

SUFFICIENT CONDITIONS FOR EXTREMUM OF FRACTIONAL VARIATIONAL PROBLEMS

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Abstract. Sufficient conditions for extremum of fractional variational problems are formulated with the help of Caputo fractional derivatives. The Euler–Lagrange equation is defined in the Caputo sense and Jacobi conditions are derived using this. Again, Wierstrass integral for the considered functional is obtained from the Jacobi conditions and the transversality conditions. Further, using the Taylor’s series expansion with Caputo fractional derivatives in the Wierstrass integral, the Legendre’s sufficient condition for extremum of the fractional variational problem is established. Finally, a suitable counterexample is presented to justify the efficacy of the fresh findings.

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1. INTRODUCTION AND PRELIMINARIES

The fractional calculus deals with the problem of extremizing functionals, which are of non integer order and are differentiable. It’s origin is more than three centuries old, and dates back to L’Hôpital’s query to Leibniz about the significance of the fractional order derivative of a function. In his reply to the above question, Leibniz indicated it to be a paradox which can lead to useful consequences in the near future [36]. The subject caught the attention of many eminent mathematicians including Euler, Fourier, Laplace, Lacroix, Abel, Riemann, Liouville and Caputo *et al.* for its advancement. Several fractional derivatives in the sense of Grünwald Letnikov, Riemann–Liouville and Caputo *et al.* are discussed in [32]. However, it gained popularity in the last three decades for its wide applications in the field of science, engineering, statistics and mathematical biology [1–6, 12, 14, 17, 19–21, 24, 28, 34, 35, 37] etc.

The fractional derivatives are used in calculus of variations to generalize it to non integer orders. Agrawal [1] formulated the Euler–Lagrange equations for fractional variational problem and proved the necessary conditions for extremum of fractional variational problems using Riemann–Liouville fractional derivatives. Legendre’s second order necessary optimality conditions for weak extremizers of the variational problems was subsequently established by Lazo and Torres [29], where the involved functionals are fractional differentiable in the sense of Riemann–Liouville. The isoperimetric problems have also been proved using the Caputo fractional derivatives [10, 30].

Keywords. Caputo fractional derivative, Jacobi conditions, transversality conditions, Wierstrass integral, Legendre’s condition.

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Variational functionals with a Lagrangian are considered by Odzijewicz *et al.* [31] using classical as well as Caputo fractional derivatives. Wang and Xiao [39] defined the fractional variational integrators in terms of Caputo derivatives. The notion of Euler–Lagrange fractional extremals is used to prove the Noether type theorem by Frederico and Torres [16]. Agrawal and Baleanu [7] studied fractional optimal control problems as an application of the Riemann–Liouville fractional derivatives.

Recently, Zhang [40] proved the necessary and sufficient optimality conditions for the fractional variational problems using the Caputo–Fabrizio fractional derivatives, and Baleanu *et al.* [11] presented suitable counterexamples to support it. Almeida [9] considered several problems of calculus of variations depending upon Lagrange function on a Caputo type fractional derivative. Again, Almeida [8], obtained the necessary and sufficient conditions, when the involved functionals are dependent upon the fractional integrals and fractional derivatives having indefinite integral and in the presence of time delay. Rahman *et al.* [33] established the generalized Riemann–Liouville fractional integrals in the sense of increasing, positive, monotone, and measurable function Ψ and also discussed the Chebyshev functionals. Goufo *et al.* [18] explored the differential fractional operators, which include Atangana–Baleanu derivative and the Caputo–Fabrizio derivative.

Several interesting applications of fractional derivatives in modeling the ecological systems are found in [23, 26, 28]. Between the Haar wavelet and Adams–Bashforth–Moulton method, a comparative investigation for the fractional Lotka–Volterra (LV) system in the Caputo sense was done by Kumar *et al.* [22]. The fractional derivatives such as the Caputo, Caputo–Fabrizio and Atangana–Baleanu derivatives are used to investigate a mathematical system numerically to figure out the possible dynamics for the spread of the diseases like HIV/AIDS and COVID-19 by Kumar *et al.* [25, 27].

In this paper, sufficient optimality conditions for fractional variational problems are investigated with the help of Legendre’s condition by using the Caputo fractional derivative. Again, the Jacobi condition and the transversality conditions are discussed to validate the aforesaid condition. With the help of fractional Taylor’s series expansion, the Legendre’s sufficient conditions for extremum of the fractional variational problems are obtained. A number of remarks are mentioned at appropriate places for verifying the proper generalizations of several classical equations and conditions. To support our new findings, a suitable counterexample is also provided.

Definition 1.1 ([15]). Let f be a real valued function in the interval $[a, b]$, α be a positive real number, n be the integer satisfying $(n - 1) \leq \alpha < n$, and Γ be the Euler gamma function. Then,

(i) the left Caputo fractional derivatives (CFD) of order α is defined by

$${}_a^C \mathcal{D}_\mu^\alpha f(\mu) = \frac{1}{\Gamma(n - \alpha)} \int_a^\mu \frac{f^{(n)}(\lambda)}{(\mu - \lambda)^{\alpha+1-n}} d\lambda,$$

and

(ii) the right CFD of order α is defined by

$${}_\mu^C \mathcal{D}_b^\alpha f(\mu) = \frac{1}{\Gamma(n - \alpha)} \int_\mu^b (-1)^n \frac{f^{(n)}(\lambda)}{(\lambda - \mu)^{\alpha+1-n}} d\lambda.$$

Remark 1.2. If $\alpha \in \mathbb{Z}$, then the derivatives discussed above can be standardized in the classical sense, *i.e.*,

$${}_a^C \mathcal{D}_\mu^\alpha f(\mu) = \left(\frac{d}{d\mu} \right)^\alpha f(\mu), {}_\mu^C \mathcal{D}_b^\alpha f(\mu) = \left(-\frac{d}{d\mu} \right)^\alpha f(\mu), \quad \alpha = 1, 2, \dots$$

Remark 1.3. The CFD of a constant is always equal to zero.

Lemma 1.4. Consider a functional of the form

$$J(\mu) = \int_{\lambda_0}^{\lambda_1} f\left(\lambda, \mu, {}_a^C \mathcal{D}_\lambda^\alpha \mu, {}_\lambda^C \mathcal{D}_b^\beta \mu\right) d\lambda,$$

defined on the set of functions $\mu(\lambda)$ with continuous left and right CFDs of order α and β respectively in $[\lambda_0, \lambda_1]$, having boundary conditions $\mu(\lambda_0) = \mu_0$ and $\mu(\lambda_1) = \mu_1$. Then $J[\mu]$ will have an extremum at $\mu(\lambda)$ if $\mu(\lambda)$ satisfies the necessary condition (1.1) i.e., the Euler–Lagrange equation:

$$\frac{\partial F}{\partial \mu} + {}_C^{\lambda} \mathcal{D}_b^{\alpha} \frac{\partial F}{\partial {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu} + {}_a^C \mathcal{D}_{\lambda}^{\beta} \frac{\partial F}{\partial {}_{\lambda}^C \mathcal{D}_b^{\beta} \mu} = 0. \quad (1.1)$$

Remark 1.5. If the set of functions $\mu(\lambda)$ have continuous left and right Riemann–Liouville derivatives of order α and β respectively, then the above Lemma 1.4 becomes Theorem 1, Page-372 of [1].

2. RESULT AND DISCUSSION

2.1. The Jacobi condition

Consider the functional $J[\mu(\lambda)] = \int_{\lambda_0}^{\lambda_1} F(\lambda, \mu, {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu, -{}_{\lambda}^C \mathcal{D}_b^{\beta} \mu) d\lambda$, where the function $F(\lambda, \mu, {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu, -{}_{\lambda}^C \mathcal{D}_b^{\beta} \mu)$ under consideration has continuous first and second order partial derivatives with respect to all its arguments. Then the necessary condition for the functional $J[\mu]$ to have an extremum is that it satisfies the Euler–Lagrange’s equation given by (1.1).

Let $\mu = \mu(\lambda, \kappa)$, be the solution curves of the Euler–Lagrange’s equation having center at the point $A(\lambda_0, \mu_0)$. The κ -discriminant curve is defined by the equations;

$$\mu = \mu(\lambda, \kappa) \quad (2.1)$$

and

$$\frac{\partial \mu(\lambda, \kappa)}{\partial \kappa} = 0. \quad (2.2)$$

The curves of this family that are close to the extremals $\mu = \mu(\lambda)$ moving through the points $A(\lambda_0, \mu_0)$ & $B(\lambda_1, \mu_1)$ and the κ -discriminant curves will intersect at points close to the tangent points of the extremal under consideration with the κ -discriminant curve. The function $\mu(\lambda)$ will have continuous left and right Caputo fractional derivatives of order α and β , respectively with the boundary points $A(\lambda_0, \mu_0)$ and $B(\lambda_1, \mu_1)$.

Now let’s consider the functional on the pencil of extremals i.e.,

$$J[\mu(\lambda, \kappa)] = \int_{\lambda_0}^{\lambda_1} F(\lambda, \mu(\lambda, \kappa), {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu(\lambda, \kappa), -{}_{\lambda}^C \mathcal{D}_b^{\beta} \mu(\lambda, \kappa)) d\lambda. \quad (2.3)$$

The Jacobi condition states that, to establish a central field of extremals containing the arc AB of extremals it is sufficient that, there does not exist any common point (apart from A) between the κ -discriminant curve and the arc AB of extremals.

Now, let us define the Jacobi condition analytically in fractional variational sense.

Denote the function $g := \frac{\partial \mu(\lambda, \kappa)}{\partial \kappa}$, where κ is given.

Hence $g'_{\lambda} = g' = \frac{\partial^2 \mu(\lambda, \kappa)}{\partial \kappa \partial \lambda}$, since g is the function of λ alone.

The functions $\mu = \mu(\lambda, \kappa)$; are solutions of the Euler–Lagrange’s equation.

∴

$$\begin{aligned} \frac{\partial F(\lambda, \mu(\lambda, \kappa), {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu(\lambda, \kappa), -{}_{\lambda}^C \mathcal{D}_b^{\beta} \mu(\lambda, \kappa))}{\partial \mu} + {}_C^{\lambda} \mathcal{D}_b^{\alpha} \frac{\partial F(\lambda, \mu(\lambda, \kappa), {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu(\lambda, \kappa), -{}_{\lambda}^C \mathcal{D}_b^{\beta} \mu(\lambda, \kappa))}{\partial {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu} \\ + {}_a^C \mathcal{D}_{\lambda}^{\beta} \frac{\partial F(\lambda, \mu(\lambda, \kappa), {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu(\lambda, \kappa), -{}_{\lambda}^C \mathcal{D}_b^{\beta} \mu(\lambda, \kappa))}{\partial {}_{\lambda}^C \mathcal{D}_b^{\beta} \mu} = 0. \end{aligned} \quad (2.4)$$

Differentiating the above equation with respect to κ in the Caputo sense, we get,

$$\begin{aligned} F_{\mu\mu}g + F_{\mu_a^C\mathcal{D}_\lambda^\alpha\mu}g' - F_{\mu_\lambda^C\mathcal{D}_b^\beta\mu}g' + {}_a^C\mathcal{D}_\lambda^\alpha\left(F_{\mu_a^C\mathcal{D}_\lambda^\alpha\mu}g + F_{\lambda^C\mathcal{D}_\lambda^\alpha\mu_a^C\mathcal{D}_\lambda^\alpha\mu}g' - F_{\lambda^C\mathcal{D}_\lambda^\alpha\mu_a^C\mathcal{D}_b^\beta\mu}g'\right) \\ + {}_a^C\mathcal{D}_\lambda^\beta\left(F_{\mu_\lambda^C\mathcal{D}_b^\beta\mu}g + F_{\lambda^C\mathcal{D}_b^\beta\mu_a^C\mathcal{D}_\lambda^\alpha\mu}g' - F_{\lambda^C\mathcal{D}_b^\beta\mu_\lambda^C\mathcal{D}_b^\beta\mu}g'\right) = 0, \end{aligned} \quad (2.5)$$

which is called the Jacobi's equation.

The solution of the above equation $g = \frac{\partial\mu(\lambda, \kappa)}{\partial\kappa}$ vanishes at the center of the extremals for $\lambda = \lambda_0$. If it does not vanishes further at any point in the interval $\lambda_0 < \lambda < \lambda_1$, then the Jacobi condition is fulfilled.

If the solution of the equation (2.2), vanishes at any other point of the interval $\lambda_0 < \lambda < \lambda_1$, then the point A^* conjugate to A is defined by the equations;

$$\mu = \mu(\lambda, \kappa_0) \text{ and } \frac{\partial\mu(\lambda, \kappa)}{\partial\kappa} = 0 \text{ or } g = 0,$$

which lies on the arc AB of the extremal.

Remark 2.1. When $\alpha = \beta = 1$, ${}_a^C\mathcal{D}_\lambda^\alpha = \frac{d}{d\lambda}$ and ${}_a^C\mathcal{D}_b^\beta = \frac{-d}{d\lambda}$ and the above equation reduces to

$$F_{\mu\mu}g + F_{\mu\mu'}g' - \frac{d}{d\lambda}(F_{\mu\mu'}g + F_{\mu'\mu'}g) = 0, \quad (2.6)$$

which is the Jacobi equation in the classical sense [13].

Note that for fractional calculus of variational problems the resulting Jacobi equation contains both the left and right CFD. This is to be expected, since the optimum function must satisfy both terminal conditions. If there exists a solution of the Jacobi equation that vanishes only at the point $\lambda = \lambda_0$ and does not vanish at any other point in $[\lambda_0, \lambda_1]$, then no points are conjugate to the arc AB; and the Jacobi condition is fulfilled.

Remark 2.2. The functional $F\left(\lambda, \mu(\lambda, \kappa), {}_a^C\mathcal{D}_\lambda^\alpha\mu(\lambda, \kappa), {}_a^C\mathcal{D}_b^\beta\mu(\lambda, \kappa)\right)$, satisfies the Euler–Lagrange's equation (1.1) but fails to satisfy the Jacobi equation (2.5) taken in right CFD sense and hence fails to satisfy the sufficient conditions for optimality. Whereas, the functional $F\left(\lambda, \mu(\lambda, \kappa), {}_a^C\mathcal{D}_\lambda^\alpha\mu(\lambda, \kappa), -{}_\lambda^C\mathcal{D}_b^\beta\mu(\lambda, \kappa)\right)$, satisfies the Euler–Lagrange's equation (1.1) as well as the Jacobi equation (2.5) in CFD sense. Therefore, the above functional is taken into consideration.

2.2. The transversality condition

Consider the functional of the form (2.3) having fixed boundary points. This functional has a solution of the form (2.1). Now consider the case where one or both of the boundary points can move. Then we can get a larger class of acceptable curves comprising of the comparison curves having common boundary points with the curve under consideration and the curves that are generated by the extension of the boundary points. We establish the following result:

Theorem 2.3. *Let $J[\mu(\lambda)]$ be a functional of the form (2.3) defined on the set of function $\mu(\lambda)$, which have continuous left and right CFD of order α and β respectively with moving boundary points. Then the extremal of the given functional satisfies the transversality condition given by*

$$\begin{aligned} \left[F + F_{\lambda^C\mathcal{D}_\lambda^\alpha\mu}\left({}_a^C\mathcal{D}_\lambda^\alpha\psi - {}_a^C\mathcal{D}_\lambda^\alpha\mu\right)\delta\lambda\right]_{\lambda_1}^{\lambda_2} &= 0 \text{ (in left CFD sense) and} \\ \left[F + F_{\lambda^C\mathcal{D}_b^\beta\mu}\left({}_\lambda^C\mathcal{D}_b^\beta\psi - {}_\lambda^C\mathcal{D}_b^\beta\mu\right)\delta\lambda\right]_{\lambda_1}^{\lambda_2} &= 0 \text{ (in right CFD sense),} \end{aligned} \quad (2.7)$$

where ψ is a function of λ .

Proof. The variation of the functional $J[\mu(\lambda, \kappa)]$ on the extremals $\mu = \mu(\lambda, \kappa)$, when the boundary point is displaced from the position (λ_1, λ_2) to the position $(\lambda_1 + \delta\lambda_1, \lambda_2 + \delta\lambda_2)$ may be calculated in the fractional sense and is given by,

$$\Delta J = \int_{\lambda_1 + \delta\lambda_1}^{\lambda_2 + \delta\lambda_2} F\left(\lambda, \mu + h, {}_a^C\mathcal{D}_\lambda^\alpha \mu + {}_a^C\mathcal{D}_\lambda^\alpha h, -\frac{C}{\lambda} \mathcal{D}_b^\beta \mu - \frac{C}{\lambda} \mathcal{D}_b^\beta h\right) d\lambda - \int_{\lambda_1}^{\lambda_2} F\left(\lambda, \mu, {}_a^C\mathcal{D}_\lambda^\alpha \mu, -\frac{C}{\lambda} \mathcal{D}_b^\beta \mu\right) d\lambda,$$

where h is the difference between the original and the perturbed curve and is given by, $h = \delta\mu - \frac{C}{a} \mathcal{D}_\lambda^\alpha \mu \delta\lambda$ (in left CFD sense) and $h = \delta\mu + \frac{C}{\lambda} \mathcal{D}_b^\beta \mu \delta\lambda$ (in right CFD sense). Then

$$\begin{aligned} \Delta J &= \int_{\lambda_1}^{\lambda_2} F\left(\lambda, \mu + h, {}_a^C\mathcal{D}_\lambda^\alpha \mu + {}_a^C\mathcal{D}_\lambda^\alpha h, -\frac{C}{\lambda} \mathcal{D}_b^\beta \mu - \frac{C}{\lambda} \mathcal{D}_b^\beta h\right) d\lambda - \int_{\lambda_1}^{\lambda_2} F\left(\lambda, \mu, {}_a^C\mathcal{D}_\lambda^\alpha \mu, -\frac{C}{\lambda} \mathcal{D}_b^\beta \mu\right) d\lambda \\ &\quad - \int_{\lambda_1}^{\lambda_1 + \delta\lambda_1} F\left(\lambda, \mu + h, {}_a^C\mathcal{D}_\lambda^\alpha \mu + {}_a^C\mathcal{D}_\lambda^\alpha h, -\frac{C}{\lambda} \mathcal{D}_b^\beta \mu - \frac{C}{\lambda} \mathcal{D}_b^\beta h\right) d\lambda \\ &\quad + \int_{\lambda_2}^{\lambda_2 + \delta\lambda_2} F\left(\lambda, \mu + h, {}_a^C\mathcal{D}_\lambda^\alpha \mu + {}_a^C\mathcal{D}_\lambda^\alpha h, -\frac{C}{\lambda} \mathcal{D}_b^\beta \mu - \frac{C}{\lambda} \mathcal{D}_b^\beta h\right) d\lambda \\ &\approx \int_{\lambda_1}^{\lambda_2} F\left(\lambda, \mu + h, {}_a^C\mathcal{D}_\lambda^\alpha \mu + {}_a^C\mathcal{D}_\lambda^\alpha h, -\frac{C}{\lambda} \mathcal{D}_b^\beta \mu - \frac{C}{\lambda} \mathcal{D}_b^\beta h\right) d\lambda - \int_{\lambda_1}^{\lambda_2} F\left(\lambda, \mu, {}_a^C\mathcal{D}_\lambda^\alpha \mu, -\frac{C}{\lambda} \mathcal{D}_b^\beta \mu\right) d\lambda \\ &\quad - F|_{\lambda=\lambda_1} \delta\lambda_1 + F|_{\lambda=\lambda_2} \delta\lambda_2. \end{aligned} \tag{2.8}$$

□

If, we expand the first term on the right hand side of the equation (2.8) by means of Taylor's expansion [38] then,

$$F\left(\lambda, \mu + h, {}_a^C\mathcal{D}_\lambda^\alpha \mu + {}_a^C\mathcal{D}_\lambda^\alpha h, -\frac{C}{\lambda} \mathcal{D}_b^\beta \mu - \frac{C}{\lambda} \mathcal{D}_b^\beta h\right) = F + F_\mu h + F_{a^C\mathcal{D}_\lambda^\alpha \mu} {}_a^C\mathcal{D}_\lambda^\alpha h - F_{\frac{C}{\lambda} \mathcal{D}_b^\beta \mu} \frac{C}{\lambda} \mathcal{D}_b^\beta h. \tag{2.9}$$

Thus,

$$\begin{aligned} &\int_{\lambda_1}^{\lambda_2} \left[F\left(\lambda, \mu + h, {}_a^C\mathcal{D}_\lambda^\alpha \mu + {}_a^C\mathcal{D}_\lambda^\alpha h, -\frac{C}{\lambda} \mathcal{D}_b^\beta \mu - \frac{C}{\lambda} \mathcal{D}_b^\beta h\right) - F\left(\lambda, \mu, {}_a^C\mathcal{D}_\lambda^\alpha \mu, -\frac{C}{\lambda} \mathcal{D}_b^\beta \mu\right) \right] d\lambda \\ &= \int_{\lambda_1}^{\lambda_2} \left(F_\mu h + F_{a^C\mathcal{D}_\lambda^\alpha \mu} {}_a^C\mathcal{D}_\lambda^\alpha h - F_{\frac{C}{\lambda} \mathcal{D}_b^\beta \mu} \frac{C}{\lambda} \mathcal{D}_b^\beta h \right) d\lambda. \end{aligned} \tag{2.10}$$

The second part of the integration of the above equation is

$$\int_{\lambda_1}^{\lambda_2} F_{a^C\mathcal{D}_\lambda^\alpha \mu} {}_a^C\mathcal{D}_\lambda^\alpha h d\lambda = \left[F_{a^C\mathcal{D}_\lambda^\alpha \mu} h \right]_{\lambda_1}^{\lambda_2} + \int_{\lambda_1}^{\lambda_2} h {}_a^C\mathcal{D}_\lambda^\alpha F_{a^C\mathcal{D}_\lambda^\alpha \mu} d\lambda, \tag{2.11}$$

and the third part of the integration is

$$\int_{\lambda_1}^{\lambda_2} F_{\frac{C}{\lambda} \mathcal{D}_b^\beta \mu} \frac{C}{\lambda} \mathcal{D}_b^\beta h d\lambda = \left[F_{\frac{C}{\lambda} \mathcal{D}_b^\beta \mu} h \right]_{\lambda_1}^{\lambda_2} + \int_{\lambda_1}^{\lambda_2} h {}_a^C\mathcal{D}_\lambda^\beta F_{\frac{C}{\lambda} \mathcal{D}_b^\beta \mu} d\lambda. \tag{2.12}$$

Hence,

$$\begin{aligned} &\int_{\lambda_1}^{\lambda_2} \left(F_\mu h + F_{a^C\mathcal{D}_\lambda^\alpha \mu} {}_a^C\mathcal{D}_\lambda^\alpha h - F_{\frac{C}{\lambda} \mathcal{D}_b^\beta \mu} \frac{C}{\lambda} \mathcal{D}_b^\beta h \right) d\lambda \\ &= \int_{\lambda_1}^{\lambda_2} \left(F_\mu + {}_a^C\mathcal{D}_\lambda^\alpha F_{a^C\mathcal{D}_\lambda^\alpha \mu} + {}_a^C\mathcal{D}_\lambda^\beta F_{\frac{C}{\lambda} \mathcal{D}_b^\beta \mu} \right) h + \left[F_{a^C\mathcal{D}_\lambda^\alpha \mu} h \right]_{\lambda_1}^{\lambda_2} - \left[F_{\frac{C}{\lambda} \mathcal{D}_b^\beta \mu} h \right]_{\lambda_1}^{\lambda_2}. \end{aligned} \tag{2.13}$$

$$\therefore \Delta J \approx \int_{\lambda_1}^{\lambda_2} \left(F_\mu + {}_C^C \mathcal{D}_b^\alpha F_a^\alpha \mathcal{D}_\lambda^\alpha \mu + {}_a^C \mathcal{D}_\lambda^\beta F_\lambda^\beta \mathcal{D}_b^\beta \mu \right) h + \left[F_a^\alpha \mathcal{D}_\lambda^\alpha \mu h \right]_{\lambda_1}^{\lambda_2} - \left[F_\lambda^\beta \mathcal{D}_b^\beta \mu h \right]_{\lambda_1}^{\lambda_2} + [F \delta \lambda]_{\lambda_1}^{\lambda_2}. \quad (2.14)$$

At the extremals,

$$\begin{aligned} \int_{\lambda_1}^{\lambda_2} \left(F_\mu + {}_C^C \mathcal{D}_b^\alpha F_a^\alpha \mathcal{D}_\lambda^\alpha \mu + {}_a^C \mathcal{D}_\lambda^\beta F_\lambda^\beta \mathcal{D}_b^\beta \mu \right) h = 0. \quad (\text{by EL equation}) \\ \therefore \Delta J = \left[F_a^\alpha \mathcal{D}_\lambda^\alpha \mu h \right]_{\lambda_1}^{\lambda_2} - \left[F_\lambda^\beta \mathcal{D}_b^\beta \mu h \right]_{\lambda_1}^{\lambda_2} + [F \delta \lambda]_{\lambda_1}^{\lambda_2} = 0. \end{aligned} \quad (2.15)$$

Now replacing the value of h in the above equation (2.15) in left CFD sense, we have,

$$\begin{aligned} \left[F_a^\alpha \mathcal{D}_\lambda^\alpha \mu (\delta \mu - {}_a^C \mathcal{D}_\lambda^\alpha \mu \delta \lambda) \right]_{\lambda_1}^{\lambda_2} + [F \delta \lambda]_{\lambda_1}^{\lambda_2} = 0 \\ \implies \left[(F - F_a^\alpha \mathcal{D}_\lambda^\alpha \mu) \delta \lambda \right]_{\lambda_1}^{\lambda_2} + \left[F_a^\alpha \mathcal{D}_\lambda^\alpha \mu \delta \mu \right]_{\lambda_1}^{\lambda_2} = 0. \end{aligned} \quad (2.16)$$

Now the basic necessary condition for an extremum; $\Delta J = 0$ has the form (2.16).

If the variations are independent, then,

$$\left[\left(F - F_a^\alpha \mathcal{D}_\lambda^\alpha \mu \right) \delta \lambda \right]_{\lambda_1}^{\lambda_2} = 0 \quad (2.17)$$

and

$$\left[F_a^\alpha \mathcal{D}_\lambda^\alpha \mu \delta \mu \right]_{\lambda_1}^{\lambda_2} = 0. \quad (2.18)$$

If the variations are dependent; let $\mu = \psi(\lambda)$,

$$\Rightarrow \delta \mu = {}_a^C \mathcal{D}_\lambda^\alpha \psi \delta \lambda. \quad (2.19)$$

By substituting the value of $\delta \mu$ from the above equation in equation (2.16);

$$\begin{aligned} \left[\left(F - F_a^\alpha \mathcal{D}_\lambda^\alpha \mu \right) \delta \lambda + F_a^\alpha \mathcal{D}_\lambda^\alpha \psi \delta \lambda \right]_{\lambda_1}^{\lambda_2} = 0 \\ \Rightarrow \left[\left(F + F_a^\alpha \mathcal{D}_\lambda^\alpha \left({}_a^C \mathcal{D}_\lambda^\alpha \psi - {}_a^C \mathcal{D}_\lambda^\alpha \mu \right) \right) \delta \lambda \right]_{\lambda_1}^{\lambda_2} = 0. \end{aligned} \quad (2.20)$$

Similarly, replacing the value of h in equation (2.15) and proceeding in the same way, we will get,

$$\left[\left(F + F_\lambda^\beta \mathcal{D}_b^\beta \left({}_a^C \mathcal{D}_\lambda^\beta \psi - {}_a^C \mathcal{D}_\lambda^\beta \mu \right) \right) \delta \lambda \right]_{\lambda_1}^{\lambda_2} = 0. \quad (2.21)$$

This establishes a relationship between the slopes of ψ and μ at the boundary points. This condition is called the transversality condition.

Remark 2.4. When $\alpha = \beta = 1$, ${}_a^C \mathcal{D}_\lambda^\alpha = \frac{d}{d\lambda}$ and ${}_a^C \mathcal{D}_\lambda^\beta = -\frac{d}{d\lambda}$, the above conditions will reduce to $[F + (\psi' - \mu')F_\mu] = 0$, which is the transversality condition in classical sense.

3. SUFFICIENT CONDITION FOR EXTREMUM

In the preceding sections we have discussed the Euler–Lagrange equation, Jacobi condition and the transversality condition in the CFD sense. The Euler–Lagrange equation is the principal necessary condition which must be satisfied by the solution of the functional to achieve the extremum. The Jacobi condition determines the limitations to the extent of the region over which the integral may be extended. The transversality condition deals with the moving boundary conditions of the functional under consideration.

Next, we are going to prove the sufficient conditions for the solution of the functional to achieve the extremum. We now prove the following:

Theorem 3.1. A problem involving an extremal of the functional of the form (2.3) having boundary points $\mu(\lambda_0) = \mu_0$ and $\mu(\lambda_1) = \mu_1$ satisfies the Jacobi condition in Caputo sense. If $\mu_0(\lambda)$ gives a relative minimum (or maximum) for $\int_{\kappa} F(\lambda, \mu, {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu, -{}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu)$ then,

$$E(\lambda, \mu, m, {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu, -{}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu) \geq 0 \text{ (or } \leq 0 \text{ for maximum),} \quad (3.1)$$

$\forall \lambda \in [\lambda_0, \lambda_1]$, and for all close permissible curve $\bar{\kappa}$. Here $m(\lambda, \mu)$ is the slope of the central field through which the extremal κ has passed.

Proof. Consider the problem involving an extremal of the functional of the form (2.3) having boundary points $\mu(\lambda_0) = \mu_0$ and $\mu(\lambda_1) = \mu_1$. Let the Jacobi conditions are satisfied. Suppose the extremal κ , passes through the points $A(\lambda_0, \mu_0)$ and $B(\lambda_1, \mu_1)$ is included in the central field with slope $m(\lambda, \mu)$. The increment ΔJ , the difference of the functional J passing from the extremal κ to some close acceptable curve $\bar{\kappa}$, is given by,

$$\Delta J = \int_{\bar{\kappa}} F(\lambda, \mu, {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu, -{}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu) d\lambda - \int_{\kappa} F(\lambda, \mu, {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu, -{}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu) d\lambda. \quad (3.2)$$

The symbols $\int_{\bar{\kappa}} F(\lambda, \mu, {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu, -{}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu) d\lambda$ and $\int_{\kappa} F(\lambda, \mu, {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu, -{}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu) d\lambda$, represent the values of the functional $J = \int_{\lambda_0}^{\lambda_1} F(\lambda, \mu, {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu, -{}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu) d\lambda$ taken along the arcs of the curves $\bar{\kappa}$ and κ , respectively. \square

Now the auxiliary functional

$$\int_{\bar{\kappa}} [F(\lambda, \mu, m) + \left({}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu - {}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu - m \right) F_m(\lambda, \mu, m)] d\lambda, \quad (3.3)$$

reduces to $\int_{\bar{\kappa}} F(\lambda, \mu, {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu, -{}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu) d\lambda$, taken along the extremal κ (when $\alpha = \beta = 1$, ${}_a^C \mathcal{D}_{\lambda}^{\alpha} = \frac{d}{d\lambda}$ and ${}_{\lambda}^C \mathcal{D}_{b}^{\beta} = \frac{-d}{d\lambda}$); since $\frac{d\mu}{d\lambda} = m$ on extremals of the field.

The auxiliary functional (3.3);

$$\begin{aligned} & \int_{\bar{\kappa}} [F(\lambda, \mu, m) + \left({}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu - {}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu - m \right) F_m(\lambda, \mu, m)] d\lambda \\ &= \int_{\bar{\kappa}} [F(\lambda, \mu, m) - m F_m(\lambda, \mu, m)] d\lambda + \int_{\bar{\kappa}} \left[{}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu F_m(\lambda, \mu, m) - {}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu F_m(\lambda, \mu, m) \right] d\lambda, \end{aligned}$$

is the integral of an exact differential.

The functional $J[\mu(\lambda)]$ is obtained by taking the differential of the function $\bar{J}(\lambda, \mu)$ having the form (by the transversality condition (2.7))

$$d\bar{J} = \left(F + {}_{\lambda}^C \mathcal{D}_{b}^{\beta} \mu F_{a^C \mathcal{D}_{\lambda}^{\alpha} \mu} + {}_a^C \mathcal{D}_{\lambda}^{\alpha} \mu F_{\lambda^C \mathcal{D}_{b}^{\beta} \mu} \right) d\lambda + \left(F_{a^C \mathcal{D}_{\lambda}^{\alpha} \mu} - F_{\lambda^C \mathcal{D}_{b}^{\beta} \mu} \right) d\mu, \quad (3.4)$$

or

$$d\bar{J} = \left(F + {}_a^C \mathcal{D}_\lambda^\alpha \mu F_{\lambda \mathcal{D}_b^\beta \mu} + {}_\lambda^C \mathcal{D}_b^\beta \mu F_{a \mathcal{D}_\lambda^\alpha \mu} \right) d\lambda + \left({}_a^C \mathcal{D}_\lambda^\alpha \mu F_{a \mathcal{D}_\lambda^\alpha \mu} + {}_\lambda^C \mathcal{D}_b^\beta \mu F_{\lambda \mathcal{D}_b^\beta \mu} \right) d\lambda. \quad (3.5)$$

Thus, on the extremal κ the integral given by (2.7) coincides with the integral,

$$\int_{\bar{\kappa}} F \left(\lambda, \mu, {}_a^C \mathcal{D}_\lambda^\alpha \mu, -{}_\lambda^C \mathcal{D}_b^\beta \mu \right) d\lambda.$$

Since the functional (2.7) is the integral of an exact differential equation, it does not depend on the path of integration. Therefore,

$$\int_{\kappa} F \left(\lambda, \mu, {}_a^C \mathcal{D}_\lambda^\alpha \mu, -{}_\lambda^C \mathcal{D}_b^\beta \mu \right) d\lambda = \int_{\bar{\kappa}} \left[F(\lambda, \mu, m) + \left({}_a^C \mathcal{D}_\lambda^\alpha \mu - {}_\lambda^C \mathcal{D}_b^\beta \mu - m \right) F_m(\lambda, \mu, m) \right] d\lambda, \quad (3.6)$$

holds not just for $\kappa = \bar{\kappa}$ but for any value of $\bar{\kappa}$ as well.

Hence,

$$\Delta J = \int_{\bar{\kappa}} F \left(\lambda, \mu, {}_a^C \mathcal{D}_\lambda^\alpha \mu, -{}_\lambda^C \mathcal{D}_b^\beta \mu \right) d\lambda - \int_{\bar{\kappa}} \left[F(\lambda, \mu, m) + \left({}_a^C \mathcal{D}_\lambda^\alpha \mu - {}_\lambda^C \mathcal{D}_b^\beta \mu - m \right) F_m(\lambda, \mu, m) \right] d\lambda. \quad (3.7)$$

The integrand is called Weierstrass function and is denoted by $E \left(\lambda, \mu, m, {}_a^C \mathcal{D}_\lambda^\alpha \mu, -{}_\lambda^C \mathcal{D}_b^\beta \mu \right)$;

$$E = \int_{\bar{\kappa}} \left[F \left(\lambda, \mu, {}_a^C \mathcal{D}_\lambda^\alpha \mu, -{}_\lambda^C \mathcal{D}_b^\beta \mu \right) - F(\lambda, \mu, m) - \left({}_a^C \mathcal{D}_\lambda^\alpha \mu - {}_\lambda^C \mathcal{D}_b^\beta \mu - m \right) F_m(\lambda, \mu, m) \right] d\lambda. \quad (3.8)$$

In this notation;

$$\Delta J = \int_{\lambda_0}^{\lambda_1} E \left(\lambda, \mu, m, {}_a^C \mathcal{D}_\lambda^\alpha \mu, -{}_\lambda^C \mathcal{D}_b^\beta \mu \right) d\lambda. \quad (3.9)$$

The functional ΔJ is minimum on the curve κ if the function $E \geq 0$, and maximum if $E \leq 0$. This condition is sufficient to determine the extremum of the given functional.

The Taylor's series expansion [38] in Caputo sense for $F \left(\lambda, \mu, {}_a^C \mathcal{D}_\lambda^\alpha \mu, -{}_\lambda^C \mathcal{D}_b^\beta \mu \right)$, is given by

$$\begin{aligned} F \left(\lambda, \mu, {}_a^C \mathcal{D}_\lambda^\alpha \mu, -{}_\lambda^C \mathcal{D}_b^\beta \mu \right) &= F(\lambda, \mu, m) + \left({}_a^C \mathcal{D}_\lambda^\alpha \mu - {}_\lambda^C \mathcal{D}_b^\beta \mu - m \right) F_m(\lambda, \mu, m) \\ &+ \frac{\left({}_a^C \mathcal{D}_\lambda^\alpha \mu - {}_\lambda^C \mathcal{D}_b^\beta \mu - m \right)^2}{2!} \left(F_{a \mathcal{D}_\lambda^{2\alpha} \mu}(\lambda, \mu, t) + F_{\lambda \mathcal{D}_b^{2\beta} \mu}(\lambda, \mu, t) \right), \end{aligned} \quad (3.10)$$

where t lies between m and μ' .

Substituting the value of $F \left(\lambda, \mu, {}_a^C \mathcal{D}_\lambda^\alpha \mu, -{}_\lambda^C \mathcal{D}_b^\beta \mu \right)$ from Taylor's series expansion in Weierstrass integral we get,

$$\Rightarrow E \left(\lambda, \mu, m, {}_a^C \mathcal{D}_\lambda^\alpha \mu, -{}_\lambda^C \mathcal{D}_b^\beta \mu \right) = \frac{\left({}_a^C \mathcal{D}_\lambda^\alpha \mu - {}_\lambda^C \mathcal{D}_b^\beta \mu - m \right)^2}{2!} \left[F_{a \mathcal{D}_\lambda^{2\alpha} \mu}(\lambda, \mu, t) + F_{\lambda \mathcal{D}_b^{2\beta} \mu}(\lambda, \mu, t) \right]. \quad (3.11)$$

The sign of the function E depends on the sign of the functions $F_{a \mathcal{D}_\lambda^{2\alpha} \mu}(\lambda, \mu, t)$ and $F_{\lambda \mathcal{D}_b^{2\beta} \mu}(\lambda, \mu, t)$. Thus $E \geq 0$ implies $F_{a \mathcal{D}_\lambda^{2\alpha} \mu}(\lambda, \mu, t) > 0$ (or $F_{\lambda \mathcal{D}_b^{2\beta} \mu}(\lambda, \mu, t) > 0$) for left (or right) CFD, where the functional achieves a weak minimum; and $E \leq 0$ implies $F_{a \mathcal{D}_\lambda^{2\alpha} \mu}(\lambda, \mu, t) < 0$ (or $F_{\lambda \mathcal{D}_b^{2\beta} \mu}(\lambda, \mu, t) < 0$) for left (or right) CFD, where the functional achieves a weak maximum. The condition $F_{a \mathcal{D}_\lambda^{2\alpha} \mu}(\lambda, \mu, t) > 0$ (or $F_{\lambda \mathcal{D}_b^{2\beta} \mu}(\lambda, \mu, t) > 0$) is called the Legendre condition.

The following example is given to understand some of our theoretical investigations considered in this paper.

Example 3.2. Test for an extremum for the functional

$$J[\mu(\lambda)] = \int_0^1 \left(\left({}_0^C \mathcal{D}_\lambda^\alpha \mu \right)^2 - \mu^2 \right) d\lambda, \text{ where } \mu(0) = 0, \mu(1) = 1 \text{ and } \frac{\partial \mu}{\partial \lambda} = \kappa \cos \lambda.$$

The Euler–Lagrange equation for left CFD is given by equation (1.1);

$$-2\mu + {}_0^C \mathcal{D}_1^\alpha \left({}_0^C \mathcal{D}_\lambda^\alpha \mu \right) = 0.$$

For $\alpha = 1$,

$$\begin{aligned} &\Rightarrow -2\mu - 2\mu'' = 0 \\ &\Rightarrow \mu'' + \mu = 0. \end{aligned}$$

It's general solution is given by

$$\mu = \kappa_1 \cos \lambda + \kappa_2 \sin \lambda.$$

Applying the initial conditions, we get, $\kappa_1 = 0$ and $\kappa_2 = 1.188396$.

The Jacobi condition for left CFD is given by,

$$\begin{aligned} F_{\mu\mu}g + F_{\mu {}_0^C \mathcal{D}_\lambda^\alpha \mu}g' + {}_0^C \mathcal{D}_1^\alpha \left(F_{\mu {}_0^C \mathcal{D}_\lambda^\alpha \mu}g + F_{_0^C \mathcal{D}_\lambda^\alpha \mu {}_0^C \mathcal{D}_\lambda^\alpha \mu}g' \right) &= 0 \\ \Rightarrow -2g + {}_0^C \mathcal{D}_1^\alpha (2g') &= 0 \\ \Rightarrow -g + {}_0^C \mathcal{D}_1^\alpha (g') &= 0. \end{aligned}$$

When $\alpha = \frac{1}{2}$,

$$\begin{aligned} {}_0^C \mathcal{D}_1^\alpha (g') = g &\Rightarrow {}_0^C \mathcal{D}_1^{-\alpha} \left({}_0^C \mathcal{D}_1^\alpha (g') \right) = {}_0^C \mathcal{D}_1^{-\alpha} (g), \\ &\Rightarrow g' = \frac{-4}{3\sqrt{\pi}} (1 - \lambda)^{\frac{3}{2}}, \\ &\Rightarrow \frac{\partial^2 \mu(\lambda, \kappa)}{\partial \kappa \partial \lambda} = \frac{-4}{3\sqrt{\pi}} (1 - \lambda)^{\frac{3}{2}}. \end{aligned}$$

Integrating the above equation with respect to κ we get,

$$\Rightarrow \frac{\partial \mu(\lambda, \kappa)}{\partial \lambda} = \frac{-4}{3\sqrt{\pi}} (1 - \lambda)^{\frac{3}{2}} \kappa + \phi(\lambda).$$

Using the initial conditions,

$$\phi(\lambda) = \kappa \cos \lambda + \frac{4}{3\sqrt{\pi}} (1 - x)^{\frac{3}{2}}.$$

Again integrating $\frac{\partial \mu(\lambda, \kappa)}{\partial \lambda} = \kappa \cos \lambda$ with respect to λ , we have

$$\mu = \kappa(-\sin \lambda) + \psi(\kappa).$$

Substituting the initial conditions $\psi(\kappa) = 0$, we obtain

$$\begin{aligned} \mu &= \kappa(-\sin \lambda), \\ \Rightarrow \mu &= \kappa_1 \sin \lambda. \end{aligned}$$

When $\alpha = 1$, ${}_0^C\mathcal{D}_\lambda^\alpha = -\frac{d}{d\lambda}$,

$$\begin{aligned} -g - \frac{d}{d\lambda}(g') &= 0 \\ \Rightarrow g'' + g &= 0, \\ \Rightarrow g &= \kappa_1 \sin \lambda, \end{aligned}$$

where κ_1 is a constant.

The function vanishes at the points $\lambda = r\pi$, where r is an integer. On the interval $0 \leq \lambda \leq 1$, the function g vanishes only at the point $\lambda = 0$ and the Jacobi equation is satisfied.

By the Legendre's condition,

$$\frac{\partial^2 F}{\partial ({}_0^C\mathcal{D}_\lambda^\alpha \mu)^2} = 2 > 0,$$

for any value of ${}_0^C\mathcal{D}_\lambda^\alpha \mu$.

It follows that on the straight line $\mu = 0$ a strong minimum is achieved for $\lambda < \pi$.

4. CONCLUSION

The sufficient conditions for fractional variational problems of the type

$$J[\mu(\lambda)] = \int_{\lambda_0}^{\lambda_1} F\left(\lambda, \mu(\lambda, \kappa), {}_a^C\mathcal{D}_\lambda^\alpha \mu(\lambda, \kappa), -{}_\lambda^C\mathcal{D}_b^\beta \mu(\lambda, \kappa)\right) d\lambda,$$

are formulated with the help of Caputo fractional derivatives. The sufficient conditions for the proposed problems are established using the Weierstrass integral and the Legendre condition. It is observed that the present work generalizes many classical investigations [13]. Again, with the help of discussed counterexample, our fresh findings have been justified. Many concepts of classical calculus of variations were found to be the extension of fractional calculus of variations with minor modifications. Thus, ample of opportunities are open for further investigation in this area.

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REFERENCES

- [1] O.P. Agrawal, Formulation of Euler–Lagrange equations for fractional variational problems. *J. Math. Anal. Appl.* **272** (2002) 368–379.
- [2] O.P. Agrawal, Analytical schemes for a new class of fractional differential equations. *J. Phys. A: Math. Theor.* **40** (2007) 5469–5477.
- [3] O.P. Agrawal, Fractional variational calculus in terms of Riesz fractional derivatives. *J. Phys. A: Math. Theor.* **40** (2007) 6287–6303.
- [4] O.P. Agrawal, A general finite element formulation for fractional variational problems. *J. Math. Anal. Appl.* **337** (2008) 1–12.
- [5] O.P. Agrawal, A formulation and numerical scheme for fractional optimal control problems. *J. Vib. Control* **14** (2008) 1291–1299.
- [6] O.P. Agrawal, Generalized variational problems and Euler–Lagrange equations. *Comput. Math.* **59** (2010) 1852–1864.
- [7] O.P. Agrawal and D. Baleanu, A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems. *J. Vib. Control* **13** (2007) 1269–1281.
- [8] R. Almeida, Fractional variational problems depending on indefinite integrals and with delay. *Bull. Malays. Math. Sci. Soc.* **39** (2016) 1515–1528.

- [9] R. Almeida, Variational problems involving a Caputo-type fractional derivative. *J. Optim. Theory Appl.* **174** (2017) 276–294.
- [10] R. Almeida and D.F.M. Torres, Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives. *Commun. Nonlinear Sci. Numer. Simul.* **16** (2011) 1490–1500.
- [11] D. Baleanu, A. Mousalou and S. Rezapour, A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo–Fabrizio derivative. *Adv. Differ. Equ.* **51** (2017) 1–12.
- [12] I.B. Bapna and N. Mathur, Application of fractional calculus in statistics. *Int. J. Contemp. Math. Sci.* **7** (2012) 849–856.
- [13] L. Elsgolts, *Differential Equations and Calculus of Variations*. Mir Publication, Moscow (1970).
- [14] Y. Feng, X. Yang and J. Liu, On overall behavior of Maxwell mechanical model by the combined Caputo fractional derivative. *Chin. J. Phys.* **66** (2020) 269–276.
- [15] G.S.F. Frederico, Fractional optimal control in the sense of Caputo and the fractional Noether's theorem. *Int. Math. Forum* **3** (2008) 479–493.
- [16] G.S.F. Frederico and D.F.M. Torres, A formulation of Noether's theorem for fractional problems of the calculus of variations. *J. Math. Anal. Appl.* **334** (2007) 834–846.
- [17] B. Ghanbari, S. Kumar and R. Kumar, A study of behaviour for immune and tumor cells in immunogenetic tumour model with non-singular fractional derivative. *Chaos Solitons Fractals* **133** (2020) 109619.
- [18] E.F.D. Goufo, S. Kumar and S.B. Mugisha, Similarities in a fifth-order evolution equation with and with no singular kernel. *Chaos Solitons Fractals* **130** (2020) 109467.
- [19] M. Klimek, Fractional sequential mechanics – models with symmetric fractional derivative. *Czech. J. Phys.* **51** (2001) 1348–1354.
- [20] M. Klimek, Lagrangean and Hamiltonian fractional sequential mechanics. *Czech. J. Phys.* **52** (2002) 1247–1253.
- [21] D. Kumar, J. Singh and D. Baleanu, Modified Kawahara equation within a fractional derivative with non-singular kernel. *Thermal Sci.* **22** (2018) 789–796.
- [22] S. Kumar, R. Kumar, R.P. Agarwal and B. Samet, A study of fractional Lotka–Volterra population model using Haar wavelet and Adams–Bashforth–Moulton methods. *Math. Methods Appl. Sci.* **43** (2020) 5564–5578.
- [23] S. Kumar, R. Kumar, C. Cattani and B. Samet, Chaotic behaviour of fractional predator-prey dynamical system. *Chaos Solitons Fractals* **135** (2020) 109811.
- [24] S. Kumar, K.S. Nisar, R. Kumar, C. Cattani and B. Samet, A new Rabotnov fractional-exponential function based fractional derivative for diffusion equation under external force. *Math. Meth. Appl. Sci.* **43** (2020) 4460–4471.
- [25] S. Kumar, R.P. Chauhan, S. Momani and S. Hadid, Numerical investigations on COVID-19 model through singular and non-singular fractional operators. *Numer. Methods Part. Differ. Equ.* (2020) 1–27. DOI: [10.1002/num.22707](https://doi.org/10.1002/num.22707).
- [26] S. Kumar, A. Kumar, A.H. Abdel-Aty and M.R. Alharthi, A study on four-species fractional population competition dynamical model. *Results Phys.* **24** (2021) 104089.
- [27] S. Kumar, R.P. Chauhan, A.H. Abdel-Aty and M.R. Alharthi, A study on transmission dynamics of HIV/AIDS model through fractional operators. *Results Phys.* **22** (2021) 1–14.
- [28] S. Kumar, A. Kumar, B. Samet and H. Dutta, A study on fractional host-parasitoid population dynamical model to describe insect species. *Numer. Methods Part. Differ. Equ.* **37** (2021) 1673–1692.
- [29] M.J. Lazo and D.F.M. Torres, The Legendre condition of the fractional calculus of variations. *Optimization* **63** (2014) 1157–1165.
- [30] A.B. Malinowska and D.F.M. Torres, Fractional calculus of variations for a combined Caputo derivative. *Fract. Calc. Appl. Anal.* **14** (2011) 523–537.
- [31] T. Odzijewicz, A.B. Malinowska and D.F.M. Torres, Fractional variational calculus with classical and combined Caputo derivatives. *Nonlinear Anal.* **75** (2012) 1507–1515.
- [32] I. Podlubny, *Fractional Differential Equations*. Academic Press, New York (1999).
- [33] G. Rahman, K. S. Nisar, B. Ghanbari and T. Abdeljawad, On generalized fractional integral inequalities for the monotone weighted Chebyshev functionals. *Adv. Differ. Equ.* **368** (2020) 1–19.
- [34] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics. *Phys. Rev.* **53** (1996) 1890–1899.
- [35] F. Riewe, Mechanics with fractional derivatives. *Phys. Rev.* **55** (1997) 3581–3592.
- [36] B. Ross, The development of fractional calculus 1695–1900. *Hist. Math.* **4** (1977) 75–89.
- [37] J. Singh, D. Kumar and J.J. Nieto, Analysis of an El Niño–Southern Oscillation model with a new fractional derivative. *Chaos Solitons Fractals* **99** (2017) 109–115.
- [38] D. Usero, Fractional Taylor series for Caputo fractional derivatives. In: *Construction of Numerical Schemes*. Universidad Complutense de Madrid. Spain (2007).
- [39] D. Wang and A. Xiao, Numerical methods for fractional variational problems depending on indefinite integrals. *ASME. J. Comput. Nonlinear Dynam.* **8** (2013) 021018.

[40] J. Zhang, X. Ma and L. Li, Optimality conditions for fractional variational problems with Caputo–Fabrizio fractional derivatives. *Adv. Differ. Equ.* **357** (2017) 1–14.

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