RAIRO-Oper. Res. 56 (2022) 1411–1427
https://doi.org/10.1051/ro/2022051

RAIRO Operations Research
www.rairo-ro.org

PRICING GAMES OF DUOPOLY SERVICE-INVENTORY SYSTEMS WITH LOST SALES

Doo Ho Lee[1] and Dae-Eun Lim[2,*]

Abstract. This study considers a duopoly market in which two competitors operate their own service-inventory systems. Both competitors determine their prices to maximize their profit while considering the inventory holding cost, ordering cost, and cost incurred by lost sales. Customers are price sensitive, and customer attractiveness is expressed by arrival rates. We use a game theory approach to formulate and analyze three types of pricing games: (i) a parallel pricing game, (ii) a sequential pricing game, and (iii) a unified pricing game. The uniqueness of equilibrium prices is analytically proven, after which, a solution procedure for obtaining equilibrium prices is outlined.

Mathematics Subject Classification. 90B22, 90B50.

Received April 8, 2021. Accepted April 6, 2022.

1. Introduction

Over the past few decades, there has been considerable research examining integrated service-inventory systems. These systems originated from the well-known assembly-like queues, wherein several types of parts are simultaneously served to manufacture a finished product [4]. Service-inventory systems assume positive handling times, including preparing, retrieving, packing, and loading, while classical inventory models neglect handling times [10]. For example, replacing auto parts requires not only new parts but also time at an auto repair shop. Queueing models are applied to service-inventory systems to devise an efficient operation method. However, even for simple cases, there have not been many related studies due to the complexity caused by the intrinsic dependency in queue-considered service-inventory systems. That is, the volatilities of demand, service processes, and lead time make it difficult to develop an optimal inventory policy, which is a major research topic in service-inventory models.

In addition to these operational issues, marketing issues – including the pricing of items – are found in service-inventory models [9]. Whitin [15] studied the newsvendor problem with price-dependent demand. Since this study, numerous models considering pricing strategies have been developed. In recent years, integrated service-inventory systems, in which a service process is frequently connected with inventory management, have been extensively studied. A service-inventory model is also referred to as a system with positive service time,

Keywords. Service-inventory system, lost sales, parallel pricing game, sequential pricing game, unified pricing game.

[1] Department of Artificial Intelligence and Software, Kangwon National University, 346 Joongang-ro, Samcheok-si, Gangwon-do 29513, Republic of Korea.

[2] Department of Industrial Engineering, Kangwon National University, 1 Gangwondaehakgil, Chuncheon-si, Gangwon-do 24341, Republic of Korea.

*Corresponding author: del@kangwon.ac.kr

and the rich body of literature on this topic was well summarized by Krishnamoorthy and Narayanan [6] and Krishnamoorthy *et al.* [7]. Marand *et al.* [9] summarized and classified inventory models in terms of the arrival processes of demand, the distributions of services and lead time, stock-out policy, and system capacity.

Many studies have investigated pricing issues in service-inventory systems. However, no study has considered the pricing of two competitors with queue-considered service-inventory systems. This paper therefore presents game-theory-based pricing games of two competitors with inventory queues, each with a service-inventory system, under a duopolistic situation. We consider three types of pricing games: (i) a parallel pricing game, (ii) a sequential pricing game, and (iii) a unified pricing game. Duopoly markets consisting of two competitors can be found in various applications including supply chains [8,16], peak-load pricing [5], and two-tier service systems. An example of two-tier service is public healthcare service, which comprises free and paid service providers [17]. Healthcare service is a type of service-inventory application because service for customers requires both resources and time. Careful pricing is required to maximize the revenue of a toll service provider – which is typically a private service provider – because customers are sensitive to price; if the price is too high, customers may select a free service even if the waiting time is long. In a two-tier service system, only one service provider considers pricing. However, this study considers the more general situation wherein two service providers consider pricing.

Berman and Kim [2] discussed a service-inventory system in which demand arrives at a system according to a Poisson process and service times are exponentially distributed. They assumed that the capacity is finite or infinite and then characterized the optimal order policy for a service facility. In a later study, Berman and Sapna [3] generalized this model to an $M/G/1$ queue, in which the reorder point was zero, and they proposed the closed-form expressions of a stationary distribution and performance measures. Schwarz *et al.* [12] derived the stationary distributions of joint queue length and inventory processes in an explicit product form for various $M/M/1$ systems. They considered the inventory under continuous review, different inventory management policies, and lost sales. Baek *et al.* [1] studied a continuous (s, Q) inventory control model with an attached $M/M/1$ queue and lost sales. They assumed that each customer leaves a system with a random number of items at the time of service completion. More recently, Marand *et al.* [9] integrated inventory control and pricing decisions. They assumed price-dependent customer arrival rates, and they continuously reviewed inventory level under the (r, Q) policy. They formulated an $M/M/1$ service-inventory system with lost sales while considering the inventory and pricing policies, then proposed algorithms for obtaining optimal solutions. Note that none of the extant studies have considered the pricing strategies of two competitors in a service-inventory system. Although the pricing strategies of service providers have been addressed in a two-tier service system, inventory policies have yet to be considered.

The rest of this paper is organized as follows: In Section 2, we describe the queueing model of a service-inventory system and derive the key performance measures. In Section 3, we describe the aforementioned three pricing games in a duopoly service-inventory market and propose a solution procedure for the games. In Section 4, we describe the comparative analysis of the three pricing games and show that unified pricing is the most advantageous in terms of maximizing profit. Finally, we conclude the paper in Section 5.

2. Model description and preliminary analysis

In the service-inventory market, the power of retailers with their own service-inventory systems has increased considerably in recent years with the emergence of large supermarkets and chain stores in recent years. These retailers provide a variety of services and products with their brands. For customers, similar products of different brands sold by different retailers are substitutable items. Therefore, these retailers must compete on price, which is a key determinant of market share. From this perspective, our study analyzes three types of pricing games in a duopoly service-inventory market: (i) a parallel pricing game, (ii) a sequential pricing game, and (iii) a unified pricing game. In this section, we present a mathematical model and the solution procedure for obtaining the equilibrium prices of retailers.

We consider the following duopoly market conditions: Two competing retailers that operate their own service-inventory systems. The two retailers sell similar products of different brands. We equivalently index the retailers

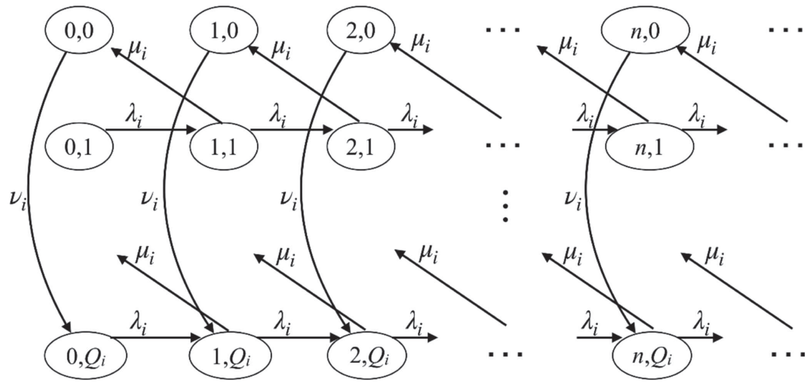


FIGURE 1. Transition diagram for the service-inventory system of a retailer.

and their products by $i, j = 1, 2$. We regard the service-inventory system of retailer i as an $M/M/1$ queue, wherein customers arrive according to a Poisson process at a rate of λ_i. The two retailers compete on price. We assume that each retailer uses a uniform pricing strategy to attract customers. Thus, the arrival rate to the service-inventory system is price dependent, and it is assumed to be a decreasing linear function of price. The arrival rate for retailer i is given by

$$\lambda_i = \alpha_i - \beta_i p_i + \gamma p_j; i = 1, 2, \text{ and } j = 3 - i. \tag{2.1}$$

In equation (2.1), α_i, β_i, and p_i indicate the potential market scale, degree of price sensitivity, and product/service price of retailer i, respectively. γ represents cross-price sensitivity, which reflects the degree of cannibalization between the two retailers. We also assume $\gamma \leq \beta_i$.

A single server serves all customers on a first-come-first-served basis, and one customer is served at a time. Service times follow an exponential distribution with a rate of μ_i. The inequality $\lambda_i < \mu_i$ must hold for the service-inventory system to be stable. Upon the completion of a service, each customer takes one item from the inventory, if any. In this context, we can regard the service time as the duration of delivering an item from the inventory. Subsequently, the customer completes the service and immediately leaves the system. When the on-hand inventory reaches zero, a replenishment order is instantly triggered. The size of the replenishment order is fixed as $Q_i < \infty$ units. The replenishment lead time is exponentially distributed with parameter ν_i. While the inventory level remains zero, any arriving customers are lost.

Let $N_i(t)$ and $K_i(t)$ be the number of customers and the inventory level, respectively, at time t in the service-inventory system of retailer i. Then, the queueing inventory process $\{(N_i(t), K_i(t)), t \geq 0\}$ becomes a two-dimensional continuous time Markov chain with the state space expressed as $\Omega_i = \{(n, k) : n \geq 0, k = 0, 1, \cdots, Q_i\}$. The corresponding transition diagram is shown in Figure 1.

Let $\pi_i(n, k)$ be the stationary joint distribution of the number of customers and the inventory level in the service-inventory system of retailer i. According to Wang and Zhang [14] and Schwarz *et al.* [12], $\pi_i(n, k)$ has the product form of $\pi_i(n, k) = \phi_i(n)\varphi_i(k)$. Concretely, $\phi_i(n)$ denotes the probability distribution of the number of customers, and it is presented by

$$\phi_i(n) = (1 - \lambda_i/\mu_i)(\lambda_i/\mu_i)^n, n \geq 0.$$

Similarly, $\varphi_i(k)$ denotes the probability distribution of the inventory level when the number of customer is i, and it is presented as follows:

$$\varphi_i(k) = (\lambda_i \mathbf{1}_{\{k=0\}} + \nu_i \mathbf{1}_{\{k \neq 0\}})/(\lambda_i + \nu_i Q_i), \ k = 0, 1, \cdots, Q_i,$$

where $\mathbf{1}_{\{\}}$ is an indicator function. In the sequence, we next use $\phi_i(n)$ and $\varphi_i(k)$ to obtain the effective arrival rate λ_i^{eff}, (customer) loss rate λ_i^{loss}, expected number of customers $E[N_i]$, and expected inventory level $E[K_i]$ as follows:

$$\lambda_i^{\text{eff}} = \lambda_i(1-\varphi_i(0)) = \frac{\lambda_i \nu_i Q_i}{\lambda_i + \nu_i Q_i},$$
$$\lambda_i^{\text{loss}} = \lambda_i - \lambda_i^{\text{eff}} = \frac{\lambda_i^2}{\lambda_i + \nu_i Q_i},$$
$$E[N_i] = \sum_{n=0}^{\infty} n\phi_i(n) = \frac{\lambda_i}{\mu_i - \lambda_i},$$
$$E[K_i] = \sum_{k=0}^{Q_i} k\varphi_i(k) = \frac{\nu_i Q_i(Q_i+1)}{2(\lambda_i + \nu_i Q_i)}\cdot$$

Let T_i be the replenishment cycle length of product i. Then, T_i is expressed as the sum of the time required for all Q_i products to run out and the replenishment lead time. Therefore, we have $E[T_i] = Q_i/\lambda_i + 1/\nu_i$. The reorder rate, o_i, can be obtained using $o_i = 1/E[T_i] = \lambda_i\nu_i/(\lambda_i + \nu_i Q_i)$.

3. Pricing games

This section describes the results of several pricing strategies intending to maximize the profits of retailers. First, we separately formulate the revenue and cost terms. Then, we add them to obtain an integrated objective function, which is the average profit per time unit.

– *Marginal revenue*: the average marginal revenue per time unit for retailer i (MR_i) is given by

$$\text{MR}_i = p_i \lambda_i^{\text{eff}} = \frac{p_i \lambda_i \nu_i Q_i}{\lambda_i + \nu_i Q_i}\cdot$$

– *Inventory holding cost*: let h_i be the holding cost for product i per time unit. The average inventory holding cost per time unit for retailer i (IC_i) can then be expressed as

$$\text{IC}_i = h_i E[K_i] = \frac{h_i \nu_i Q_i (Q_i+1)}{2(\lambda_i + \nu_i Q_i)}\cdot$$

– *Ordering cost*: the system incurs a fixed cost c_i each time it creates a replenishment order. One replenishment order is created during each cycle. Thus, the ordering cost per time unit for retailer i (OC_i) is given by

$$\text{OC}_i = c_i o_i = \frac{c_i \lambda_i \nu_i}{\lambda_i + \nu_i Q_i}\cdot$$

– *Loss of goodwill cost*: we assume that lost sales negatively affect the image of the retailer in the long term. Hence, we consider the loss of the goodwill cost per unit l_i for lost sales. The average loss of the goodwill cost per time unit for retailer i (LC_i) is given by

$$\text{LC}_i = l_i \lambda_i^{\text{loss}} = \frac{l_i \lambda_i^2}{\lambda_i + \nu_i Q_i}\cdot$$

– *Objective function*: the long-term average profit function for retailer i is expressed as a function of product prices. Using equation (2.1), we have

$$\Pi_i(p_i, p_j) = \text{MR}_i - \text{IC}_i - \text{OC}_i - \text{LC}_i = \frac{\Pi_i^C(p_i,p_j)}{E[T_i]}, i = 1,2, \text{ and } j = 3-i, \tag{3.1}$$

where $E[T_i] = Q_i/(\alpha_i - \beta_i p_i + \gamma p_j) + 1/\nu_i$ and

$$\Pi_i^C(p_i, p_j) = p_i Q_i - c_i - \frac{h_i Q_i (Q_i + 1)}{2(\alpha_i - \beta_i p_i + \gamma p_j)} - \frac{l_i(\alpha_i - \beta_i p_i + \gamma p_j)}{\nu_i}. \tag{3.2}$$

$\Pi_i^C(p_i, p_j)$ in equation (3.2) indicates the expected profit during each replenishment cycle. The objective function in equation (3.1) is used to solve the pricing games. The following assumptions are made for the convenience of the analysis:

Assumption 3.1. *As it is irrelevant to study pricing decisions with no positive profit in practice, there is at least one feasible point that satisfies $\Pi_i(p_i, p_j) > 0$, where $i = 1, 2$ and $j = 3 - i$.*

Assumption 3.2. *For the pricing problem of retailer i, it should be ensured that $p_i > 0$ and $\lambda_i > 0$. Thus, p_i is defined in the interval $\Theta_i = [0, \bar{p}_i]$, where $\bar{p}_i = (\alpha_i + \gamma p_j)/\beta_i$, $i = 1, 2$, and $j = 3 - i$.*

3.1. Parallel pricing game

In the parallel pricing game, two competing retailers establish retail prices at the same time. We adapt a concept from game theory to obtain equilibrium prices. The retailers are the players of this strategic game, and they have complete information. The outcome of the game is referred to as the Nash equilibrium (NE), in which none of the players can benefit by changing only his/her strategy. Hence, the strategy of each player is optimal against that of the other. In other words, each retailer maximizes its profit given the price of the competitor. For notational simplicity, we denote $\Pi_i(p_i, p_j)$ and $\Pi_i^C(p_i, p_j)$ as Π_i and Π_i^C, respectively.

Proposition 3.3. *In the parallel pricing mechanism, there exists a unique NE under the price of retailer 1, p_1^{NE}, and under the price of retailer 2, p_2^{NE}. Further, the Karush–Kuhn–Tucker (KKT) conditions are necessary and sufficient for finding the unique NE price.*

Proof. Consider the pricing problem of retailer 1 as an example. Given the price of retailer 2, we have

$$\frac{\partial^2 E[T_1]}{\partial p_1^2} = \frac{2\beta_1^2 Q_1}{(\alpha_1 - \beta_1 p_1 + \gamma p_2)^3} > 0,$$

which implies that $E[T_1]$ is strictly convex with respect to p_1. Based on Assumption 3.1 and the strict positivity of $E[T_1]$, there is an area within which $\Pi_1^C > 0$. From

$$\frac{\partial^2 \Pi_1^C}{\partial p_1^2} = -\frac{h_1 \beta_1^2 Q_1 (Q_1 + 1)}{(\alpha_1 - \beta_1 p_1 + \gamma p_2)^3} < 0,$$

we can observe that Π_1^C is strictly concave with respect to p_1. Thus, there must be at most two roots – *i.e.*, p_1^L and p_1^U – for which it is the case that $\Pi_1^C > 0$ for $p_1 > p_1^L$, $p_1 < p_1^U$, or $p_1^L < p_1 < p_1^U$. Let us define a new feasible region for p_1 for our problem: $\max(0, p_1^L) = \underline{p}_1^{\mathrm{new}} \le p_1 \le \bar{p}_1^{\mathrm{new}} = \min(\bar{p}_1, p_1^U)$. It is clear that any $p_1 \in \Theta_1^{\mathrm{new}} = [\underline{p}_1^{\mathrm{new}}, \bar{p}_1^{\mathrm{new}}]$ satisfies $p_1 \in \Theta_1$. Because of the strict concavity and positivity of Π_1^C on Θ_1^{new} and the strict convexity and positivity of $E[T_1]$, condition K in Schaible and Ibaraki [11] is met, and the profit maximization problem of retailer 1,

$$p_1^{\mathrm{NE}} = \underset{p_1 \in \Theta_1}{\operatorname{argmax}}\, \Pi_1 = \underset{p_1 \in \Theta_1^{\mathrm{new}}}{\operatorname{argmax}} \frac{\Pi_1^C}{E[T_1]},$$

given p_2, is a concave fractional programming problem in Θ_1^{new}. According to Proposition 2 in Schaible and Ibaraki [11], Π_1 is strictly pseudo concave, which implies that Π_1 is unimodal. Further, based on Proposition 3 in Schaible and Ibaraki [11] and Assumption 3.1, a point that satisfies the KKT conditions and yields the best objective value is the unique optimal p_1^{NE}. The same argument holds for the pricing problem of retailer 2. According to the KKT conditions, we have

$$\begin{aligned}\frac{\partial \Pi_1}{\partial p_1} = 0 \Leftrightarrow p_1(p_2) = \frac{\alpha_1 + \gamma p_2 + \nu_1 Q_1}{\beta_1} - M_1(p_2),\\ \frac{\partial \Pi_2}{\partial p_2} = 0 \Leftrightarrow p_2(p_1) = \frac{\alpha_2 + \gamma p_1 + \nu_2 Q_2}{\beta_2} - M_2(p_1),\end{aligned} \tag{3.3}$$

where

$$M_i(p_j) = \sqrt{\frac{\nu_i Q_i[\beta_i h_i(Q_i+1) + 2\nu_i\{Q_i(\alpha_i + \gamma p_j + \nu_i Q_i) - \beta_i c_i\}]}{2\beta_i^2(\beta_i l_i + \nu_i Q_i)}}, i = 1, 2 \text{ and } j = 3 - i.$$

The equation system in equation (3.3) can easily be solved by employing numerical methods using mathematics software. □

Remark 3.4. The effect of retail prices on the optimal replenishment cycle can be determined through the following relationships:

$$\begin{aligned}E[T_i] &= \frac{Q_i}{\lambda_i} + \frac{1}{\nu_i} = \frac{Q_i}{\alpha_i - \beta_i p_i + \gamma p_j} + \frac{1}{\nu_i},\\ \frac{\partial E[T_i]}{\partial p_i} &= \frac{\beta_i Q_i}{(\alpha_i - \beta_i p_i + \gamma p_j)^2} > 0,\\ \frac{\partial E[T_i]}{\partial p_j} &= -\frac{\gamma Q_i}{(\alpha_i - \beta_i p_i + \gamma p_j)^2} < 0.\end{aligned}$$

Therefore, the optimal replenishment cycle of a retailer increases with the price of that retailer; however, the optimal replenishment cycle of a retailer decreases as the retail price of a competitor increases. These results can be intuitively explained as follows: An increase in the price of a retailer decreases the arrival rate to his/her service-inventory system, and the replenishment cycle becomes longer as the arrival rate decreases. Conversely, the arrival rate to the service-inventory system of a retailer increases with the price of a competitor; the replenishment cycle becomes shorter as the arrival rate increases.

3.2. Sequential pricing game

In the sequential pricing game, two competing retailers have different levels of power in pricing. This type of pricing game is generally modeled as a duopoly Stackelberg game, where one retailer is a pricing leader and the other is a pricing follower. For convenience, we regard retailers 1 and 2 as the pricing leader and the pricing follower, respectively. Therefore, retailer 2 maximizes his/her profit based on the price of retailer 1, and retailer 1 maximizes his/her profit based on the best response function of retailer 2.

Proposition 3.5. *In the sequential pricing mechanism, the price of retailer 1, p_1^{ST}, and the price of retailer 2, p_2^{ST}, form a unique equilibrium solution for the Stackelberg pricing game. In addition, p_1^{ST} and p_2^{ST} are obtained through Algorithm 1.*

Proof. From Proposition 3.3, the profit function of retailer 2 is strictly pseudo concave and unimodal in p_2. Thus, given p_1, the optimal price of retailer 2, $p_2^{\mathrm{ST}}(p_1)$, can be obtained by solving the first-order condition, $\partial\Pi_2/\partial p_2 = 0$. First, we substitute $p_2^{\mathrm{ST}}(p_1)$ into the profit function of retailer 1. The strict pseudo concavity and unimodality of Π_1 ensures the existence of a unique equilibrium for this Stackelberg pricing game. The first-order condition, $\partial\Pi_1/\partial p_1 = 0$, is solved to find p_1^{ST}. After calculating p_1^{ST}, p_2^{ST} is calculated from equation (3.3). This solution procedure is summarized in Algorithm 1. □

Algorithm 1. Calculation of p_1^{ST} and p_2^{ST}.

1: **Set** $p_2 = (\alpha_2 + \gamma p_1 + \nu_2 Q_2)\beta_2^{-1} - M_2(p_1)$ using equation (3.3);
2: **Set** $\Pi_1 = \Pi_1(p_1, p_2)$ using equation (3.1);
3: **Solve** $\partial\Pi_1/\partial p_1 = 0$ with respect to p_1; and then
Set $p_1^{\text{ST}} = \{p_1 : \partial\Pi_1/\partial p_1 = 0, p_1 \in \Theta_1\}$;
4: **Calculate** $p_2^{\text{ST}} = \left(\alpha_2 + \gamma p_1^{\text{ST}} + \nu_2 Q_2\right)\beta_2^{-1} - M_2\left(p_1^{\text{ST}}\right)$

3.3. Unified pricing game

In the unified pricing game, it is assumed that both substitutable products are sold by a monopolist, and that p_1 and p_2 are jointly determined to maximize profit. This is similar to a case where two retailers cooperate and set their prices to maximize their total profit and then share the added benefit according to an agreed mechanism. This unified pricing game is expressed as follows:

$$\left(p_1^{\text{UN}}, p_2^{\text{UN}}\right) = \underset{(p_1,p_2)\in\Theta_1\times\Theta_2}{\operatorname{argmax}} \quad \Pi_1 + \Pi_2 \tag{3.4}$$

The objective function in equation (3.4) may not be unimodal, because there is no guarantee that the sum of two strictly pseudo concave functions is also strictly pseudo concave. However, Shen and Yu [13] recently introduced an iterative approach for solving a concave-convex fractional programming (CCFP) problem such as that in equation (3.4). The CCFP defined by Shen and Yu [13] has the standard form of

$$x^* = \underset{x}{\operatorname{argmax}} \sum_{m=1}^{M} \frac{A_m(x)}{B_m(x)}, \tag{3.5}$$

where the following conditions are satisfied: (i) all $A_m(x)$ are concave, (ii) all $B_m(x)$ are convex, and (iii) constraint set x is a nonempty convex set in the standard form expressed by a finite number of inequality constraints. Then, the CCFP problem in equation (3.5) is equivalent to

$$(x^*, y^*) = \underset{x,y}{\operatorname{argmax}} \sum_{m=1}^{M} \left(2y_m\sqrt{A_m(x)} - y_m^2 B_m(x)\right), \tag{3.6}$$

where y denotes a collection of auxiliary variables $\{y_1, \cdots, y_M\}$. Briefly, the iterative approach for solving equation (3.6) is as follows: When x is fixed, the optimal y_m can be found in the closed form as $y_m = \sqrt{A_m(x)}/B_m(x)$, $\forall m = 1, \cdots, M$. When y_m is fixed, owing to the concavity of each $A_m(x)$, the convexity of each $B_m(x)$, and the fact that the square-root function is concave and increasing, the quadratic transform, $2y_m\sqrt{A_m(x)} - y_m^2 B_m(x)$, is concave in x. Therefore, the problem in equation (3.6) becomes a concave maximization problem over x. As a result, the optimal x can be efficiently obtained through a numerical convex optimization process such as the Newton–Raphson method. More details on the CCFP problem can be found in Shen and Yu [13] and the references therein. Based on all of the above results, we introduce the final proposition.

Proposition 3.6. *In the unified pricing mechanism, the price of retailer 1, p_1^{UN}, and the price of retailer 2, p_2^{UN}, form the concave maximization problem in equation (3.7). In addition, p_1^{UN} and p_2^{UN} are obtained through Algorithm 2.*

Proof. The objective function in equation (3.4) is rewritten as $\Pi_1 + \Pi_2 = \Pi_1^C/E[T_1] + \Pi_2^C/E[T_2]$ using equation (3.1). From Proposition 3.3, Π_i^C and $E[T_i]$ are strictly concave and convex, respectively. The constraint set, $\Theta_1 \times \Theta_2$, is a nonempty convex set. Therefore, the maximization problem in equation (3.4) is converted to the following quadratic transformed CCFP problem:

$$\max \quad \sum_{i=1}^{2} \left(2y_i\sqrt{\Pi_i^C} - y_i^2 E[T_i]\right)$$

$$\text{subject to } p_i \in \Theta_i, 1, 2,$$
$$y_i \in \mathbb{R}, i = 1, 2. \tag{3.7}$$

The solution algorithm for equation (3.7) is provided in Algorithm 2. The convergence of equation (3.7) to a stationary point has been proven in Theorem 3 in Shen and Yu [13]; therefore, it is omitted here. □

Algorithm 2. Calculation p_1^{UN} and p_2^{UN}.

1: **Initialize** $(p_1^0, p_2^0) \in \Theta_1 \times \Theta_2$;
2: **Set** $t = 0$; $//t$ is an iteration index
3: **Repeat**
  Update $y_i^{t+1} = \sqrt{\Pi_i^C(p_i^t, p_j^t)}/E[T_i], i = 1, 2$;
  Update p_i^{t+1} by solving problem in equation (3.7);
  Set $t = t + 1$;
4: **until** convergence;
5: **Set** $(p_1^{\text{UN}}, p_2^{\text{UN}}) = (p_1^t, p_2^t)$;

4. Numerical experiments

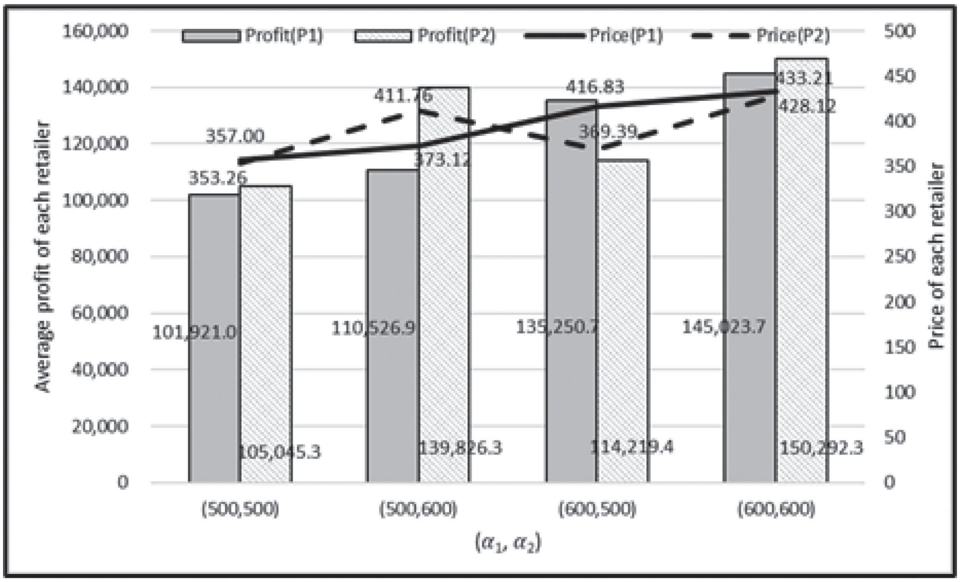
We investigate the effect of changes in various parameters on each retailer's pricing game through various numerical examples. The main experimental parameters are classified into three types. First, α_i, β_i and γ_i are used to determine the arrival rates λ_i where $i = 1, 2$. Second, Q_i and ν_i $(i = 1, 2)$ are related to the operation strategies of the two retailers. Finally, the operating costs are calculated using h_i, l_i and $c_i(i = 1, 2)$. The parameters vary according to their respective values as listed in Table 1.

First, we investigate the effects of the potential market scales denoted by α_1 and α_2. The other parameters are fixed as follows: $\beta_1 = \beta_2 = 1, \nu_1 = \nu_2 = 10, Q_1 = Q_2 = 400, h_1 = h_2 = 5, l_1 = l_2 = 10$, and $c_1 = c_2 = 50$. We obtain the following figures by choosing values between 500 and 600 for α_1 and α_2, and values between 0.5 and 0.8 for γ. Figure 2 shows the results of parallel games. P1 and P2 respectively corresponds to retailer 1 and retailer 2. The horizontal axis denotes the combinations of α_1 and α_2, while the bar charts and the line graphs respectively denote the average profit $(\Pi_1^{\text{NE}}, \Pi_2^{\text{NE}})$ and prices of two retailers (p_1, p_2), respectively. Further, the results of the sequential and unified games are presented in Figures 3 and 4, respectively.

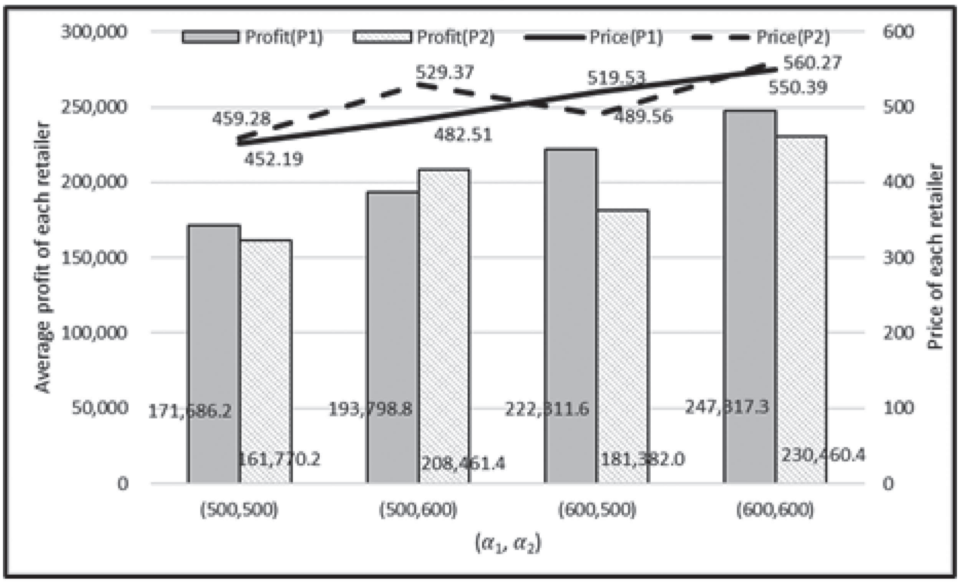
In general, the figures are showing that as the market scale increases, retailers can obtain higher profits by raising prices. A retailer of a larger market scale is more profitable. When the market scale is the same, the equilibrium price of retailer 2 is determined to be lower than that of retailer 1, and it thus attracts more

Table 1. Values of parameters.

Parameter	Values
α_1, α_2	300, 400, 500, 600
β_1, β_2	0.8, 1.0, 1.2, 1.4
γ	0.2, 0.5, 0.7, 0.8
ν_1, ν_2	5, 10
Q_1, Q_2	300, 400, 500
h_1, h_2	5
l_1, l_2	5, 10, 15
c_1, c_2	30, 50, 60



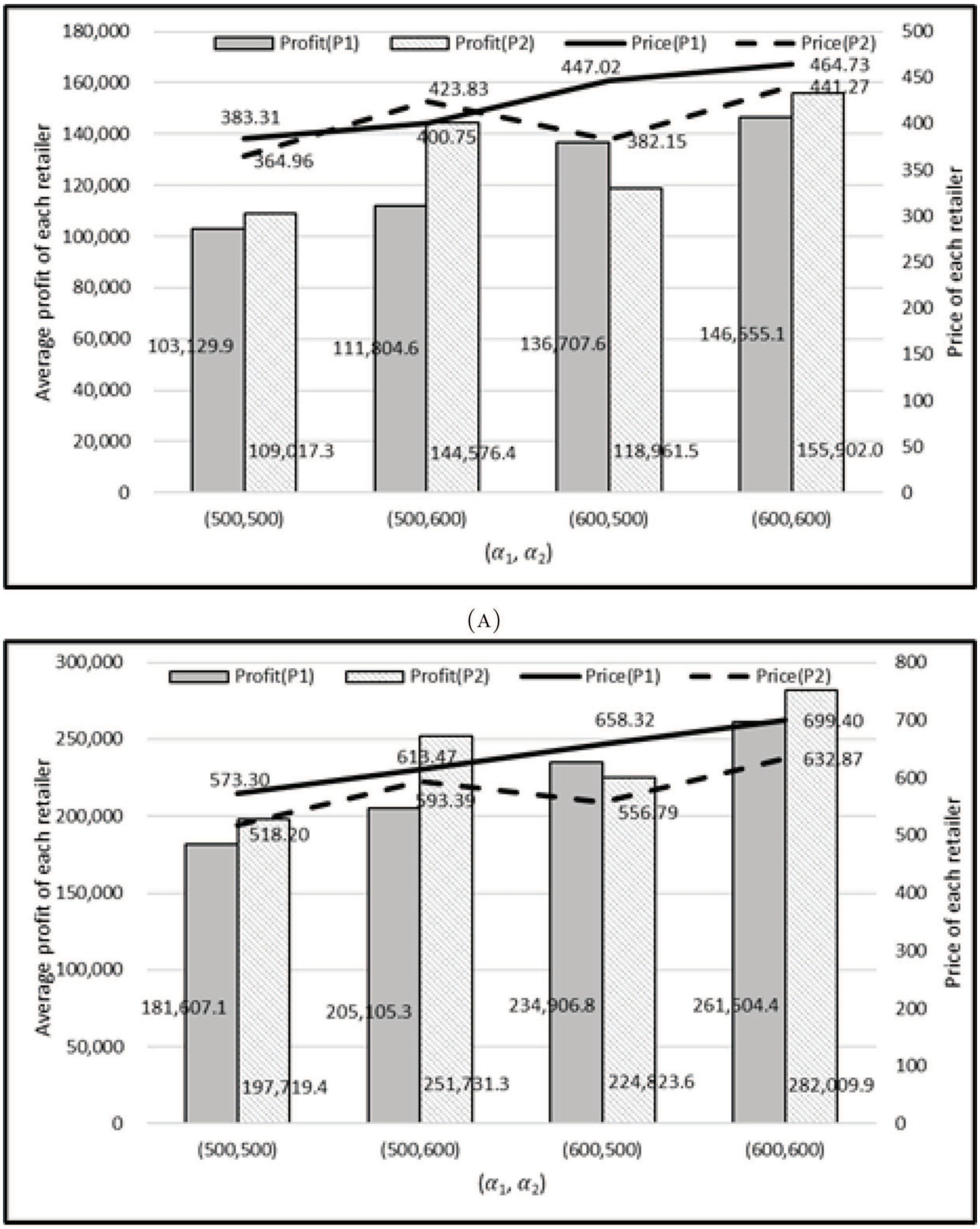
(A)



(B)

FIGURE 2. Results of parallel games when $\alpha_1, \alpha_2 \in \{500, 600\}$. (A) $\gamma = 0.5$ case. (B) $\gamma = 0.8$ case.

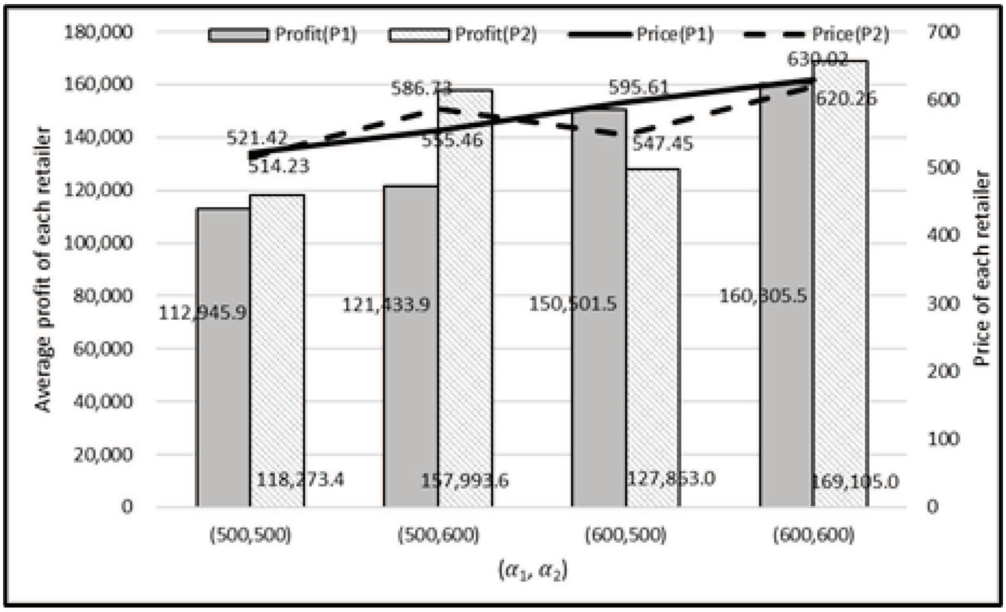
customers (*i.e.*, $\lambda_1 < \lambda_2$). This is because, even though the values of Q_1 and Q_2 are the same, $E[T_i]$ decrease as λ_i increases, the profit per unit time can increase. It is also observed that the relation $\Pi_i^{\mathrm{NE}} < \Pi_i^{\mathrm{ST}} < \Pi_i^{\mathrm{UN}}$, where $i = 1, 2$, holds. If the value of γ is large, then the degree of influence from the price of the other retailer increases, so the price is determined to be higher than it is when the value is low. This results in increased profit.



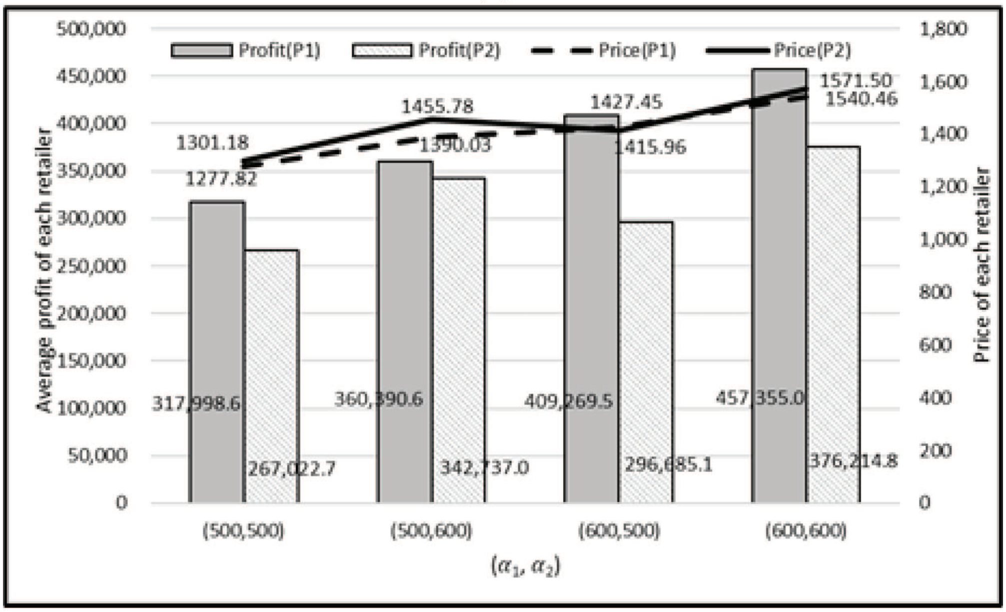
(A)

(B)

FIGURE 3. Results of sequential games when $\alpha_1, \alpha_2 \in \{500, 600\}$. (A) $\gamma = 0.5$ case. (B) $\gamma = 0.8$ case.

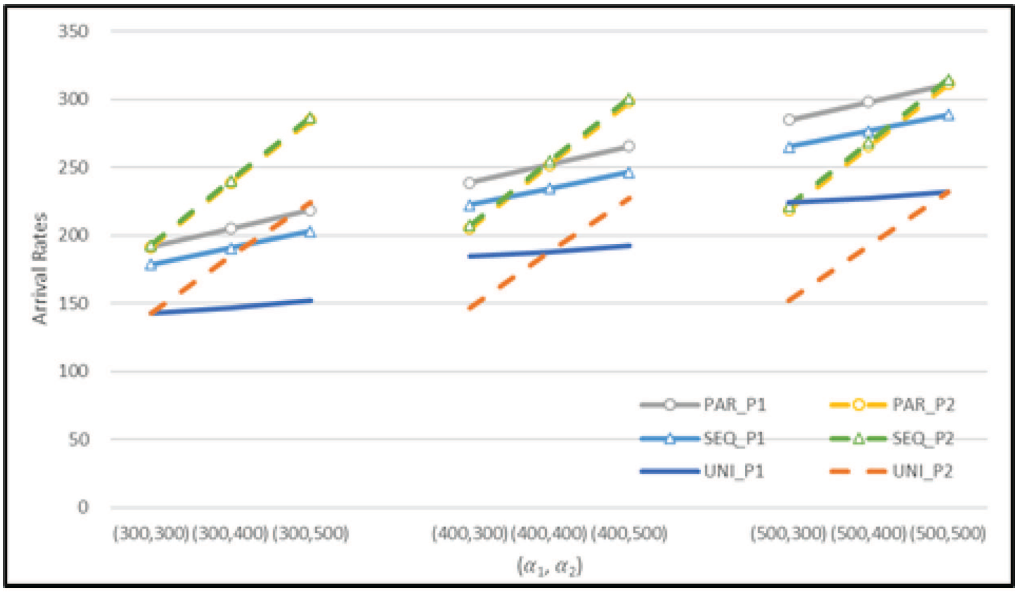


(A)

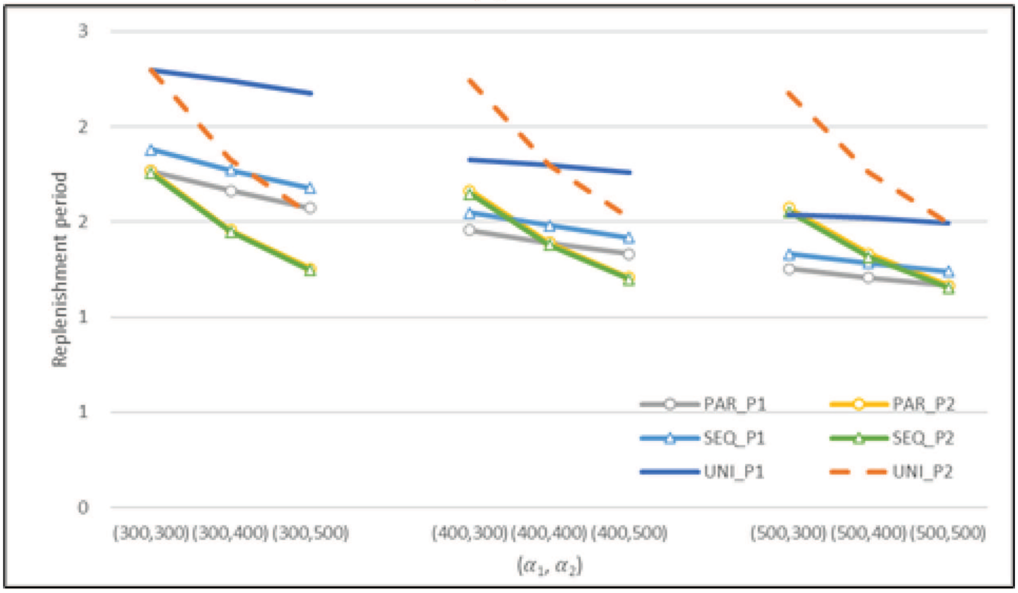


(B)

FIGURE 4. Results of unified games when $\alpha_1, \alpha_2 \in \{500, 600\}$. (A) $\gamma = 0.5$ case. (B) $\gamma = 0.8$ case.



(A)



(B)

FIGURE 5. Arrival rates and expected replenishment cycle when $\alpha_1, \alpha_2 \in \{500, 600\}$. (A) Arrival rates ($\gamma = 0.5$). (B) Expected replenishment cycle $E[T_i]\gamma = 0.5$.

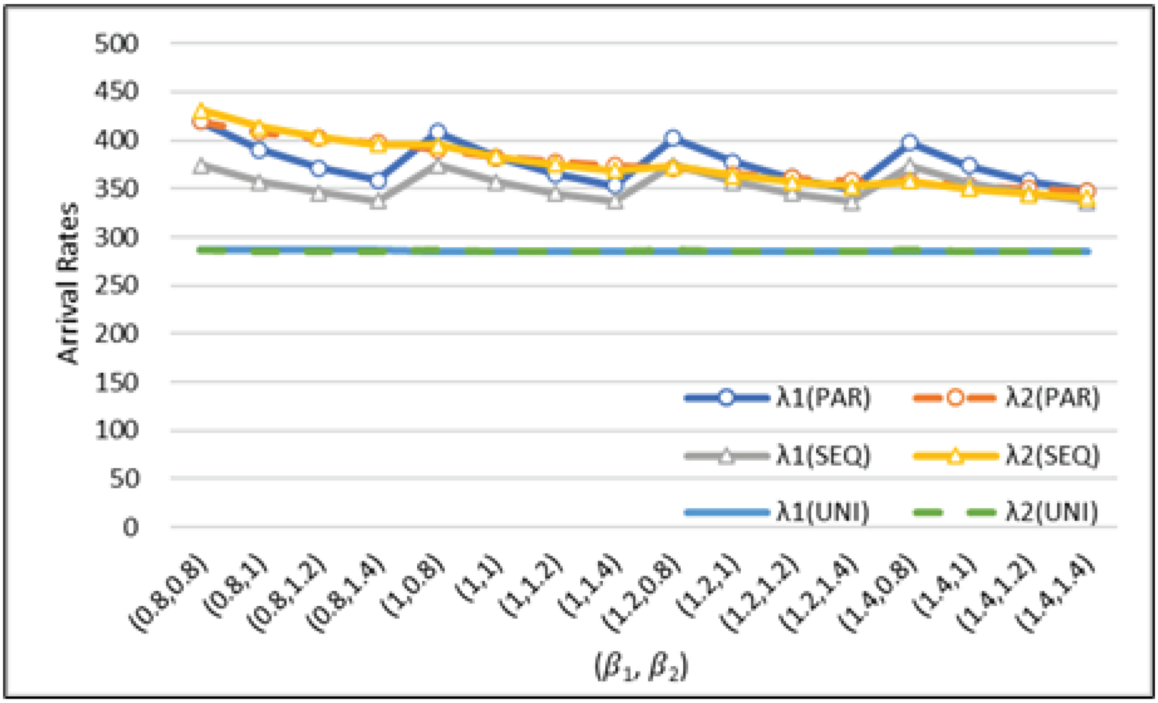


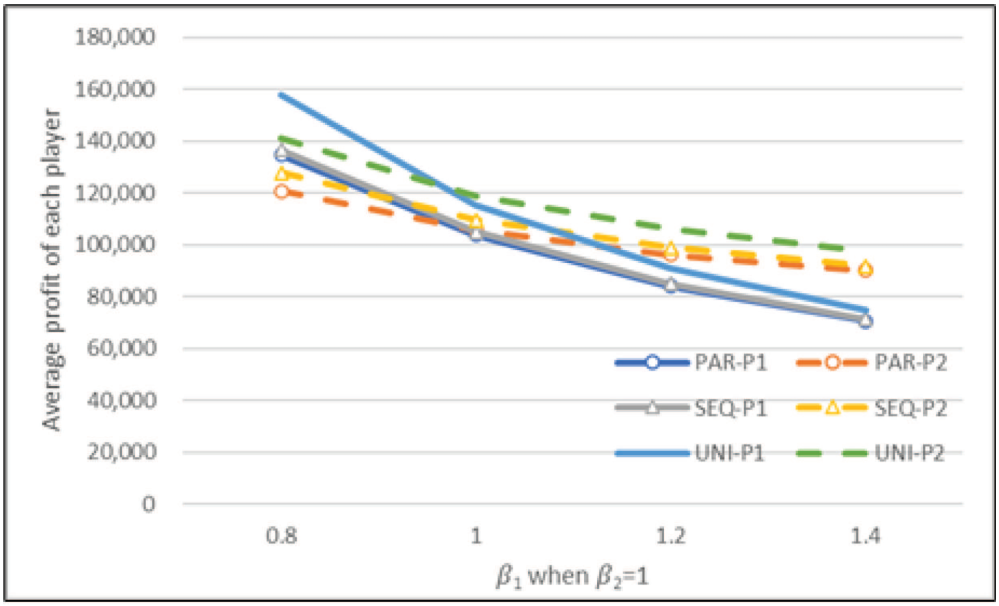
FIGURE 6. Arrival rates on various combinations of β_1 and β_2.

Figure 5a shows that the arrival rates increase as the values of $\alpha_i (i = 1, 2)$ increase. The arrival rate of retailer 1 has the smallest change when the game is of the unified type. Figure 5b presents the decrease in $E[T_i] (i = 1, 2)$ as the arrival rates increase. "PAR", "SEQ" and "UNI" are used to respectively denote the parallel game, sequential game, and unified game.

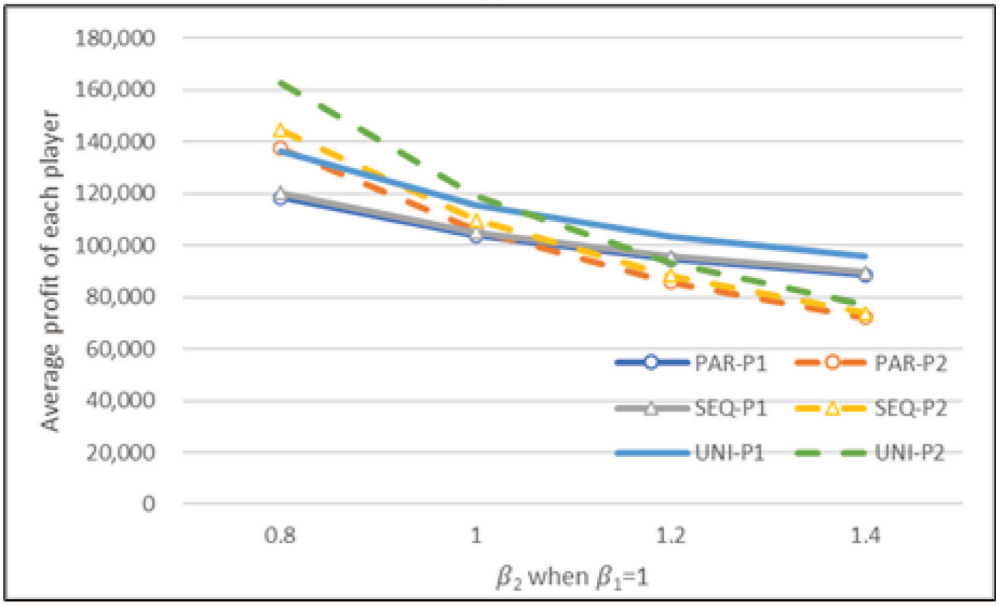
In the results of the unified game, it is found that λ_1 (or λ_2) is insensitive to the values of β_1 and β_2, given that the other parameters are fixed. Figure 6 shows an example of an almost constant arrival rates, while the values of β_1 and β_2 vary. Although each retailer has almost the same arrival rate, their profit and price are determined by the value of β_1 (or β_2).

The influence of β_1 and β_2 are also investigated. We present Figure 7 by fixing β_1 and varying β_2, and vice versa. The profit of each retailer is shown in the vertical axis. The equilibrium prices of both retailers decrease as β_1 increases. This result can be intuitively justified as follows: As β_1 increases, a higher price of retailer 1 has a stronger effect on decreasing the demand of retailer 1. Thus, retailer 1 reduces his/her price to minimize this effect. In addition, retailer 2 observes the reduction in the price of retailer 1 and decreases his/her price to compete. Furthermore, an increase in β_1 increases the replenishment cycles and decreases the profit of both retailers. Figure 7 also shows that the relation $\Pi_i^{\mathrm{NE}} < \Pi_i^{\mathrm{ST}} < \Pi_i^{\mathrm{UN}}$ holds.

Figure 8 shows the profits of the retailers when the value of γ is low: we set γ to 0.2. We observe that the profits of the retailers are almost the same regardless of the game types. When the value of γ decreases, the profit of each retailer also decreases. Table 2 compares the profits in the cases of $\gamma = 0.2$ and $\gamma = 0.8$. For example, for the parallel game when $\beta_1 = 1.4$ and $\gamma = 0.2$, the profit decreases by 51.8% compared to when $\gamma = 0.8$.

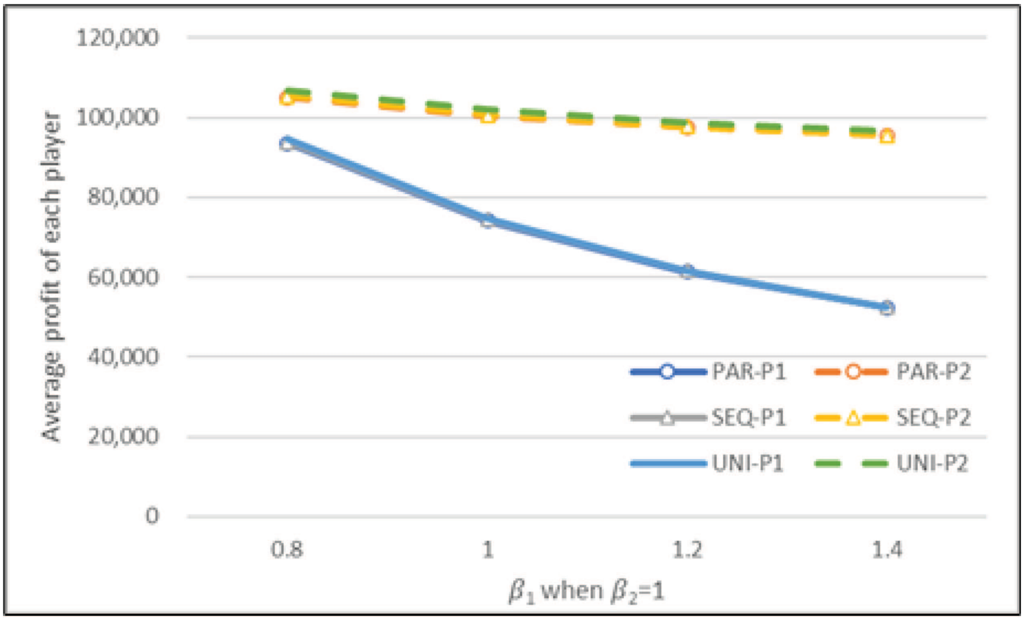


(A)

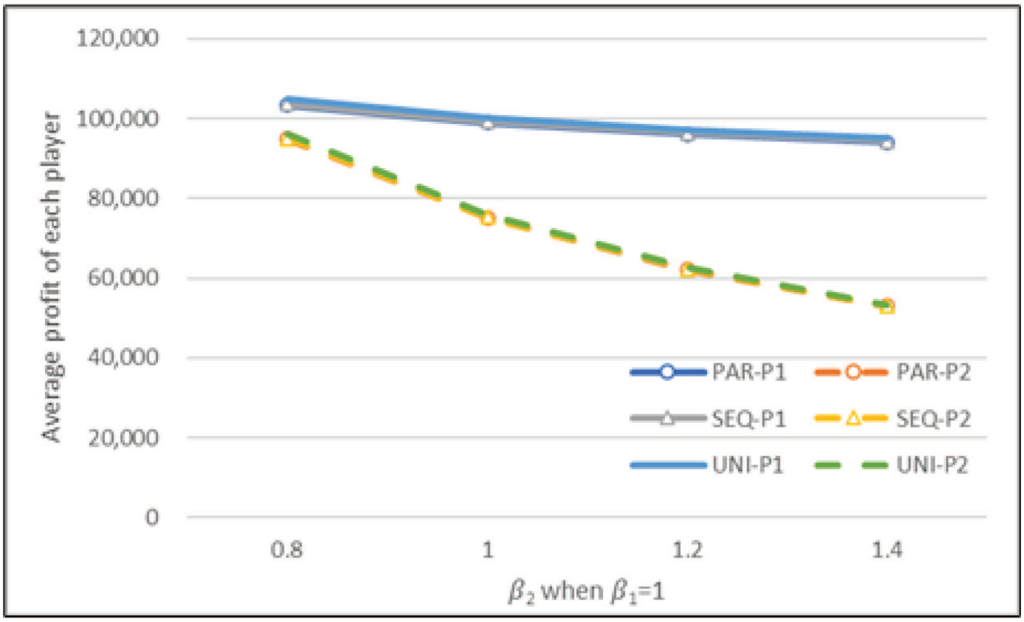


(B)

FIGURE 7. Profits of two retailers for the combination of the values of β_1 and β_2. (A) Profits when β_2 is fixed. (B) Profits when β_1 is fixed.



(A)



(B)

FIGURE 8. Profits of two retailers with low γ value. (A) Profits when $\gamma = 0.2$ and β_2 is fixed. (B) Profits when $\gamma = 0.2$ and β_1 is fixed.

TABLE 2. Results of pricing games for the values of $\gamma = 0.2, 0.8$ and $\beta_2 = 1$.

Game type	β_1	p_1 $\gamma = 0.2$	p_1 $\gamma = 0.8$	p_2 $\gamma = 0.2$	p_2 $\gamma = 0.8$	$\Pi_1 + \Pi_2$ $\gamma = 0.2$	$\Pi_1 + \Pi_2$ $\gamma = 0.8$
Parallel pricing	0.8	368.81	638.99	348.09	584.39	198 537.6	536 885.9
	1	294.39	481.05	340.19	515.38	174 675.3	405 256.0
	1.2	245.02	386.01	334.95	474.14	158 946.9	331 992.3
	1.4	209.86	322.46	331.22	446.69	147 799.6	285 538.3
Sequential pricing	0.8	374.12	872.75	353.35	692.06	199 163.5	655 789.8
	1	297.86	600.87	345.25	572.21	175 113.9	457 913.5
	1.2	247.48	459.21	339.91	510.45	159 278.4	361 339.6
	1.4	211.72	371.95	336.12	472.66	148 062.8	304 190.6
Unified pricing	0.8	420.81	3153.39	392.83	2828.45	201 575.1	1 497 591.2
	1	333.31	1402.21	375.37	1428.58	176 749.1	711 377.2
	1.2	275.99	901.76	363.93	1028.68	160 483.9	486 745.6
	1.4	235.52	664.69	355.85	839.28	149 002.6	380 341.4

5. Conclusion

We evaluate three pricing games of two competitors in a duopoly market. The inventory holding cost, ordering cost, and loss of the goodwill cost are considered to maximize the revenue of the competitors. The service-inventory system of each competitor is modeled as an $M/M/1$ queue with variable arrival rates, which are price dependent. A decreasing linear function of price is assumed to attract customers. Three types of games are analyzed. The uniqueness of the equilibrium prices is proven, and the solution procedure for obtaining equilibrium prices is outlined. Finally, Table 2 lists the performance measures and characteristics of each game. The results of the pricing games for the values of $\gamma = 0.2, 0.8$ and $\beta_2 = 1$ are presented. We provide managerial implications for each pricing game and show that the unified pricing strategy is the most advantageous for both competitors.

In the future, the pricing games of two competitors, each with a service-inventory system, can be analyzed by considering other distributions for the service time and replenishment lead time. The backorder assumption can be applied instead of lost sales. As this assumption appears to make the problem difficult to solve, other computer-based experiments can be applied to investigate the pricing games.

Acknowledgements. This work was supported by Basic Science Research through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A3B07040887). This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2020R1I1A3A04037238). This study has been worked with the support of a research grant of Kangwon National University in 2021.

References

[1] J.W. Baek, Y.H. Bae, H.W. Lee and S. Ahn, Continuous-type (s, Q)-inventory model with an attached $M/M/1$ queue and lost sales. *Perform. Eval.* **125** (2018) 68–79.

[2] O. Berman and E. Kim, Stochastic models for inventory management at service facilities. *Commun. Stat. Stoch. Models* **15** (1999) 695–718.

[3] O. Berman and K.P. Sapna, Inventory management at service facilities for systems with arbitrarily distributed service times. *Commun. Stat. Stoch. Models* **16** (2000) 343–360.

[4] J. M. Harrison, Assembly-like queues. *J. Appl. Probab.* **10** (1973) 354–367.

[5] J.-Y. Kim, M.H. Lee and N. Berg, Peak-load pricing in duopoly. *Econ. Modell.* **57** (2016) 47–54.

[6] A. Krishnamoorthy and V.C. Narayanan, Production inventory with service time and vacation to the server. *IMA J. Manage. Math.* **22** (2011) 33–45.

[7] A. Krishnamoorthy, R. Manikandan, R. Manikandan, B. Lakshmy and B. Lakshmy, A revisit to queueing-inventory system with positive service time. *Ann. Oper. Res.* **233** (2015) 221–236.
[8] D.H. Lee, Pricing decisions in a competitive closed-loop supply chain with duopolistic recyclers. *Math. Prob. Eng.* **2020** (2020) 1–22.
[9] A.J. Marand, H. Li and A. Thorstenson, Joint inventory control and pricing in a service-inventory system. *Int. J. Prod. Econ.* **209** (2019) 78–91.
[10] M. Saffari, S. Asmussen and R. Haji, The $M/M/1$ queue with inventory, lost sale, and general lead times. *Queue. Syst.* **75** (2013) 65–77.
[11] S. Schaible and T. Ibaraki, Fractional programming. *Eur. J. Oper. Res.* **12** (1983) 325–338.
[12] M. Schwarz, C. Sauer, H. Daduna, R. Kulik and R. Szekli, $M/M/1$ queueing systems with inventory. *Queue. Syst.* **54** (2006) 55–78.
[13] K. Shen and W. Yu, Fractional programming for communication systems-part I: power control and beamforming. *IEEE Trans. Signal Process.* **66** (2018) 2616–2630.
[14] J. Wang and X. Zhang, Optimal pricing in a service-inventory system with delay-sensitive customers and lost sales. *Int. J. Prod. Res.* **55** (2017) 6883–6902.
[15] T.M. Whitin, Inventory control and price theory. *Manage. Sci.* **2** (1955) 61–68.
[16] S.-L. Yang and Y.-W. Zhou, Two-echelon supply chain models: Considering duopolistic retailers' different competitive behaviors. *Int. J. Prod. Econ.* **103** (2006) 104–116.
[17] Z.G. Zhang and X. Yin, Information and pricing effects in two-tier public service systems. *Int. J. Prod. Econ.* **231** (2021) 107897.