RAIRO-Oper. Res. 56 (2022) 1969–1994
https://doi.org/10.1051/ro/2022068

RAIRO Operations Research
www.rairo-ro.org

COST OPTIMIZATION INVENTORY MODEL FOR DETERIORATING ITEMS WITH TRAPEZOIDAL DEMAND RATE UNDER COMPLETELY BACKLOGGED SHORTAGES IN CRISP AND FUZZY ENVIRONMENT

Boina Anil Kumar and Susanta Kumar Paikray*

Abstract. Recently, various deterministic inventory models were developed for deteriorating items with the uniform demand pattern (either increasing or decreasing) throughout the cycle. However, such types of models are not suitable for many real business problems. In particular, the demand patterns of various items are not steady throughout the cycle. In many inventory models, ordinarily, the demand rises first, then it becomes static and finally decreases, and such types of demands can be portrayed by considering trapezoidal functions. Moreover, the costs associated with the inventory become imprecise due to several socio-economical factors. As a result, the optimal solution obtained by the classical inventory model may not fit the actual scenario. Keeping this in view, we develop here an inventory model for deteriorating items having the trapezoidal type of demand function in both crisp and fuzzy environments by considering three possible cases of shortages which are completely backlogged. Furthermore, in view of the comparative study of both scenarios, different data sets of constraints are examined for optimal results. Also, it is observed that the optimal results of the fuzzy model are more appropriate to real-world inventory problems. Finally, in order to strengthen the present investigation, the managerial insight of fluctuation in parameters is presented analytically *via* sensitivity analysis.

Mathematics Subject Classification. 90B05, 03E72.

Received May 22, 2021. Accepted May 10, 2022.

1. Introduction

The main objective of traders is to earn profit by running the business smoothly without any interruption. Simultaneously, they wish to increase their goodwill and brand value to attract more customers by implementing proper managerial strategies. In general, to have a smooth business affair, the proper inventory management of items is highly desirable. The inventory management depends on several constraints like demand, deterioration, shortages, inflation, and different inventory costs. Also, it differs from business to business. Many inventory models have been developed in this direction by several researchers. Among all, Ford Whitman Harris was the first to propose an EOQ model as mentioned in [10].

The deterioration and decay of items usually prevent them from their original use. The deteriorating nature is varied with respect to the type of items, storage facility, environmental conditions, etc. For instance, fruits,

Keywords. Inventory model, trapezoidal demand, signed distance method.

Department of Mathematics, Veer Surendra Sai University of Technology, Burla 768018, Odisha, India.
*Corresponding author; skpaikray_math@vssut.ac.in

dry fruits, vegetables, different types of agricultural products, milk products, and food products have various deteriorations. Also, cold drinks, health drinks, medicines, photographic films, radioactive substances, electronic goods, fashionable products, clothes, plastic products, jewelry have different life spans. Moreover, the deterioration of items also affects the inventory cost. So, the inventory management of deteriorating items is a challengeable task for inventory managers to minimize the total cost. Ghare and Schrader [11] have obtained the optimal results for retailer's inventory having constant deteriorating items in the early days of 1960.

Again, the demand of the items has also an important role for proper maintaining of the inventory. Different items have various demand patterns, and the demand patterns of items varies with consumers as well as business cycles and so on. The demand for items may increases, decreases, or constant depending on the need of the customers. Also, in some scenarios, it depends on time, selling price, advertisement cost, trade credit financing, and many other factors. Moreover, the demand for the same item may vary from cycle to cycle and market to market. On the other hand, transportation and communication between countries are developing day by day with technological advancement. So, many substitute products from other countries are arriving into the market, and it affects the demand of the items. So, many researchers used to develop inventory models for different demanding conditions along with other constraints.

The shortages of the items in an inventory cycle are quite a general phenomenon and are occurring when the supplier provides limited items for a business cycle or the storage capacity of the retailer's stock point is smaller than the actual requirement. Furthermore, shortages may arise in the inventory due to the fuzziness in demand and deterioration. In such cases, all the customers or some of them may wait till the arrival of the next stock or may not wait. Then, different backlogging rates are to be considered for shortages in the inventory depending upon the business environment. Thus, shortages and backlogging also affect the inventory cost.

In classical inventory models, researchers considered the demand, deterioration, and other constraints as deterministic in nature. These constraints are formulated by the experts based on existing data and other parameters. But, these data may inaccurate or inadequate to formulate the constraints of the model due to the impreciseness of available data and rapid changes in market conditions. For instance, the costs associated with inventory are fuzzy in nature due to uncertainty in global economic conditions. Thus, the optimal results obtained in classical inventory models may be inappropriate for their applications. That is, the ordering quantity, total cost, and cycle time are different from the actual requirements. As a result, the retailers may lose or get a reduction in their profit by applying the results of classical inventory models. Hence, it becomes a critical task for inventory managers to obtain optimal results for inventory models having imprecise costs and other parameters. In such scenarios, the fuzzy set theory is the best tool to deal with the impreciseness of costs and other parameters in inventory models. Initially, Lee and Yao [26] used the fuzzy concepts to deal with the impreciseness in demand and production quantity in their inventory model. Later on, the concepts of fuzzy set theory have been used by many researchers in developing inventory models involving the impreciseness of parameters to find more accurate optimal results.

In real business sectors, most of the inventories follow a trapezoidal demand pattern. That is, it has a growing demand at the beginning of a cycle, followed by a constant demand rate, and then declines towards end of the cycle. For instance, fashionable items, seasonal goods, new products, and many more have such types of demand. Moreover, the shortage of items and their fulfillment is quite natural in any business cycle. Thus, in relevance to these real scenarios, we first consider an inventory model for constant deteriorating items with trapezoidal type demand rate under fully backlogged shortages with deterministic costs and parameters. Furthermore, as the classical approach is not sufficient to deal with real-world inventory problems as discussed in the earlier paragraph, we devise the proposed inventory problem in a fuzzy environment by incorporating imprecise costs to obtain the optimal decisions. We have considered three different possible cases of shortages arising under trapezoidal demand in both crisp and fuzzy environments. Also, we present various numerical examples to discuss the effect of imprecise costs on the optimum results.

In view of our proposed investigation, the review of existing literature is presented in Section 2. Section 3, provides the assumptions and notations used in the model. Section 4 describes the formulation of the model in different scenarios. In Section 5, we explain the solution procedure for the crisp model. The corresponding fuzzy model with its solution procedure is discussed in Section 6. The effect of imprecise costs on the optimal results is discussed through numerical illustrations in Section 7. Section 8 provides managerial insights into the sensitivity behavior of different parameters. Finally, Section 9 describes the brief conclusion with managerial suggestions and implications.

2. Literature review

The objective of inventory management is to fulfill the demand of the customers at any time under various constraints and minimize the total inventory cost. All the items in nature do not have the same kind of demand. So, many researchers developed inventory models for various demand items with different constraints under several settings intending for minimal cost and other perspectives.

In recent years, the constant demand for imperfect quality having constant deterioration of items were considered by Khanna *et al.* [20], Tiwari *et al.* [49], and Jaggi *et al.* [15] in their inventory models. But, Yu [52] considered constant demand for constant deteriorating items to obtain the optimal results for his inventory problem. The stock dependent demand inventory models corresponding to time dependent deteriorating items and constant deteriorating items were studied by Indrajitsingha *et al.* [13], Shaikh *et al.* [45] respectively. Further, the inventory problems with price dependent demand were discussed by Barik *et al.* [3] and Routray *et al.* [38] for deteriorating items under different scenarios. Whereas, both the stock and price dependent demand were assumed by Mishra *et al.* [33] for constant deteriorating items in their proposed inventory problem. Also, the optimal cost estimated for imperfect quality and constant deteriorating items when demand depends upon both time and stock in a business cycle were presented by Khurana [22]. However, the stock, price and time-dependent demand having constant deteriorating inventory problems were examined by Chen *et al.* [8]. Furthermore, Banerjee and Agarwal [5], Singh and Kumar [47], Jaggi *et al.* [16], and Indrajitsingha *et al.* [14] found optimal results for constant deteriorating items having selling price dependent demand. Nevertheless, the selling price dependent demand assumed by Sahoo *et al.* [40] for linear deteriorating items, and Khanna *et al.* [21] for imperfect quality and constant deteriorating items in their proposed problems. Moreover, both advertising cost and price dependent demand were discussed by Chanda and Kumar [7] for fashionable products, Shaikh *et al.* [43, 44] for Weibull and constant deteriorating items, and Kumar *et al.* [23] for defective products. Rajan and Uthayakumar [36], and Kaliraman *et al.* [18] developed inventory models for constant deteriorating items under exponentially growing demand. Also, the exponentially declining demand for an inventory of imperfect quality havin constant deterioration was earlier taken into account by Jaggi *et al.* [17]. For different inventory models with different demand and deteriorations under various inventory constraints, the attention of interested readers is drawn towards the works of Barik *et al.* [1, 2], Routray *et al.* [37], and Mishra *et al.* [28–31].

The demand considered by different authors cited above is same throughout the cycle. But, in reality, it may not be the case for all the items in every cycle. Subsequently, some researchers considered ramp-type demand (that means, the demand increases up to a certain point of time and thereafter, it becomes stable) for their inventory problems. The ramp-type demand was considered by Sharma *et al.* [46] for constant deteriorating items and Chakraborty *et al.* [6] for Weibull deteriorating items. Furthermore, the trapezoidal-type demand (that is, in the cycle, the first phase has increased demand, the second phase has constant demand and the third phase has decreased demand) was considered by Mishra *et al.* [27], Wu *et al.* [50, 51] and Singh *et al.* [48] for their inventory models. Whereas, quadratic trapezoidal demand was taken by Debata *et al.* [9], and Price and Time dependent trapezoidal demand was assumed by Kaushik and Sharma [19].

The shortages in an inventory system affect the inventory cost as well as the business, which may lead to the loss in both the business and brand value of the product. In some cases, the shortages are filled partially or fully at the beginning of the subsequent cycle. In the literature discussed above, the researchers developed inventory models having shortages with different backlogging rates. Also, some inventory models are discussed without shortages. The details are outlined in the Table 1 below.

Most of the inventory parameters were taken as deterministic based on available past data. However, this data may inaccurate or inappropriate in the real sense. Moreover, due to various factors the associated costs under

TABLE 1. Summary of literature review.

Author(s)	Year	Demand type	Deterioration type	Backlogging type	Fuzzy model
Debata *et al.* [9]	2015	Quadratic trapezoidal	Constant	Partial	No
Khurana [22]	2015	Time and stock dependent	Time dependent	Completely backlogged	No
Mishra *et al.* [27]	2015	Trapezoidal-type	Constant	Lost sale	No
Rajan and Uthayakumar [36]	2015	Exponential increasing	Constant	No shortages	No
Shabani *et al.* [42]	2015	Fuzzy demand	Fuzzy deterioration	No shortages	Yes
Khanna *et al.* [20]	2016	Constant	Constant defectiveness	Fully	No
Tiwari *et al.* [49]	2016	Constant	Constant	Partially backlogged	No
Wu *et al.* [50]	2016	Trapezoidal-type	Time-dependent	(i) No shortages (ii) Partial	No
Benerjee and Agrawal [5]	2017	Price dependent	Constant	Lost sale	No
Chanda and Kumar [7]	2017	Price and advertisement cost	No	No shortages	Yes
Jaggi *et al.* [16]	2017	Selling price dependent	Constant	Completely	No
Jaggi *et al.* [15]	2017	Constant	Constant	No shortages	No
Kaliraman *et al.* [18]	2017	Exponential	Constant	No shortages	No
Khanna *et al.* [21]	2017	Selling price	Constant	Fully	No
Mishra *et al.* [33]	2017	Price and stock dependent	Constant	(i) Partial (ii) Fully	No
Sharma *et al.* [46]	2017	Ramp type		Partial	Yes
Chakraborty *et al.* [6]	2018	Ramp-type	Weibull	Partial	No
Indrajitsingha *et al* [13]	2018	Stock dependent	Time dependent	No shortages	Yes
Jaggi *et al.* [17]	2018	Exponentially declining	Constant	Partially	No
Shaikh *et al.* [43]	2018	Advertisement selling price	Constant	Partial	Yes
Singh and Kumar [47]	2018	Selling price	Constant	Partial	No
Wu *et al.* [51]	2018	Trapezoidal	Time dependent	Partial	No
Chen *et al.* [8]	2019	Stock,time, price dependent	Constant	No sfhortages	No
Indrajitsingha *et al.* [14]	2019	Selling price	Constant	Partial	Yes
Sahoo *et al.* [40]	2019	Selling price	Linear	Partial	No
Shaikh *et al.* [45]	2019	Stock dependent	Constant	Partial	No
Shaikh *et al.* [44]	2019	Price and advertisement cost	Weibull	Partial	No
Yu [52]	2019	Constant	Constant	Completely	No
Kaushik and Sharma [19]	2020	Price and time dependent Trapezoidal	Partial	No	
Kumar *et al.* [24]	2020	Fuzzy Exponential	Fuzzy Constant	No shortages	Yes
Singh *et al.* [48]	2021	Trapezoidal	Weibull	Completely	No
This paper		Trapezoidal	Constant	Fully	Yes

different inventory constraints may fuzzy in nature. That is to say, the introduction of substitute products, foreign products, etc. affects the demand. Similarly, the environmental conditions, available facilities, advancement of preservation technology, and other scientific methods have an impact on the deterioration rate. Moreover, the global economical condition affects the different costs involved in the inventory system. Thus, many researchers in the present era have considered the impreciseness of the parameters for the inventory requirements in the fuzzy environment. For some developments in this direction, see the recent works of Chanda and Kumar [7], Sharma *et al.* [46], Indrajitsingha *et al.* [12–14], Mishra *et al.* [32], Shaikh *et al.* [43], Kumar *et al.* [23–25], Nayak *et al.* [35], Routray *et al.* [39] and Shabani *et al.* [42]. Apart from these, the government and various companies are changing their policies from time to time, which also lead to fuzziness in inventory models. Also, while planning a project, the managers asses the risk factors of different components including inventory management. In this context, we refer the interested readers to the current works of Mitsumori [34], Seyedimany [41], and Baylan [4].

Motivated essentially by the above mentioned works, we develop here an inventory model for deteriorating items having trapezoidal type of demand function in both crisp and fuzzy environments by considering three possible cases of shortages which are completely backlogged. In the fuzzy environment, the imprecise costs are characterized by the trapezoidal fuzzy numbers, and the signed distance method is used for defuzzification of fuzzy total costs. Furthermore, in view of the comparative study of both scenarios, different data sets of constraints are examined for optimal results. Also, it is observed that the optimal results of the fuzzy model are more appropriate to real-world inventory problems. Finally, in order to strengthen the present investigation, the managerial insight of fluctuation in parameters is presented analytically *via* sensitivity analysis.

3. Assumptions and notations

3.1. Assumptions

(i) The infinite replenishment facility is available, thus the replenishment occurs instantaneously.
(ii) The associated costs are imprecise in nature.
(iii) The items in the inventory follows trapezoidal type demand $R(t)$,

that is, $$R(t)=\begin{cases} c_1+d_1t, & 0\le t\le \nu_1,\\ D, & \nu_1\le t\le \nu_2,\\ c_2-d_2t, & \nu_2\le t\le T\le \frac{c_2}{d_2}.\end{cases}$$

Here, the demand increases linearly in the first interval, is stable in the second interval and linearly decreases in the third interval. The demand changes from increasing to stable at the point ν_1 and from stable to decreasing at ν_2 (see Fig. 1).

(iv) The rate of deterioration of the items in the inventory is constant.
(v) Shortages occur in the inventory and are backlogged completely to the next cycle.
(vi) Imprecise costs are considered in the fuzzy environment and are characterized by trapezoidal fuzzy numbers.

3.2. Notations

(i) The inventory at any time t, $(0\le t\le T)$ is denoted by $x(t)$.
(ii) The inventory becomes empty at time $t=t_1$.
(iii) The inventory cycle terminates at time $t=T$.
(iv) The rate of deterioration of the items in the inventory is η.
(v) The ordering cost per cycle is $C0$.
(vi) The deterioration cost per unit item is k_1.
(vii) The holding cost per unit item per unit time is k_2.
(viii) The shortage cost per unit item per unit time is k_3.
(ix) The initial inventory of the cycle is Π, that is, $\Pi=x(0)$.

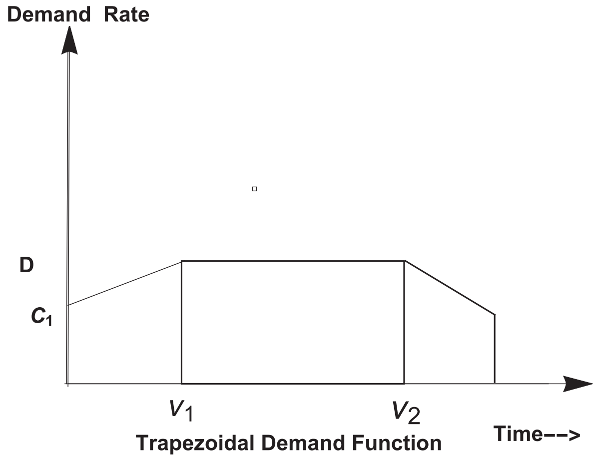

FIGURE 1. Demand function.

(x) The optimal ordering quantity per cycle is W.
(xi) The total average cost per unit time is T_c.
(xii) The total average cost per unit time in the fuzzy environment is ST_c.
(xiii) Costs in the fuzzy environment are $\overline{k_1}$, $\overline{k_2}$, and $\overline{k_3}$ corresponding to the costs k_1, k_2 and, k_3 in the crisp environment respectively.

4. Formulation of mathematical model

Let the ordering quantity W be received at time $t = 0$. After this, the back ordered quantity B of the previous cycle is delivered to the customer and the inventory cycle starts with remaining Π items. From beginning onwards the inventory depletes due to trapezoidal demand $R(t)$ and constant deterioration η till it reaches zero at $t = t_1$. Then, the shortages occur in the inventory during the cycle from $t = t_1$ to $t = T$ and are back ordered to the next replenishment. The behavior of the inventory level $x(t)$ at any time during the cycle $[0, T]$ is explained by the following differential equations

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \eta x(t) = -R(t) \quad (0 < t < t_1) \tag{4.1}$$

and

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -R(t) \quad (t_1 < t < T) \tag{4.2}$$

with the condition $x(t_1) = 0$.

Depending upon the values of t_1, ν_1, and ν_2, there arise three cases:

Case 1 ($0 < t_1 \leq \nu_1$)

Due to trapezoidal demand and constant deterioration, the inventory level diminishes gradually and reaches zero at $t = t_1$ for $0 < t_1 \leq \nu_1$ (see Fig. 2). Then, the corresponding differential equations are

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \eta x(t) = -(c_1 + d_1 t) \quad (0 < t < t_1) \tag{4.3}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -(c_1 + d_1 t) \quad (t_1 < t < \nu_1) \tag{4.4}$$

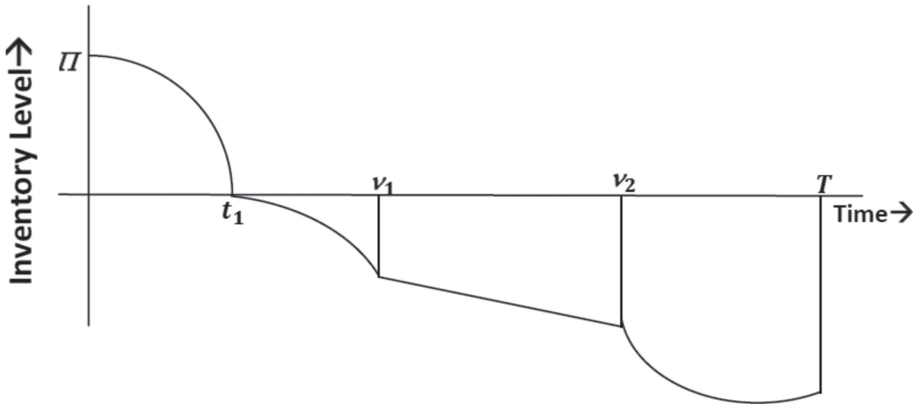

FIGURE 2. Inventory level $x(t)$ in Case 1 ($0 < t_1 \leq \nu_1$).

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -D \qquad (\nu_1 < t < \nu_2) \tag{4.5}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -(c_2 - d_2 t) \quad (\nu_2 < t < T) \tag{4.6}$$

with the boundary condition $x(t_1) = 0$.

The solutions of the above differential equations are

$$x(t) = \left(\frac{c_1 + d_1 t_1}{\eta} - \frac{d_1}{\eta^2}\right) e^{\eta(t_1 - t)} - \left(\frac{c_1 + d_1 t}{\eta} - \frac{d_1}{\eta^2}\right) \quad (0 \leq t <\leq t_1) \tag{4.7}$$

$$x(t) = c_1(t_1 - t) + \frac{d_1}{2}\left(t_1^2 - t^2\right) \qquad (t_1 \leq t \leq \nu_1) \tag{4.8}$$

$$x(t) = c_1 t_1 - Dt + \frac{d_1}{2}\left(t_1^2 + \nu_1^2\right) \qquad (\nu_1 \leq t \leq \nu_2) \tag{4.9}$$

$$x(t) = c_1 t_1 - c_2 t + \frac{d_2}{2}\left(t^2 + \nu_2^2\right) + \frac{d_1}{2}\left(t_1^2 + \nu_1^2\right) \qquad (\nu_2 \leq t \leq T). \tag{4.10}$$

Initial inventory

The inventory starts with Π items, where

$$\Pi = x(0) = \left(\frac{c_1}{\eta} - \frac{d_1}{\eta^2}\right)(e^{\eta t_1} - 1) + \frac{d_1}{\eta} t_1 e^{\eta t_1}. \tag{4.11}$$

Deteriorating cost

The total deterioration cost during the cycle due to deterioration of items is η_T, where

$$\begin{aligned} \eta_T &= k_1\left[\Pi - \int_0^{t_1} R(t)\,\mathrm{d}t\right] = k_1\left[\Pi - \int_0^{t_1} (c_1 + d_1 t)\,\mathrm{d}t\right] \\ &= k_1\left[\left(\frac{c_1}{\eta} - \frac{d_1}{\eta^2}\right)(e^{\eta t_1} - 1) + \frac{d_1}{\eta} t_1 e^{\eta t_1} - c_1 t_1 - \frac{d_1 t_1^2}{2}\right]. \end{aligned} \tag{4.12}$$

Inventory carrying cost

The total carrying cost of the items during the interval $[0, t_1]$ is

$$C_T = k_2\left[\int_0^{t_1} x(t)\,\mathrm{d}t\right] = k_2\left[\left(\frac{c_1 + d_1 t_1}{\eta^2} - \frac{d_1}{\eta^3}\right)(e^{\eta t_1} - 1) - \left(\frac{(c_1\eta - d_1)t_1}{\eta^2} + \frac{d_1 t_1^2}{2\eta}\right)\right]. \tag{4.13}$$

Shortage cost

The opportunity cost due to shortage of items during the interval $[t_1, T]$ is

$$\begin{aligned} S_T &= k_3\left[-\int_{t_1}^{T} x(t)\,\mathrm{d}t\right] = k_3\left[-\int_{t_1}^{\nu_1} x(t)\,\mathrm{d}t - \int_{\nu_1}^{\nu_2} x(t)\,\mathrm{d}t - \int_{\nu_2}^{T} x(t)\,\mathrm{d}t\right] \\ &= k_3\left[\frac{c_1}{2}(t_1-\nu_1)(t_1+\nu_1-2T) + \frac{d_1}{6}\left(2t_1^3 - 2\nu_1^3 + 3T\nu_1^2 - 3Tt_1^2\right) + \frac{c_2}{2}(\nu_2 - T)^2 \right. \\ &\quad \left. + \frac{d_2}{6}\left(3T\nu_2^2 - T^3 - 2\nu_2^3\right) + \frac{D}{2}(\nu_1-\nu_2)(\nu_1+\nu_2-2T)\right]. \end{aligned} \tag{4.14}$$

Total cost

The total average cost of the inventory is

$$\begin{aligned} T_{c1} &= \frac{1}{T}\{C0 + \eta_T + C_T + S_T\} \\ &= \frac{1}{T}\left\{C0 + k_1\left[\left(\frac{c_1}{\eta} - \frac{d_1}{\eta^2}\right)\left(e^{\eta t_1} - 1\right) + \frac{d_1}{\eta}t_1 e^{\eta t_1} - c_1 t_1 - \frac{d_1 t_1^2}{2}\right] + k_2\left[\left(\frac{c_1 + d_1 t_1}{\eta^2} - \frac{d_1}{\eta^3}\right)\left(e^{\eta t_1} - 1\right)\right.\right. \\ &\quad \left. - \left(\frac{(c_1\eta - d_1)t_1}{\eta^2} + \frac{d_1 t_1^2}{2\eta}\right)\right] + k_3\left[\frac{c_1}{2}(t_1-\nu_1)(t_1+\nu_1-2T) + \frac{d_1}{6}\left(2t_1^3 - 2\nu_1^3 + 3T\nu_1^2 - 3Tt_1^2\right)\right. \\ &\quad \left.\left. + \frac{c_2}{2}(\nu_2 - T)^2 + \frac{d_2}{6}\left(3T\nu_2^2 - T^3 - 2\nu_2^3\right) + \frac{D}{2}(\nu_1-\nu_2)(\nu_1+\nu_2-2T)\right]\right\}. \end{aligned} \tag{4.15}$$

Back ordered quantity

The amount of backlogging, which is to be fulfilled in subsequent cycle is

$$\begin{aligned} B &= \int_{t_1}^{T} R(t)\,\mathrm{d}t = \int_{t_1}^{\nu_1} R(t)\,\mathrm{d}t + \int_{\nu_1}^{\nu_2} R(t)\,\mathrm{d}t + \int_{\nu_2}^{T} R(t)\,\mathrm{d}t \\ &= c_1(\nu_1 - t_1) + \frac{d_1}{2}\left(\nu_1^2 - t_1^2\right) + D(\nu_2 - \nu_1) + c_2(T - \nu_2) - \frac{d_2}{2}\left(T^2 - \nu_2^2\right). \end{aligned} \tag{4.16}$$

Economic ordering quantity

The total Economic Ordering Quantity for the inventory management is

$$\begin{aligned} W_1 = \Pi + B &= \left(\frac{c_1}{\eta} - \frac{d_1}{\eta^2}\right)\left(e^{\eta t_1} - 1\right) + \frac{d_1}{\eta}t_1 e^{\eta t_1} \\ &\quad + c_1(\nu_1 - t_1) + \frac{d_1}{2}\left(\nu_1^2 - t_1^2\right) + D(\nu_2 - \nu_1) + c_2(T - \nu_2) - \frac{d_2}{2}\left(T^2 - \nu_2^2\right). \end{aligned} \tag{4.17}$$

Case 2 ($\nu_1 < t_1 \leq \nu_2$)

Due to trapezoidal demand and constant deterioration, the inventory level diminishes gradually and reaches zero at $t = t_1$ for $\nu_1 < t_1 \leq \nu_2$ (see Fig. 3). Then, the corresponding differential equations are

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \eta x(t) = -(c_1 + d_1 t) \quad (0 < t < \nu_1) \tag{4.18}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \eta x(t) = -D \quad (\nu_1 < t < t_1) \tag{4.19}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -D \quad (t_1 < t \leq \nu_2) \tag{4.20}$$

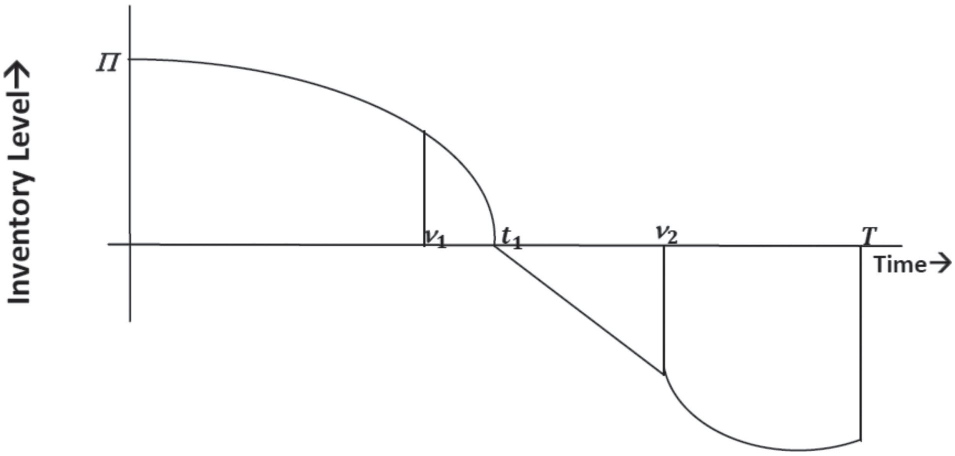

FIGURE 3. Inventory level $x(t)$ in Case 2 ($\nu_1 < t_1 \leq \nu_2$).

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -(c_2 - d_2 t) \quad (\nu_2 < t < T) \tag{4.21}$$

with the boundary condition $x(t_1) = 0$.

The solutions of the above differential equations are

$$x(t) = \left(\frac{De^{\eta t_1}}{\eta} - \frac{d_1 e^{\eta\nu_1}}{\eta^2}\right)e^{-\eta t} + \frac{d_1}{\eta^2} - \frac{c_1 + d_1 t}{\eta} \quad (0 \leq t \leq \nu_1) \tag{4.22}$$

$$x(t) = \frac{D}{\eta}\left(e^{\eta(t_1 - t)} - 1\right) \quad (\nu_1 \leq t \leq t_1) \tag{4.23}$$

$$x(t) = D(t_1 - t) \quad (t_1 \leq t \leq \nu_2) \tag{4.24}$$

$$x(t) = Dt_1 - c_2 t + \frac{d_2}{2}\left(t^2 + \nu_2^2\right) \quad (\nu_2 \leq t \leq T). \tag{4.25}$$

Initial inventory

The inventory starts with Π items, where

$$\Pi = x(0) = \frac{De^{\eta t_1}}{\eta} - \frac{d_1 e^{\eta\nu_1}}{\eta^2} + \frac{d_1}{\eta^2} - \frac{c_1}{\eta}. \tag{4.26}$$

Deteriorating cost

The total deterioration cost during the cycle due to deterioration of items is η_T, where

$$\begin{aligned} \eta_T &= k_1\left[\Pi - \int_0^{t_1} R(t)\,\mathrm{d}t\right] = k_1\left[\Pi - \int_0^{\nu_1} (c_1 + d_1 t)\,\mathrm{d}t - \int_{\nu_1}^{t_1} D\,\mathrm{d}t\right] \\ &= k_1\left[\frac{De^{\eta t_1}}{\eta} - \frac{d_1 e^{\eta\nu_1}}{\eta^2} + \frac{d_1}{\eta^2} - \frac{c_1}{\eta} - Dt_1 + \frac{d_1\nu_1^2}{2}\right]. \end{aligned} \tag{4.27}$$

Inventory carrying cost

The total carrying cost of the items during the interval $[0, t_1]$ is

$$C_T = k_2\left[\int_0^{t_1} x(t)\,\mathrm{d}t\right] = k_2\left[\int_0^{\nu_1} x(t)\,\mathrm{d}t + \int_{\nu_1}^{t_1} x(t)\,\mathrm{d}t\right]$$

$$= k_2\left[\frac{De^{\eta t_1}}{\eta^2} - \frac{d_1 e^{\eta\nu_1}}{\eta^3} + \frac{d_1}{\eta^3} - \frac{c_1}{\eta^2} - \frac{Dt_1}{\eta} + \frac{d_1\nu_1^2}{2\eta}\right]. \tag{4.28}$$

Shortage cost

The opportunity cost due to shortage of items during the interval $[t_1, T]$ is

$$S_T = k_3\left[-\int_{t_1}^{T} x(t)\,\mathrm{d}t\right] = k_3\left[-\int_{t_1}^{\nu_2} x(t)\,\mathrm{d}t - \int_{\nu_2}^{T} x(t)\,\mathrm{d}t\right]$$
$$= k_3\left[\frac{D}{2}(\nu_2 - t_1)^2 + \frac{c_2}{2}(T-\nu_2)^2 + \frac{d_2}{6}\left(3T\nu_2^2 - T^3 - 2\nu_2^3\right) + D(\nu_2 - t_1)(T-\nu_2)\right]. \tag{4.29}$$

Total cost

The total average cost of the inventory is

$$T_{c2} = \frac{1}{T}\{C0 + \eta_T + C_T + S_T\}$$
$$= \frac{1}{T}\left\{C0 + k_1\left[\frac{De^{\eta t_1}}{\eta^2} - \frac{d_1 e^{\eta\nu_1}}{\eta^3} + \frac{d_1}{\eta^3} - \frac{c_1}{\eta^2} - \frac{Dt_1}{\eta} + \frac{d_1\nu_1^2}{2\eta}\right] + k_2\left[\frac{De^{\eta t_1}}{\eta^2} - \frac{d_1 e^{\eta\nu_1}}{\eta^3} + \frac{d_1}{\eta^3} - \frac{c_1}{\eta^2} - \frac{Dt_1}{\eta}\right.\right.$$
$$\left.\left. + \frac{d_1\nu_1^2}{2\eta}\right] + k_3\left[\frac{D}{2}(\nu_2 - t_1)^2 + \frac{c_2}{2}(T-\nu_2)^2 + \frac{d_2}{6}(3T\nu_2^2 - T^3 - 2\nu_2^3) + D(\nu_2 - t_1)(T-\nu_2)\right]\right\}. \tag{4.30}$$

Back ordered quantity

The amount of backlogging, which is to be fulfilled in subsequent cycle is

$$B = \int_{t_1}^{T} R(t)\,\mathrm{d}t = \int_{t_1}^{\nu_2} R(t)\,\mathrm{d}t + \int_{\nu_2}^{T} R(t)\,\mathrm{d}t = D(\nu_2 - t_1) + c_2(T-\nu_2) - \frac{d_2}{2}\left(T^2 - \nu_2^2\right). \tag{4.31}$$

Economic Ordering Quantity

The total Economic Ordering Quantity for the inventory management is

$$W_2 = \Pi + B = \frac{De^{\eta t_1}}{\eta} - \frac{d_1 e^{\eta\nu_1}}{\eta^2} + \frac{d_1}{\eta^2} - \frac{c_1}{\eta} + D(\nu_2 - t_1) + c_2(T-\nu_2) - \frac{d_2}{2}\left(T^2 - \nu_2^2\right). \tag{4.32}$$

Case 3 ($\nu_2 < t_1 \le T$)

Due to trapezoidal demand and constant deterioration, the inventory level diminishes gradually and reaches zero at $t = t_1$ for $\nu_2 < t_1 \le T$ (see Fig. 4). Then, the corresponding differential equations are

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \eta x(t) = -(c_1 + d_1 t) \quad (0 < t < \nu_1) \tag{4.33}$$
$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \eta x(t) = -D \quad (\nu_1 < t < \nu_2) \tag{4.34}$$
$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \eta x(t) = -(c_2 - d_2 t) \quad (\nu_2 < t \le t_1) \tag{4.35}$$
$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -(c_2 - d_2 t) \quad (t_1 < t < T) \tag{4.36}$$

with the boundary condition $x(t_1) = 0$.

The solutions of the above differential equations are

$$x(t) = \left[\left(\frac{c_2 - d_2 t_1}{\eta} + \frac{d_2}{\eta^2}\right)e^{\eta t_1} - \frac{d_1}{\eta^2}e^{\eta\nu_1} - \frac{d_2}{\eta^2}e^{\eta\nu_2}\right]e^{-\eta t} + \frac{d_1}{\eta^2} - \frac{c_1 + d_1 t}{\eta} \qquad (0 \le t \le \nu_1) \tag{4.37}$$

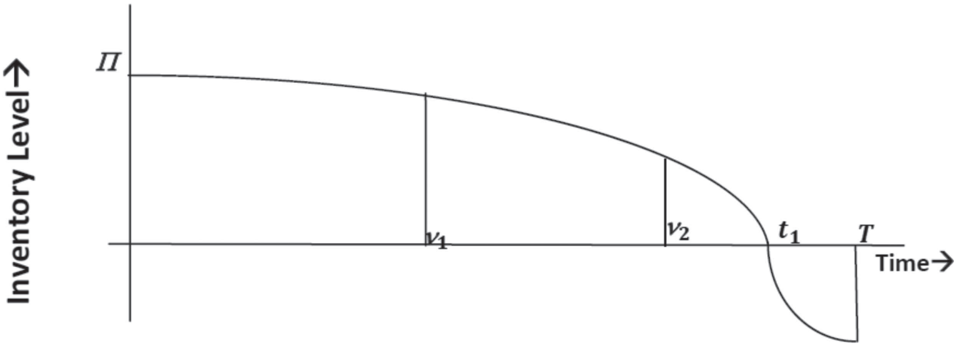

FIGURE 4. Inventory level $x(t)$ in Case 3 ($\nu_2 < t_1 \le T$).

$$x(t) = \frac{D}{\eta}\left(e^{\eta(\nu_2 - t)} - 1\right) + \left(\frac{c_2 - d_2 t_1}{\eta} + \frac{d_2}{\eta^2}\right)e^{\eta(t_1 - t)} - \left(\frac{D}{\eta} + \frac{d_2}{\eta^2}\right)e^{\eta(\nu_2 - t)} \quad (\nu_1 \le t \le \nu_2) \tag{4.38}$$

$$x(t) = \left(\frac{c_2 - d_2 t_1}{\eta} + \frac{d_2}{\eta^2}\right)e^{\eta(t_1 - t)} - \frac{d_2}{\eta^2} - \frac{c_2 - d_2 t}{\eta} \quad (\nu_2 \le t \le t_1) \tag{4.39}$$

$$x(t) = c_2(t_1 - t) + \frac{d_2}{2}(t^2 - t_1^2) \quad (t_1 \le t \le T). \tag{4.40}$$

Initial inventory

The inventory starts with Π items, where

$$\Pi = x(0) = \left(\frac{c_2 - d_2 t_1}{\eta} + \frac{d_2}{\eta^2}\right)e^{\eta t_1} - \frac{d_1}{\eta^2}e^{\eta\nu_1} - \frac{d_2}{\eta^2}e^{\eta\nu_2} + \frac{d_1 - c_1\eta}{\eta^2}. \tag{4.41}$$

Deteriorating cost

The total deterioration cost during the cycle due to deterioration of items is η_T, where

$$\begin{aligned}\eta_T &= k_1\left[\Pi - \int_0^{t_1} R(t)\,\mathrm{d}t\right] = k_1\left[\Pi - \int_0^{\nu_1}(c_1 + d_1 t)\,\mathrm{d}t - \int_{\nu_1}^{\nu_2} D\,\mathrm{d}t - \int_{\nu_2}^{t_1}(c_2 - d_2 t)\right] \\ &= k_1\left[\left(\frac{c_2 - d_2 t_1}{\eta} + \frac{d_2}{\eta^2}\right)e^{\eta t_1} - \frac{d_1}{\eta^2}e^{\eta\nu_1} - \frac{d_2}{\eta^2}e^{\eta\nu_2} + \frac{d_1 - c_1\eta}{\eta^2} + \frac{d_1\nu_1^2}{2} - c_2 t_1 + \frac{d_2}{2}\left(t_1^2 + \nu_2^2\right)\right].\end{aligned} \tag{4.42}$$

Inventory carrying cost

The total carrying cost of the items during the interval $[0, t_1]$ is

$$\begin{aligned}C_T &= k_2\left[\int_0^{t_1} x(t)\,\mathrm{d}t\right] = k_2\left[\int_0^{\nu_1} x(t)\,\mathrm{d}t + \int_{\nu_1}^{\nu_2} x(t)\,\mathrm{d}t + \int_{\nu_2}^{t_1} x(t)\,\mathrm{d}t\right] \\ &= k_2\left[\left(\frac{c_2 - d_2 t_1}{\eta^2} + \frac{d_2}{\eta^3}\right)e^{\eta t_1} - \frac{d_1}{\eta^3}e^{\eta\nu_1} - \frac{d_2}{\eta^3}e^{\eta\nu_2} + \frac{d_1 - c_1\eta}{\eta^3} + \frac{d_1\nu_1^2}{2\eta} - \frac{c_2 t_1}{\eta} + \frac{d_2}{2\eta}\left(t_1^2 + \nu_2^2\right)\right].\end{aligned} \tag{4.43}$$

Shortage cost

The opportunity cost due to shortage of items during the interval $[t_1, T]$ is

$$S_T = k_3\left[-\int_{t_1}^{T} x(t)\,\mathrm{d}t\right] = k_3\left[\frac{c_2}{2}(T - t_1)^2 + \frac{d_2 t_1^2}{2}(T - t_1) + \frac{d_2}{6}\left(t_1^3 - T^3\right)\right]. \tag{4.44}$$

Total cost

The total average cost of the inventory is

$$
\begin{aligned}
T_{c3} &= \frac{1}{T}\{C0 + \eta_T + C_T + S_T\} \\
&= \frac{1}{T}\Bigg\{C0 + k_1\Bigg[\left(\frac{c_2 - d_2 t_1}{\eta} + \frac{d_2}{\eta^2}\right)e^{\eta t_1} - \frac{d_1}{\eta^2}e^{\eta\nu_1} - \frac{d_2}{\eta^2}e^{\eta\nu_2} + \frac{d_1 - c_1\eta}{\eta^2} + \frac{d_1\nu_1^2}{2} - c_2 t_1 + \frac{d_2}{2}\left(t_1^2 + \nu_2^2\right)\Bigg] \\
&\quad + k_2\Bigg[\left(\frac{c_2 - d_2 t_1}{\eta^2} + \frac{d_2}{\eta^3}\right)e^{\eta t_1} - \frac{d_1}{\eta^3}e^{\eta\nu_1} - \frac{d_2}{\eta^3}e^{\eta\nu_2} + \frac{d_1 - c_1\eta}{\eta^3} + \frac{d_1\nu_1^2}{2\eta} - \frac{c_2 t_1}{\eta} + \frac{d_2}{2\eta}\left(t_1^2 + \nu_2^2\right)\Bigg] \\
&\quad + k_3\Bigg[\frac{c_2}{2}(T - t_1)^2 + \frac{d_2 t_1^2}{2}(T - t_1) + \frac{d_2}{6}\left(t_1^3 - T^3\right)\Bigg]\Bigg\}. \qquad (4.45)
\end{aligned}
$$

Back ordered quantity

The amount of backlogging, which is to be fulfilled in subsequent cycle is

$$B = \int_{t_1}^{T} R(t)\,\mathrm{d}t = c_2(T - t_1) - \frac{d_2}{2}(T^2 - t_1^2). \qquad (4.46)$$

Economic Ordering Quantity

The total Economic Ordering Quantity for the inventory management is

$$W_3 = \Pi + B = \left(\frac{c_2 - d_2 t_1}{\eta} + \frac{d_2}{\eta^2}\right)e^{\eta t_1} - \frac{d_1}{\eta^2}e^{\eta\nu_1} - \frac{d_2}{\eta^2}e^{\eta\nu_2} + \frac{d_1 - c_1\eta}{\eta^2} + c_2(T - t_1) - \frac{d_2}{2}(T^2 - t_1^2). \qquad (4.47)$$

5. Solution of the model

The following procedure is explained to get the minimum average cost of the model.

(i) Find $t_1^{(1)}$ such that, $\frac{\mathrm{d}T_{c1}t_1^{(1)}}{\mathrm{d}t_1} = 0$ and $\frac{d^2 T_{c1}t_1^{(1)}}{\mathrm{d}t_1^2} > 0$.

(ii) Find $t_1^{(2)}$ such that, $\frac{\mathrm{d}T_{c2}t_1^{(2)}}{\mathrm{d}t_1} = 0$ and $\frac{d^2 T_{c2}t_1^{(2)}}{\mathrm{d}t_1^2} > 0$.

(iii) Find $t_1^{(3)}$ such that, $\frac{\mathrm{d}T_{c3}t_1^{(3)}}{\mathrm{d}t_1} = 0$ and $\frac{d^2 T_{c3}t_1^{(3)}}{\mathrm{d}t_1^2} > 0$.

(iv) If $0 \le t_1^{(1)} \le \nu_1$, then calculate $W_1\left(t_1^{(1)}\right)$ and $T_{c1}\left(t_1^{(1)}\right)$.

(v) If $\nu_1 \le t_1^{(2)} \le \nu_2$, then calculate $W_2\left(t_1^{(2)}\right)$ and $T_{c2}\left(t_1^{(2)}\right)$.

(vi) If $\nu_2 \le t_1^{(3)} \le T$, then calculate $W_3\left(t_1^{(3)}\right)$ and $T_{c3}\left(t_1^{(3)}\right)$.

(vi) Let $t_1^{(j)} = \arg\min\left\{T_{c1}\left(t_1^{(1)}\right), T_{c2}\left(t_1^{(2)}\right), T_{c3}\left(t_1^{(3)}\right)\right\}$. Then, set $t_1 = t_1^{(j)}$, $W = W_j\left(t_1^{(j)}\right)$ and $T_c = \min\left\{T_{c1}\left(t_1^{(1)}\right), T_{c2}\left(t_1^{(2)}\right), T_{c3}\left(t_1^{(3)}\right)\right\}$.

Thus, t_1 is the optimal positive inventory time, W is the optimal economic ordering quantity and T_c is the optimal total inventory cost.

The Flow Chart of the crisp model is presented in the Figure 5.

6. Fuzzy model

Due to the globalization of business, many multinational companies (MNC) having different types of new products are opening their retail outlets in the existing and new marketplaces. As a result, space scarcity increases and it affect the holding cost. Also, due to the rapid growth in technological advancement and unpredictable climate conditions, the deterioration cost is fuzzy in nature. Similarly, different inventory costs are

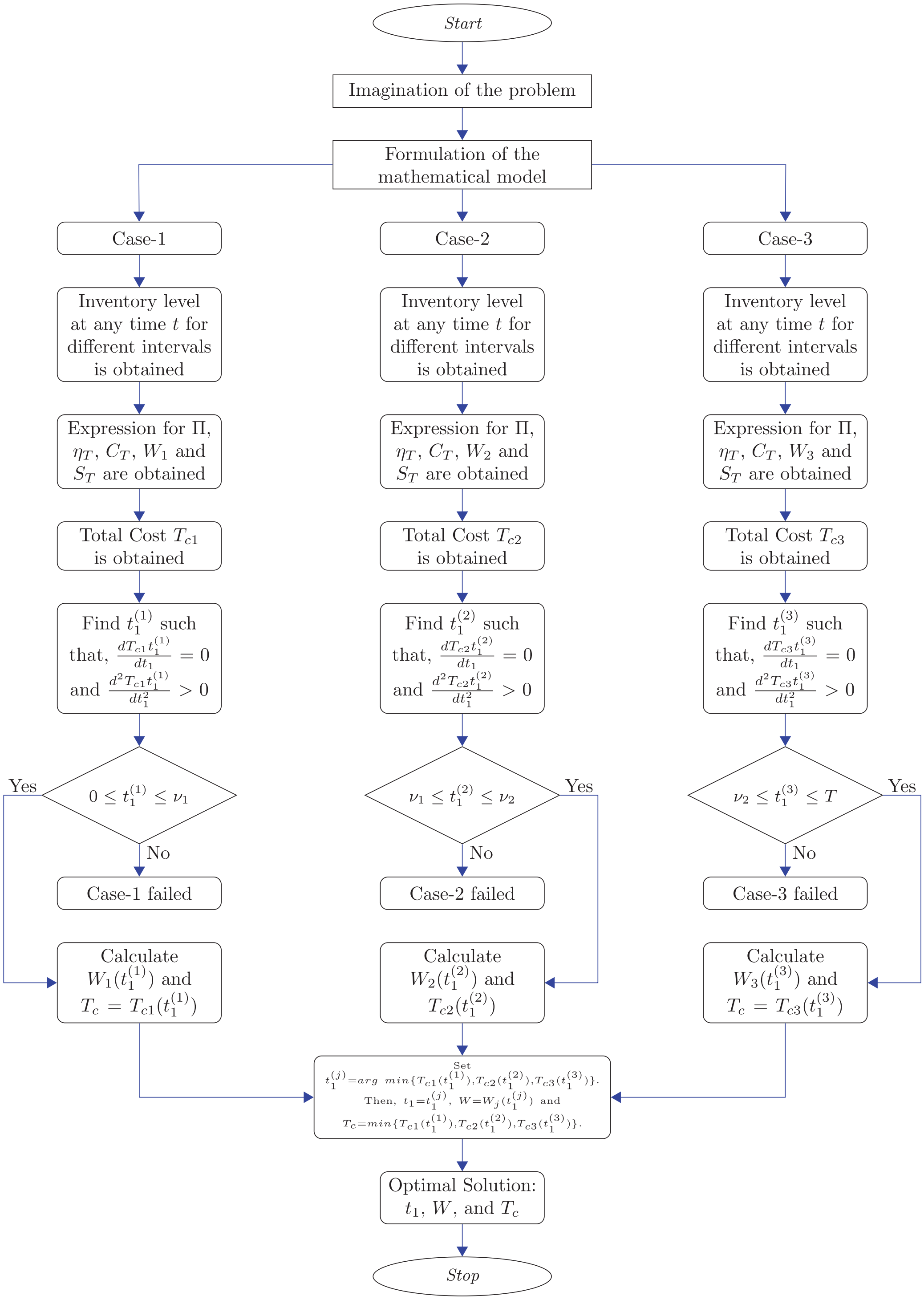

FIGURE 5. Flow chart of crisp model.

imprecise in nature due to many social factors. Hence, we considered deteriorating cost (k_1), holding cost (k_2) and shortage cost (k_3) as fuzzy parameters $\overline{k_1}$, $\overline{k_2}$ and $\overline{k_3}$ respectively. Then by proceeding in the similar fashions of the crisp model, we get the following fuzzy total costs:

Case 1 ($0 < t_1 \leq \nu_1$)

$$\begin{aligned}\overline{T_{c1}} &= \frac{1}{T}\{C0 + \eta_T + C_T + S_T\}\\ &= \frac{1}{T}\bigg\{C0 + \overline{k_1}\bigg[\left(\frac{c_1}{\eta} - \frac{d_1}{\eta^2}\right)(e^{\eta t_1} - 1) + \frac{d_1}{\eta}t_1 e^{\eta t_1} - c_1 t_1 - \frac{d_1 t_1^2}{2}\bigg] + \overline{k_2}\bigg[\left(\frac{c_1 + d_1 t_1}{\eta^2} - \frac{d_1}{\eta^3}\right)(e^{\eta t_1} - 1)\\ &\quad - \left(\frac{(c_1\eta - d_1)t_1}{\eta^2} + \frac{d_1 t_1^2}{2\eta}\right)\bigg] + \overline{k_3}\bigg[\frac{c_1}{2}(t_1 - \nu_1)(t_1 + \nu_1 - 2T) + \frac{d_1}{6}\left(2t_1^3 - 2\nu_1^3 + 3T\nu_1^2 - 3Tt_1^2\right)\\ &\quad + \frac{c_2}{2}(\nu_2 - T)^2 + \frac{d_2}{6}(3T\nu_2^2 - T^3 - 2\nu_2^3) + \frac{D}{2}(\nu_1 - \nu_2)(\nu_1 + \nu_2 - 2T)\bigg]\bigg\}. \end{aligned} \tag{6.1}$$

Case 2 ($\nu_1 < t_1 \leq \nu_2$)

$$\begin{aligned}\overline{T_{c2}} &= \frac{1}{T}\{C0 + \eta_T + C_T + S_T\}\\ &= \frac{1}{T}\bigg\{C0 + \overline{k_1}\bigg[\frac{De^{\eta t_1}}{\eta^2} - \frac{d_1 e^{\eta\nu_1}}{\eta^3} + \frac{d_1}{\eta^3} - \frac{c_1}{\eta^2} - \frac{Dt_1}{\eta} + \frac{d_1\nu_1^2}{2\eta}\bigg] + \overline{k_2}\bigg[\frac{De^{\eta t_1}}{\eta^2} - \frac{d_1 e^{\eta\nu_1}}{\eta^3} + \frac{d_1}{\eta^3} - \frac{c_1}{\eta^2} - \frac{Dt_1}{\eta}\\ &\quad + \frac{d_1\nu_1^2}{2\eta}\bigg] + \overline{k_3}\bigg[\frac{D}{2}(\nu_2 - t_1)^2 + \frac{c_2}{2}(T - \nu_2)^2 + \frac{d_2}{6}(3T\nu_2^2 - T^3 - 2\nu_2^3) + D(\nu_2 - t_1)(T - \nu_2)\bigg]\bigg\}. \end{aligned} \tag{6.2}$$

Case 3 ($\nu_2 < t_1 \leq T$)

$$\begin{aligned}\overline{T_{c3}} &= \frac{1}{T}\{C0 + \eta_T + C_T + S_T\}\\ &= \frac{1}{T}\bigg\{C0 + \overline{k_1}\bigg[\left(\frac{c_2 - d_2 t_1}{\eta} + \frac{d_2}{\eta^2}\right)e^{\eta t_1} - \frac{d_1}{\eta^2}e^{\eta\nu_1} - \frac{d_2}{\eta^2}e^{\eta\nu_2} + \frac{d_1 - c_1\eta}{\eta^2} + \frac{d_1\nu_1^2}{2} - c_2 t_1 + \frac{d_2}{2}\left(t_1^2 + \nu_2^2\right)\bigg]\\ &\quad + \overline{k_2}\bigg[\left(\frac{c_2 - d_2 t_1}{\eta^2} + \frac{d_2}{\eta^3}\right)e^{\eta t_1} - \frac{d_1}{\eta^3}e^{\eta\nu_1} - \frac{d_2}{\eta^3}e^{\eta\nu_2} + \frac{d_1 - c_1\eta}{\eta^3} + \frac{d_1\nu_1^2}{2\eta} - \frac{c_2 t_1}{\eta} + \frac{d_2}{2\eta}\left(t_1^2 + \nu_2^2\right)\bigg]\\ &\quad + \overline{k_3}\bigg[\frac{c_2}{2}(T - t_1)^2 + \frac{d_2 t_1^2}{2}(T - t_1) + \frac{d_2}{6}\left(t_1^3 - T^3\right)\bigg]\bigg\}. \end{aligned} \tag{6.3}$$

6.1. Defuzzification

The fuzzy parameters are characterized by trapezoidal fuzzy numbers as follows: $\overline{k_1} = (k_{11}, k_{12}, k_{13}, k_{14})$, $\overline{k_2} = (k_{21}, k_{22}, k_{23}, k_{24})$, and $\overline{k_3} = (k_{31}, k_{32}, k_{33}, k_{34})$. Using these trapezoidal fuzzy numbers and Signed Distance Method, the defuzzified total costs are:

$$\mathrm{ST}_{c1} = \frac{1}{4}[T_{c11} + T_{c12} + T_{c13} + T_{c14}] \quad (0 < t_1 \leq \nu_1) \tag{6.4}$$

$$\mathrm{ST}_{c2} = \frac{1}{4}[T_{c21} + T_{c22} + T_{c23} + T_{c24}] \quad (\nu_1 < t_1 \leq \nu_2) \tag{6.5}$$

$$\mathrm{ST}_{c3} = \frac{1}{4}[T_{c31} + T_{c32} + T_{c33} + T_{c34}] \quad (\nu_2 < t_1 \leq T). \tag{6.6}$$

Here, T_{cij} are obtained by replacing $\overline{k_1}$, $\overline{k_2}$, and $\overline{k_3}$ by k_{1j}, k_{2j}, and k_{3j} in $\overline{T}_{ci}$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$.

Moreover, the solution of the fuzzy model can be obtained in similar lines to the crisp model, so we skip the details involved.

The Flow Chart of the fuzzy model is presented in the Figure 6.

7. Numerical illustration

Based on our proposed solution procedure and by using Mathematica 11.1.1 software, the numerical illustrations of both models are presented below.

Example 7.1 (Case 1 ($0 < t_1 \leq \nu_1$)).

(a) Crisp model

The values of the parameters involved in demand function are $c_1 = 750$, $d_1 = 50$, $c_2 = 1500$, $d_2 = 150$, and $D = 900$, the time at which demand pattern changes in the cycle are $\nu_1 = 3$, and $\nu_2 = 4$, the total cycle time is $T = 5$, the ordering cost is $C0 = 2000$, the rate of deterioration of items in the inventory is $\eta = 0.32$, and the deterioration cost, holding cost, and shortage cost per unit item are $k_1 = 6$, $k_2 = 4$, and $k_3 = 8$ respectively.

Solution

Using the above data and the solution procedure explained in Section 5, we obtain the feasible solutions as follow:

$$\begin{aligned} t_1^{(1)} &= 2.37219 \qquad W_1 = 5180.49 \qquad T_{c1} = 8921.58 \\ t_1^{(2)} &= 2.37219 \qquad W_2 = 5167.81 \qquad T_{c2} = 8912.79 \\ t_1^{(3)} &= 2.37219 \qquad W_3 = 4857.57 \qquad T_{c3} = 8427.93, \end{aligned}$$

and the optimal solution: $t_1 = 2.37219 \quad W = 5180.49 \quad T_c = 8921.58$.

That is, on ordering $W = 5180.49$ items initially, the minimized total cost of inventory is $T_c = 8921.58$ and the inventory becomes empty at $t_1 = 2.37219$.

(b) Fuzzy model

The values of the parameters involved in demand function are $c_1 = 750$, $d_1 = 50$, $c_2 = 1500$, $d_2 = 150$, and $D = 900$, the time at which demand pattern changes in the cycle are $\nu_1 = 3$, and $\nu_2 = 4$, the total cycle time is $T = 5$, the ordering cost is $C0 = 2000$, the rate of deterioration of items in the inventory is $\eta = 0.32$, and the fuzzy deterioration, holding and shortage costs are $\overline{k_1} = (k_{11}, k_{12}, k_{13}, k_{14}) = (5, 6, 7, 8)$, $\overline{k_2} = (k_{21}, k_{22}, k_{23}, k_{24}) = (3, 4, 5, 6)$, and $\overline{k_3} = (k_{31}, k_{32}, k_{33}, k_{34}) = (7, 8, 9, 10)$ respectively.

Solution

Using the above data, first, we obtain the defuzzified costs as explained in Section 6.1. Then, following the solution procedure explained in Section 5 for defuzzified costs, we obtain the feasible solutions as follow:

$$\begin{aligned} t_1^{(1)} &= 2.42686 \qquad W_1 = 5235.44 \qquad \mathrm{ST}_{c1} = 8206.36 \\ t_1^{(2)} &= 2.42686 \qquad W_2 = 5224.65 \qquad \mathrm{ST}_{c2} = 8200.25 \\ t_1^{(3)} &= 2.37219 \qquad W_3 = 4857.57 \qquad \mathrm{ST}_{c3} = 7807.0, \end{aligned}$$

and the optimal solution: $t_1 = 2.42686 \quad W = 5235.44 \quad \mathrm{ST}_c = 8206.36$.

That is, on ordering $W = 5235.44$ items initially, the minimized total cost of inventory is $\mathrm{ST}_c = 8206.36$ and the inventory becomes empty at $t_1 = 2.42686$.

Example 7.2 (Case 1 ($0 < t_1 \leq \nu_1$)).

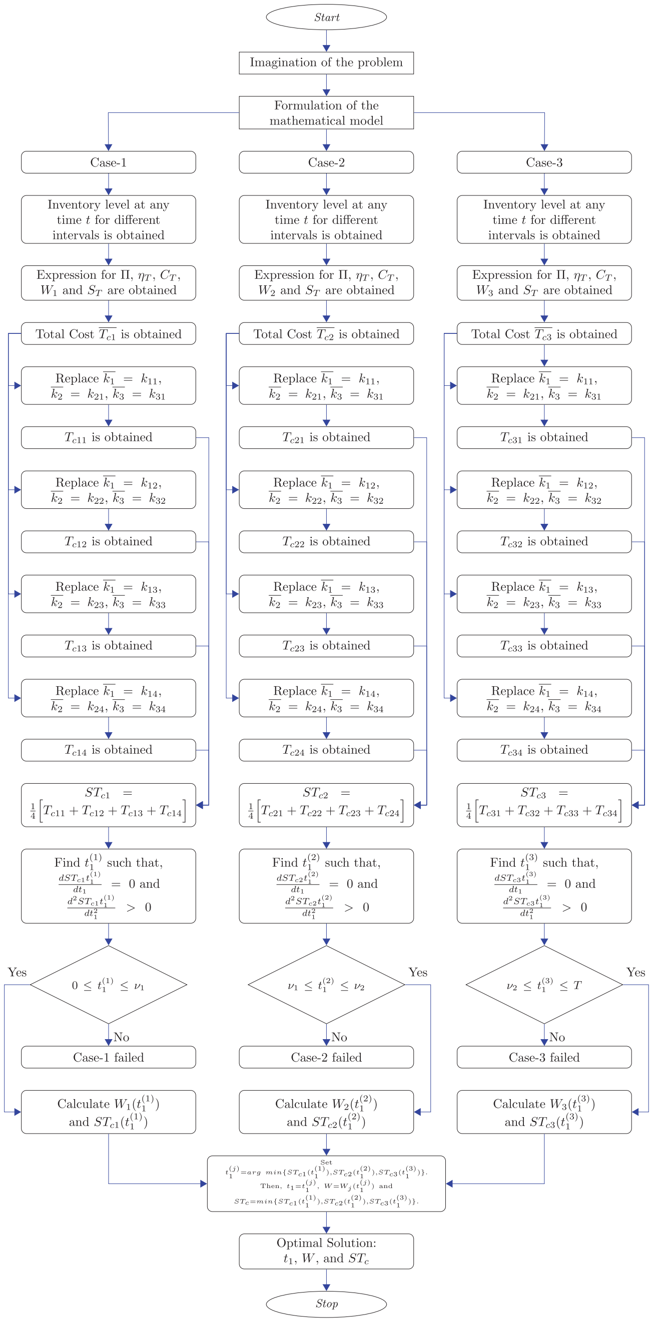

FIGURE 6. Flow chart of fuzzy model.

(a) Crisp model

The values of the parameters involved in demand function are $c_1 = 450$, $d_1 = 5$, $c_2 = 522.5$, $d_2 = 15$, and $D = 462.5$, the time at which demand pattern changes in the cycle are $\nu_1 = 2.5$, and $\nu_2 = 4$, the total cycle time is $T = 5$, the ordering cost is $C0 = 1800$, the rate of deterioration of items in the inventory is $\eta = 0.25$, and the deterioration cost, holding cost, and shortage cost per unit item are $k_1 = 13$, $k_2 = 7$, and $k_3 = 10$ respectively.

Solution

Using the above data and the solution procedure explained in Section 5, we obtain the feasible solutions as follow:

$$
\begin{aligned}
t_1^{(1)} &= 2.12524 \qquad W_1 = 2600.03 \qquad T_{c1} = 6722.63\\
t_1^{(2)} &= 2.12524 \qquad W_2 = 2599.76 \qquad T_{c2} = 6722.38\\
t_1^{(3)} &= 2.12524 \qquad W_3 = 2573.37 \qquad T_{c3} = 6624.57,
\end{aligned}
$$

and the optimal solution: $t_1 = 2.12524 \quad W = 2600.03 \quad T_c = 6722.63$.

That is, on ordering $W = 2600.03$ items initially, the minimized total cost of inventory is $T_c = 6722.63$ and the inventory becomes empty at $t_1 = 2.12524$.

(b) Fuzzy model

The values of the parameters involved in demand function are $c_1 = 450$, $d_1 = 5$, $c_2 = 522.5$, $d_2 = 15$, and $D = 462.5$, the time at which demand pattern changes in the cycle are $\nu_1 = 2.5$, and $\nu_2 = 4$, the total cycle time is $T = 5$, the ordering cost is $C0 = 1800$, the rate of deterioration of items in the inventory is $\eta = 0.25$, and the fuzzy deterioration, holding and shortage costs are $\overline{k_1} = (k_{11}, k_{12}, k_{13}, k_{14}) = (12, 13, 14, 15)$, $\overline{k_2} = (k_{21}, k_{22}, k_{23}, k_{24}) = (6, 7, 8, 9)$, and $\overline{k_3} = (k_{31}, k_{32}, k_{33}, k_{34}) = (9, 10, 11, 12)$ respectively.

Solution

Using the above data, first, we obtain the defuzzified costs as explained in Section 6.1. Then, following the solution procedure explained in Section 5 for defuzzified costs, we obtain the feasible solutions as follow:

$$
\begin{aligned}
t_1^{(1)} &= 2.11435 \qquad W_1 = 2596.52 \qquad \mathrm{ST}_{c1} = 7068.56\\
t_1^{(2)} &= 2.11435 \qquad W_2 = 2596.24 \qquad \mathrm{ST}_{c2} = 7068.28\\
t_1^{(3)} &= 2.12524 \qquad W_3 = 2573.37 \qquad \mathrm{ST}_{c3} = 6963.36,
\end{aligned}
$$

and the optimal solution: $t_1 = 2.11435 \quad W = 2596.52 \quad \mathrm{ST}_c = 7068.56$.

That is, on ordering $W = 2596.52$ items initially, the minimized total cost of inventory is $\mathrm{ST}_c = 7068.56$ and the inventory becomes empty at $t_1 = 2.11435$.

Example 7.3 (Case 2 ($\nu_1 < t_1 \leq \nu_2$)).

(a) Crisp model

The values of the parameters involved in demand function are $c_1 = 350$, $d_1 = 25$, $c_2 = 537.5$, $d_2 = 50$, and $D = 387.5$, the time at which demand pattern changes in the cycle are $\nu_1 = 1.5$, and $\nu_2 = 3$, the total cycle time is $T = 5$, the ordering cost is $C0 = 1000$, the rate of deterioration of items in the inventory is $\eta = 0.2$, and the deterioration cost, holding cost, and shortage cost per unit item are $k_1 = 5$, $k_2 = 4$, and $k_3 = 8$ respectively.

Solution

Using the above data and the solution procedure explained in Section 5, we obtain the feasible solutions as follow:

$$t_1^{(1)} = 2.7301 \qquad W_1 = 2166.98 \qquad T_{c1} = 3397.52$$
$$t_1^{(2)} = 2.7301 \qquad W_2 = 2155.76 \qquad T_{c2} = 3422.53$$
$$t_1^{(3)} = 2.7301 \qquad W_3 = 2154.38 \qquad T_{c3} = 3421.98,$$

and the optimal solution: $t_1 = 2.7301 \quad W = 2155.76 \quad T_c = 3422.53$.

That is, on ordering $W = 2155.76$ items initially, the minimized total cost of inventory is $T_c = 3422.53$ and the inventory becomes empty at $t_1 = 2.7301$.

(b) Fuzzy model

The values of the parameters involved in demand function are $c_1 = 350$, $d_1 = 25$, $c_2 = 537.5$, $d_2 = 50$, and $D = 387.5$, the time at which demand pattern changes in the cycle are $\nu_1 = 1.5$, and $\nu_2 = 3$, the total cycle time is $T = 5$, the ordering cost is $C0 = 1000$, the rate of deterioration of items in the inventory is $\eta = 0.2$, and the fuzzy deterioration, holding and shortage costs are $\overline{k_1} = (k_{11}, k_{12}, k_{13}, k_{14}) = (4, 5, 6, 7)$, $\overline{k_2} = (k_{21}, k_{22}, k_{23}, k_{24}) = (3, 4, 5, 6)$, and $\overline{k_3} = (k_{31}, k_{32}, k_{33}, k_{34}) = (7, 8, 9, 10)$ respectively.

Solution

Using the above data, first, we obtain the defuzzified costs as explained in Section 6.1. Then, following the solution procedure explained in Section 5 for defuzzified costs, we obtain the feasible solutions as follow:

$$t_1^{(1)} = 2.67245 \qquad W_1 = 2149.74 \qquad \mathrm{ST}_{c1} = 3697.66$$
$$t_1^{(2)} = 2.67245 \qquad W_2 = 2139.76 \qquad \mathrm{ST}_{c2} = 3721.2$$
$$t_1^{(3)} = 2.7301 \qquad W_3 = 2154.38 \qquad \mathrm{ST}_{c3} = 3722.55,$$

and the optimal solution:$t_1 = 2.67245 \quad W = 2139.76 \quad \mathrm{ST}_c = 3721.2$.

That is, on ordering $W = 2139.76$ items initially, the minimized total cost of inventory is $\mathrm{ST}_c = 3721.2$ and the inventory becomes empty at $t_1 = 2.67245$.

Example 7.4 (Case 3 ($\nu_2 < t_1 \le T$))**.**

(a) Crisp model

The values of the parameters involved in demand function are $c_1 = 350$, $d_1 = 25$, $c_2 = 505$, $d_2 = 50$, and $D = 355$, the time at which demand pattern changes in the cycle are $\nu_1 = 0.2$, and $\nu_2 = 3$, the total cycle time is $T = 7$, the ordering cost is $C0 = 1000$, the rate of deterioration of items in the inventory is $\eta = 0.2$, and the deterioration cost, holding cost, and shortage cost per unit item are $k_1 = 5$, $k_2 = 4$, and $k_3 = 8$ respectively.

Solution

Using the above data and the solution procedure explained in Section 5, we obtain the feasible solutions as follow:

$$t_1^{(1)} = 3.64581 \qquad W_1 = 2794.7 \qquad T_{c1} = 3588.26$$
$$t_1^{(2)} = 3.64581 \qquad W_2 = 2695.41 \qquad T_{c2} = 3997.43$$
$$t_1^{(3)} = 3.64581 \qquad W_3 = 2685.12 \qquad T_{c3} = 4003.21,$$

and the optimal solution: $t_1 = 3.64581 \quad W = 2685.12 \quad T_c = 4003.21$.

That is, on ordering $W = 2685.12$ items initially, the minimized total cost of inventory is $T_c = 4003.21$ and the inventory becomes empty at $t_1 = 3.64581$.

(b) Fuzzy model

The values of the parameters involved in demand function are $c_1 = 350$, $d_1 = 25$, $c_2 = 505$, $d_2 = 50$, and $D = 355$, the time at which demand pattern changes in the cycle are $\nu_1 = 0.2$, and $\nu_2 = 3$, the total cycle time is $T = 7$, the ordering cost is $C0 = 1000$, the rate of deterioration of items in the inventory is $\eta = 0.2$, and the fuzzy deterioration, holding and shortage costs are $\overline{k_1} = (k_{11}, k_{12}, k_{13}, k_{14}) = (4, 5, 6, 7)$, $\overline{k_2} = (k_{21}, k_{22}, k_{23}, k_{24}) = (3, 4, 5, 6)$, and $\overline{k_3} = (k_{31}, k_{32}, k_{33}, k_{34}) = (7, 8, 9, 10)$ respectively.

Solution

Using the above data, first, we obtain the defuzzified costs as explained in Section 6.1. Then, following the solution procedure explained in Section 5 for defuzzified costs, we obtain the feasible solutions as follow:

$$\begin{aligned}
t_1^{(1)} &= 3.56874 \qquad W_1 = 2758.83 \qquad \mathrm{ST}_{c1} = 3945.71\\
t_1^{(2)} &= 3.56874 \qquad W_2 = 2666.48 \qquad \mathrm{ST}_{c2} = 4360.79\\
t_1^{(3)} &= 3.56874 \qquad W_3 = 2658.67 \qquad \mathrm{ST}_{c3} = 4365.08,
\end{aligned}$$

and the optimal solution: $t_1 = 3.56874 \quad W = 2658.67 \quad \mathrm{ST}_c = 4365.08$.

That is, on ordering $W = 2658.67$ items initially, the minimized total cost of inventory is $\mathrm{ST}_c = 4365.08$ and the inventory becomes empty at $t_1 = 3.56874$.

Example 7.5 (Case 3 ($\nu_2 < t_1 \leq T$)).

(a) Crisp model

The values of the parameters involved in demand function are $c_1 = 600$, $d_1 = 35$, $c_2 = 1155$, $d_2 = 130$, and $D = 635$, the time at which demand pattern changes in the cycle are $\nu_1 = 1$, and $\nu_2 = 4$, the total cycle time is $T = 8$, the ordering cost is $C0 = 1600$, the rate of deterioration of items in the inventory is $\eta = 0.25$, and the deterioration cost, holding cost, and shortage cost per unit item are $k_1 = 5$, $k_2 = 3$, and $k_3 = 9$ respectively.

Solution

Using the above data and the solution procedure explained in Section 5, we obtain the feasible solutions as follow:

$$\begin{aligned}
t_1^{(1)} &= 4.32281 \qquad W_1 = 6467.17 \qquad T_{c1} = 7624.64\\
t_1^{(2)} &= 4.32281 \qquad W_2 = 6220.7 \qquad T_{c2} = 8140.97\\
t_1^{(3)} &= 4.32281 \qquad W_3 = 6208.04 \qquad T_{c3} = 8142.91,
\end{aligned}$$

and the optimal solution: $t_1 = 4.32281 \quad W = 6208.04 \quad T_c = 8142.91$.

That is, on ordering $W = 6208.04$ items initially, the minimized total cost of inventory is $T_c = 8142.91$ and the inventory becomes empty at $t_1 = 4.32281$.

(b) Fuzzy model

The values of the parameters involved in demand function are $c_1 = 600$, $d_1 = 35$, $c_2 = 1155$, $d_2 = 130$, and $D = 635$, the time at which demand pattern changes in the cycle are $\nu_1 = 1$, and $\nu_2 = 4$, the total cycle time is $T = 8$, the ordering cost is $C0 = 1600$, the rate of deterioration of items in the inventory is $\eta = 0.25$, and the fuzzy deterioration, holding and shortage costs are $\overline{k_1} = (k_{11}, k_{12}, k_{13}, k_{14}) = (4, 5, 6, 7)$, $\overline{k_2} = (k_{21}, k_{22}, k_{23}, k_{24}) = (2, 3, 4, 5)$, and $\overline{k_3} = (k_{31}, k_{32}, k_{33}, k_{34}) = (8, 9, 10, 11)$ respectively.

TABLE 2. Effect of change in values of parameters on optimal results.

Sensitivity of η				Sensitivity of T			
η	t_1	W	ST_c	T	t_1	W	ST_c
0.2	3.56874	2658.67	4365.08	7	3.56874	2658.67	4365.08
0.3	3.16979	2837.39	4865.56	7.1	3.61128	2688.46	4402.6
0.4	2.83815	2951.33	5302.73	7.2	3.65361	2717.83	4439.22
0.5	2.56256	3021.66	5680.58	7.3	3.69574	2746.77	4474.94
0.6	2.33206	3062.43	6007.84	7.4	3.73766	2775.27	4509.76
Sensitivity of ν_1				Sensitivity of ν_2			
ν_1	t_1	W	ST_c	ν_2	t_1	W	ST_c
0.2	3.56874	2658.67	4365.08	3	3.56874	2658.67	4365.08
0.3	3.56874	2679.63	4399.28	3.1	3.56874	2680.85	4410.56
0.4	3.56874	2700.32	4433.42	3.2	3.56874	2702.11	4454.33
0.5	3.56874	2720.74	4467.47	3.3	3.56874	2722.42	4496.3
0.6	3.56874	2740.88	4501.42	3.4	3.56874	2741.76	4536.4

Solution

Using the above data, first, we obtain the defuzzified costs as explained in Section 6.1. Then, following the solution procedure explained in Section 5 for defuzzified costs, we obtain the feasible solutions as follow:

$$\begin{aligned} t_1^{(1)} &= 4.47896 \qquad W_1 = 6703.25 \qquad \mathrm{ST}_{c1} = 6714.4 \\ t_1^{(2)} &= 4.47896 \qquad W_2 = 6419.51 \qquad \mathrm{ST}_{c2} = 7253.55 \\ t_1^{(3)} &= 4.32281 \qquad W_3 = 6208.04 \qquad \mathrm{ST}_{c3} = 7276.67, \end{aligned}$$

and the optimal solution: $t_1 = 4.32281 \quad W = 6208.04 \quad \mathrm{ST}_c = 7276.67$.

That is, on ordering $W = 6208.04$ items initially, the minimized total cost of inventory is $\mathrm{ST}_c = 7276.67$ and the inventory becomes empty at $t_1 = 4.32281$.

Remark. In view of the above mentioned examples, we have the following outcomes.

(i) In Example 7.1, the values of positive inventory time t_1 and economic ordering quantity W of fuzzy model are greater than the crisp model. But, the total inventory cost ST_c in fuzzy model is less than the T_c of crisp model.

(ii) In Examples 7.2–7.4, the values of positive inventory time t_1 and economic ordering quantity W of fuzzy model are less than the crisp model. But, the total inventory cost ST_c in fuzzy model is greater than the T_c of crisp model.

(iii) In Example 7.5, the values of positive inventory time t_1 and economic ordering quantity W of both crisp and fuzzy models are equal. But, the total inventory cost ST_c in fuzzy model is less than the T_c of crisp model.

8. Sensitivity analysis

It is very important in an inventory system for a retailer to know the behavior of the system parameters which impact the optimal strategies. Thus, in order to illustrate the applicability of the model and to locate some significant managerial ramifications in the model, the sensitivity analysis with a variety of different parameters is to be carried out. In view of this, here we consider Example 7.4 to study the effect of different parameters on optimal results of the inventory.

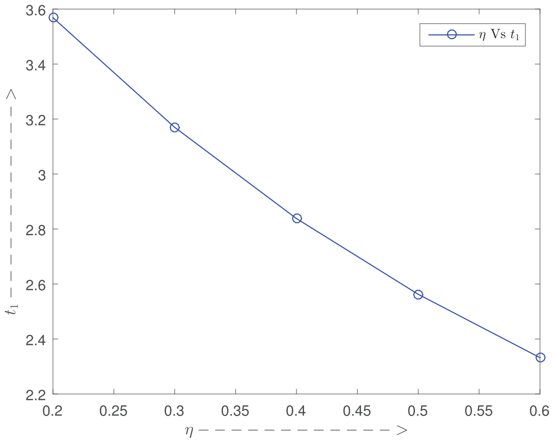

FIGURE 7. η *vs.* t_1.

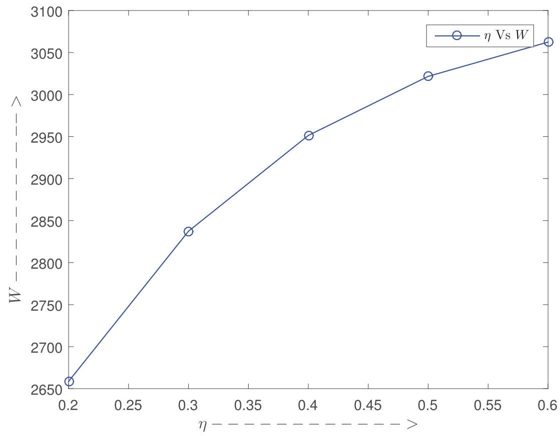

FIGURE 8. η *vs.* W.

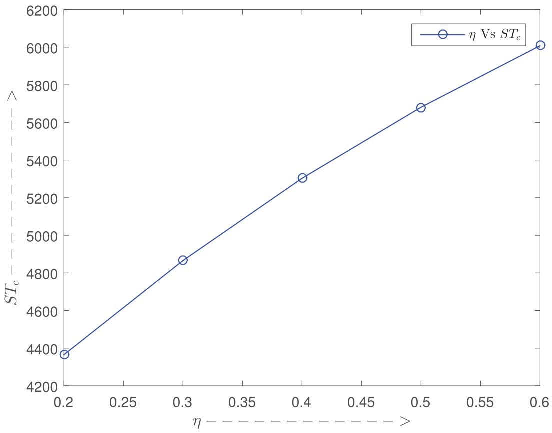

FIGURE 9. η *vs.* ST$_c$.

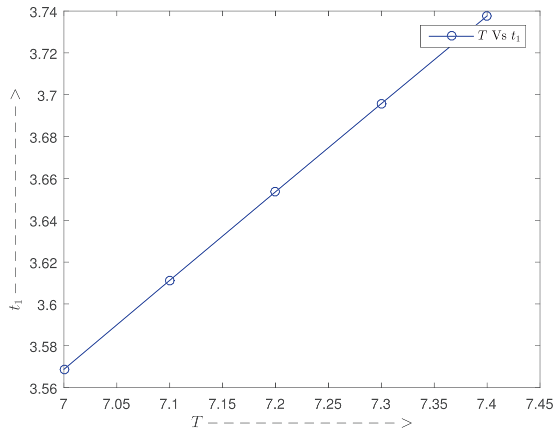

FIGURE 10. T *vs.* t_1.

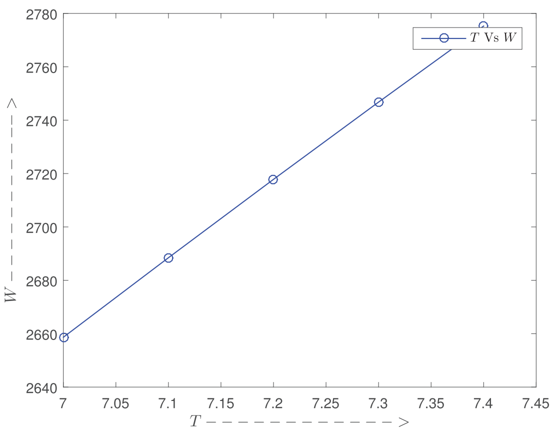

FIGURE 11. T *vs.* W.

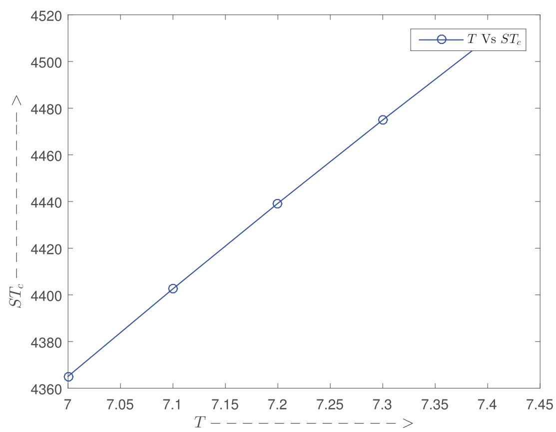

FIGURE 12. T *vs.* ST$_c$.

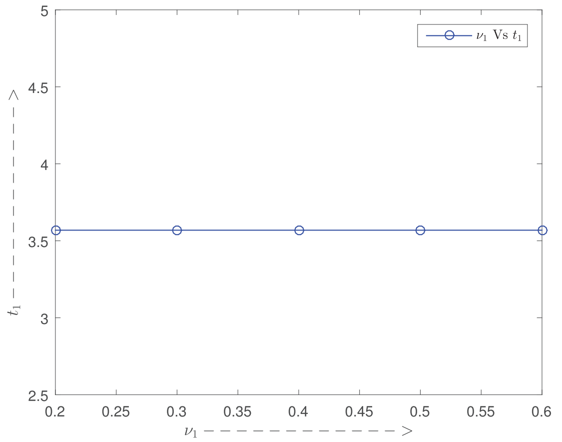

FIGURE 13. ν_1 *vs.* t_1.

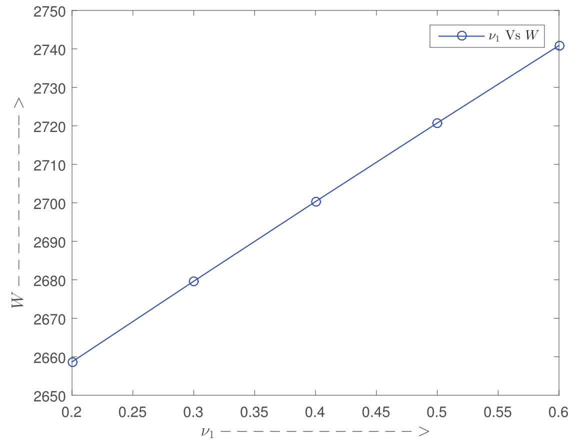

FIGURE 14. ν_1 *vs.* W.

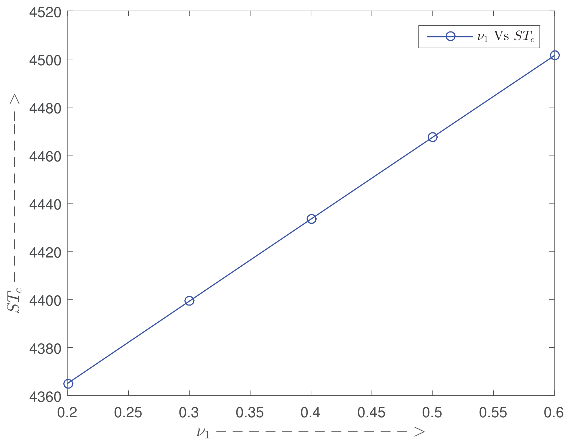

FIGURE 15. ν_1 *vs.* ST_c.

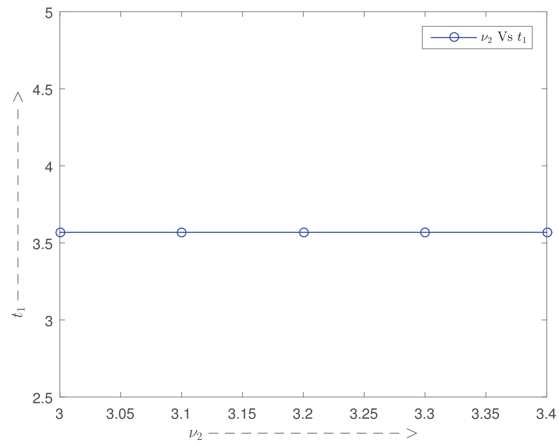

FIGURE 16. ν_2 *vs.* t_1.

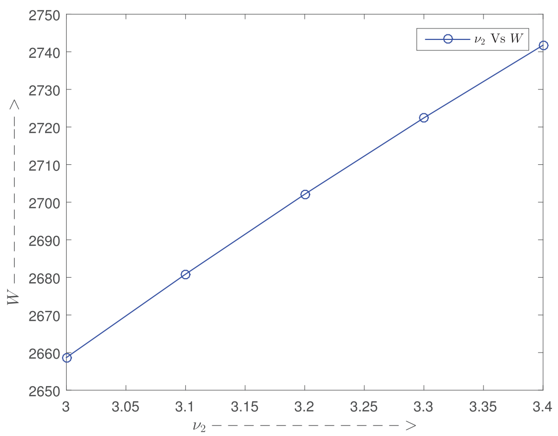

FIGURE 17. ν_2 *vs.* W.

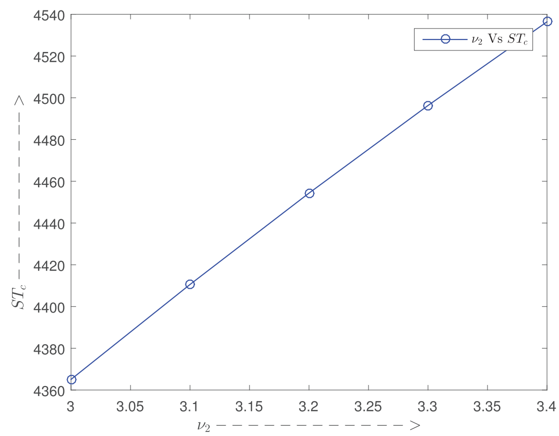

FIGURE 18. ν_2 *vs.* ST_c.

From Table 2 and Figures 7–18, we draw the following conclusions:

(η) The increase of deterioration η rate results, decrease in positive inventory time t_1, and increase in economic ordering quantity W and total cost ST_c in fuzzy environment (see Tab. 2 and Figs. 7–9).

(T) The increase of total cycle time T results, an increase in positive inventory time t_1, economic ordering quantity W and total cost ST_c in fuzzy environment (see Tab. 2 and Figs. 10–12).

(ν_1) The increase in increasing demand period ν_1 results, increase in economic ordering quantity W and total cost ST_c both in fuzzy environment. But, there is no change in positive inventory time t_1 (see Tab. 2 and Figs. 13–15).

(ν_2) The increase in constant demand period ν_2 results, increase in economic ordering quantity W and total cost ST_c both in fuzzy environment. But, there is no change in positive inventory time t_1 (see Tab. 2 and Figs. 16–18).

Managerial insights

The following managerial insights have been found from the sensitivity examination of various parameters (refer Tab. 2).

(i) Increases in the rate of deterioration lead to a significant increase in inventory cost, order quantity, and the number of inventory shortages (refer Figs. 7–9). This result suggests the inventory managers should take extra precautions to prevent the deterioration from intensifying in order to keep the total inventory cost under control.

(ii) Inventory cost, order quantity, and positive inventory time all rise when the planning horizon is extended (refer Figs. 10–12). Hence, retailers must pay more attention to the duration of the planning horizon and acquire and store items accordingly in order to keep their business costs under control.

(iii) The cost of inventory and order quantity rise as the duration of linear growing demand or steady demand rises. But, there will be no change in positive inventory time (refer Figs. 13–18). This means that when the length of increasing demand or stable demand increases, retailers will need to order more inventory and have to spend more money on inventory upkeep to match the demand.

9. Conclusion

Inventory management of decaying products with trapezoidal demand is a crucial part of most of the firms in the present real-world business. On the other hand, the imprecision of various costs has a substantial impact on the inventory's ideal performance. In this context, we built an inventory model with trapezoidal demand with constant deterioration in both crisp and fuzzy environments by integrating fully backlogged shortages. The crisp model is developed by considering deterministic costs, and the fuzzy model is developed by considering imprecised costs. Further, the impreciseness in costs was taken as trapezoidal fuzzy numbers, and the resultant total inventory costs was defuzzified by using the signed distance method. In addition, the procedure for obtaining optimal strategy was explained for both crisp and fuzzy situations. Thereafter, several inventory constraint sets were examined numerically for the validation of the proposed model in both scenarios. Further, the managerial insights have been drawn using sensitivity analysis to deal with the situations that may arise due to the variation in parameters in a business cycle. Moreover, we found the following managerial suggestions and managerial implications from the numerical experiments.

Managerial suggestions

The optimal strategies of both classical and fuzzy models are different for the same set of constraints. Because, the crisp model doesn't consider the impreciseness of parameters where as the fuzzy model considers it. Further, it is evident from the numerical illustration that the difference in strategies of crisp and fuzzy models are not identical and are varies with constraint sets. That is, the values for inventory cost, shortage time point, and order quantity in the fuzzy model may less or more or equal to the corresponding crisp model depending upon the constraint set. Thus, it is concluded that the optimal strategies obtained in the crisp model are suitable for inventory problems where the parameters are known with complete certainty and the optimal strategies obtained in the fuzzy model are suitable for inventory problems where the parameters are known with uncertainty. That means, the optimal strategy obtained in the crisp model is inadequate to implement in most of the real-life inventory problems as they have impreciseness in costs and other parameters. Hence, we suggest the inventory managers to adopt the fuzzy environment by taking account of impreciseness in costs and other parameters in order to get more accuracy in optimal strategy for real-world inventory problems.

Managerial implications

In the present paper, we have used trapezoidal fuzzy numbers to characterize the impreciseness of cost parameters, and the Signed Distance method was employed for defuzzification of fuzzified costs. But, the trapezoidal fuzzy numbers may not be adequate to quantify the impreciseness of parameters in all the scenarios. Also, other defuzzification methods may give better optimal solutions. However, there are a variety of fuzzy numbers to represent impreciseness, for instance, triangular, pentagonal, Hexagonal fuzzy numbers, etc. Also, there are different defuzzification methods available. Hence, researchers can use the work in this paper to obtain a more accurate optimal strategy for real-world inventory issues with varying restrictions by using appropriate fuzzy numbers and defuzzification techniques.

Moreover, it has been identified from the sensitivity analysis that the retailers have to order more quantities and spend more cost on inventory maintenance subject to the increase in parametric values. In view of some similar works (see, [27, 50, 51]) in this direction, our study enhance the scope for more real world inventory problems involving imprecise costs with completely backlogged shortages. Basically, the main contribution of this article is that, the fuzzy approach has been adopted to obtain the optimal decision policy for the proposed model with imprecise costs under three different cases.

Future research directions

The present model can be extended with no shortages, shortages without any backlogging for various inventory items under different demands and deteriorations. Again, one may consider the production inventory model under a similar approach. The model can further be extended by incorporating trade credit financing under various inventory constraints. Another possible extension can be done by implementing preservation techniques for deteriorating inventories.

References

[1] S. Barik, S.K. Paikray and U.K. Misra, Inventory model of deteriorating items for nonlinear holding cost with time dependent demand. *J. Adv. Math.* **9** (2014) 2705–2709.

[2] S. Barik, S. Mishra, S.K. Paikray and U.K. Misra, An inventory model for deteriorating items under time varying demand condition. *Int. J. Appl. Eng. Res.* **10** (2015) 35770–35773.

[3] S. Barik, S. Mishra, S.K. Paikray and U.K. Misra, A deteriorating inventory model with shortages under price dependent demand and inflation. *Asian J. Math. Comput. Res.* **2016** (2016) 14–25.

[4] E.B. Baylan, A novel project risk assessment method development via AHP-TOPSIS hybrid algorithm. *Emerging Sci. J.* **4** (2020) 390–410.

[5] S. Benerjee and S. Agrawal, Inventory model for deteriorating items with freshness and price dependent demand: optimal discounting and ordering policies. *Appl. Math. Model.* **52** (2017) 53–64.

[6] D. Chakraborty, D.K. Jana and T.K. Roy, Two-warehouse partial backlogging inventory model with ramp type demand rate, three-parameter weibull distribution deterioration under inflation and permissible delay in payments. *Comput. Ind. Eng.* **123** (2018) 157–179.

[7] U. Chanda and A. Kumar, Optimisation of fuzzy EOQ model for advertising and price sensitive demand model under dynamic ceiling on potential adoption. *Int. J. Syst. Sci.: Oper. Logist.* **4** (2017) 145–165.

[8] L. Chen, X. Chen, M.F. Keblis and G. Li, Optimal pricing and replenishment policy for deteriorating inventory under stock-level-dependent, time-varying and price-dependent demand. *Comput. Ind. Eng.* **135** (2019) 1294–1299.

[9] S. Debata, M. Acharya and G.C. Samanta, An inventory model for perishable items with quadratic trapezoidal type demand under partial backlogging. *Int. J. Ind. Eng. Comput.* **6** (2015) 185–198.

[10] D. Erlenkotter, Ford Whitman Harris and the economic order quantity model. *Oper. Res.* **38** (1990) 937–946.

[11] P.M. Ghare and G.P. Schrader, A model for exponentially decaying inventory. *J. Ind. Eng.* **14** (1963) 238–243.

[12] S.K. Indrajitsingha, S.S. Routray, S.K. Paikray and U. Misra, Fuzzy economic production quantity model with time dependent demand rate. *Log Forum* **12** (2016) 193–198.

[13] S.K. Indrajitsingha, P.N. Samanta and U.K. Misra, A fuzzy inventory model for deteriorating items with stock dependent demand rate. *Int. J. Logist. Syst. Manag.* **30** (2018) 538–555.

[14] S.K. Indrajitsingha, P.N. Samanta and U.K. Misra, A fuzzy two-warehouse inventory model for single deteriorating item with selling-price-dependent demand and shortage under partial-backlogged condition. *Appl. Appl. Math.* **14** (2019) 511–536.

[15] C.K. Jaggi, L.E. Cárdenas-Barrón, S. Tiwari and A. Shafi, Two-warehouse inventory model for deteriorating items with imperfect quality under the conditions of permissible delay in payments. *Sci. Iran. Trans. E.* **24** (2017) 390–412.

[16] C.K. Jaggi, S. Tiwari and S.K. Goel, Credit financing in economic ordering policies for non-instantaneous deteriorating items with price dependent demand and two storage facilities. *Ann. Oper. Res.* **248** (2017) 253–280.

[17] C.K. Jaggi, P. Gautam and A. Khanna, Inventory decisions for imperfect quality deteriorating items with exponential declining demand under trade credit and partially backlogged shortages. In: Quality, IT and Business Operations, edited by P. Kapur, U. Kumar and A. Verma. *Springer Proceedings in Business and Economics*. Springer, Singapore (2018) 213–229.

[18] N.K. Kaliraman, R. Raj, S. Chandra and H. Chaudhary, Two warehouse inventroy model for deteriorating items with exponential demand rate and permissible delay in payment. *Yugoslav J. Oper. Res.* **27** (2017) 109–124.

[19] J. Kaushik and A. Sharma, Inventory model for the deteriorating items with price and time dependent trapezoidal type demand rate. *Int. J. Adv. Sci. Technol.* **29** (2020) 1617–1629.

[20] A. Khanna, M. Mittal, P. Goutam and C.K. Jaggi, Credit financing for deteriorating imperfect quality items with allowable shortages. *Decis. Sci. Lett.* **5** (2016) 45–60.

[21] A. Khanna, P. Goutam and C.K. Jaggi, Inventory modeling for deteriorating imperfect quality items with selling price dependent demand and shortage backordering under credit financing. *Int. J. Math. Eng. Manag. Sci.* **2** (2017) 110–124.

[22] D. Khurana, Two warehouse inventory model for deteriorating items with time dependent demand under inflation. *Int. J. Comput. Appl.* **114** (2015) 34–38.

[23] B.A. Kumar, S.K. Paikray, S. Mishra and S. Routray, A fuzzy inventory model of defective items under the effect of inflation with trade credit financing. In: Recent Advances in Intelligent Information Systems and Applied Mathematics, ICITAM 2019. Studies in Computational Intelligence, edited by O. Castillo, D. Jana, D. Giri and A. Ahmed. Vol. 863. Springer (2019).

[24] B.A. Kumar, S.K. Paikray and H. Dutta, Cost optimization model for items having fuzzy demand and deterioration with two-warehouse facility under the trade credit financing. *AIMS Math.* **5** (2020) 1603–1620.

[25] B.A. Kumar, S.K. Paikray and U. Misra, Two-Storage fuzzy inventory model with time dependent demand and holding cost under acceptable delay in payment. *Math. Model. Anal.* **25** (2020) 441–460.

[26] H.M. Lee and J.S. Yao, Economic production quantity for fuzzy demand and fuzzy production quantity. *Eur. J. Oper. Res.* **109** (1998) 203–211.

[27] U. Mishra, An inventory model for deteriorating items under trapezoidal type demand and controllable deterioration rate. *Prod. Eng. Res. Devel.* **9** (2015) 351–365.

[28] S. Mishra, U.K. Misra, S. Barik and S.K. Paikray, Optimal control of an inventory system for weibull ameliorating, deteriorating items under the influence of inflation. *Bull. Pure Appl. Sci.* **30** (2011) 85–94.

[29] S. Mishra, M. Mallick, U.K. Misra and S.K. Paikray, An EOQ model for both ameliorating and deteriorating items under the influence of inflation and time-value of money. *J. Comput. Model.* **1** (2011) 101–113.

[30] S. Mishra, U.K. Misra, G. Mishra, S. Barik and S.K. Paikray, An inventory model for inflation induced demand and weibull deteriorating items. *Int. J. Adv. Eng. Tech.* **4** (2012) 176–182.

[31] S. Mishra, L.K. Raju, U.K. Misra and G. Misra, Optimal control of an inventory system with variable demand and ameliorating deteriorating items. *Asian J. Current Eng.* **1** (2012) 154–157.

[32] S. Mishra, S. Barik, S.K. Paikray and U.K. Misra, An inventory control model of deteriorating items in fuzzy environment. *Global J. Pure Appl. Math.* **11** (2015) 1301–1312.

[33] U. Mishra, L.E. Cárdenas-Barrón, S. Tiwari, A.A. Shaikh and G. Treviño-Garza, An inventory model under price and stock dependent demand for controllable deterioration rate with shortages and preservation technology investment. *Ann. Oper. Res.* **254** (2017) 165–190.

[34] Y. Mitsumori, An analysis of the transformation of mega-pharma's business model toward the emerging market. *Emerging Sci. J.* **4** (2020) 253–262.

[35] D.K. Nayak, S.S. Routray, S.K. Paikray and H. Dutta, A fuzzy inventory model for weibull deteriorating items under completely backlogged shortages. *Discrete Contin. Dyn. Syst. Ser. S* **14** (2020) 2435–2453.

[36] R.S. Rajan and R. Uthayakumar, A two-warehouse inventory model for deteriorating items with permissible delay under exponentially increasing demand. *Int. J. Suppl. Oper. Manag.* **2** (2015) 662–682.

[37] S.S. Routray, S.K. Paikray and U.K. Misra, A note on optimal order level deteriorating items with uniform demand rate. *Proc. Jangjeon Math. Soc.* **17** (2014) 403–409.

[38] S.S. Routray, S. Mishra, S.K. Paikray and U.K. Misra, A model on deteriorating items with price dependent demand rate. *Int. J. Res. Rev. Appl. Sci.* **25** (2015) 44–49.

[39] S.S. Routray, S.K. Paikray, S. Mishra and U.K. Misra, Fuzzy inventory model with single item under time dependent demand and holding cost. *Int. J. Adv. Res. Sci.* **6** (2017) 1604–1618.

[40] A.K. Sahoo, S.K. Indrajitsingha, P.N. Samanta and U.K. Misra, Selling price dependent demand with allowable shortages model under partially backlogged – deteriorating items. *Int. J. Appl. Comput. Math.* **5** (2019) 1–13.

[41] A. Seyedimany, Stock price reactions on NASDAQ stock exchange for special dividend announcements. *Emerging Sci. J.* **3** (2019) 382–388.

[42] S. Shabani, A. Mirzazadeh and E. Sharifi, A two-warehouse inventory model with fuzzy deterioration rate and fuzzy demand rate under conditionally permissible delay in payment. *J. Ind. Prod. Eng.* **33** (2016) 134–142.

[43] A.A. Shaikh, L.E. Cárdenas-Barrón and L. Sahoo, A fuzzy inventory model for a deteriorating item with variable demand, permissible delay in payments and partial backlogging with shortage follows inventory (SFI) policy. *Int. J. Fuzzy Syst.* **20** (2018) 1606–1623.

[44] A.A. Shaikh, L.E. Cárdenas-Barrón, A.K. Bhunia and S. Tiwari, An inventory model of a three parameter weibull distributed deteriorating item with variable demand dependent on price and frequency of advertisement under trade credit. *RAIRO-Oper. Res.* **53** (2019) 903–916.

[45] A.A. Shaikh, L.E. Cárdenas-Barrón and S. Tiwari, A two-warehouse inventory model for non-instantaneous deteriorating items with interval valued inventory costs and stock dependent demand under inflationary conditions. *Neural Comput. Appl.* **31** (2019) 1931–1948.

[46] A. Sharma, U. Sharma and C. Singh, A robust replenishment model for deteriorating items considering ramp-type demand and inflation under fuzzy environment. *Int. J. Logist. Manag.* **28** (2017) 287.

[47] D. Singh and A. Kumar, Two-warehouses partial backlogging inventory model for deteriorating items with selling price dependent demand under trade credit in the inflationary environment. *Int. J. Pure Appl. Math.* **118** (2018) 1447–1457.

[48] T. Singh, N.N. Sethy and A.K. Nayak, An optimal policy for deteriorating items with generalized deterioration, trapezoidal-type demand, and shortages. *Int. J. Inf. Syst. Supply Chain Manag.* **14** (2021) 23–54.

[49] S. Tiwari, L.E. Cárdenas-Barrón, A. Khanna and C.K. Jaggi, Impact of trade credit and inflation on retailer's ordering policies for non-instantaneous deteriorating items in a two-warehouse environment. *Int. J. Prod. Econ.* **176** (2016) 154–169.

[50] J. Wu, K. Skouri, J-T. Teng and Y. Hu, Two inventory systems with trapezoidal-type demand rate and time-dependent deterioration and backlogging. *Expert Syst. Appl.* **46** (2016) 367–379.

[51] J. Wu, J.-T. Teng and K. Skouri, Optimal inventory policies for deteriorating items with trapezoidal-type demand patterns and maximum lifetimes under upstream and downstream trade credits. *Ann. Oper. Res.* **264** (2018) 459–476.

[52] J.C.P. Yu, Optimizing a two-warehouse system under shortage backordering, trade credit, and decreasing rental conditions. *Int. J. Prod. Econ.* **209** (2019) 147–155.