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CONGESTION AND NON-CONGESTION AREAS: IDENTIFY AND MEASURE CONGESTION IN DEA

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Abstract. Detecting the weak and strong congestion statuses of decision-making units (DMUs) and measuring them *via* data envelopment analysis (DEA) is an important issue that has been discussed in several studies and with different views. The efficiency frontier is a concept derived from the underlying production possibility set (PPS), and the congestion concept is related to them. Still, researchers have defined congestion for each DMU in many previous studies and ignored that congestion is linked with the underlying production technology. In the congestion measurement matter, this paper presents two new insights into a congestion area and non-congestion area for production technology and two new mathematical definitions of congestion based on the PPS properties and detecting the weak and strong congestion status of DMUs (CSOD). We prove our definitions are equal to the original definition of congestion. First, we describe the congestion and non-congestion areas based on the PPS; then provide full details of how to measure congestion built on these new insights. Our approaches are very accurate and fast to calculate; they are theoretically elementary and efficient in performance. Our proposed methods can deal with both non-negative and negative data. Finally, an empirical example is provided to illustrate our approaches.

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1. Introduction

If the resources of a decision-making unit (DMU) are overused, this excess consumption may adversely affect its performance. When units use resources in this way, without considering optimality, they only pay attention to the benefits/costs and ignore other adverse effects they have on the company, such as congestion. So, to improve the performance of a DMU, it is essential to identify and detect the presence of congestion as a first step and to plan and formulate appropriate scientific and practical solutions to eliminate such congestion as the second step in production analysis. For this purpose, Färe and Svensson [12] introduced congestion insights in the economy.

Inefficiency is a necessary condition for congestion. Congestion is an extreme of inefficiency. It is necessary to note that congestion is a production frontiers' property and not a DMU's. So, congestion is a characteristic of PPS, not a DMU. To improve the DMU's performance, it is essential to recognize the existence of inefficiency, and

Keywords. Data envelopment analysis (DEA), congestion starting point (CSP), efficiency, congestion area (CA), non-congestion area (NCA), production possibility set (PPS).

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since congestion is a kind of inefficiency (and even worse, causes a reduction in DMU's output) so congestion causes its poor performance. Therefore, appropriate solutions to eliminate the causes of congestion-related inefficiency should be presented and implemented. Therefore, identifying congestion as the first step and then measuring its exact value as the second step of assessing the production situation is one of the most important aspects of data analysis. So, in a nutshell, we can say that: Excessive use of input sources (overuse) may adversely affect the performance of a DMU.

After developing the non-parametric data envelopment analysis (DEA) by the Charnes *et al.* [2], the concept of congestion, which was an economic concept, has been widely studied in the DEA framework. As one of the first congestion measurement studies, Färe and Grosskopf [10] have proposed the radial model methods to measure congestion in DEA as the difference between technologies under weak and strong congestion. However, this model could not successfully detect congestion with a single input-output. In the next studies, Färe *et al.* [13] and Cooper *et al.* [3, 4] developed this concept.

In the DEA literature, since 1980, the concept of congestion by many researchers has been investigated to date. According to this, weak congestion occurs if reductions in some inputs are associated with increases in some outputs without changing other inputs or outputs, and strong congestion occurs if the reduction in all inputs increases all outputs. To the best of our knowledge, except for the original definition of congestion proposed by Cooper *et al.* [3, 9], most of the congestion definitions and measures that have been developed thereafter are for the efficient DMUs on the congestion-based production possibility set (PPS) T_{convex}, *e.g.*, Färe and Grosskopf [11], Tone and Sahoo [23] and Wei and Yan [24, 25], Sueyoshi and Sekitani [22], Khodabakhshi *et al.* [15], Noura *et al.* [20], Khoveyni *et al.* [16], Mehdiloozad *et al.* [19], among others. some of the most important of these researches is the congestion assessment methods by Cooper *et al.* [4–9], Tone and Sahoo [23], Khoveyni *et al.* [17], and Mehdiloozad *et al.* [19].

Note that, same as returns to scale, congestion is a production frontier concept, and any computation and its measurements are dependent on DMUs located on it (inefficient DMUs projected to their efficient frontier). The basis of the Tone and Sahoo [23] is based on the possible projection point sets. In Tone and Sahoo [23], and under convex and strong output disposability assumptions, the concepts of congestion-based technology that classified DMUs into two separate sets as efficient and inefficient sets, was developed. But, by Sueyoshi and Sekitani [22] the problem occurs when each DMU has multiple projection points and management is required to choose one of them, randomly. It seems, that since different projection points of inefficient DMUs produce different results in terms of congestion status, therefore, their definition in multiple projection points condition does not be efficient. Though Sueyoshi and Sekitani [22] have proposed an approach to deal with multiple projections (in improving [23]), their method can't identify the difference between weak congestion and strong congestion.

In improving previous approaches, besides more accurate modeling, Khoveyni *et al.* [17] and Mehdiloozad *et al.* [19] suggested identifying weak and strong congestion with both non-negative and negative data. Out of its theoretical attraction, and admirable from a mathematical computation point of view, their model has high accuracy in identifying the congestion status of DMUs (CSOD). This method has a computational complexity for calculating the maximal element for large-size issues (Khoveyni *et al.* [16] have proposed an approach to deal with multiple projections, but they did not provide evidence to prove their claim).

Mehdiloozad *et al.* [19] thought about dealing with multiple projections while determining the congested DMU. Their research was carried out by linking an insight developed by Mehdiloozad *et al.* [18] as the global reference set (GRS) with the congestion. So, by finding a maximal element of a non-negative convex set of all possible optimal intensity vectors of their congestion-identification model, the GRS can be identified. Finally, with the help of results from Mehdiloozad *et al.* ([18, 19]), they developed an approach (a unique single-stage LP model) to identify the MAX-projection by using the GRS.

Before Mehdiloozad *et al.* [19] we have not identified an accurate, reliable, and good approach to finding CSOD (weak and strong) in the presence of multiple projections and negative data. Mehdiloozad *et al.* [19] have developed a well-definition single-stage LP model to fill in this null. They work on Tone and Sahoo [23] and developed their method. So, they have developed definitions of the weak and strong congestions, but their definition of strong congestion was different from Tone and Sahoo [23]. To identify the CSOD, first time, they defined the non-negative convex set of input-output slack vectors that correspond to DMUs in the congestion-based DEA technology, that all dominate the DMU under evaluation. Next time, they proved that the problem under consideration reduces to finding a maximal element an element with the maximum number of positive components of this set. So, they developed a single-stage LP model to determine congestion. But, despite the great accuracy and power in identifying CSOD, yet, their method is unable to measure the amount of congestion.

While Mehdiloozad *et al.* [19] identify weak and strong CSOD with indicator $\alpha_i^{\max}$ $(i = 1, \ldots, n)$ without measuring the values of the input's congestion. This is the best reason to provide our proposed method. So, in improving the performance of Mehdiloozad *et al.* [19], we develop an approach; exactly equivalent to Mehdiloozad *et al.* [19] (occur and efficient in CSOD) and identify weak and strong CSOD by measuring the amount of input congestion. For this aim, we work on input space. Therefore, we formulate the two non-negative convex sets of inputs space as congestion area (X_{C})/non-congestion area (X_{NCA}) by defining specific DMUs named CSP, which, this CSP concept has some special properties. So, our aim for developing this paper is to fill in this null.

This paper provides two new mathematical definitions for congestion and offers two alternative approaches to the discussion. Briefly, this paper focuses on identifying and measuring congestion in the DEA measurement framework. We define two new concepts in the inputs space as congestion area (CA) and non-congestion area based on PPS property and made our definitions based on these two concepts. Also, we present a new CSP concept and formulate them, then, in inputs space formulate X_{C} based on CSP's input. So, to find CSOD and measure congested DMU's congestion, we formulate our suggested model. Finally, we present two numerical examples to explain new insights and compare the results of our method with Cooper *et al.* [9], Noura *et al.* [20], Tone and Sahoo [23], and Mehdiloozad *et al.* [19].

One of the congestion area advantages is that it does not need to calculate congestion for all DMUs when management wants to determine; which DMUs have congestion or not? Which DMUs work well or not? One advantage of the mathematical definition of congestion is that; can be used in modeling and calculations because the traditional definition of congestion is theoretical and cannot be used in calculations. This definition is also accurate only for DMUs on the production function frontier and inaccurate for inefficient DMUs that are not on the production frontier. That is the mean for inefficient DMUs, and they can also increase some or all inputs and reduce some or all outputs simultaneously. So, what is the difference between congestion and inefficiency? Therefore, the mathematical definitions we provide are practical and accurate for all efficient and inefficient DMUs. Note that the traditional definition of congestion in economics is accurate and acceptable because, in economics, they work only with the production frontier. Still, in DEA, researchers work with all DMUs (on the frontier or below the frontier) (Tab. 1).

2. Preliminary assumption and definitions

Data Envelopment Analysis (DEA) is an efficiency evaluation technique with extensive use in efficiency analysis over the past three decades. With theoretical and practical progress in this branch of management science, DEA has become a perfect tool for assessing the performance and efficiency of decision-making units (DMUs) of a different variety. Since basic DEA does not involve calculating the efficiency frontier, congestion in DMU inputs cannot be determined in that way. Throughout non-parametric DEA, using the production possibility set (PPS) instead of the production frontier, one could investigate the existence of congestion. Suppose that it is n homogenous $\mathrm{DMU}_j (j = 1, \ldots, n)$ in a constant time interval, and $X_j(x_{1j}, \ldots, x_{mj})$ and $Y_j = (y_{1j}, \ldots, y_{rj})$ are the input and output vectors of DMU_j, respectively. As mentioned in the introduction, congestion is a production-related situation that can be viewed as a severe technical inefficiency case. Before proceeding to the main discussion, we need to define two critical concepts based on Cooper *et al.* [4] and Cooper *et al.* [6].

Definition 2.1 (Technical efficiency)**.** A $\mathrm{DMU}_o (o \in j)$ is a technically efficient DMU if the optimal solution of the following BBC model (2.1a) [1] is $\varphi^* = 1$ to evaluate it.

$$\theta^* = \max \theta$$

Table 1. Comparison of the advantages of our method over previous models.

Methods Comparison cases	Mehdiloozad *et al.* [19]	Tone and Sahoo [23]	Noura *et al.* [20]	Cooper *et al.* [9]	Khoveyni *et al.* [17]	Our methods
Simplicity in mathematical modeling and computation	Has complex mathematical calculations	Has complex mathematical calculations	Simplicity in calculations	Simplicity in calculations	Has complex mathematical calculations	Simplicity in calculations
Weak status congestion detection	All DMUs	Some DMUs	Some DMUs	Some DMUs	Some DMUs	All DMUs
Strong status congestion detection	All DMUs	Some DMUs	Some DMUs	Some DMUs	fails to individuate between the weakly but not strongly DMUs	All DMUs
Type of congestion definition	Mathematical definition	Mathematical definition	Economic definition	Economic definition	Economic definition	Mathematical definition
Measuring the value of congestion (weak and strong)	There is no measure of congestion value	There is no measure of congestion value	Measuring the congestion value of identified congested DMUs	Measuring the congestion value of identified congested DMUs	Measuring the weak congestion value of identified weak congested DMUs	Measuring the congestion value of identifying all congested DMUs
Identify congested DMUs without having to perform all steps	For identification, requires all article calculations	For identification, requires all article calculations	For identification, requires all article calculations	For identification, requires all article calculations	For identification, requires all article calculations	By solving a model, it can be said whether the DMU has congestion or not
Identify CA-NCA in PPS	No	No	No	No	No	Yes

$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_j \leq x_{io} & i = 1, 2, \ldots, m \\ \sum_{j=1}^{n} \lambda_j y_j \geq \theta y_{ro} & r = 1, 2, \ldots, s \\ \sum_{j=1}^{n} \lambda_j = 1 \\ \lambda_j \geq 0 \end{cases} \tag{2.1a}$$

$$\varphi^* = \max \theta^* - \varepsilon \left(\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^- \right)$$

$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} + s_{io}^- = x_{io} & i = 1, 2, \ldots, m \\ \sum_{j=1}^{n} \lambda_j y_{rj} - s_{ro}^+ = \varphi y_{ro} & r = 1, 2, \ldots, s \\ \sum_{j=1}^{n} \lambda_j = 1 \\ \lambda_j \geq 0,\ s_r^+ \geq 0,\ s_i^- \geq 0. \end{cases} \tag{2.1b}$$

Notice that in the model (2.1b) above $\varepsilon > 0$ is a non-Archimedean number.

Definition 2.2. Congestion occurs when with T_{convex} technology, increasing one or more inputs reduces one or more outputs without improvement in any other inputs or outputs. Conversely, decreasing one or more inputs increases one or more outputs, worsening no other input or output (this is an economic definition of congestion).

In the last part of the introduction as explained, in DEA, this is not an accurate definition and needs improvement (Because it does not distinguish between inefficient and congested DMUs, also this definition is not rigorous enough since it ignores that congestion should be a concept focusing on the frontier). Thus, we introduce our mathematical definition of congestion to fill this gap.

2.1. Technology set

The technology T is defined as the following,

$$T = \{(x, y)|x \text{ can produce } y\}. \tag{2.2}$$

Notice that the technologies T_{convex} and T_v were defined in the upper section. Assume that the outputs and inputs are non-negative. So, T_{convex} is the decussating of all Technologies $T \subset R$ that satisfy the syllogism of (i) involving observations, (ii) strong disposability of outputs, and (iii) convexity of PPS, and T_v is the decussating of all technologies $T \subset R$ that satisfy the syllogism of (i) involving of observations, (ii) strong disposability of outputs, (ii) strong disposability of inputs, and (iii) convexity of PPS.

The congestion measurement is discussed in output-oriented DEA models. The congestion concept does not fit the traditional DEA literature, because, the congestion concept was started by economic researchers. Furthermore, congestion is modeled and measured by replacing the axiom of strong input disposability with the axiom of weak input disposability. In other words, congestion occurs when the strong disposability of inputs is not available.

Therefore, to model congestion, this principle must be omitted from the fundamental principles of DEA. A PPS called T_{convex} should be constructed to act as the reference for the congestion measurement in DEA. The explicit DEA-base representations of T_v and T_{convex} in the VRS are then defined:

$$T_v = \left\{ (x_o, y_o) \middle| \exists \lambda : \sum_{j=1}^{n} \lambda_j x_j \leq x_o, \sum_{j=1}^{n} \lambda_j y_j \geq y_o, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0; j = 1, \ldots, n \right\}, \tag{2.3}$$

$$T_{\text{convex}} = \left\{ (x_o, y_o) \middle| \exists \lambda : \sum_{j=1}^{n} \lambda_j x_j = x_o, \sum_{j=1}^{n} \lambda_j y_j \geq y_o, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0; j = 1, \ldots, n \right\}. \tag{2.4}$$

By definition, technology T_{convex} satisfies all the syllogisms that define technology T_v. Because T_v is the smallest technology that satisfies this syllogism, so $T_{\text{convex}} \subseteq T_v$.

Definition 2.3. Assume that $\left(\varphi^*, s_{io}^{-*}, s_{ro}^{+*}\right)$ is an optimum solution of the model (2.1b). We called DMU_o is efficient if and only if $\varphi^* = 1$, $\left(s_{io}^{-*}, s_{ro}^{+*}\right) = (0, 0)$. Otherwise, DMU_o is inefficient.

3. Background of congestion

In compaction research within the DEA framework, there are two basic ideas that we will describe below.

In the first step, they proved that the relative interior points of the minimum face (a dimension that contains all the projections) have congestion if one of the vertices spanning this face has congestion, and each of the points displays the equivalent congestion (weak or strong). In the next step, they have defined DMU's congestion at its MAX-projection (By congestion-identification model and maximal element of the non-negative convex set of all possible optimal intensity vectors MAX-projection can be achieved) the GRS can be identified. Finally, with the help of results from Mehdiloozad *et al.* ([18, 19]), they developed an approach (a unique single-stage LP model) to identify the MAX-projection by using the GRS. The calculations of Mehdiloozad *et al.* [19] are summarized as follows: From solving model (4.5) for input and output values, the columns representing the maximal elements (alpha-maximal (in the number of inputs), beta-maximal (in the number of outputs)) are obtained (refer to paragraph 9 below model (4.5) and Cor. 1.3 from [19]) that, only the positive component is important not their value, that is, suppose we have two inputs and one output, if all three components are positive (two alpha-maxima for two inputs and one Beta-maximum for output) is strong congestion. Also, if the third component (Beta-maximal) is positive but the first or second component (two alpha-maximal) is zero, the congestion is weak.

As we have already explained, the basis of our new definition of congestion is the decomposition of the PPS (and, more precisely, the decomposition of the DMUs' inputs area). Therefore, we divide the inputs' area into two special subsets of the congestion area and the non-congestion area. Then define each area and describe the properties and each area. These properties provide two mathematical definitions of congestion (unlike the previous definition of congestion, which is an economic definition).

This research focused on detecting and measuring congestion in the DEA measurement framework. The authors proposed a mathematical definition of congestion and congestion zone linked to the production possibility set instead of each DMU. The advantage of our paper over others is the identification and measurement of the amount of congestion in the construction of the congestion area. That means, without talking about the congestion of each DMU, we specify an area where the inputs of each DMU in that area have congestion. In this case, identifying congested DMUs and also measuring their congestion can be easily identified and measured.

4. Congestion area

Congestion is a frontier concept (like returns to scale), and any argument on its evaluation is bounded to the DMUs, which are efficient only. We define the congestion of one inefficient DMU at its efficient projection that it is obtained by congestion, based on the DEA model. Before discussing our proposed method, we must present the definitions of some new congestion-related concepts and propose a new mathematical definition for the concept of congestion itself.

Definition 4.1 (Congestion area)**.** A convex area created by DMU's congested inputs is called the congestion area, So, the congestion area is an area in the inputs space that the congested inputs located in this area and denoted by $X_{\text{congestion}}$. So, $\text{DMU}_o : (x_o, y_o)^t$ has congestion, thus, $x_o \in X_{\text{congestion}}$.

Concerning Definition 4.1 which includes congested inputs of congested DMUs, it refers to equation (4.14) and does not build on the preceding Definition 2.2 of congestion, but is directly related to it. This area does not include all DMUs. As we will show in the following, this area is infinite and convex in a way that, from the bottom, it is bounded to no congestion and technically efficient DMUs, such as $\text{DMU}_p (p \in j)$. Such points are called congestion starting points (CSPs). CSPs are a type of technically efficient DMUs – that are weakly but not strongly efficient DMUs – with the highest input consumption among the efficient (with maximum output). The most important feature of the CSP is that the production frontier, after CSP, by increasing some inputs leads to reducing some outputs, provided that other inputs and outputs remain unchanged, or by increasing all inputs leads to reducing all outputs. Hence, there are supporting hyperplanes on T_{convex} at the CSP, that all components have negative partial derivatives. We can use the following model (4.4) to distinguish (1) CSPs and (2) supporting hyperplanes from the technically efficient DMUs (the CSPs). Assume that DMU_p is a technically efficient DMU in evaluating with the model (2.1a) and $(u_1^*, \ldots, u_s^*, v_1^*, \ldots, v_m^*, u_0^*)$ is an optimal solution for the model (4.4). Note that the condition of congestion for a DMU is that; in evaluation by BCC model (2.1a), this DMU should not be technically efficient (otherwise, model (4.4) will be infinite, and the supporting hyperplane will be Fig. 1b). So, not only the model (4.4) can obtain the supporting hyperplanes of T_{convex} (Figs. 1a and 1b) but also can find CSPs. So, for any technically efficient point (on the model (2.1a)), we run the following model to obtain CSPs and T_{convex} supporting hyperplanes. Suppose the point $(x_p, y_p)^t$ is located on the upper frontier of T_{convex}. In this sense, for each distinct point in T_{convex} such as $(x, y)^t$ we have $y \ngeq y_p$.

By eliminating the input principle's disposability in the PPS, increasing some inputs will not necessarily result in increasing or remaining unchanged outputs; somehow, this increase in inputs will also reduce some outputs to be fully described in the next sections. This issue is also debatable, to reduce some inputs. Reducing some inputs of the technically efficient DMUs can lead to a decrease, remain unchanged, or even increase some outputs (congestion).

Hence, the supporting hyperplanes on the T_{convex} on the point $(x_p, y_p)^t$ will be as follows,

$$\sum_r u_r y_r - \sum_i v_i x_i + u_0 = 0,\ u_r \geq 0,\ r = 1, \ldots, s. \tag{4.1}$$

So,

$$\frac{\partial y_r}{\partial x_i} = \begin{cases} \frac{v_i}{u_r} & u_r > 0 \\ 0 & u_r = 0. \end{cases} \tag{4.2}$$

That specifies the rate and change type y_r relative to the change x_i. Obviously, an increased x_i decrease y_r when $v_i = \frac{\partial y_r}{\partial x_i} < 0$, that is an indicator for entering the CA. Suppose the point $(x_p, y_p)^t$ is located on the upper frontier of T_{convex}. The component x_i has congestion if there are supporting hyperplanes on the T_{convex} on the $(x_p, y_p)^t$ as

$$\sum_r u_r y_{rp} - \sum_i v_i x_{ip} + u_0 = 0\,, u_r \geq 0,\ v_i < 0, r = 1, \ldots, s, . \tag{4.3}$$

Then, we say that the point $(x_p, y_p)^t$ related to the ith component of input is a CSP. DMU has congestion if and only if it has congestion in at least one of its input components. So, to determine the technical efficient points (CSPs) as $(x_p, y_p)^t$ and upper supporting hyperplanes on the T_{convex} (upper frontier of T_{convex}) we used the following model:

$$\begin{aligned} & u_0^* = \max\ u_0 \\ & \text{s.t.} \\ & \qquad \sum_r u_r y_{rp} - \sum_i v_i x_{ip} + u_0 = 0, \qquad p, \\ & \qquad \sum_r u_r y_{rj} - \sum_i v_i x_{ij} + u_0 \leq 0, \qquad j = 1, \ldots, n, \\ & \qquad u_r \geq 0, v_i \text{ is free, } u_0 \text{ is free,} \qquad r = 1, \ldots, s. \end{aligned} \tag{4.4}$$

Suppose that $(U^*, V^*, u_0) = (u_1^*, \ldots, u_s^*, v_1^*, \ldots, v_m^*, u_0^*)$ is an optimal solution to model (4.4) and $\sum_r u_r^* y - \sum_i v_i^* x + u_0^* = 0$ is one of the supporting hyperplanes on the T_{convex} in the technically efficient $(x_p, y_p)^t$. If the optimal solution (U^*, V^*, u_0^*) has $v_i^* = 0\,(\forall i)$, so $(x_p, y_p)^t$ not only the DMU$_p$ does not have congestion but also not CSP. We have $u_0^* = \infty$. Otherwise, if the optimal solution (U^*, V^*, u_0) has negative components of v_i^*, then not only the technical efficient DMU$_p(x_p, y_p)^t$ is the landing on T_{convex} frontier, but also $(x_p, y_p)^t$ is a CSP. We have $u_0^* > 0$.

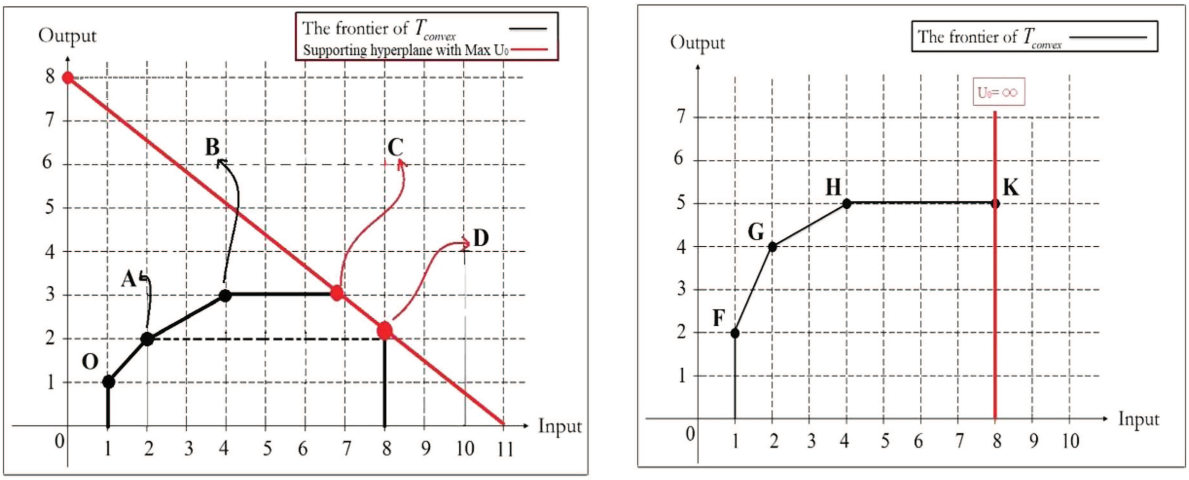
Highlight 1. The technically efficient DMU$_p$ $(x_p, y_p)^t$ (that is weakly technically efficient DMU$_p$) is a CSP if and only if in evaluating with the model (4.4) we have $u_0^* \geq 0, u_0^* \neq 0$. In other words, $(x_p, y_p)^t$ is a CSP, if on multiple model (4.4) there exists an optimal solution such as (U^*, V^*, u_0^*) that $V^* \ngeq 0$.

By solving model (4.4), the supporting hyperplanes of T_{convex} can be obtained. Because of the T_{convex} constituent supporting hyperplanes that envelopment all observation and including CSPs have maximum u_0 compared to all computable supporting hyperplanes (consider Figs. 1a and 1b).

Highlight 2. There is another point; the authors have assumed DMU$_p(x_p, y_p)^t$ as a technically efficient DMU, and introduced model (4.4). However, it is unknown that of the technically efficient DMUs should be considered DMU$_p$ in the model (4.4)? Several technically efficient DMUs can be used as DMU$_p$ in the model (4.4). It is clear; first, we run models (2.1a), (2.1b) and calculated the set of all technically efficient DMUs. Also, sometimes, the observation sets of technically efficient DMUs may not include weakly efficient and DMUs are strongly efficient, so, to find CSP in this case, the all technically efficient DMUs which have the maximum inputs are CSP.

So, to find CSPs between the set of technically efficient DMUs, and the supporting hyperplanes corresponding optimal solution (U^*, V^*, u_0) with maximum u_0 as T_{convex} frontier, we run model (4.4) $(u_0^* \geq 0, u_0^* \neq 0)$.

So, the coordinate of the CSP input vector is located on the lower bound of $X_{\text{congestion}}$ frontier. Besides, the necessary condition that $(x_p, y_p)^t$ as a landing location of the DEA production function frontier is $u_0^* > 0$. It



(a) (b)

FIGURE 1. (a) The visual description of supporting hyperplane of model (4.12). (b) The visual description of supporting hyperplane of model (4.12) (k; technical sufficient in T_v).

should be noted that, if in some optimal solutions of the above model $v_i^* < 0$, then the production frontier after the point $(x_p, y_p)^t$ with increasing one or more inputs more than x_p decreases some outputs without improving (worsening) other inputs or outputs. Figures 1a and 1b make a visual description of different situations of optimal solutions of model (4.4) under T_{convex} technology. It means displaying different scenarios of the optimal solutions of model (4.4) and their supporting hyperplanes.

Remark 4.2. If production technology has congestion property, then there is at least one CSP. Alternatively, *vice versa*, there is at least one CSP so, production technology has congestion properties.

Remark 4.3. In solving the model (4.4) with the index of technical efficient units, it is necessary to point out that $u_0 = 0, v_i = 0, u_r = 0$ is a feasible solution to the model (4.4). Furthermore, if the model has an optimal solution, it is either zero, positive, or unbounded. In other words, it will never be negative.

Remark 4.4. In T_{convex} at least one observation is the CSP (provided that there is at least one congested observation in T_{convex} technology based on Cooper *et al.* [9] congestion definition). In other words, the set of CSPs is not empty if and only if the optimal solution of model (4.4) is not equal to zero or infinite (see Fig. 1b).

As described in the introduction, the economic definition of congestion (Def. 2.2) has some wrong in DEA. Still, using the set of CSPs, this definition can be expressed:

Remark 4.5. A DMU has congestion if an increase in one or more inputs over one or more CSPs input. It is associated with decreases in one or more outputs without improving (worsening) other inputs or outputs.

Remark 4.6. A DMU has weak congestion if an increase in some or (all) inputs over some (all) inputs of CSPs, is associated with decreases in some or (all) outputs without improving (worsening) other inputs or outputs.

Remark 4.7. A DMU has strong congestion if an increase in all inputs is more than all CSP's inputs, It is associated with decreases in all outputs without improving (worsening) other inputs or outputs. As described above, CSPs themselves have no congestion.

Remark 4.8. CSPs are a type of technically efficient DMU – that are weakly but not strongly efficient DMU – with maximum outputs; thus, they do not have congestion.

In the DEA literature, the concept of congestion is expressed for the unit under evaluation, while any obvious point does not necessarily refer to the production frontier. To address the problem, instead of doing calculations on observed points, the calculations are done on the production frontier's projection points. To calculate the congestion of the observed point (x_o, y_o) its projection point $\left(x_o, \varphi^* y_o + s^{+^*}\right)$ is used, that $\left(\varphi^*, s^{+^*}\right)$ is an optimal solution for a model (2.1b). To specify whether one DMU has congestion or not (without congestion measurement on inputs components), the model (4.5) based on the congestion definition of Cooper *et al.* [4,6] and the property of the convex PPS is presented:

$$\eta = \min\left(\sum_{j=1}^{n} t_{ro}^{+} + \sum_{j=1}^{n} t_{io}^{-}\right)$$
$$\text{s.t.}\begin{cases}\sum_{j=1}^{n}\lambda_j x_{ij} + t_{io}^{-} = x_{io} & i = 1,\dots,m\\ \sum_{j=1}^{n}\lambda_j y_{rj} - t_{ro}^{+} = (\varphi^* y_{ro} + s_{ro}^{+*}) & r = 1,\dots,s\\ \sum_{j=1}^{n}\lambda_j = 1 & \\ \lambda_j \geq 0, t_i^{-} \geq 0, t_r^{+} \geq 0. & \end{cases} \tag{4.5}$$

t_r^+ is an output slack of the model (4.5). Also, φ^* and s_{ro}^+ are an optimal solution and the optimum output slack of the model (2.1b). This model can identify the congested DMUs (from inefficient DMUs evaluated by model (2.1b)) and does not talk about their congestion size. So,

(a) $\eta^* = 0$, then inefficient (x_o, y_o) is not congested.
(b) $\eta^* > 0$, then inefficient (x_o, y_o) is congested.

Definition 4.9 (New definition of congestion)**.** For some optimal solutions of the model (2.1b) being $\left(\varphi^*, \lambda_j^*, s_i^{-*}, s_r^{+*}\right)$, the point $(x, y) \in T_v$ is said to have congestion if $(x, \varphi^* y + s_r^{+*}) \notin T_{\text{convex}}$.

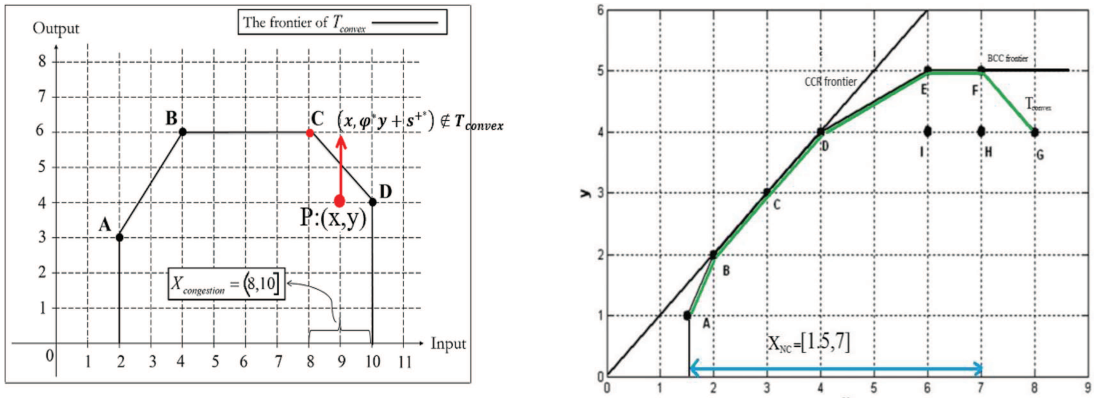
Notice the definition of $X_{\text{congestion}}(X_c)$, given this definition and its inequality; it is clear that the inclusion set is an open interval (a, ∞) as well for bounded convex production technology (T_{convex}) as Figures 2a and 2b; we can say that $X_c = (a, b]$. See, *e.g.*, in Figures 2a and 2b below, that the convex production technology (T_{convex}) is composed of five DMUs. As can be seen in the Figure 2, DMU$_C$ is CSP because an increase in input more than the input of DMU$_c$ causes a decreased output from the maximum level. Also, DMU$_C$ is technically efficient in T_v. As per Definition 4.9, DMU$_C$ is not congested since DMU$_C$ $= (x_C = 8, y_C = 6)$ and $x_C = 8 \notin X_c$. But DMU$_P$ is congested since DMU$_P$ $= (x_P = 9, y_P = 4) \in T_{\text{convex}}$ and also $\left(x_P = 8, \varphi^* y_P + s^{+^*} = 6\right) \notin T_{\text{convex}}$. Using Definition 4.9, DMU$_P$ is identified as congested. The congestion area defined in the previous section can be expressed ($X_c = X_{\text{congestion}}$):

$$X_c = \{\text{The area consists of congested inputs created by Convex combination of CSPs}\}. \tag{4.6}$$

Suppose that mth DMU of the observed DMUs are the CSPs, DMU$_{(m)}$ $(1 \leq m \leq n-1)$, and without completing the whole argument, assume we re-number the number of CSPs with the symbol (j), it means the member of the set (j) is the type of CSPs. So, DMU$_{(1)}, \dots,$ DMU$_{(m)}$ are the type of CSP. Now, we define the congestion area as follows ((J) is an index of CSPs DMUs, which means $(j) = \{(1), \dots, (m)\}$), so, the congestion area be built:

$$X_c = \left\{x_i \,\middle|\, \exists\lambda_{(j)} : x_i > \sum_{(j)} \lambda_{(j)} x_{(j)},\ \sum_{(j)} \lambda_{(j)} = 1,\ \lambda_{(j)} \geq 0,\ i = 1, \dots, m\right\}. \tag{4.7}$$

Consider the example and Figure 2b chosen from Noura *et al.* [20]. The set of efficient DMUs is $E = \{A, B, C, D, E, F\}$. DMU$_F$ has the highest input consumption among the efficient DMUs, *i.e.*, $x_F = x^* = 7$.



(a) (b)

FIGURE 2. (a) The visual description of X_c and Definition 4.9. (b) The numerical visual description of T_{convex}, BCC and CCR frontiers and X_{NC} from Noura *et al.* [20].

Using Noura *et al.* [20] method, they compare the inputs of the inefficient DMU_s with $x_F = x^* = 7$. Then, using Definition 2 of Noura *et al.* [20], DMU_H and DMU_I exhibit no congestion, *i.e.*, $x_I - x^* = 6 - 7 = -1 < 0$ and $x_H - x^* = 7 - 7 = 0$. While DMU_G exhibits congestion of one unit $x_G - x^* = 8 - 7 = 1 > 0$. Also, efficient DMU_F has no congestion because this is an efficient DMU also $x_F - x^* = 7 - 7 = 0$. According to our Definition 4.10, the point $(x, y) \in T_{\text{convex}}$ is said to have congestion if and only if $(x, y); x \notin X_{\text{NC}}$. See Figure 2b, that based on the Definition 4.10, we have $X_c = (7, 8]$. So, it is clear that $x_F = 8 \notin X_c, x_H = 8 \notin X_c$ so DMU_F and DMU_H actually exhibit no congestion and that $x_G = 8 \in X_c, (x_G = 8, \varphi^* y_P + s^{+*} = 4) \notin T_{\text{convex}}$ so DMU_G actually exhibits congestion.

As is clear, DMU_c from Figure 2a and DMU_F and DMU_H from Figure 2b, have the same conditions, *i.e.*, all have no congestion, and all are CSPs.

Definition 4.10 (Single CSP)**.** Consider to single input-output $\text{DMU}_p : (x_p, y_p)^t$. The input of DMU_p is CSP if, $\forall \text{DMU}_p : (x_p, y_p) \in T_v, \exists \varepsilon > 0 : x_p \notin X_c, y_o = y^{\max} \Rightarrow (x_p + \varepsilon) \in X_c, y_p < y^{\max}$.

Theorem 4.11. *Consider* $\text{DMU}o : (x_o, y_o) \in T_v$*. So, the following definitions are equivalent:*

(A) *Consider the optimal solution of the model* (2.1b) *as* $(\varphi^*, \lambda_j^*, s_i^{-*}, s_r^{+*})$*. The point* (x_o, y_o) *does have congestion if* $(x_o, \varphi^* y_o + s_r^{+*}) \notin T_{\text{convex}}$.

(B) *There is a* $\lambda_{(j)} = (\lambda_{(1)}, \ldots, \lambda_{(m)}) \geq 0$ *such that:* $\sum_{j=(1)}^{(m)} \lambda_{(j)} x_{(j)} \leq x_o, \sum_{j=(1)}^{(m)} \lambda_{(j)} x_{(j)} \neq x_o, \sum_{j=(1)}^{(m)} \lambda_{(j)} = 1$.

Proof. Suppose that, $(x_o, y_o) \in T_v$ but $(x_o, \varphi^* y_o + s_r^{+*}) \notin T_{\text{convex}}$, it is enough to prove that there is a $(\lambda_{(1)}, \ldots, \lambda_{(m)}) \geq 0$ such that $\sum_{j=(1)}^{(m)} \lambda_{(j)} x_{(j)} \leq x_o, \sum_{j=(1)}^{(m)} \lambda_{(j)} x_{(j)} \neq x_o, \sum_{j=(1)}^{(m)} \lambda_{(j)} = 1$. Cause $(x_o - s_i^-, \varphi^* y_o + s_r^{+*}) \in T_v$ is an efficient point and there is $s_i^c \geq 0$ [9] and $(j) \in \{(1), (2), \ldots, (m)\}$ such that:

$$(x_{(j)}, y_{(j)}) = (x_o - s_i^- + s_i^c, \varphi y_o + s_r^+), -s_i^- + s_i^c > 0. \tag{4.8}$$

Therefore: $x_o > x_o + (s_i^c - s_i^-) = x_{(j)}$. Consider $\lambda_{(j)} = 1, \lambda_{(j')} = 0; (j) \neq (j')$, that the statement is correct. On the contrary, suppose that there is a $(\lambda_{(1)}, \ldots, \lambda_{(m)}) \geq 0$ such that $\sum_{j=(1)}^{(m)} \lambda_{(j)} x_{(j)} \leq x_o, \sum_{j=(1)}^{(m)} \lambda_{(j)} x_{(j)} \neq x_o, \sum_{j=(1)}^{(m)} \lambda_{(j)} = 1$. We prove that $(x_o, \varphi y_o + s_r^+) \notin T_{\text{convex}}$. Therefore, since there is a $(j') \in \{(1), (2), \ldots, (m)\}$, such that, $x_{(j')o} \geq x_{(j')(j')}$. So, $(x_o, \varphi y_o + s_r^+)$ is Output-Oriented BCC-efficient (why?), and according to (4.4), $(x_{(j')}, y_{(j')})$ is the last point of T_{convex} along the axis (j'). Hence, $(x_o, \varphi^* y_o + s_r^{+*}) \notin T_{\text{convex}}$. $\square$

Theorem 4.12. *$X_{\text{congestion}}(X_c)$ is convex.*

Proof. Assume that $x^1, x^2 \in X_{\mathrm{C}}$, and $\mu \in (0,1)$. Under these assumptions, it must be proved that $x(\lambda) \in X_{\mathrm{C}}$, where $x(\lambda) = \lambda x^1 + (1-\lambda)x^2$. Since $x^i \in X_c; (i = 1, 2)$ then, there is $y^i > 0$ such that:

$$\begin{cases} (x^i, \varphi^* y^i + s^{i+*}) \notin T_{\text{convex}} \\ (x^i, y^i) \in T_{\text{convex}}. \end{cases} \tag{4.9}$$

Then we have $(x^1, \varphi^{*1} y^1 + s^{1+*}) \notin T_{\text{convex}}$.

So, for every $\lambda^i_{(j)}$ that $\sum_{j=(1)}^{(m)} \lambda^1_{(j)} = 1$ and $x^1 = \sum_{j=(1)}^{(m)} \lambda^1_{(j)} x_{(j)}$ are correct, therefore,

$$\varphi^1 y^{1+} + s^{1+} > \sum_{j=(1)}^{(m)} \lambda^1_{(j)} y_{(j)}. \tag{4.10}$$

Also, for every $\lambda^i_{(j)}$ that $\sum_{j=(1)}^{(m)} \lambda^2_{(j)} = 1$ and $x^2 = \sum_{j=(1)}^{(m)} \lambda^2_{(j)} x_{(j)}$ are correct, therefore,

$$\varphi^2 y^{2+} + s^{2+} > \sum_{j=(1)}^{(m)} \lambda^2_{(j)} y_{(j)}. \tag{4.11}$$

It is clear that

$$\begin{aligned} &\begin{cases} x(\lambda) = \sum_{j=(1)}^{(m)} \left(\mu\lambda^1_{(j)} + (1-\mu)\lambda^2_{(j)}\right) x_{(j)} \\ 1 = \sum_{j=(1)}^{(m)} \left(\mu\lambda^1_{(j)} + (1-\mu)\lambda^2_{(j)}\right) \end{cases} \\ &\mu\big(\varphi^1 y^1 + s^{+1}\big) + (1-\mu)\big(\varphi^2 y^2 + s^{+2}\big) = \mu\varphi^1 y^1 + (1-\mu)\varphi^2 y^2 + \mu s^{1+} + (1-\mu) s^{2+} \\ &\qquad > \sum_{j=(1)}^{(m)} \left(\mu\lambda^1_{(j)} + (1-\mu)\lambda^2_{(j)}\right) y_{(j)}. \end{aligned} \tag{4.12}$$

Consider that

$$\begin{cases} y(\lambda) = \mu\varphi^1\lambda^1_{(j)} + (1-\mu)\varphi^2\lambda^2_{(j)} \\ \varphi(\mu) = 1 \\ s(\mu) \ = \mu s^{1+} + (1-\mu) s^{2+}. \end{cases} \tag{4.13}$$

So, $(x(\mu), \varphi(\mu) y(\mu) + s^+(\mu)) \notin T_{\text{convex}}$. Hence, it is enough to prove that $(x(\mu), y(\mu)) \in T_{\text{convex}}$.

If the relation is true, then the theorem proved, assume, the relation is not correct then, according to the output disposability, there are $0 < \beta < 1$ such that $(x(\mu), \beta y(\mu)) \in T_{\text{convex}}$

Hence, it is enough:

$$\begin{cases} y(\lambda) = \frac{1}{\beta}\left(\mu\varphi^1\lambda^1_{(j)} + (1-\mu)\varphi^2\lambda^2_{(j)}\right) \\ \varphi(\mu) = \beta \\ s(\mu) = \mu s^{1+} + (1-\mu) s^{2+}. \end{cases} \tag{4.14}$$

So, $(x(\mu), \varphi(\mu) y(\mu) + s^+(\mu)) \notin T_{\text{convex}}$. □

This is obvious that $X_{\text{congestion}}$ is a polytope, but a polytope may be convex (or maybe non-convex), so this above theorem must be proved that $X_{\text{congestion}}(X_c)$ is convex.

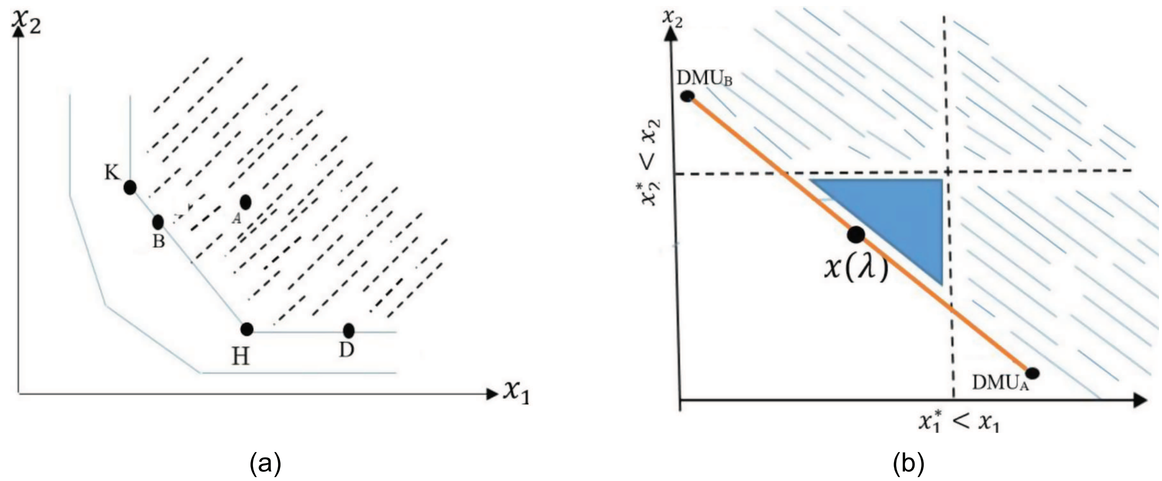


Figure 3. (a) A view of the congestion area (hatched area) and the projection of PPS on the inputs space. (b) Congestion area obtained from Noura *et al.* [20] for two outputs (hatched area).

Theorem 4.13. *If* DMU$_o$ *does has congestion, then* $x_o \in X_c$.

Proof. This is obvious according to the definition of X_c. □

A necessary condition for the DMU to be congested is that the input's vector should be inside X_c. Note that Theorem 4.13 does not constitute sufficient conditions for the DMUs to be congested. Indeed, it could be positioned by the frontier of the set X_c that represents congestion relative to some but not all inputs; that is the definition of weak congestion. The remark below provides the necessary and sufficient conditions for a DMU to be recognized as having congestion.

Figure 3a shows the frontier and congestion area in the input space of DMUs and PPS projection on the input planes. Note that this projection of PPS is not from Farrell's frontier. Indeed, some points within the projection area could be technically efficient. As shown in Figure 3a, the congestion area is a subset of the input space of the PPS. According to the above theorems, any point such as DMU$_A$ that is positioned within the congestion area has congestion.

Note that the boundaries of the congestion area do not necessarily indicate the presence of congestion. In Figure 3a, for example, DMU$_B$ does not have congestion because it is located on the lower border that includes technically efficient points as CSPs (it means DMU$_B$ is a technically efficient DMU and CSP). Also, DMU$_D$ does not have congestion, and it is only technically efficient DMU and CSP. So, CSPs are technically efficient points and have no congestion because each point $(x_p, y_p) \in T_v$ on the border is a convex combination of at least two CSPs, so any point on the border is technically efficient and has no congestion also, the points $(x_p, y_p) \in T_v$ located on the border between congestion areas (X_c) and non-congestion area have no congestion and any points on this border is CSP. Therefore, the points on the PPS can be divided into three classes:

(1) without congestion,
(2) with weak congestion,
(3) with strong congestion.

Remark 4.14. If a DMU does has weak congestion, some but not all, of the inputs, are within X_c.

Remark 4.15. If a DMU does has strong congestion, all inputs are within X_c.

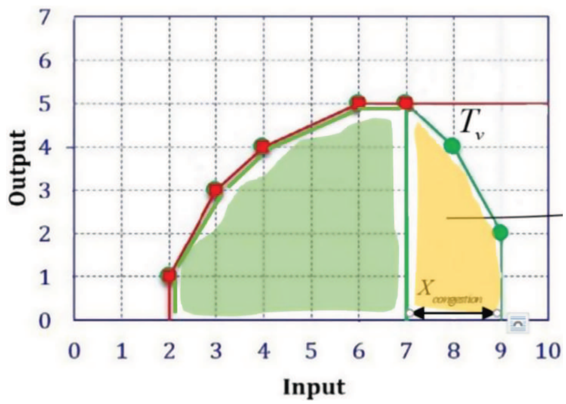


FIGURE 4. A view of the congestion zone (hatched zone) and the projection of PPS on the inputs space.

Remark 4.16. Each congested DMU is within the congestion area. Since the congestion area is convex and was made by inputs of congested DMUs. So, the convex composition of both points within the area and each DMUs is congested (Fig. 4).

That is important to note that the blue area in Figure 3b is a part of congestion area and some previous congestion measurement methods based on DMUs (not production frontier) same as Noura *et al.* [20] lack congestion in DMUs located this area. It means these methods cannot identify congestion from DMUs whose inputs are located in this blue area. If the DMU does has congestion, the distance of a DMU from the congestion area bottom frontier can represent its congestion. Model (19) computes the so-called non-radial distance of a DMU with congestion. This model derived from the additive DEA model can determine the congestion of DMU's components.

$$\begin{aligned} &\text{Max } 1s^c \\ &\text{s.t.} \begin{cases} x_o - s^c \in X_c \\ s^c \geq 0. \end{cases} \end{aligned} \tag{4.15}$$

In model (4.15), $1 = (1_1, 1_2, \ldots, 1_m)^t$ and $s^c = (s_1^c, s_2^c, \ldots, s_m^c)^t$ is the congestion measure for the DMU's input components. Here, the optimal solution of the objective function is zero. Under two scenarios, a DMU does not have congestion; First, when the above optimal solution is zero, and second when the model is infeasible. Assume that the optimal solution is s^{c*}:

(1) DMU_o does has strong congestion if $s^{c*} > 0$.
(2) DMU_o does has weak congestion if $s^{c*} \underset{\neq}{>} 0$.

If we had used the radial model to compute congestion, it would be impossible to obtain component congestion. Similarly, the SBM model (for efficiency evaluation) can construct a model for measuring relative and non-radial congestion of DMUs. The relative congestion model is presented below:

$$\begin{aligned} &\alpha = \text{Max } \frac{1}{m}\sum_{i=1}^{m}\frac{s_i^c}{x_{io}} \\ &\text{s.t.} \begin{cases} \sum_{(j)} \lambda_{i(j)} x_{i(j)} = x_{io} - s_i^c & (j) = (1), \ldots, (m) \\ \sum_{(j)} \lambda_{(j)} = 1 \\ s_i^c \geq 0 \\ \lambda_{(j)} \geq 0. \end{cases} \end{aligned} \tag{4.16}$$

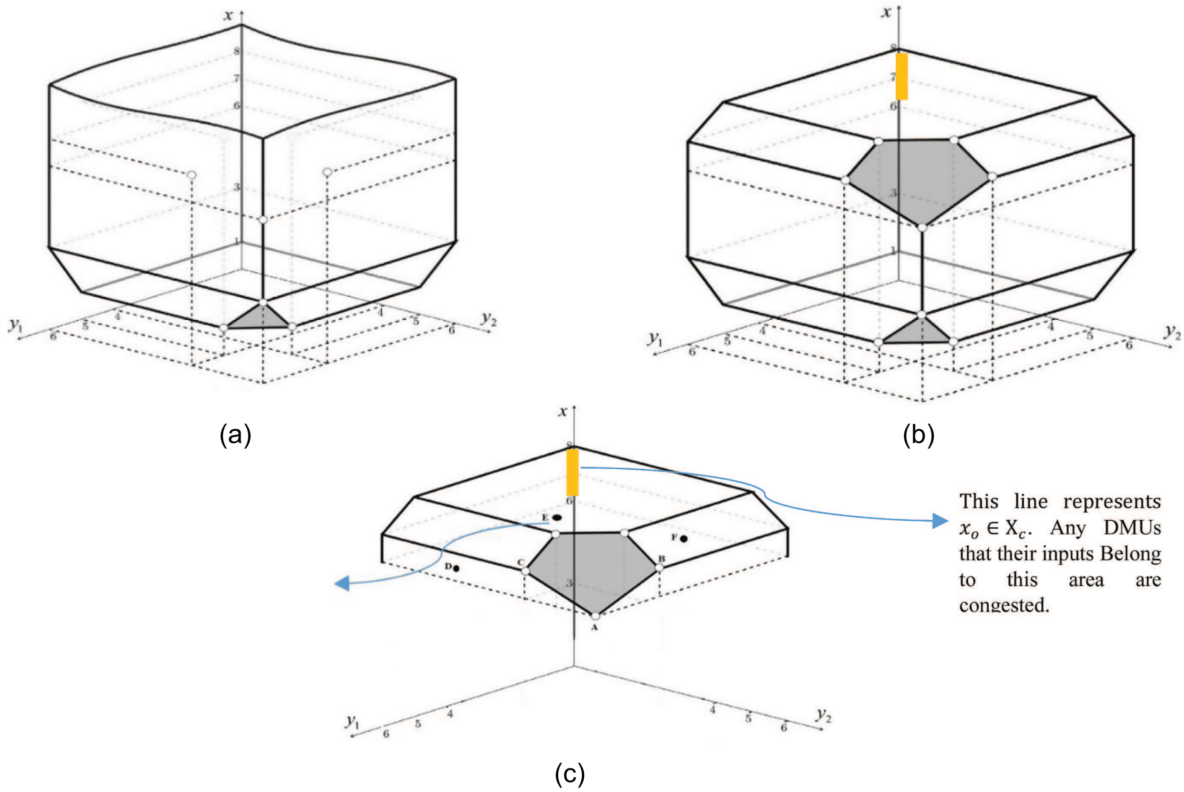


FIGURE 5. (a) The technology T_v of Mehdiloozad *et al.* [19]. (b) The technology T_{convex} of Mehdiloozad *et al.* [19]. (c) The area $(x, y_1, y_2) \subset$ Figure 5b which $x \in X_c$, DMU:(x, y_1, y_2) is a congested. Also, this area created by the DMUs that their inputs are member of X_c. All DMUs located in this area are congested.

So, congestion's degree of DMU$_o$ represented by α is defined: $\alpha = \begin{cases} \alpha \text{ model is feasible} \\ 0 \text{ model is infeasible} \end{cases}$. It is clear that $0 \leq \alpha \leq 1$. Meanwhile, if $\alpha = 0$, therefore DMU$_o$ does not has congestion, otherwise DMU$_o$ does has congestion. If $\alpha > 0$, and $s^c = (s_1^c, s_2^c, \ldots, s_m^c)^t \ngtr 0$ DMU$_o$ does has weak congestion.

According to Figure 5c, each DMU whose input is located in this area has congestion (weak or strong). The DMUs A, B, and C are CSP. The DMU$_D$ is also a technically efficient DMU and, DMU$_E$ and DMU$_F$ are strongly congested. The following MOLP model (4.17) is an equivalent model to determine the technical efficient points (CSPs) as $(x_p, y_p)^t$. So,

Definition 4.17. A BCC-o technical efficient DMU$_p$ is a CSP if and only if there is i $(i = 1, \ldots, m)$ so that $v_i^* < 0$, therefore:

$$
\begin{aligned}
& v_i^* = \min\ v_i \\
& \text{s.t.} \\
& \qquad \sum_r u_r y_{rp} - \sum_i v_i x_{ip} + u_0^* = 0, \qquad p, \\
& \qquad \sum_r u_r y_{rj} - \sum_i v_i x_{ij} + u_0^* \leq 0, \qquad j = 1, \ldots, n,\ i = 1, \ldots, m,
\end{aligned}
$$

$$u_r \geq 0,\ v_i \text{ is free}, \qquad r = 1, \ldots, s. \tag{4.17}$$

That u_0^* is an optimal solution of model (4.12). So, assumed $\hat{V}^* = (\hat{v}_1^*, \ldots \hat{v}_m^*)$ it results in,

(a) $(x_p, y_p)^t$ is a CSP, if $\hat{V}^* \ngeq 0$, $(\forall i, v_i^* \ngeq 0, i = 1, \ldots, m)$.

To solve model (4.17), we efficiently use the following mth models;

$$\begin{aligned} &\min\ \hat{v}_i^* \\ &\text{s.t. } v_i \geq \hat{v}_i^* \\ &\qquad v_i \text{ is free}, \qquad i = 1, \ldots, m. \end{aligned} \tag{4.18}$$

So, there is at least one CSP if we have at least one negative v_i^*. If all the components are negative, $v_i^* \ngeq 0$, then all the components of the unit under evaluation are CSP, which means DMU has congestion in all components. Also, if all components are positive, $v_i^* > 0$, then the unit under evaluation is not CSP. To explain the congestion area and CSP in the face of multiple inputs, noted the following explanations. The optimal solution of model (4.18) may be unbounded, so it is clear that $\hat{v}_i^* < 0$.

It is necessary to mention that; Congestion occurs in the input vectors of the DMUs. Therefore, when we say DMU_A has congestion, which means, the input of this DMU has congestion. So, when we say a DMU is CSP, it means the input of this DMU is CSP (consider inputs space). This is easy to understand when there is only one input-output, but when dealing with multiple inputs-outputs, it can be described as follows:

Definition 4.18 (CSP)**.** Consider to $\text{DMU}_p : (x_p, y_p)^t, x_p = (x_{1p}, \ldots, x_{mp})$ and $y_p = (y_{1p}, \ldots, y_{rp})$. The input of DMU_p is CSP if, $\forall \text{DMU}_p : (x_p, y_p) \in T_v$, $\exists \varepsilon > 0 : x_p \notin X_c, y_o = y^{\max} \Rightarrow (x_{ip} + \varepsilon) \in X_c, y_{ro} < y_r^{\max}$, then we called that DMU_p is full CSP. Also, if, $\forall \text{DMU}_p : (x_p, y_p) \in T_v$, $\exists \varepsilon > 0 : x_{ip} \notin X_c, y_{rp} = y^{\max} \Rightarrow \forall i \neq m : (x_{ip} + \varepsilon) \in X_c, y_{rp} < y_r^{\max}$, then we called DMU_p weak CSP.

Consider 5 DMUs with 5 inputs components, $\text{DMU}^i = \left(x_1^i, x_2^i, x_3^i, x_4^i, x_5^i\right), (i = 1, 2, 3, 4, 5)$ and $(a^1, b^1, c^1, d^1, e^1)$ is congestion measures of inputs components of DMU^1, $\left(a^2\right)$ congestion measure of first input of DMU^2, (a^3, b^3) congestion measure of first and second input DMU^3, $\left(c^4, d^4\right)$ third and fourth inputs congestion measure of DMU^4 and $\left(e^5\right)$ congestion measure of five inputs of DMU^5, respectively. Other input components have no congestion. DMU^1 has strong congestion, which means all 5 inputs components have congestion, DMU^2 has weak congestion only in x_1^2, DMU^3 has weak congestion in x_1^3, x_2^3, DMU^4 has weak congestion in x_3^4, x_4^4 and DMU^5 has weak congestion in x_5^5.

$$\begin{aligned} \text{DMU}^1 &= \left(x_1^1 - a^1, x_2^1 - b^1, x_3^1 - c^1, x_4^1 - d^1, x_5^1 - e^1\right) = (x_1', x_2', x_3', x_4', x_5') \\ \text{DMU}^2 &= \left(x_1^2 - a^2, x_2^2, x_3^2, x_4^2, x_5^2\right) = \left(x_1', x_2^2, x_3^2, x_4^2, x_5^2\right) \\ \text{DMU}^3 &= \left(x_1^3 - a^3, x_2^3 - b^3, x_3^3, x_4^3, x_5^3\right) = \left(x_1', x_2', x_3^3, x_4^3, x_5^3\right) \\ \text{DMU}^4 &= \left(x_1^4, x_2^4, x_3^4 - c^4, x_4^4 - d^4, x_5^4\right) = \left(x_1^4, x_2^4, x_3', x_4', x_5^4\right) \\ \text{DMU}^5 &= \left(x_1^5, x_2^5, x_3^5, x_4^5, x_5^5 - e^5\right) = \left(x_1^5, x_2^5, x_3^5, x_4^5, x_5'\right) \end{aligned}$$

Virtual $\text{DMU}_{\text{CSP}} = (x_1', x_2', x_3', x_4', x_5')$ obtained by subtracting congestion values from their corresponding input components is the CSP unit. More precisely, the input components of this virtual unit are the congestion boundary and with the turbulence of each of these components plus epsilon, the said component will be congested. Then, in multiple inputs, components of input (some or all) will play the role of CSP. To describe precisely, in the above example, for the first component x_1', the second component x_2', the third x_3', the fourth x_4' and the five components x_5' are CSPs. So, in DMU^1 all inputs components $x_1', x_2', x_3', x_4', x_5'$ are CSP, x_1' in the DMU^2, x_1' and x_2' in the DMU^3, x_3' and x_4' in the DMU^4 and x_5' in the DMU^5 are CSP.

If in a collection of congested DMUs, one or more DMUs have strong congestion, then all their input components are CSP, otherwise, only the congested input component of the DMUs are CSP. The X_c is also created by the same CSP components. Of course, there may not be strong congestion, in which case the congestion zone will be made up of a combination of compact components that make up CSP and not all inputs components. Congested DMUs are not CSPs because CSPs are technically efficient points. Also, CSPs are a border between congestion and non-congestion. In single input (dimension one) all CSPs are one point and this border is one point, in two input all points located on the convex composition of two CSPs are CSP and this border is a line, and for 3 input this border is a hyperplane and for more than 3 inputs this is a convex multi-dimensional.

5. Second insight of congestion measurement; non-congestion area

This section describes new insights into congestion based on non-congestion DMU's inputs space and proves some fundamental theorems and a second new mathematical definition of congestion. Before discussing our proposed method, we must present the definitions of several new congestion-related concepts and propose a new mathematical definition for the concept of congestion itself.

Definition 5.1 (New definition of congestion)**.** For some optimal solutions of the model (2.1b) being $\left(\varphi^*, \lambda_j^*, s_i^{-*}, s_r^{+*}\right)$, the point $(x, y) \in T_v$ is said to do has congestion if $(x, \varphi^* y + s_r^{+*}) \notin T_{\text{convex}}$.

Building the non-congestion area is enough to find DMUs efficiently in the BCC-O and get the convex composition of these efficient DMUs. The DMUs in this compound are convex within the non-congestion area and lack congestion. Other DMUs are congested.

Theorem 5.2. *X_{NC} is convex.*

To prove that the X_{NC} is convex, we prove that the convex combination of both points of this area is within X_{NC}. Assume that $(\bar{x}, \bar{y})$ and $(\hat{x}, \hat{y})$ does not have congestion, and $\alpha \in (0, 1)$, and we have $(x(\alpha), y(\alpha)) = \alpha(\bar{x}, \bar{y}) + (1 - \alpha)(\hat{x}, \hat{y})$. So $(x(\alpha), y(\alpha))$ does not have congestion.

Proof. Assume that $(x(\alpha), y(\alpha))$ has congestion And $\varphi_\alpha^* \geq 1$ is an optimum solution of BCC-o. So $(x(\alpha), \varphi_\alpha^* y(\alpha)) \notin T_{\text{convex}}$. While we know that, $(x(\alpha), y(\alpha)) = \alpha(\bar{x}, \bar{y}) + (1 - \alpha)(\hat{x}, \hat{y})$.

Moreover,

$$(\bar{x}, \bar{y}) \in T_{\text{convex}} \Rightarrow \begin{cases} \bar{x} = \sum_j \bar{\lambda}_j x_j \\ \bar{y} \leq \sum_j \bar{\lambda}_j y_j \\ \sum_j \bar{\lambda}_j = 1 \end{cases}, \text{ and } (\hat{x}, \hat{y}) \in T_{\text{convex}} \Rightarrow \begin{cases} \hat{x} = \sum_j \hat{\lambda}_j x_j \\ \hat{y} \leq \sum_j \hat{\lambda}_j y_j \\ \sum_j \hat{\lambda}_j = 1. \end{cases} \tag{5.1}$$

So, $x(\alpha) = \sum_j (\alpha\bar{\lambda}_j + (1 - \alpha)\bar{\lambda}_j)(\hat{x})$ and $y(\alpha) = \sum_j (\alpha\bar{\lambda}_j + (1 - \alpha)\bar{\lambda}_j)(\hat{y})$. With assumption $\mu_j = (\alpha\bar{\lambda}_j + (1 - \alpha)\bar{\lambda}_j)$, first, $\mu_j \geq 0$, $(j = 1, \ldots, n)$ so $\sum_j \mu_j = \alpha \sum_j \bar{\lambda}_j + (1 - \alpha) \sum_j \bar{\lambda}_j = 1$, then $(x(\alpha), y(\alpha)) \in T_{\text{convex}}$. Besides, $\varphi_\alpha^* y(\alpha) \geq y(\alpha)$. So

$$\begin{cases} x(\alpha) = \sum_j \mu_j x_j \\ \varphi_\alpha^* y(\alpha) \geq \sum_j \mu_j y_j \\ \sum_j \mu_j = 1 \end{cases} \Rightarrow (x(\alpha), \varphi_\alpha^* y(\alpha)) \in T_{\text{convex}}. \tag{5.2}$$

This is inconsistent with the assumption. $\square$

According to this theorem, we proved that the convex combination of both non-congestion area points is in the non-congestion area: the convex combination of two points that are not congested, not congested.

We know that if the DMU is ($\varphi^*_{\text{BCC-o}}$) efficient, it does not have congestion. So, the convex combination of the ($\varphi^*_{\text{BCC-o}}$) efficient points do not have congestion. Then we consider the convex combination of ($\varphi^*_{\text{BCC-o}}$) effective points and define that any point outside this area has congestion.

Note that to identify congested DMUs, we divide the PPS into two areas of congestion and non-congestion areas. We also classified the DMUs into two groups congested and non-congested (NC). Then, by computing the non-congestion area and a non-radial model, we identify the congested DMUs and calculate congestion of the congested DMUs simply and accurately. Given that compaction is a feature of the production frontier, and we also use post-PPS properties to identify congested DMUs, we can say that our method is also very accurate and fast. We test our model with the best methods available to identify congestion DMUs and show that our method is the most accurate to prove our claim.

We can express the congestion area defined in the previous section:

$$X_{\text{NCA}} = \{\text{The non-congested inputs area made by inputs of DMUs that, } \varphi^*_{\text{BCC-o}} = 1\}. \tag{5.3}$$

Definition 5.3 (NCA)**.** An area in the inputs space where; inputs of DMUs without congestion are located inside it and denoted X_{NC}.

Suppose $\varphi^*_{\text{BCC-o}}$ is an optimal solution of the output-oriented BCC-o model (2.1a), then consider set J_E as follows,

$$J_E = \{j | \varphi^*_{\text{BCC-o}} = 1\}. \tag{5.4}$$

So, the non-congestion area creates:

$$X_{\text{NC}} = \left\{ x_o \middle| \exists\lambda : x_o = \sum_{j\in J_E} \lambda_j x_j,\ \sum_{j\in J_E} \lambda_j = 1, \lambda_j \geq 0, j \in J_E \right\}. \tag{5.5}$$

The minimum distance of the DMU$_o$ from the non-congestion area's frontier represents the amount of its congestion measure. The model (5.6) computes the non-radial distance of congested DMU$_o$ from the non-congestion area. This model, derived from the additive DEA model, can determine the amount of congestion in the presence of both negative and non-negative data.

$$\begin{aligned}
&\varphi = \min \sum_{i=1}^{m} s_i^c \\
&\text{s.t.} \\
&\begin{cases}
\sum_{j\in J_E} \lambda_j x_j = x_o - s_i^c, & i = 1,\dots,m, \\
\sum_{j\in J_1}^{n} \lambda_j = 1, & \\
\lambda_j \geq 0, & j \in J_E, \\
s_i^c \geq 0, & i = 1,\dots,m.
\end{cases}
\end{aligned} \tag{5.6}$$

That, $s_i^{c*} = (s_1^{c*}, s_2^{c*}, \dots, s_m^{c*})^t$ is the amount of congestion of the DMU$_o$. Assume that the optimal solution is s_i^{c*}:

(1) DMU$_o$ does has congestion if and only if $s_i^{c*} \geq 0, s_i^{c*} \neq 0$. More precisely, DMU$_o$ has strong congestion if and only if $s_i^{c*} > 0$ and does has weak congestion if and only if $s_i^{c*} \geq_{\neq} 0$

(2) DMU$_o$ does not have congestion if and only if $s_i^{c*} = 0$.

Suppose that there is n DMU$_j (j = 1,\dots,n)$ to be analyzed. Each DMU$_j$ has m inputs and s outputs, that are denoted by $x_{ij} (i = 1,\dots,m)$ and $y_{rj} (r = 1,\dots,s)$, respectively, in the presence of both negative and non-negative data at least one is non-zero, also DMU$_o$ is evaluated and denoted by DMU$_o$.

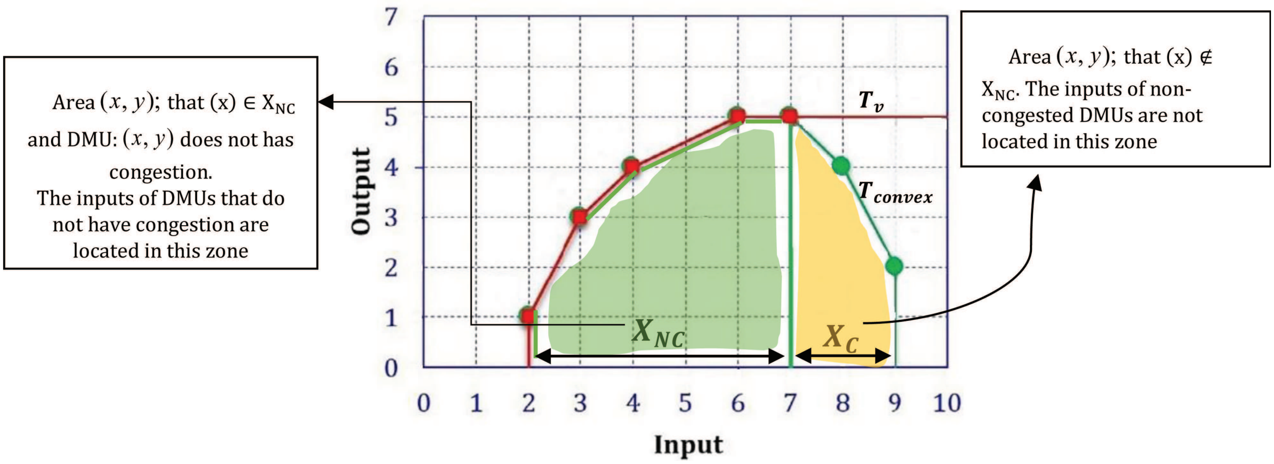


FIGURE 6. The visual description of X_{NC} and X_{C} in Technologies T_{convex} (green line) and T_v (red line).

Definition 5.4 (New definition of congestion)**.** The observation $\mathrm{DMU}_j; (x_j, y_j) \in T_{\text{convex}}$ is said to do has congestion if and only if $(x_j) \notin X_{\mathrm{NC}}$.

Note that, if $(x_j) \notin X_{\mathrm{NC}}$, so (x_j) belongs to the area where congested DMUs' inputs are located in this area (the area made up of the convex combination of non-congested DMUs' inputs), that we denoted by X_c. In other words, if $(x_j) \in X_{\mathrm{NC}}$ (Fig. 6).

Remark 5.5. If DMU_o does not has congestion, then $x_o \in X_{\mathrm{NC}}$. In other words, if DMU_o does has congestion, thus $(x_o) \notin X_{\mathrm{NC}}$.

Remark 5.6. For some optimal solutions of the model (2.1b) being $\left(\varphi^*, \lambda_j^*, s_i^{-*}, s_r^{+*}\right)$, for any point $(x_o, y_o) \in T_v$ if $(x_o, \varphi^* y_o + s_r^{+*}) \in T_{\text{convex}}$ then $(x_o) \in X_{\mathrm{NC}}$. Also, for some optimal solutions of the model (2.1b) being $\left(\varphi^*, \lambda_j^*, s_i^{-*}, s_r^{+*}\right)$, for any point $(x_o, y_o) \in T_v$, if $(x_o, \varphi^* y_o + s_r^{+*}) \notin T_{\text{convex}}$ then $(x_o) \notin X_{\mathrm{NC}}$.

Note that X_{C} and X_{NC} are areas in inputs space. But to better represent, also showing the DMUs which are not on the border, we symbolically display it as PPS, to maintain its integrity. The yellow color in Figure 7a is represent $(x_o) \notin X_{\mathrm{NC}}$ and green color in Figure 7b $(x_o) \in X_{\mathrm{NC}}$.

Note that the set of efficient DMUs is $E = \{A, B, C\}$. In Definition 5.4, the point $(x, y) \in T_{\text{convex}}$ is said to do has congestion if and only if $(x, y); x \notin X_{\mathrm{NC}}$. See Figure 8a, that we have $X_{\mathrm{NC}} = [2, 8]$, so $x_A = 2 \in X_{\mathrm{NC}}, x_B = 4 \in X_{\mathrm{NC}}$ then DMU_A and DMU_B exhibit no congestion. Also, based on the result of Definition 4.10. $(X_c) = (8, 10]$. So, $x_C = 8 \in X_{\mathrm{NC}}$. So DMU_C exhibits no congestion. But, $x_D = 10 \notin X_{\mathrm{NC}}$, so DMU_D exhibits congestion, and its size is $s_D^{c*} = 2$. Now we compare the result of our method with Noura *et al.* [20]. Noura *et al.* [20] compare the inputs of the inefficient DMU_s with $x_C = x^* = 8$. Then, by Definition 2 of Noura *et al.* [20] (Thm. 5.2 above), DMU_A and DMU_B exhibit no congestion, *i.e.*, $x_A - x^* = 2 - 8 = -6 < 0$ and $x_B - x^* = 4 - 8 = -4 < 0$. While DMU_D exhibits congestion of one unit $x_D - x^* = 10 - 8 = 3 > 0$. Also, efficient DMU_C does not have congestion because this is a technical efficient DMU also $x_C - x^* = 8 - 8 = 0$. By Remark 4.16, DMU_C is a CSP.

Consider the example and Figure 8b chosen from Noura *et al.* [20]. The set of efficient DMUs is $E = \{A, B, C, D, E, F\}$. As seen, DMU_F has the highest input consumption among the efficient DMUs, *i.e.*, $x_F = x^* = 7$. Using Noura *et al.* [20] method, they compare the inputs of the inefficient DMU_s with $x_F = x^* = 7$. Then, using Definition 2 of Noura *et al.* [20], DMU_H and DMU_I exhibit no congestion, *i.e.*, $x_I - x^* = 6 - 7 = -1 < 0$ and $x_H - x^* = 7 - 7 = 0$. While DMU_G exhibits congestion of one unit $x_G - x^* = 8 - 7 = 1 > 0$. Also, efficient

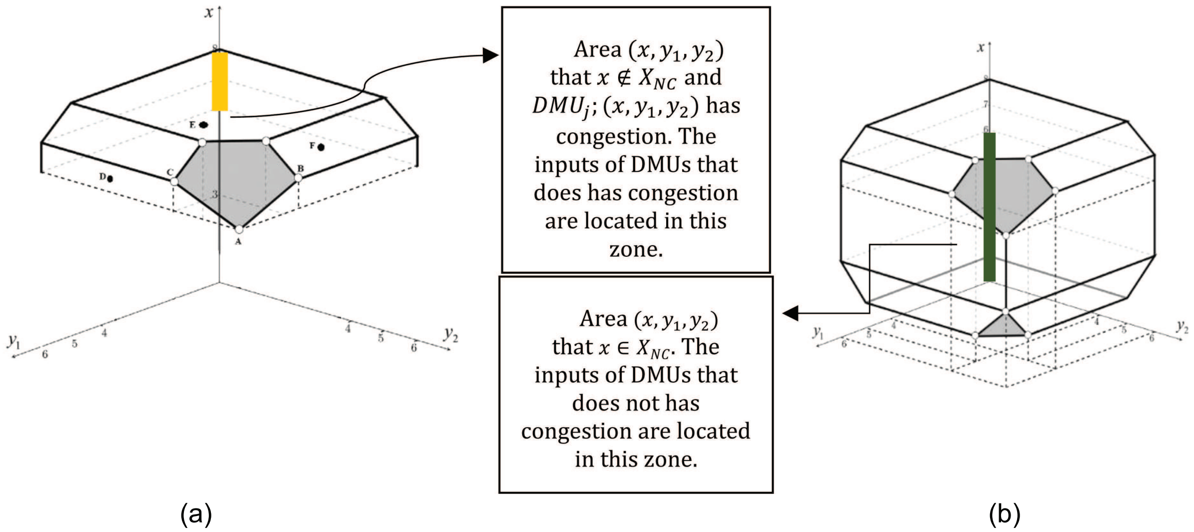


FIGURE 7. (a) The area (x, y_1, y_2) that $x \notin X_{\mathrm{NC}}$ does has congestion Mehdiloozad *et al.* [19]. (b) T_{convex} Technology from Mehdiloozad *et al.* [19].

DMU_F does not have congestion because this is an efficient DMU also $x_F - x^* = 7 - 7 = 0$. According to our Definition 5.4, the point $(x, y) \in T_{\text{convex}}$ is said to do has congestion if and only if $(x, y); x \notin X_{\mathrm{NC}}$. See Figure 3b, that we have $X_{\mathrm{NC}} = [1.5, 7]$, and based on the Definition 5.4. $(X_c) = (7, 8]$. So, $x_F = 8 \in X_{\mathrm{NC}}, x_H = 8 \in X_{\mathrm{NC}}$, so DMU_F and DMU_H exhibit no congestion, and that $x_G = 8 \notin X_{\mathrm{NC}}, \left(x_G = 8, \varphi^* y_P + s^{+^*} = 4\right) \notin T_{\text{convex}}$ so DMU_G exhibits congestion.

As is clear, DMU_c from Figure 8a and DMU_F and DMU_H (after a project to technical efficient frontier) from Figure 8b have the same conditions, *i.e.*, all do not have congestion, and all are CSPs.

Note that there are n $\mathrm{DMU}_j (j = 1, \ldots, n)$ that, each DMU_j have m inputs and s outputs that are denoted by $x_{ij} (i = 1, \ldots, m)$ and $y_{rj} (r = 1, \ldots, s)$, respectively, in the presence of both negative and non-negative data,

Remark 5.7. If a DMU does has weak congestion, some not all inputs are not in the X_{NC}.

Remark 5.8. If a DMU does has strong congestion, all inputs are not in the X_{NC}.

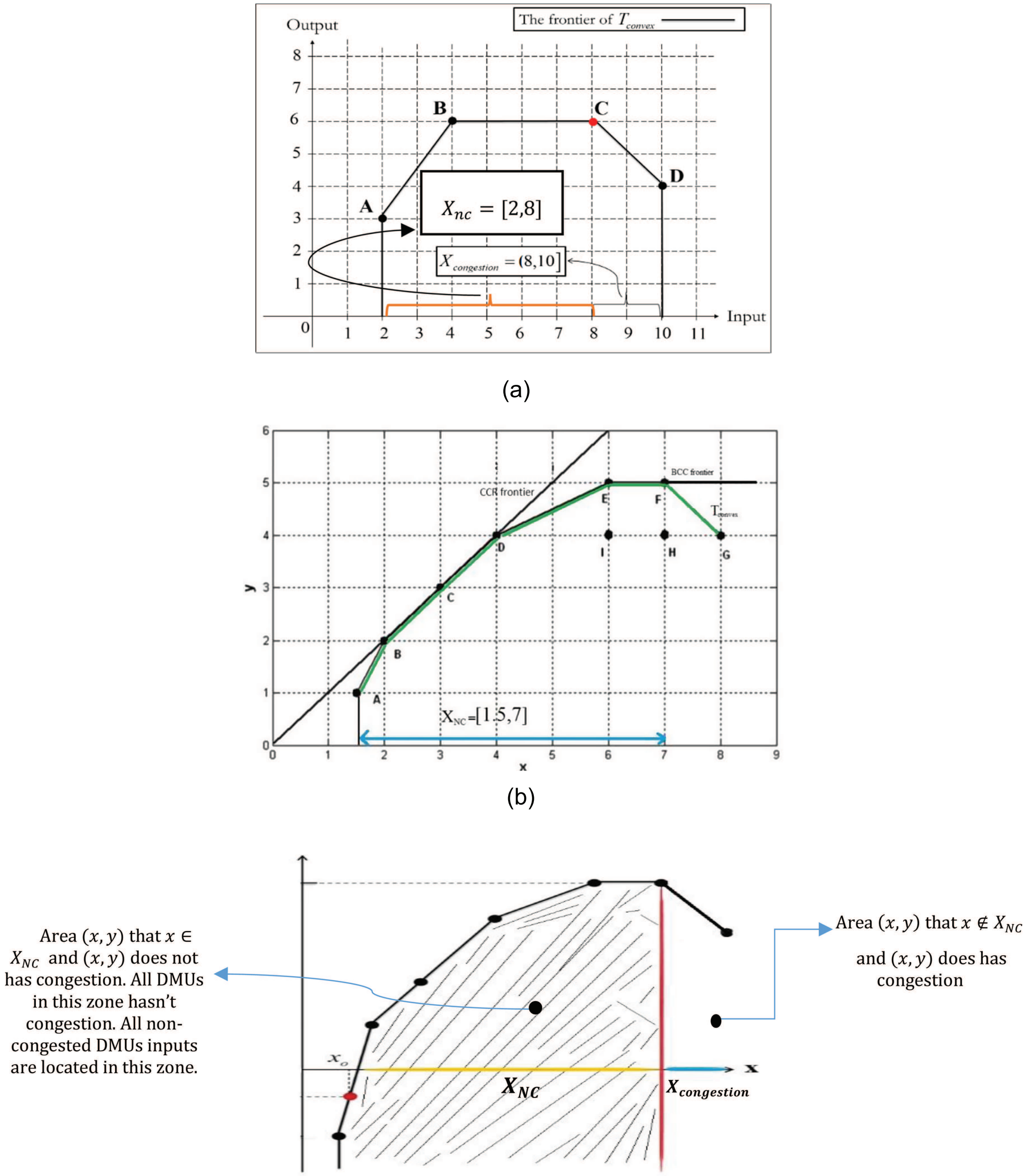
Remark 5.9. According to the definition of the X_C and X_{NC}, our two approaches are exactly equivalent to each other.

6. NUMERICAL EXAMPLE (THE TEXTILE INDUSTRIES AND AUTO INDUSTRIES IN CHINA)

Table 2 shows the data about China's textile and automobile industries from 1981 through 1997, which Cooper *et al.* (2001) compiled. In this example, variables are defined: Y is the output measured in units of one million renminbi at 1991 prices, K is the capital price measured in units of one million renminbi at 1991 prices, and L is the work done in units of 1000 people.

The results are summarized in Tables 3 and 4, which are obtained by solving the model of Cooper *et al.* [9], Noura *et al.* [20], Mehdiloozad *et al.* [19], and our proposed method using the data in Table 2. By solving model (2.1a) BCC-o, for the Textile industry data, we arrive at the following efficient DMUs set $E = \{\mathrm{DMU}_j | \varphi^*_{\text{BCC-o}} = 1\}$. So,

$$E = \{\mathrm{DMU}_1, \mathrm{DMU}_3, \mathrm{DMU}_4, \mathrm{DMU}_5, \mathrm{DMU}_{11}, \mathrm{DMU}_{14}, \mathrm{DMU}_{16}, \mathrm{DMU}_{17}\}. \tag{6.1}$$



(c)

FIGURE 8. (a) The numerical visual description of X_{NC} and X_{NC} Technologies. (b) The numerical visual description of T_{convex}, BCC and CCR frontiers and X_{NC} from Noura *et al.* [20]. (c) Technologies T_{convex} and T_v with PPS decomposition in the presence of both negative and non-negative data.

TABLE 2. The data from the textile industries and auto industries in China (1981–1997).

DMU year	Textile L	Textile K	Textile Y	Auto L	Auto K	Auto Y
DMU_1 1981	389.00	19.86	856.02	90.43	3.81	70.47
DMU_2 1982	412.30	21.16	866.85	94.28	4.13	82.07
DMU_3 1983	423.50	17.08	956.04	104.66	5.56	117.78
DMU_4 1984	417.30	18.10	1082.94	121.24	9.50	168.29
DMU_5 1985	570.00	12.61	1273.20	140.72	21.44	273.99
DMU_6 1986	600.50	13.45	1230.72	129.08	20.95	212.89
DMU_7 1987	641.10	15.91	1410.66	134.83	30.99	273.19
DMU_8 1988	715.30	23.72	1728.16	150.58	41.29	407.29
DMU_9 1989	736.00	25.97	2109.57	157.07	37.88	481.02
DMU_{10} 1990	745.00	18.24	2291.08	156.53	41.30	492.49
DMU_{11} 1991	756.00	14.40	2533.27	170.39	58.93	704.48
DMU_{12} 1992	743.00	17.50	2899.16	184.87	102.75	1191.05
DMU_{13} 1993	684.00	25.08	3520.74	193.26	164.27	1792.00
DMU_{14} 1994	691.00	25.45	4949.93	196.88	198.77	2183.10
DMU_{15} 1995	673.00	29.35	4604.00	195.25	231.34	2530.87
DMU_{16} 1996	634.00	23.05	4722.29	195.06	194.90	2399.09
DMU_{17} 1997	596.00	25.02	4760.28	197.81	203.96	2668.69

In the next step, by running model (4.4) for the set E, we try to find technically efficient DMUs (CSPs) with maximum u_0^*. So, the set (J) will be identified.

$$\begin{aligned} & u_i^* = \max\ u_i \\ & \text{s.t.} \\ & \quad \sum_r u_r y_{rp} - \sum_i v_i x_{ip} + u_0 = 0, && p = 1,3,4,5,11,14,16,17, \\ & \quad \sum_r u_r y_{rj} - \sum_i v_i x_{ij} + u_0 \le 0, && j = 1,\ldots,17,\ i = 1,2, \\ & \quad u_r \ge 0,\ v_i \text{ is free},\ u_0 \text{ is free}, && r = 1. \end{aligned} \tag{6.2}$$

So, $(J) = \{j|11,14\}$ that it means, $u_{11}^* = u_{14}^* = 1675.3$ are maximum u_0^* between the members of set E. It means, the inputs of DMU_{11} and DMU_{14} are CSP (because both components of DMUs have congestion and also these DMUs have strong congestion so we called DMUs are CSP) and we can create X_c as follows:

$$X_c = \left\{ x_{io} | x_{io} \ge \sum_{(J)} \lambda_{(J)} x_{i(J)}, \lambda_{11} + \lambda_{14} = 1, \lambda_{11}, \lambda_{14} \ge 0, (J) = 11, 14, i = 1,2 \right\}. \tag{6.3}$$

Therefore, to identify congested inputs components and according to the definition of X_c and solving the model (4.16) for DMUs outside of set E (the members of set E are efficient). So, the inputs of $\text{DMU}_2, \text{DMU}_6, \text{DMU}_7, \text{DMU}_8, \text{DMU}_9, \text{DMU}_{10}, \text{DMU}_{12}, \text{DMU}_{13}, \text{DMU}_{15}$ are congested. Also, we use model (4.16) to calculate the congestion measure of each DMUs (The results of the calculations, as well as the congestion values, are presented in Tab. 4).

We analyzed the sample data with three approaches Mehdiloozad *et al.* [19], Noura *et al.* [20], and Cooper *et al.* [9]. Also, to check in accuracy and performance of our method and for comparison, the results of our method with the results of solving other methods are presented in Tables 4 and 5. It should be noted that the values under the columns of Noura *et al.* [20] and Cooper *et al.* [9] show the congestion value of each component,

Table 3. Congestion results for the Textile industries (T) by using Cooper *et al.*'s approach [9], Noura *et al.* [20], Mehdiloozad *et al.* [19].

DMU	Cooper *et al.* [9]		Noura *et al.* [20]		Mehdiloozad *et al.* [19]		
	s_1^{c*}	s_2^{c*}	s_1^{c*}	s_2^{c*}	$\alpha_1^{\max}$	$\alpha_2^{\max}$	$\beta_1^{\max}$
DMU_{01}	0.00	0.00	0.00	0.00	0.00	0.000	0.000
DMU_{02}	0.00	0.72	0.00	0.00	22.67	1.284	1.000
DMU_{03}	0.00	0.00	0.00	0.00	0.00	0.000	0.000
DMU_{04}	0.00	0.00	0.00	0.00	0.00	0.000	0.000
DMU_{05}	0.00	0.00	0.00	0.00	0.00	0.000	0.000
DMU_{06}	0.00	0.00	0.00	0.00	30.50	0.84	0.84
DMU_{07}	0.00	0.00	0.00	0.00	1.00	1.000	1.000
DMU_{08}	65.39	0.00	0.00	0.00	259.52	4.741	1.000
DMU_{09}	45.00	0.00	0.00	0.00	257.51	6.472	1.000
DMU_{10}	43.16	0.00	0.00	0.00	154.48	1.000	1.000
DMU_{11}	0.00	0.00	0.00	0.00	0.00	0.000	0.000
DMU_{12}	30.72	0.00	0.00	0.00	20.26	1.000	1.000
DMU_{13}	0.00	0.00	0.00	0.00	121.49	3.663	1.000
DMU_{14}	0.00	0.00	0.00	0.00	0.00	0.000	0.000
DMU_{15}	0.00	3.98	0.00	3.90	1.00	4.252	1.000
DMU_{16}	0.00	0.00	0.00	0.00	0.00	0.000	0.000
DMU_{17}	0.00	0.00	0.00	0.00	0.00	0.000	0.000

but since the Mehdiloozad *et al.* [19] do not specify the congestion value, these numbers are not related to the congestion value. The columns represent the maximal element (alpha-maximal, beta-maximal) obtained from solving model (4.5) in Mehdiloozad *et al.* [19] (refer to the paragraph below model (4.5) and result from 1.3). Only the positivity of the component is important, not its value, that is, if all three components are positive, the congestion is strong. Also, if the third component (output) is positive but the first or second component (inputs) is zero, the congestion is weak. So, in our example, DMUs 2, 6, 7, 8, 9, 10, 12, 13, and 15 have strong congestion. The results of our two methods are shown in Table 4. Also, Table 5 is the result of our methods and comparison with four methods Cooper *et al.* [9] and Noura *et al.* [20], Mehdiloozad *et al.* [19], and Tone and Sahoo [23].

Using the data of Tables 4 and 5 and comparing the congestion values calculated by our proposed algorithm with the information obtained from the Noura *et al.* [20], Cooper *et al.* [9], and Mehdiloozad *et al.* [19] models. According to Table 3 and the results of the Noura *et al.* [20] model, DMU9 and DMU15 have weak congestion. According to the results of the Cooper *et al.* [9] model, DMU02 on x_1, and DMU08, DMU09, DMU10, DMU12, and DMU15 on x_2 have weak congestion. Also, according to the results of Mehdiloozad *et al.* [19], DMU02, DMU06, DMU08, DMU09, DMU10, DMU12, DMU13, and DMU15 have strong congestion. According to Table 5, the result of our two proposed methods is equal to Mehdiloozad *et al.* [19] exactly (in the weak and strong CSOD identification). According to Table 4, our method's results (value of congested DMUs) are illustrated. To find the CSP, congestion area, congested DMUs, and congestion in each input component, we run our purposed algorithm. In addition, according to the results specified in Table 4, the results of our two proposed methods are completely consistent with each other, so it can be said that the methods are equivalent.

Also, to identify the congestion state of DMUs by X_{NC}, consider to following steps related to relation (5.4)–(5.6). The results are presented in Table 5,

$$J_E = \{j|\varphi^*_{\text{BCC-o}} = 1\} \tag{6.4}$$

$$J_E = \{\text{DMU01}, \text{DMU03}, \text{DMU04}, \text{DMU05}, \text{DMU11}, \text{DMU14}, \text{DMU16}, \text{DMU17}\} \tag{6.5}$$

TABLE 4. Congestion result of our two proposed approaches.

DMU year	Congestion area		Non-Congestion area		Congestion states
	s_1^{c*}	s_2^{c*}	s_1^{c*}	s_2^{c*}	
DMU_{01} 1981	0.00	0.00	0.00	0.00	
DMU_{02} 1982	38.67	3.177	38.67	3.177	Strong congestion
DMU_{03} 1983	0.00	0.00	0.00	0.00	
DMU_{04} 1984	0.00	0.00	0.00	0.00	
DMU_{05} 1985	0.00	0.00	0.00	0.00	
DMU_{06} 1986	30.50	0.84	30.50	0.84	Strong congestion
DMU_{07} 1987	0.00	0.00	0.00	0.00	
DMU_{08} 1988	291.80	6.64	291.80	6.64	Strong congestion
DMU_{09} 1989	312.50	8.89	312.50	8.89	Strong congestion
DMU_{10} 1990	175.16	5.63	175.16	5.63	Strong congestion
DMU_{11} 1991	0.00	0.00	0.00	0.00	
DMU_{12} 1992	173.00	4.89	173.00	4.89	Strong congestion
DMU_{13} 1993	260.50	8.00	260.50	8.00	Strong congestion
DMU_{14} 1994	0.00	0.00	0.00	0.00	
DMU_{15} 1995	249.50	3.98	249.50	3.98	Strong congestion
DMU_{16} 1996	0.00	0.00	0.00	0.00	
DMU_{17} 1997	0.00	0.00	0.00	0.00	

TABLE 5. Compare the results of the CSOD. CSOD? Yes/no. If yes, then strong or weak.

DMU year	Our two approaches	Cooper's method	Noura's method	Mehdiloo's method	Sahoo and Tone method
DMU1	No	No	No	No	No
DMU2	Strong	Weak	No	Strong	Weak
DMU3	No	No	No	No	No
DMU4	No	No	No	No	No
DMU5	No	No	No	No	No
DMU6	Strong	No	No	Strong	No
DMU7	No	No	No	No	No
DMU8	Strong	Weak	No	Strong	Weak
DMU9	Strong	Weak	Weak	Strong	Weak
DMU10	Strong	Weak	No	Strong	Strong
DMU11	No	No	No	No	No
DMU12	Strong	Weak	No	Strong	Weak
DMU13	Strong	No	No	Strong	No
DMU14	No	No	No	No	No
DMU15	Strong	Weak	Weak	Strong	Strong
DMU16	No	No	No	No	No
DMU17	No	No	No	No	

$$X_{\mathrm{NC}} = \left\{ x_{io} | x_{io} = \sum_{(J_E)} \lambda_{(J_E)} x_{i(J_E)}, \sum_{J_E} \lambda_{J_E} = 1, \lambda_{J_E} \geq 0, (J_E) = 1,3,4,5,11,14,16,17, i = 1,2 \right\}. \tag{6.6}$$

Therefore, to identify congested input components and measure their congestion the model (5.6) is running (Results in Tab. 4).

7. Conclusion

This paper focuses on detecting and measuring congestion in the DEA measurement framework. First, this paper aims at investigating the weak and strong congestion statuses of decision-making units (DMUs) by identifying a congestion zone (CZ) and non-congestion zone (NC) for production technology. Second, unlike the methods of Tone and Sahoo (2004) which considered multiple projections of DMU_o, Mehdiloozad *et al.* [19] thought of projections of DMU_o, maximal elements and used definition and properties of the face, minimum face, convex hull, and cone, we want to provide the new mathematical well-defined and efficient perspective of congestion relying on the concept of PPS only. The alternative approaches, based on the congestion and non-congestion areas, can identify weak and strong CSOD and efficiently calculate the amount of congestion of each input component.

Recently, Khoveyni *et al.* [17] have created a new method to develop the Tone and Sahoo [23] approach in identifying the weak and strong CSOD with both negative-positive data. However, as mentioned, their methods do not identify the strong congestion status of DMUs. Correctly. Meanwhile, they did not attempt been made to deal with the incidence of multiple optimal projections. However, as reasoned by Sueyoshi and Sekitani [22], specifying the weak-strong CSOD (inefficient DMUs) is probably problematic under such an incidence. So, this paper is interested in the precise recognition of the weak and strong CSOD under the decomposition of PPS with both non-negative and negative data. Since Tone and Sahoo's [23] description of strong congestion includes commensurate changes in inputs-outputs, based on this definition, we create a new mathematical definition and new insight of congestion to create it stable with both negative and non-negative data.

In this paper, we proved that the convex combination of both points of the non-congestion area is in the non-congestion area; the convex combination of two not congested points lacks congestion. Also, we explain that if the DMU is $((\varphi^*_{\text{BCC-o}}))$ efficient; it does not have congestion and the convex combination of the $((\varphi^*_{\text{BCC-o}}))$ efficient points do not have congestion. Then we consider the convex combination of $((\varphi^*_{\text{BCC-o}}))$ effective points and define that any point outside this area has congestion. In this paper, first, define a new definition of congestion as the point $(x, y) \in T_{\text{convex}}$ does has congestion if and only if $(x, y) \in X_{\text{NC}}$ and second, we prove that the necessary and sufficient condition for a DMU_o to does has congestion is that its input vector should not be within X_{NC}. DMU_o has congestion if $x_o \notin X_{\text{NC}}$. In this paper, to identify congested DMUs, we divide the PPS into two areas of congestion and non-congestion area. We also classified the DMUs into two groups congested and non-congested DMUs. Finally, we identify the congested DMUs and calculate congested DMUs simply and accurately by computing the non-congestion area and a non-radial model. Given that congestion is a feature of the production frontier, and we also use post-PPS properties to identify congested DMUs, it can be said that our method is also very accurate and fast. We test our model with the best methods available to identify congestion DMUs and show that our method is the most accurate to prove our claim.

8. Algorithm

(A) First, we run model (2.1a) and (2.1b) BCC-o to create a set of technically efficient DMUs. So, assume that we have an optimal solution as $\left(\varphi^*, \lambda_j^*, s_i^{-*}, s_r^{+*}\right)$.

  (a) If $\varphi^* = 1, s_i^{-*} = 0, s_r^{+*} = 0$, then (x_o, y_o) is strongly efficient DMU and not congested and but, maybe CSP (if weak efficiency or technically efficient frontier does not exist).

  (b) If $\varphi^* = 1, s_i^{-*} \neq 0$ or $s_r^{+*} \neq 0$, then (x_o, y_o) is technically efficient DMU and not congested and also, this is a candidate for CSP. So we go to Section C.

(B) If $\varphi^* \neq 1$, then (x_o, y_o) is not efficient DMU, and we run model (4.5). We run the model (4.5) to get the congested DMUs. According to our new definition of congestion, this model specifies congested DMUs without specifying each input component's congestion measures.

  (a) If $\eta^* > 0$, then (x_o, y_o) is congested DMU and not a candidate for CSP.

  (b) If $\eta^* = 0$, then (x_o, y_o) is not congested DMU.

(C) We run model (4.4) for the set of DMUs obtained from step A-b. The set of solutions obtained from this model's running gives rise to the index (J) (the set of CSP DMUs). If in the optimal solution, u_o be equal to zero; there is no CSP, and, finally, there is no congestion area. Otherwise, if some u_o be equal to positive, thus, and these DMUs are CSP and upper supporting hyperplanes (upper frontier) of T_{convex} so, we can create the CF and go to step 3. Also, we make the set of (J) with u_o that are positive (CSP DMUs). It means (J) is the index of DMUs (CSPs) that have positive u_o. finally, If in the optimal solution, u_o be equal unbounded; there is no CSP and congestion area. then we go to D,

(D) We creat X_c with DMUs that their index creat (J). Then go to E,

(E) Run the model (4.16) to achieve the congestion measure in the input components of the congested DMUs.

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