vertices x_i, x_i' and y_i for each $x_i \in X$, c_i for each $c_i \in C$ and also create vertices a, a', a_1, a_2, a_3 and b. Add edges (x_i, y_i) for each $x_i \in X$, (a_i, a) for each a_i, (x_i', b) for each x_i', (c_j, x_i) if $x_i \in c_j$ and (b, a'). Next add edges (c_j, a) and (c_j, a') for each c_j. Also add edges by joining each c_j to every x_i'. The graph constructed is shown in the Figure 3. Let $A = \{a, a'\} \cup \{x_i, x_i' : 1 \leq i \leq 3q\}$ and $B = V \setminus A$. Assume, the set A induces a comb with elements $\{x_i' : 1 \leq i \leq 3q\} \cup \{a'\}$ as backbone and $\{x_i : 1 \leq i \leq 3q\} \cup \{a\}$ as teeth, as shown in the Figure 4, and the neighbors of each element in B induce a subtree of the comb. Therefore G is a comb convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3C in polynomial time. Next, we show that, X3C has a solution if and only if G has a R3DF with weight at most $7q + 5$.

Suppose C' is a solution for X3C with $|C'| = q$. We define a function $f : V \to \{0, 1, 2, 3\}$ as follows.

$$f(v) = \begin{cases} 3, & \text{if } v = a \\ 2, & \text{if } v \in \{y_i : 1 \leq i \leq 3q\} \cup \{b\} \\ 1, & \text{if } v \in C' \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

It can be easily verified that f is a R3DF of G and $f(V) = 7q + 5 = k$.

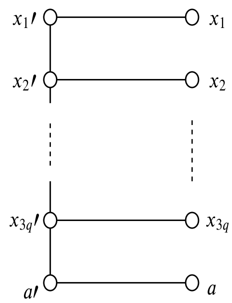

FIGURE 4. Comb graph.

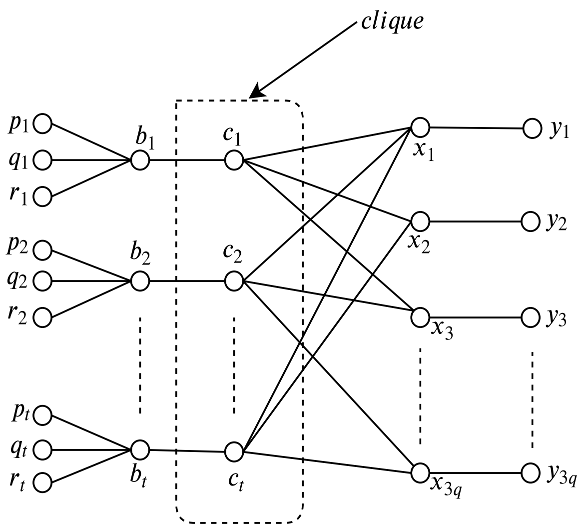

FIGURE 5. An illustration to the construction of chordal graph from an instance of X3C.

Conversely, suppose that G has a R3DF g with weight k. By contradiction, it can be easily shown that $g(b) \geq 2$ and $g(x'_i) = 0$, for $1 \leq i \leq 3q$. The rest of the proof is obtained with similar arguments as in the converse proof of the Theorem 2.1. □

The following corollary is immediate from Theorems 2.1 and 2.3.

Corollary 2.4. *R3DP is NP-complete for tree convex bipartite graphs.*

Theorem 2.5. *R3DP is NP-complete for chordal graphs.*

Proof. Clearly, R3DP is a member of NP. Now we show that R3DP is NP-hard for chordal graphs by transforming an instance $\langle X, C\rangle$ of X3C, where $X = \{x_1, x_2, . . ., x_{3q}\}$ and $C = \{c_1, c_2, . . ., c_t\}$, to an instance $\langle G, k\rangle$ of R3DP as follows.

Create vertices x_i, y_i for each $x_i \in X$, c_i, b_i, p_i, q_i and r_i for each $c_i \in C$. Add edges (x_i, y_i) for each $x_i \in X$, (b_i, c_i), (b_i, p_i), (b_i, q_i), (b_i, r_i) for each b_i and (c_j, x_i) if $x_i \in c_j$. Also add edges (c_i, c_j), $\forall c_i, c_j \in C$, where $i \neq j$. The graph constructed is shown in the Figure 5. Since G admits a PEO $(y_1, y_2, \ldots, y_{3q}, x_1, x_2, \ldots, x_{3q}, p_1, p_2, \ldots, p_t, q_1, q_2, \ldots, q_t, r_1, r_2, \ldots, r_t, b_1, b_2, \ldots, b_t, c_1, c_2, \ldots, c_t)$, it is a chordal graph and the construction of G can be accomplished in polynomial time.

Next we show that, X3C has a solution if and only if G has a R3DF with weight at most $7q + 3t$. Let $k = 7q+3t$. Suppose C' is a solution for X3C with $|C'| = q$. We define a function $f : V \to \{0, 1, 2, 3\}$ as follows.

$$f(v) = \begin{cases} 3, & \text{if } v \in \{b_i : 1 \le i \le t\} \\ 2, & \text{if } v \in \{y_i : 1 \le i \le 3q\} \\ 1, & \text{if } v \in C' \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

It can be easily verified that f is a R3DF of G and $f(V) = 7q + 3t = k$.

Conversely, suppose that G has a R3DF g with weight k. Clearly, $\forall i, 1 \le i \le t, g(p_i)+g(q_i)+g(r_i)+g(b_i) \ge 3$. Hence $g(V) \ge 3t$. The following claim holds.

Claim 2.6. If $g(V) = k$ then for each pair of vertices $\{x_i, y_i\}$, $g(x_i) = 0$ and $g(y_i) = 2$.

Proof. The proof is obtained with similar arguments as in the proof of Claim 2.2. □

Since each c_i has exactly three neighbors in X, clearly, there exist at least q number of c_i's with weight at least 1 such that $\left(\bigcup_{g(c_i)\ge 1} N_G(c_i)\right) \cap X = X$. Consequently, $C' = \{c_i : g(c_i) = 1\}$ is an exact cover for C. □

Next, we show that R3DP is NP-complete for planar graphs by giving a polynomial time reduction from Planar Exact Cover by 3-Sets (Planar X3C) [18], which is a NP-complete problem and is defined as follows.

Planar Exact Cover by 3 Sets (Planar X3C)
INSTANCE: A finite set $X = \{x_1, x_2, \ldots, x_{3q}\}$ and a collection $C = \{c_1, c_2, \ldots, c_t\}$ of 3-element subsets of X such that (i) every element of X occurs in at most three subsets and (ii) the induced graph is planar. (This induced graph $H(V, E)$ is defined as the graph such that $V = X \cup C$ and $E = \{(x_i, c_j) \text{ if } x_i \in c_j\}$).
QUESTION: Is there a subcollection C' of C such that every element of X appears in exactly one member of C'?

Theorem 2.7. *R3DP is NP-complete for planar graphs.*

Proof. Clearly, R3DP is a member of NP. We transform an instance $\langle X, C\rangle$ of Planar X3C, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{c_1, c_2, \ldots, c_t\}$, to an instance $\langle G, k\rangle$ of R3DP same as in Theorem 2.5.

Clearly, G is a planar graph and can be constructed from the given instance $\langle X, C\rangle$ of Planar X3C in polynomial time. Next we show that, Planar X3C has a solution if and only if G has a R3DF with weight at most $7q + 3t$.

Suppose C' is a solution for Planar X3C with $|C'| = q$. We construct a R3DF f, on G, same as in equation (2.3). Clearly, $f(V) = 7q + 3t = k$.

The proof of the converse is similar to the proof given in Theorem 2.5. □

3. Threshold graphs

In this section, we determine the Roman $\{3\}$-domination number of threshold graphs. A *threshold graph* is a graph that can be constructed from one vertex graph by repeated applications of the following two operations: (i) Addition of a single isolated vetrex to the graph. (ii) Addition of a single dominating vertex to the graph. For the graph to be connected the last vertex added must be a dominating vertex. Since every threshold graph is a split graph, $V = C \cup I$, where C is a clique constituting all dominating vertices and I is an independent set constituting all isolated vertices. Let $C = \{c_1, c_2, \ldots, c_n\}$ and $I = \{i_1, i_2, \ldots, i_m\}$. If the clique vertices are added in the order $c_1, c_2, \ldots, c_n$ and the independent vertices are added in the order $i_1, i_2, \ldots, i_m$ then by the definition it follows that $N_G[c_1] \subseteq N_G[c_2] \subseteq N_G[c_3] \subseteq \ldots \subseteq N_G[c_n]$ and $N_G(i_1) \supseteq N_G(i_2) \supseteq N_G(i_3) \supseteq \ldots \supseteq N_G(i_m)$ [16]. If $|V| = 1$ then, clearly, $\gamma_{\{R3\}}(G) = 2$. Otherwise, the following theorem holds.

Theorem 3.1. *Let G be a threshold graph. Then,*

$$\gamma_{\{R3\}}(G) = \begin{cases} 2k, & \text{if } |E(G)| = 0 \\ 2k+1, & \text{otherwise,} \end{cases} \tag{3.1}$$

where k is the number of connected components in G.

Proof. If a threshold graph G has k connected components but no edges, it implies G has k isolated vertices and the result follows. Otherwise, let G be a threshold graph with n clique vertices such that $N_G[c_1] \subseteq N_G[c_2] \subseteq N_G[c_3] \subseteq \ldots \subseteq N_G[c_n]$. Now, define a function $g : V \to \{0,1,2,3\}$ on G as follows.

$$g(v) = \begin{cases} 2, & \text{if } deg(v) = 0 \\ 3, & \text{if } v = c_n \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Clearly, g is a R3DF and $\gamma_{\{R3\}}(G) \leq 2k+1$.

Let $G_1, G_2, \ldots, G_k$ be the k components of G. Let G_1 be the component with at least one edge. From the definition of threshold graphs, it follows that each G_i for $2 \leq i \leq k$ is a single vertex graph. Clearly, $\gamma_{\{R3\}}(G_1) \geq 3$ and $\gamma_{\{R3\}}(G_i) = 2$ for $2 \leq i \leq k$. Hence $\gamma_{\{R3\}}(G) \geq 3 + 2(k-1) = 2k+1$. ☐

Now, the following result is immediate from Theorem 3.1.

Theorem 3.2. *MR3DP can be solvable in linear time for threshold graphs.*

Proof. Since the ordering of the vertices of the clique in a threshold graph can be determined in linear time [16], the result follows. ☐

4. Chain graphs

In this section, we determine the Roman {3}-domination number of chain graphs. A bipartite graph $G = (X,Y,E)$ is called a *chain graph* if the neighborhoods of the vertices of X form a *chain*, that is, the vertices of X can be linearly ordered, say $x_1, x_2, \ldots, x_p$, such that $N_G(x_1) \subseteq N_G(x_2) \subseteq \ldots \subseteq N_G(x_p)$. If $G = (X,Y,E)$ is a chain graph, then the neighborhoods of the vertices of Y also form a chain. An ordering $\alpha = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q) = (x_1, x_2, \ldots,$ of $X \cup Y$ is called a *chain ordering* if $N_G(x_1) \subseteq N_G(x_2) \subseteq \ldots \subseteq N_G(x_p)$ and $N_G(y_1) \supseteq N_G(y_2) \supseteq \ldots \supseteq N_G(y_q)$. Every chain graph admits a chain ordering [28]. The following proposition has been proved in [17].

Proposition 4.1 ([17])**.** *For any complete bipartite graph we have*

(1) $\gamma_{\{R3\}}(K_{1,n}) = \gamma_{dR}(K_{1,n}) = 3$,
(2) $\gamma_{\{R3\}}(K_{2,n}) = \gamma_{dR}(K_{2,n}) = 4$,
(3) $\gamma_{\{R3\}}(K_{3,n}) = 5$ *and* $\gamma_{dR}(K_{3,n}) = 6$, *for* $n \geq 3$,
(4) $\gamma_{\{R3\}}(K_{m,n}) = \gamma_{dR}(K_{m,n}) = 6$, *for* $m, n \geq 4$.

If G is a complete bipartite graph then $\gamma_{\{R3\}}(G)$ is obtained directly from Proposition 4.1. Otherwise, the following theorem holds.

Theorem 4.2. *Let $G\,(\neq K_{r,s})$ be a connected chain graph. Then,*

$$\gamma_{\{R3\}}(G) = \begin{cases} 5, & \text{if } |X| = 2 \text{ or } |Y| = 2 \\ 6, & \text{otherwise.} \end{cases} \tag{4.1}$$

Proof. If $G \cong K_1$ then $\gamma_{\{R3\}}(G) = 2$. Otherwise, let $G(X, Y, E)$ be a connected chain graph with $|X| = p$ and $|Y| = q$ where $p, q \geq 2$. Now, define a function $f : V \to \{0, 1, 2, 3\}$ as follows.

Case 1. $|X| \geq 2$ and $|Y| = 2$ then $f(v) = \begin{cases} 3, & \text{if } v = y_1 \\ 2, & \text{if } v = y_2 \\ 0, & \text{otherwise.} \end{cases}$

Case 2. $|X| = 2$ and $|Y| > 2$ then $f(v) = \begin{cases} 3, & \text{if } v = x_2 \\ 2, & \text{if } v = x_1 \\ 0, & \text{otherwise} \end{cases}$.

Clearly, f is a R3DF and $\gamma_{\{R3\}}(G) \leq 5$. From the definition of R3DF, it follows that $\gamma_{\{R3\}}(G) \geq 5$. Therefore $\gamma_{\{R3\}}(G) = 5$.

Case 3. $|X| > 2$ and $|Y| > 2$ then $f(v) = \begin{cases} 3, & \text{if } v \in \{x_p, y_1\} \\ 0, & \text{otherwise} \end{cases}$.

Clearly, f is a R3DF and $\gamma_{\{R3\}}(G) \leq 6$. By contradiction, it can be easily verified that $\gamma_{\{R3\}}(G) \geq 6$. Therefore $\gamma_{\{R3\}}(G) = 6$.

□

If the chain graph G is disconnected with k connected components $G_1, G_2, \ldots, G_k$ then it is easy to verify that $\gamma_{\{R3\}}(G) = \sum_{i=1}^{k} \gamma_{\{R3\}}(G_i)$. Now, the following result is immediate from Theorem 4.2.

Theorem 4.3. *MR3DF problem can be solvable in linear time for chain graphs.*

Proof. Since the chain ordering can be computed in linear time [25], the result follows. □

5. Bounded tree-width graphs

Let G be a graph, T be a tree and v be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T. The pair (T, v) is called a *tree-decomposition* of G if it satisfies the following three conditions: (i) $V(G) = \bigcup_{t \in V(T)} V_t$, (ii) for every edge $e \in E(G)$ there exists a $t \in V(T)$ such that both ends of e lie in V_t and (iii) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_1, t_2, t_3 \in V(T)$ and t_2 is on the path in T from t_1 to t_3. The *width* of (T, v) is the number $max\{|V_t| - 1 : t \in T\}$, and the *tree-width* $tw(G)$ of G is the minimum width of any tree-decomposition of G. By Courcelle's Theorem, it is already established that every graph problem that can be modelled as counting monadic second-order logic (CMSOL) is solvable in linear-time for bounded tree-width graphs, given a tree decomposition as input [8]. We show that R3DP can be expressed in CMSOL.

Theorem 5.1 (Courcelle's theorem [8])**.** *Let P be a graph property expressible in CMSOL and k be a constant. Then, for any graph G of tree-width at most k, it can be checked in linear-time whether G has property P.*

Theorem 5.2. *Given a graph G and a positive integer k, R3DP can be expressed in CMSOL.*

Proof. Let $g : V \to \{0, 1, 2, 3\}$ be a function on a graph $G(V, E)$, where $V_i = \{v | f(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. The CMSOL formula for the R3DP is expressed as follows.

Rom_3_Dom$(V) = (g(V) \leq k) \wedge \exists V_0, V_1, V_2, V_3, \forall p((p \in V_0 \wedge ((\exists q, r, s \in V_1 \wedge adj(p, q) \wedge adj(p, r) \wedge adj(p, s)) \vee ((\exists t \in V_1 \wedge \exists u \in V_2 \wedge adj(p, t) \wedge adj(p, u)) \vee (\exists q, r \in V_2 \wedge adj(p, q) \wedge adj(p, r)) \vee (\exists v \in V_3 \wedge adj(p, v))))) \vee (p \in V_1 \wedge (\exists w, x \in V_1 \wedge adj(p, w) \wedge adj(p, x)) \vee (\exists y \in (V_2 \cup V_3) \wedge adj(p, y))) \vee (p \in V_2) \vee (p \in V_3))$, where $adj(p, q)$ is the binary adjacency relation which holds if and only if, p, q are two adjacent vertices of G.

ROM_3_Dom(V) ensures that for every vertex $p \in V$, either (i) $p \in V_2$ or (ii) $p \in V_3$, or (iii) if $p \in V_0$ then either there exist three vertices $q, r, s \in V_1$ such that p is adjacent to q, r and s, or there exists two vertives $t \in V_1$, $u \in V_2$ such that p is adjacent to both t and u, or there exist two vertices $q, r \in V_2$ such that p is

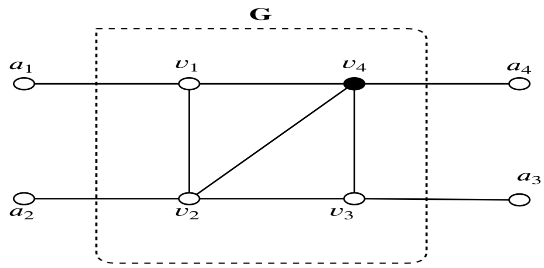

FIGURE 6. An illustration to the construction of G' from G.

adjacent to both q and r, or there exist a vertex $v \in V_3$ such that p is adjacent to v (iv) if $p \in V_1$ then either there exists two vertices $w, x \in V_1$ such that p is adjacent to both w and x or there exists a vertex $y \in V_2 \cup V_3$ such that p is adjacent to y. □

Now, the following result is immediate from Theorems 5.1 and 5.2.

Theorem 5.3. *MR3DP can be solvable in linear time for bounded tree-width graphs.*

6. Approximation results

In this section, we obtain a lower and an upper bound on the approximation ratio of the MR3DP. We also show that the MR3DP is in APX-complete for graphs with maximum degree 4.

6.1. Lower bound on approximation ratio

To obtain a lower bound, we provide an approximation preserving reduction from the MINIMUM DOMINATING SET problem, which has the following lower bound.

Theorem 6.1 ([6])**.** *For a graph $G = (V, E)$, the MINIMUM DOMINATING SET cannot be approximated within $(1-\epsilon)\ln|V|$ for any $\epsilon > 0$ unless* $\mathrm{NP} \subseteq \mathrm{DTIME}(|V|^{O(\log\log|V|)})$.

The following theorem provides a lower bound for approximation ratio of MR3DP.

Theorem 6.2. *For a graph $G = (V, E)$, the MR3DP cannot be approximated within a factor of $(1-\epsilon)\ln|V|$ for any $\epsilon > 0$ unless* $\mathrm{NP} \subseteq \mathrm{DTIME}(|V|^{O(\log\log|V|)})$.

Proof. In order to prove the theorem, we propose the following approximation preserving reduction. Let $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$ be an instance of the MINIMUM DOMINATING SET problem. From this, we construct an instance $G' = (V', E')$ of MR3DP as follows.

Create a vertex set $\{a_1, a_2, \dots, a_n\}$. Add the edges $\{(v_i, a_i) : 1 \le i \le n\}$. Example construction of G' from G is shown in Figure 6. First we need to prove the following claim.

Claim 6.3. If G' is the graph obtained from a graph $G = (V, E)$ ($|V| = n$) then $\gamma_{\{R3\}}(G') = 2n + \gamma(G)$.

Proof. Let $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$ be a graph and $G' = (V', E')$ is a graph constructed from G.

Let D^* be a minimum dominating set of G *i.e.*, $|D^*| = \gamma(G)$ and $f : V \to \{0, 1, 2, 3\}$ be a function on graph G', defined as

$$f(v) = \begin{cases} 3, & \text{if } v \in D^* \\ 2, & \text{if } v \in \{a_i : v_i \notin D^*\} \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

Clearly, f is a R3DF and $\gamma_{\{R3\}}(G') \le 2n + |D^*|$.

Next, we show that $\gamma_{\{R3\}}(G') \geq 2n + |D^*|$. Let g be a R3DF on graph G'. Clearly, $g(v_i) + g(a_i) \geq 2$, if $g(v_i) = 0$ then $g(a_i) \geq 2$, if $g(a_i) = 0$ then $g(v_i) = 3$ and if $|E(G')| = 1$ then $g(a_i) + g(v_i) \geq 3$. Therefore $\gamma_{\{R3\}}(G') \geq 2n + |D^*|$. Hence $\gamma_{\{R3\}}(G') = 2n + \gamma(G)$. □

Suppose that the MR3DP has an approximation algorithm P which runs in polynomial time with approximation ratio α, where $\alpha = (1-\epsilon)\ln|V|$ for some fixed $\epsilon > 0$. Let k be a fixed positive integer. Next, we design an approximation algorithm, say DOM-SET-APPROX which runs in polynomial time to find a dominating set of a given graph G.

Algorithm 1. DOM-SET-APPROX (G).

Require: A simple and undirected graph G.
Ensure: A dominating set D of G.

```
if there exists a dominating set D' of size at most k then
    D ← D'
else
    Construct the graph G'
    Compute a R3DF g on G' by using algorithm P
    Let D = {v_i : g(v_i) + g(a_i) ≥ 3}. (from Claim 6.3)
end if
return D.
```

Clearly, DOM-SET-APPROX runs in polynomial time. It can be noted that if D is a minimum dominating set of size at most k, then it is optimal. Otherwise, let D^* be a minimum dominating set of G and f be a R3DF of G' with $f(V') = \gamma_{\{R3\}}(G')$. Clearly $f(V') \geq k$. If D is a dominating set of G produced by the algorithm DOM-SET-APPROX, then $|D| \leq g(V') \leq \alpha(f(V')) \leq \alpha(2n + |D^*|) = \alpha\left(1 + \frac{2n}{|D^*|}\right)|D^*|$. Therefore, DOM-SET-APPROX approximates a dominating set within a ratio $\alpha\left(1 + \frac{2n}{|D^*|}\right)$. If $\frac{1}{|D^*|} < \epsilon/2$, then the approximation ratio becomes $\alpha\left(1 + \frac{2n}{|D^*|}\right) < (1-\epsilon)(1+n\epsilon)\ln n = (1-\epsilon')\ln n$, where $\epsilon' = n\epsilon^2 + \epsilon - n\epsilon$. Hence DOM-SET-APPROX approximates minimum dominating set within $(1-\epsilon')\ln|V|$. So by Theorem 6.1 and the fact that $\ln(2|V|) \approx \ln|V'|$ for $|V| \to \infty$, unless NP $\subseteq$ DTIME$(|V|^{O(\log\log|V|)})$, MR3DP cannot be approximated within a ratio of $(1-\epsilon)\ln|V|$ for any $\epsilon > 0$. □

6.2. Approximation algorithm

In this subsection, we design an approximation algorithm for optimization version of Roman $\{3\}$-domination problem based on the approximation result known for MINIMUM DOMINATION problem, which is given below.

MINIMUM DOMINATION
Instance: A simple, undirected graph $G = (V, E)$.
Solution: Minimum cardinality dominating set D of G.
Measure: Cardinality of D.

Now, we propose a $3(1+\ln(\Delta+1))$-approximation algorithm for MR3DP. The following approximation result has been obtained in [14] for MINIMUM DOMINATION problem.

Theorem 6.4 ([14]). *The MINIMUM DOMINATION problem in a graph with maximum degree Δ can be approximated with an approximation ratio of $1 + \ln(\Delta + 1)$.*

By Theorem 6.4, let APP-DOM-SET be an approximation algorithm that gives a dominating set D of a graph G such that $|D| \leq (1 + \ln(\Delta + 1))\gamma(G)$, where Δ is the maximum degree of the graph G.

Next, we propose an algorithm APP-R3D to compute an approximate solution of MR3DP. In our algorithm, first we compute a dominating set D of the input graph G using the approximation algorithm APP-DOM-SET. Next, we construct a quadruple Q_r in which every vertex in D will be assigned with weight 3 and the remaining vertices will be assigned with weight 0.

Now, let $Q_r = (D', \emptyset, \emptyset, D)$ be the quadruple obtained by using the APP-R3D algorithm. It can be easily seen that every vertex $v \in V$ is assigned with weight either 0 or 3. Since D is a dominating set of G, every vertex $v \in D'$ with weight 0 is adjacent to a vertex $u \in D$ with weight 3. Thus, Q_r gives a R3DF of G.

Algorithm 2. APP-R3D (G).

Input: A simple, undirected graph G.
Output: A Roman {3}-dominating quadruple Q_r of G.

```
1: D ← APP-DOM-SET(G)
2: Q_r ← (V \ D, ∅, ∅, D)
3: return Q_r.
```

We note that the algorithm APP-R3D computes a Roman {3}-dominating quadruple Q_r of the given graph G in polynomial time. Hence, we have the following result.

Theorem 6.5. *The MR3DP in a graph with maximum degree* Δ *can be approximated with an approximation ratio of* $3(1 + \ln(\Delta + 1))$.

Proof. Let D be the dominating set produced by the algorithm APP-DOM-SET, Q_r be the Roman {3}-dominating quadruple produced by the algorithm APP-R3D and W_r be the weight of Q_r.

It can be observed that $W_r = 3|D|$. It is known that $|D| \leq (1 + \ln(\Delta + 1))\gamma(G)$. Therefore, $W_r \leq 3(1 + \ln(\Delta + 1))\gamma(G)$. Since $\gamma(G) \leq \gamma_{\{R3\}}(G)$ [17], it follows that $W_r \leq 3(1 + \ln(\Delta + 1))\gamma_{\{R3\}}(G)$. □

We have the following corollary of Theorem 6.5.

Corollary 6.6. *MR3DP problem for bounded degree graphs is in APX.*

6.3. APX-completeness

In this subsection, we prove that the MR3DP is APX-complete for graphs with maximum degree 4. This can be proved using the L-reduction, which is defined as follows.

Definition 6.7 (L-reduction [21])**.** Given two NP optimization problems F and G and a polynomial time transformation f from instances of F to instances of G, one can say that f is an L-*reduction* if there exists positive constants α and β such that for every instance x of F

(1) $opt_G(f(x)) \leq \alpha . opt_F(x)$.
(2) For every feasible solution y of $f(x)$ with objective value $m_G(f(x), y) = c_2$ in polynomial time one can find a solution y' of x with $m_F(x, y') = c_1$ such that $|opt_F(x) - c_1| \leq \beta |opt_G(f(x)) - c_2|$.

Here, $opt_F(x)$ represents the size of an optimal solution for an instance x of the NP optimization problem F. An optimization problem π is APX-complete if:

(1) $\pi \in$ APX, and
(2) $\pi \in$ APX-hard, *i.e.*, there exists an L-reduction from some known APX-complete problem to π.

To show the APX-hardness of MR3DP, we give an L-reduction from MINIMUM DOMINATING SET problem in graphs with maximum degree 3 (DOM-3) which has been proved as APX-complete [2].

Theorem 6.8. *The MR3DP is APX-complete for graphs with maximum degree 4.*

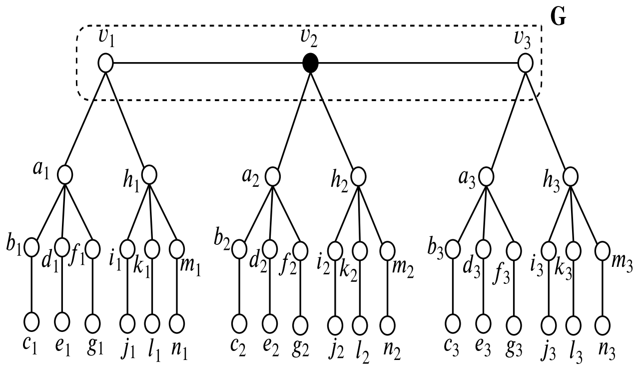

FIGURE 7. An illustration to the construction of GP graph from G.

Proof. By using Corollary 6.6, we can say that MR3DP is in APX for graphs with maximum degree 4. Given an instance $G = (V, E)$ of DOM-3, where $V = \{v_1, v_2, \dots, v_n\}$, we construct an instance $G' = (V', E')$ of MR3DP same as in Section 6.1. Note that G' is a graph with maximum degree 4. We make use of the Claim 6.3 to complete the proof.

Let D^* be a minimum dominating set of G and $f : V' \to \{0, 1, 2, 3\}$ be a minimum R3DF of G'. It is known that for any graph $G = (V, E)$ with maximum degree Δ, $\gamma(G) \geq \frac{n}{\Delta+1}$, where $n = |V|$. Thus, $|D^*| \geq \frac{n}{4}$. From Claim 6.3, it is evident that $f(V') = 2n + |D^*| \leq 8|D^*| + |D^*| = 9|D^*|$.

Now consider a R3DF $g : V' \to \{0, 1, 2, 3\}$ of G'. Clearly, the set $D = \{v_i : g(v_i) + g(a_i) \geq 3\}$ is a dominating set of G. Therefore, $|D| \leq g(V') - 2n$. Hence, $|D| - |D^*| \leq g(V') - 2n - |D^*| \leq g(V') - f(V')$. This implies that there exists an L-reduction with $\alpha = 9$ and $\beta = 1$. □

7. Complexity contrast between domination and Roman {3}-domination problems

Although Roman {3}-domination is one of the several variants of domination problem, these two differ in computational complexity. In particular, there exist graph classes for which the decision version of the first problem is polynomial-time solvable whereas the second problem is NP-complete and *vice versa*. Similar study has been made between domination and other domination parameters in [12, 19, 20].

We construct a new class of graphs in which the MR3DP can be solved trivially, whereas the decision version of the DOMINATION problem is NP-complete, which is defined as follows.

DOMINATION DECISION PROBLEM
INSTANCE: A simple, undirected graph G and a positive integer k.
QUESTION: Does there exist a dominating set of size at most k in G?

Definition 7.1 (GP graph)**.** A graph is *GP graph* if it can be constructed from a connected graph $G = (V, E)$ where $|V| = n$ and $V = \{v_1, v_2, \dots, v_n\}$, in the following way:

(1) Create six copies of P_2 graphs such as $b_i - c_i, d_i - e_i, f_i - g_i, i_i - j_i, k_i - l_i$ and $m_i - n_i$, for each i.
(2) Consider $2n$ additional vertices $\{a_1, a_2, \dots, a_n, h_1, h_2, \dots, h_n\}$.
(3) Add edges $\{(v_i, a_i), (a_i, b_i), (a_i, d_i), (a_i, f_i), (v_i, h_i), (h_i, i_i), (h_i, k_i), (h_i, m_i) : 1 \leq i \leq n\}$.

General GP graph construction is shown in Figure 7.

Theorem 7.2. *If G' is a GP graph obtained from a graph $G = (V, E)$ $(|V| = n)$, then $\gamma_{\{R3\}}(G') = 16n$.*

Proof. Let $G' = (V', E')$ is a GP graph constructed from G. Let $f : V' \to \{0, 1, 2, 3\}$ be a function on graph G', which is defined as below

$$f(v) = \begin{cases} 2, & \text{if } v \in \{a_i, h_i, c_i, e_i, g_i, j_i, l_i, n_i : 1 \le i \le n\} \\ 0, & \text{otherwise.} \end{cases} \tag{7.1}$$

Clearly, f is an R3DF and $\gamma_{\{R3\}}(G') \le 16n$.

Next, we show that $\gamma_{\{R3\}}(G') \ge 16n$. Let g be a R3DF on graph G'. Then following claim holds.

Claim 7.3. If $g(V) = 16n$ then for each $v_i \in V$, $g(v_i) = 0$.

Proof by contradiction. Assume $g(V) = 16n$ and there exist m (≥ 1) v_i's such that $g(v_i) \ne 0$. Clearly, each $\langle \{a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, i_i, j_i, k_i, l_i, m_i, n_i : 1 \le i \le n\} \rangle$, requires a weight of at least 16. Hence $g(V) \ge 16n + m > 16n$, a contradiction. Therefore for each $v_i \in V$, $g(v_i) = 0$. ☐

Clearly, $g(a_i) + g(b_i) + g(c_i) + g(d_i) + g(e_i) + g(f_i) + g(g_i) \ge 8$ and $g(h_i) + g(i_i) + g(j_i) + g(k_i) + g(l_i) + g(m_i) + g(n_i) \ge 8$, where $1 \le i \le n$. Hence $g(V) \ge 16n$. Therefore $g(V) = 16n$. ☐

Lemma 7.4. *Let G' be a GP graph constructed from a graph $G = (V, E)$. Then G has a dominating set of size at most k if and only if G' has a dominating set of size at most $k + 6n$.*

Proof. Suppose D be dominating set of G of size at most k, then it is clear that $D \cup \{b_i, d_i, f_i, i_i, k_i, m_i : 1 \le i \le n\}$ is a dominating set of G' of size at most $k + 6n$.

Conversely, suppose D' is a dominating set of G' of size at most $k + 6n$. Then at least one vertex from each pair of the vertices $\{b_i, c_i\}$, $\{d_i, e_i\}$, $\{f_i, g_i\}$, $\{i_i, j_i\}$, $\{k_i, l_i\}$, $\{m_i, n_i\}$ must be included in D'. Let D'' be the set formed by replacing all a_i's or h_i's in D' by the corresponding v_i's. Clearly, D'' is a dominating set of G of size at most k. Hence the lemma. ☐

The following result is well known for the DOMINATION DECISION problem.

Theorem 7.5 ([10])**.** *The DOMINATION DECISION problem is NP-complete for general graphs.*

From Theorem 7.5 and Lemma 7.4, it follows that DOMINATION DECISION problem is NP-hard for GP graphs. Hence the following theorem.

Theorem 7.6. *The DOMINATION DECISION problem is NP-complete for GP graphs.*

8. Integer Linear Programming formulation

Let G be a graph with $V(G) = \{1, 2, \dots, n\}$ and f be a R3DF on G. The MR3DP can now be modeled as an Integer Linear Program (ILP). The variables for this ILP are

$$a_v = \begin{cases} 1, & \text{if } f(v) = 0 \\ 0, & \text{otherwise} \end{cases} \qquad b_v = \begin{cases} 1, & \text{if } f(v) = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$c_v = \begin{cases} 1, & \text{if } f(v) = 2 \\ 0, & \text{otherwise} \end{cases} \qquad d_v = \begin{cases} 1, & \text{if } f(v) = 3 \\ 0, & \text{otherwise.} \end{cases}$$

The only constant in the ILP is n.

The ILP model of the MR3DP can now be formulated as

$$\text{Determine : } \min \left(\sum_{v \in V} b_v + 2 \sum_{v \in V} c_v + 3 \sum_{v \in V} d_v \right) \tag{8.1}$$

subject to

$$1-(a_v+b_v)+\sum_{u\in N_G[v]} b_u+2c_u+3d_u \geq 3, \qquad \forall\, v\in V \tag{8.2}$$

$$a_v+b_v+c_v+d_v=1, \qquad \forall\, v\in V \tag{8.3}$$

$$a_v,b_v,c_v,d_v\in\{0,1\}, \qquad \forall\, v\in V. \tag{8.4}$$

In the above ILP formulation, the objective function (8.1) minimizes the weight of a R3DF. The constraint in (8.2), guarantees that the sum of labels of vertices in the closed neighborood of a vertex with label zero or one is at least three. The condition in (8.3), guarantees that exactly one label is assigned to a vertex. The condition in (8.4) ensures that the decision variables are binary in nature. The number of variables in the ILP formulated for a graph with n vertices are $4n$ and and the number of constraints are $2n$.

9. Conclusion

In this paper, we have shown that the R3DP is NP-complete for star convex bipartite graphs, comb convex bipartite graphs, chordal graphs and planar graphs. Investigating the algorithmic complexity of these problems for other subclasses of bipartite and chordal graphs remains open. Next, it is also shown that MR3DP is solvable in linear time for threshold graphs, chain graphs and graphs with bounded tree-width. From the approximation point of view, it has been shown that MR3DP for graphs with maximum degree 4 is APX-complete. The complexity status of these problems are still open for graphs with maximum degree other than 4. We have shown that the domination and Roman {3}-domination problems are not equivalent in computational complexity aspects by constructing a new class of graphs called GP graphs. Thus, there is a scope to study each of these problems on its own for particular graph classes. Finally, we have proposed an ILP formulation for the MR3DP. Designing better ILP formulation methods for the MR3DP is an interesting direction for future work.

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