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SHARP LAGRANGE MULTIPLIERS FOR SET-VALUED OPTIMIZATION PROBLEMS

Abdessamad Oussarhan[1,*] and Tijani Amahroq[2]

Abstract. In this paper, we give a comparison among some notions of weak sharp minima introduced in Amahroq *et al.* [*Le matematiche J.* **73** (2018) 99–114], Durea and Strugariu [*Nonlinear Anal.* **73** (2010) 2148–2157] and Zhu *et al.* [*Set-Valued Var. Anal.* **20** (2012) 637–666] for set-valued optimization problems. Besides, we establish sharp Lagrange multiplier rules for general constrained set-valued optimization problems involving new scalarization functionals based on the oriented distance function. Moreover, we provide sufficient optimality conditions for the considered problems without any convexity assumptions.

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1. Introduction

The concept of sharp minimizer has been investigated for different types of optimization problems: real-valued, vector-valued as well as set-valued optimization problems. For real-valued optimization problems, Auslender [6] has established necessary and sufficient optimality conditions for a local sharp minimizer of order $\gamma \in \{1, 2\}$ where the objective function is locally lipschitzian and the feasible set is closed. To the same problem, Studniarski [34] comes to extend the results of Auslender [6] for any extended real-valued objective function (not necessary locally lipschitzian) and the feasible set not necessary closed where the order of sharp minimizer ($\gamma \geq 2$). Ward [36] follows the line of Studniarski with different way.

For vector-valued optimization problems, Jiménez [19] has introduced the notion of sharp minimizer of order γ, in addition, he has developed with Novo in Jiménez [20] and Jiménez and Novo [21] the theory on minimizer of order ($\gamma \geq 1$ integer) considering different frameworks. Two years after, Bednarczuk [8] has defined the notion of weak sharp minimizer of order γ where the ordering cone is assumed to be closed, convex, and pointed. This concept was used to prove conditions for upper Hölderness continuity and Hölder calmness of the solution mappings to parametric vector optimization problems. Later, Studniarski [35] introduced the notion of weak ψ-sharp local minima in vector optimization problems. Besides, he has extended some necessary and sufficient optimality conditions obtained by Jiménez [19].

Keywords. Set-valued optimization, sharp minimizers, oriented distance function, sharp Fritz-John multipliers, sharp Karush–Kuhn–Tucker multipliers, optimality conditions.

[1] LIMATI Laboratory, Sultan Moulay Slimane University, Poly-disciplinary Faculty, B.P. 592 Beni Mellal, Morocco.

[2] LAMAI Laboratory, Cadi Ayyad University, Faculty of Sciences and Techniques, B.P. 549 Marrakech, Morocco.

[*]Corresponding author: abdessamad.oussa@gmail.com.

To shed light on the study of sharp minimality in set-valued optimization problems we may refer to the papers [5, 13, 14, 40]. In [14] Flores-Bazán and Jiménez introduced the concept of sharp minima for a set-valued optimization problem and provided some optimality conditions. In connection with the paper of Durea and Strugariu [13], the sharp minimizer was introduced by means of the oriented distance function and its necessary optimality conditions are established with the use of the Mordukhovich generalized differentiation. Later, Zhu *et al.* [40] proposed the concept of the sharp minimizer by means of the distance function, they have extended the Fermat rules for the local minimizer of the constrained set-valued optimization problem to sharp and weak sharp minimizers in Banach spaces or Asplund spaces by means of the Mordukhovich generalized differentiation and the normal cone. Very recently, Amahroq *et al.* [5] introduced this notion in set-valued optimization problems without recourse to the use of distances adopted in Durea and Strugariu [13] and Zhu *et al.* [40]. They have established necessary and sufficient optimality conditions involving set-valued derivatives, besides they have provided optimality conditions in terms of Fritz-John multipliers under convexity assumptions on the objective set-valued mapping using the classical separation theorem. A new concept of sharp minima in set-valued optimization problems by means of the pseudo-relative interior, namely pseudo-relative ϕ-sharp minimizer, is proposed and studied in Amahroq and Oussarhan [1].

The importance of the study of weak sharp minima arises in the stability analysis, the sensitivity analysis, and in the study of the convergence of iterative numerical procedures, for instance, see [6, 10, 12, 15, 27, 38]. It is worth also to mention that the study of weak sharp minimizers is closely related to the study of the error bound in optimization, for more details we refer to Bednarczuk [8], Zheng *et al.* [39] and the references therein.

The tools used in the paper of Durea and Strugariu [13] to derive necessary optimality conditions in terms of multiplier rules require that the function ψ given in Definition 2.2 be Frchet differentiable at 0 and that

$$\nabla\psi(0) > 0,$$

which is not the case for $\psi(t) = t^\gamma$ with $\gamma \neq 1$. In this paper, we will study three notions of weak sharp minima those introduced in Amahroq *et al.* [5], Durea and Strugariu [13], Zhu *et al.* [40] and we will provide a comparison among them. Due to the concept of sharp minimizer given in Amahroq *et al.* [5], we will generalize the results of Amahroq *et al.* [5] and those of Durea and Strugariu [13] when $\psi(t) = t^\gamma$ and γ is an integer, by establishing Lagrange multiplier rules to the general constrained and explicit constrained set-valued optimization problems in terms of Fritz-John as well as Karush–Kuhn–Tucker multipliers, named, sharp Fritz-John as well as sharp Karush–Kuhn–Tucker multipliers. To do this, we will introduce some scalarization techniques which are suitable for sharp minima based on the oriented distance function. Moreover, we will provide sufficient optimality conditions for global sharp minimizers of order $\gamma > 0$ that have not been done in Durea and Strugariu [13].

The rest of the paper is organized as follows: In Section 2, we recall some definitions and we prove some preliminary results needed in the sequel of the paper. In Sections 3 and 4, we establish sharp Fritz-John multipliers as well as sharp Karush–Kuhn–Tucker multipliers of order $\gamma = 1$ in the weak sense. In Section 5, we derive necessary optimality conditions in terms of multiplier rules for sharp minimizers of higher order $\gamma \geq 2$ (γ integer) in the weak sense. Necessary optimality conditions for sharp minima in the strong sense are established in Section 6. In Section 7, we provide sufficient optimality conditions for global sharp minima ($\gamma > 0$) in the weak sense without any convexity assumptions. In addition, we show that necessary optimality conditions obtained in Sections 3–5 may be sufficient optimality conditions under suitable assumptions.

2. Preliminaries

Let F be a set-valued map between Banach spaces X and Y, $K_Y \subset Y$ be a pointed (*i.e.*, $K_Y \cap (-K_Y) = \{0\}$) closed solid (*i.e.*, with nonempty interior, $\mathrm{int}(\mathrm{K_Y}) \neq \emptyset$) convex cone and G be a set-valued map from X to a Banach space Z which is ordered by the pointed closed convex cone $K_Z \subset Z$. We write $\|(x, y)\| = \|x\| + \|y\|$ for the norm on the product space $X \times Y$. In the sequel the domain and the graph of F are respectively given by

$$\mathrm{Dom}(F) := \{x \in X \mid F(x) \neq \emptyset\},$$

$$\mathrm{gr}(F) := \{(x,y) \in X \times Y \mid y \in F(x)\}.$$

If A is a nonempty subset of X and B is a nonempty subset of Y, then $F(A) = \cup_{x\in A} F(x)$ and $F^{-}(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$.

Throughout this paper, X^*, Y^* and Z^* denote the continuous duals of X, Y and Z respectively, and we write $\langle \cdot,\cdot\rangle$ for the canonical bilinear forms with respect to the dualities $\langle X^*, X\rangle$, $\langle Y^*, Y\rangle$ and $\langle Z^*, Z\rangle$.

For a nonempty subset $S \subset Y$, let us recall the oriented distance function Δ_S (see [37]) which is defined by

$$\Delta_S(y) := d(y,S) - d(y,(Y\backslash S)) = \begin{cases} -d(y,(Y\backslash S)) & \text{if } y \in S \\ d(y,S) & \text{otherwise,} \end{cases} \tag{2.1}$$

where $d(\cdot,S)$ is the usual distance function

$$d(y,S) = \inf_{s\in S} \|y-s\|, \qquad \text{for all } y \in Y.$$

In the next proposition we collect some useful properties of Δ_S.

Proposition 2.1 ([37])**.** *Let $S \subset Y$ be nonempty and $S \neq Y$. Then the following assertions hold:*

(i) *Δ_S is real-valued and 1-Lipschitzian.*
(ii) *Δ_S is convex if S is convex.*
(iii) *Δ_S is positively homogenous if S is a cone.*
(iv) *$\Delta_S(y) < 0$, if and only if, $y \in \mathrm{int}(S)$.*
(v) *$\Delta_S(y) > 0$, if and only if, $y \in \mathrm{int}(Y\backslash S)$.*
(vi) *$\Delta_S(y) = 0$, if and only if, $y \in \mathrm{bd}(S)$, where $\mathrm{bd}(S)$ is the boundary of S.*
(vii) *For all $y \in Y$, $0 \notin \partial\Delta_S(y)$ if S is a convex cone with nonempty interior, where ∂ is the subdifferential in the sense of Clarke and convex analysis since Δ_S is convex.*
(viii) *Δ_S satisfies the triangle inequality when S is a convex cone, i.e.,*

$$\Delta_S(y_1+y_2) \le \Delta_S(y_1) + \Delta_S(y_2), \quad \text{for any } y_1, y_2 \in Y.$$

(ix) *If S is closed, then it holds that $S = \{y \in Y \mid \Delta_S(y) \le 0\}$.*

We consider the following set-valued optimization problem

$$(\mathrm{SP}_1) \quad \begin{cases} \text{Minimize} & F(x) \\ \text{subject to} & x \in C, \end{cases}$$

where C is a nonempty subset of X. It is said that $(\bar{x},\bar{y}) \in \mathrm{gr}(F) \cap (C \times Y)$ is a local weak Pareto minimizer for (SP_1) if there exists a neighborhood U of $\bar{x}$ such that

$$(F(U\cap C) - \bar{y}) \cap (-\mathrm{int}(K_Y)) = \emptyset.$$

Let us recall the following notions of weak sharp minima for (SP_1) those introduced respectively in Durea and Strugariu [13], Zhu *et al.* [40] and Amahroq *et al.* [5].

Definition 2.2 ([13])**.** Let $\epsilon > 0$ and $\psi : (-\epsilon, +\infty) \to \mathbb{R}$ be a nondecreasing function on $[0,+\infty[$ with the property that $\psi(t) = 0$ if and only if $t = 0$. One says that a point $(\bar{x},\bar{y}) \in \mathrm{gr}(F) \cap (C \times Y)$ is a local weak ψ-sharp Pareto minimizer for (SP_1), if there exist $c > 0$ and a neighborhood U of $\bar{x}$ such that for every $x \in U \cap C$, $y \in F(x)$ one has

$$c\psi\Big(d\Big(x,\widetilde{C}\Big)\Big) \le \Delta_{-K_Y}(y-\bar{y}), \tag{2.2}$$

where $\widetilde{C} = \{x \in C \mid \bar{y} \in F(x)\}$. If $\widetilde{C} = \{\bar{x}\}$ and one takes $\psi(t) = t$, then relation (2.2) becomes: for every $x \in U \cap C$, $y \in F(x)$ one has

$$c\|x-\bar{x}\| \le \Delta_{-K_Y}(y-\bar{y}),$$

and in this case one says that $(\bar{x},\bar{y})$ is a local sharp minimizer for (SP_1).

Definition 2.3 ([40])**.** $(\bar{x}, \bar{y}) \in \mathrm{gr}(F) \cap (C \times Y)$ is a local weak sharp minimizer for (SP$_1$), if there exist a neighborhood U of $\bar{x}$ and real numbers c, $\eta > 0$ such that

$$cd\Big(x, \widetilde{C}\Big) \leq d(y - \bar{y}, -K_Y) + \eta d(x, C), \quad \forall x \in U,\ \forall y \in F(x), \tag{2.3}$$

where $\widetilde{C} = \{x \in C \mid \bar{y} \in F(x)\}$. Specially, if $U = X$, then $(\bar{x}, \bar{y})$ is said to be a global weak sharp minimizer for (SP$_1$).

Definition 2.4 ([5])**.** Let $\gamma > 0$. It is said that $(\bar{x}, \bar{y}) \in \mathrm{gr}(F) \cap (C \times Y)$ is a local sharp minimizer of order γ in the strong sense (resp. in the weak sense) for (SP$_1$), if there exist $c > 0$ and a neighborhood U of $\bar{x}$ such that for all $x \in U \cap C$

$$F(x) + c\|x - \bar{x}\|^{\gamma}\overline{\mathbb{B}}_Y \subset \bar{y} + (Y \backslash (-K_Y)) \cup \{0\}, \tag{2.4}$$

$$(\text{resp.}\quad F(x) + c\|x - \bar{x}\|^{\gamma}\overline{\mathbb{B}}_Y \subset \bar{y} + (Y \backslash (-\mathrm{int}(K_Y))) \tag{2.5}$$

where $\overline{\mathbb{B}}_Y$ is the closed unit ball in Y. When (2.4) (resp. (2.5)) holds for all $x \in C$, then $(\bar{x}, \bar{y})$ is said to be a global sharp minimizer of order γ in the strong sense (resp. in the weak sense) for (SP$_1$).

Remark 2.5. It is easy to see that,

(i) a sharp minimizer of order γ in the strong sense is a sharp minimizer of order γ in the weak sense. Hence, each necessary condition for the existence of sharp minima in the weak sense is also a necessary condition for the existence of sharp minima in the strong sense.
(ii) for $\psi(t) = t$, a weak ψ-sharp minimizer in the sense of Definition 2.2 is a weak sharp minimizer in the sense of Definition 2.3.
(iii) for $\psi(t) = t^{\gamma}$, a local minimizer of Definition 3.1 from [14] is a local sharp minimizer of order γ in the strong sense.
(iv) weak sharp minimizers in the sense of Definitions 2.2 and 2.4 are weak Pareto minimizers for (SP$_1$).

Note that, Definition 2.2 also works in the case when $\mathrm{int}(K_Y) = \emptyset$; while the weak part in Definition 2.4 does not. In fact, the word "weak" in these definitions refers to different things: in Definition 2.2 it signifies the fact that the set $\widetilde{C}$ can have more than one element, while in Definition 2.4 it indicates exactly the presence of $\mathrm{int}(K_Y)$. In the next proposition we give some links between these two definitions when $\widetilde{C} = \{\bar{x}\}$ and $\psi(t) = t^{\gamma}$.

Proposition 2.6. *Let $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$ with $\bar{x} \in C$ and $\gamma > 0$. Assume that $\widetilde{C} = \{\bar{x}\}$ and $\psi(t) = t^{\gamma}$. The following assertions hold*:

(i) *$(\bar{x}, \bar{y})$ is a local sharp minimizer of order γ in the weak sense for* (SP$_1$) *if and only if $(\bar{x}, \bar{y})$ is a local weak ψ-sharp Pareto minimizer for* (SP$_1$) *in the sense of Definition 2.2.*
(ii) *If $(\bar{x}, \bar{y})$ is a local weak ψ-sharp Pareto minimizer for* (SP$_1$) *in the sense of Definition 2.2 and $\bar{y} \in \mathrm{Min}F(\bar{x})$, that is*

$$(F(\bar{x}) - \bar{y}) \cap (-K_Y) = \{0\},$$

then $(\bar{x}, \bar{y})$ is a local sharp minimizer of order γ in the strong sense for (SP$_1$).

Proof. (i) Since $\mathrm{int}(K_Y) \neq \emptyset$, by applying Proposition 3 of [1] for $\phi = \psi$ and $W = \widetilde{C} = \{\bar{x}\}$, together with Theorem 2.12 of [9], we conclude the required equivalence.
(ii) By assumption, there exist $c > 0$ and a neighborhood U of $\bar{x}$ such that for all $x \in U \cap C$ and $y \in F(x)$ one has

$$\Delta_{-K_Y}(y - \bar{y}) \geq c\|x - \bar{x}\|^{\gamma}.$$

Hence

$$d(y-\bar{y},-K_Y) \geq c\|x-\bar{x}\|^{\gamma}, \quad \text{for all } x \in (U \cap C)\backslash\{\bar{x}\} \text{ and } y \in F(x).$$

This equivalent to

$$F(x)-\bar{y}+c\|x-\bar{x}\|^{\gamma}\overline{\mathbb{B}}_Y \subset Y\backslash(-K_Y), \quad \text{for all } x \in (U \cap C)\backslash\{\bar{x}\}.$$

Since $\bar{y} \in \operatorname{Min}F(\bar{x})$, it follows that

$$F(x)-\bar{y}+c\|x-\bar{x}\|^{\gamma}\overline{\mathbb{B}}_Y \subset (Y\backslash(-K_Y)) \cup \{0\}, \quad \text{for all } x \in U \cap C.$$

Whence $(\bar{x},\bar{y})$ is a local sharp minimizer of order γ in the strong sense for (SP_1).

□

The following examples give a comparison among the above notions of sharp minimizers.

Example 2.7. Let $X=Y=\mathbb{R}$, $K_Y=\mathbb{R}_+$, $C=\mathbb{R}$ and $F: X \rightrightarrows Y$ defined by

$$F(x)=\begin{cases} [x,x+1] & \text{if } x<-1 \\ [-|x|,0] & \text{if } x \in [-1,1] \\ [-x,-x+1] & \text{if } x>1. \end{cases}$$

Here we observe that $(\bar{x},\bar{y})=(0,0)$ is a local weak sharp minimizer for (SP_1) in the sense of Definition 2.3. However, $(\bar{x},\bar{y})$ is not a local weak sharp minimizer for (SP_1) neither in the sense of Definition 2.2 nor in the sense of Definition 2.4. Also, $(\bar{x},\bar{y})$ is not a weak Pareto minimizer for (SP_1). Thus the inclusion in Remark 2.5(ii) is strict.

Example 2.8. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $K_Y=\mathbb{R}^2_+$, $C=\mathbb{R}$ and $F: X \rightrightarrows Y$ defined by

$$F(x)=\begin{cases} \left[(0,-1),\left(|x|,-\frac{1}{2}\right)\right] & \text{if } x \neq 0 \\ \{(0,0)\} & \text{if } x=0, \end{cases}$$

where $[(a,b),(c,d)]$ is the line segment between (a,b) and (c,d). Here we observe that $(\bar{x},\bar{y})=(0,(0,0))$ is a local weak Pareto minimizer for (SP_1) but not a local weak sharp minimizer neither in the sense of Definition 2.2 nor in the sense of Definition 2.3. Whence the inclusions in Remark 2.5(iv) are strict.

Remark 2.9. From the above examples we observe that,

(i) the notion of weak Pareto minimizer and weak sharp minimizer in the sense of Definition 2.3 are distinct, so a weak sharp minimizer in the sense of Definition 2.3 is not necessary a weak Pareto minimizer.
(ii) weak sharp minimizers in the sense of Definitions 2.2 and 2.4 are necessarily weak Pareto minimizers. Therefore, they seem as natural extensions of the notion of weak sharp minimizer to set-valued maps.

In the sequel we shall establish necessary optimality conditions for sharp minimizers in the weak sense for the problem (SP_1) and the following explicit constrained set-valued optimization problem (SP_2)

$$(\mathrm{SP}_2) \quad \begin{cases} \text{Minimize} \;\; F(x) \\ \text{subject to} \quad x \in C,\; G(x) \cap (-K_Z) \neq \emptyset. \end{cases}$$

Now we start with our first preliminary results which will be crucial steps in the sequel.

Proposition 2.10. *Let $(\bar{x},\bar{y}) \in \mathrm{gr}(F)$ with $\bar{x} \in C$ and $\gamma > 0$. If $(\bar{x},\bar{y})$ is a local sharp minimizer of order γ in the weak sense for* (SP$_1$) *then there exists $c > 0$ such that $(\bar{x},\bar{y})$ is a local minimal solution for the scalar problem $(\tilde{S})$:*

$$(\tilde{S}) \quad \begin{cases} \text{Minimize} \quad \phi(x,y) \\ \text{subject to} \quad (x,y) \in \mathrm{gr}(F) \cap (C \times Y), \end{cases}$$

where

$$\phi(x,y) := \Delta_{-K_Y}(y-\bar{y}) + c\|x-\bar{x}\|^{\gamma}.$$

Furthermore, if $\gamma \in \mathbb{N}^*$ then $(\bar{x},\bar{y})$ is a local minimal solution for the unconstrained scalar problem

$$(S') \quad \begin{cases} \text{Minimize} \quad \phi(x,y) + \bar{c}d((x,y), \mathrm{gr}(F) \cap (C \times Y)) \\ \text{subject to} \quad (x,y) \in X \times Y, \end{cases}$$

where $\bar{c} = \max(1, c\gamma)$.

Proof. By assumption there exist $c > 0$ and a neighborhood U of $\bar{x}$ such that, for all $x \in U \cap C$ and $y \in F(x)$ one has

$$\left(y - \bar{y} + c\|x-\bar{x}\|^{\gamma}\overline{\mathbb{B}}_Y\right) \cap (-\mathrm{int}(K_Y)) = \emptyset,$$

by Proposition 2.1(iv), Δ_{-K_Y} is positive on $Y \backslash (-\mathrm{int}(K_Y))$, then

$$\Delta_{-K_Y}(y - \bar{y} + c\|x-\bar{x}\|^{\gamma} b) \geq 0, \qquad \text{for all } b \in \overline{\mathbb{B}}_Y.$$

Since K_Y is a convex cone, Proposition 2.1(iii) and (viii) give that

$$\Delta_{-K_Y}(y-\bar{y}) + c\|x-\bar{x}\|^{\gamma}\Delta_{-K_Y}(b) \geq 0, \qquad \text{for all } b \in \overline{\mathbb{B}}_Y.$$

From the fact that $\Delta_{-K_Y}(0) = 0$ and Δ_{-K_Y} is 1-Lipschitz, it follows that $(\bar{x},\bar{y})$ solves locally the problem $(\tilde{S})$.

On the other hand, to show that $(\bar{x},\bar{y})$ is a local minimal solution of (S') it suffices to show that the function ϕ is $\bar{c}$-Lipschitz around $(\bar{x},\bar{y})$. Let $(x_1,y_1),(x_2,y_2) \in \mathbb{B}(\bar{x},1) \times Y$ we have

$$\begin{aligned} |\phi(x_1,y_1) - \phi(x_2,y_2)| &= |\Delta_{-K_Y}(y_1-\bar{y}) - \Delta_{-K_Y}(y_2-\bar{y}) + c\|x_1-\bar{x}\|^{\gamma} - c\|x_2-\bar{x}\|^{\gamma}| \\ &\leq |\Delta_{-K_Y}(y_1-\bar{y}) - \Delta_{-K_Y}(y_2-\bar{y})| + c|\|x_1-\bar{x}\|^{\gamma} - \|x_2-\bar{x}\|^{\gamma}| \\ &\leq \|y_1-y_2\| + c|\|x_1-\bar{x}\| - \|x_2-\bar{x}\|| \sum_{i=1}^{\gamma} \|x_1-\bar{x}\|^{\gamma-i}\|x_2-\bar{x}\|^{i} \\ &\leq \|y_1-y_2\| + c\gamma\|x_1-x_2\| \\ &\leq \bar{c}\|(x_1,y_1)-(x_2,y_2)\|. \end{aligned}$$

Then ϕ is $\bar{c}$-Lipschitz around $(\bar{x},\bar{y})$, and hence by the Clarke penalization ([11], Prop. 2.4.3), $(\bar{x},\bar{y})$ is a local minimal solution of (S'). $\square$

The following scalarization result will be useful to establish necessary conditions in terms of Fritz-John multipliers for (SP$_2$).

Proposition 2.11. *Let $(\bar{x},\bar{y}) \in \mathrm{gr}(F)$ with $\bar{x} \in C$ and $\gamma \in \mathbb{N}^*$. If $(\bar{x},\bar{y})$ is a local sharp minimizer of order γ in the weak sense for* (SP$_2$) *then there exists $c > 0$ such that for all $\bar{z} \in G(\bar{x}) \cap (-K_Z)$, $(\bar{x},\bar{y},\bar{z})$ is a local minimal solution of the unconstrained scalar problem (S_2) :*

$$(S_2) \quad \begin{cases} \text{Minimize} \quad g(x,y,z) + \bar{c}d((x,y,z), \mathrm{gr}(F,G) \cap (C \times Y \times Z)) \\ \text{subject to} \quad (x,y,z) \in X \times Y \times Z, \end{cases}$$

where

$$g(x,y,z) := \max(\Delta_{-K_Y}(y-\bar{y}) + c\|x-\bar{x}\|^\gamma, \Delta_{-K_Z}(z)),$$

$\bar{c} = \max(1, c\gamma)$, *and* (F,G) *is the set-valued mapping defined from* X *to* $Y \times Z$ *by* $(F,G)(x) := (F(x), G(x)) = F(x) \times G(x)$, *for all* $x \in X$.

Proof. It suffices to show that $(\bar{x}, \bar{y}, \bar{z})$ is a local minimal solution of the problem

$$\begin{cases} \text{Minimize} \quad g(x,y,z) \\ \text{subject to} \quad (x,y,z) \in \operatorname{gr}(F,G) \cap (C \times Y \times Z). \end{cases}$$

Indeed. Suppose the contrary. Then there exists a sequence $(x_n, y_n, z_n) \in \operatorname{gr}(F,G) \cap (C \times Y \times Z)$ such that

$$(x_n, y_n, z_n) \to (\bar{x}, \bar{y}, \bar{z}) \quad \text{and} \quad g(x_n, y_n, z_n) < g(\bar{x}, \bar{y}, \bar{z}) = 0 \quad \text{for all } n \in \mathbb{N}.$$

Hence, for all $n \in \mathbb{N}$, we get that

$$\Delta_{-K_Y}(y_n - \bar{y}) + c\|x_n - \bar{x}\|^\gamma < 0 \quad \text{and} \quad \Delta_{-K_Z}(z_n) < 0,$$

this implies that, for all $b \in \overline{\mathbb{B}}_Y$

$$\Delta_{-K_Y}(y_n - \bar{y}) + c\|x_n - \bar{x}\|^\gamma \Delta_{-K_Y}(b) < 0 \quad \text{and} \quad z_n \in G(x_n) \cap (-K_Z).$$

Proposition 2.1(iii) and (viii) give that

$$\Delta_{-K_Y}(y_n - \bar{y} + c\|x_n - \bar{x}\|^\gamma b) < 0 \quad \text{and} \quad z_n \in G(x_n) \cap (-K_Z),$$

and then

$$y_n - \bar{y} + c\|x_n - \bar{x}\|^\gamma b \in -\operatorname{int}(K_Y), \quad y_n \in F(x_n) \quad \text{and} \quad z_n \in G(x_n) \cap (-K_Z).$$

This contradicts the fact that $(\bar{x}, \bar{y})$ is a local sharp minimizer of order γ in the weak sense for (SP_2). Since g is $\bar{c}$-Lipschitz, the Clarke penalization ([11], Prop. 2.4.3) completes the proof. □

3. Sharp Fritz-John multipliers (for $\gamma = 1$)

In the sequel, for a closed cone K of Y, K° will be the polar cone of K defined by

$$K^\circ = \{y^* \in Y^* \mid \langle y^*, k\rangle \le 0, \quad \forall k \in K\}.$$

For a Lipschitzian function h on X, we will denote by $\partial h(\bar{x})$ the Clarke subdifferential of h at $\bar{x} \in X$.

Let C_i, $i = 1, 2, \ldots, n$, be nonempty subsets of X. Recall [16,31] that the sets C_i satisfy the metric inequality at $\bar{x} \in C_1 \cap \cdots \cap C_n$, if there are $\alpha > 0$ and a neighborhood U of $\bar{x}$ such that for each $x \in U$

$$d(x, C_1 \cap \cdots \cap C_n) \le \alpha[d(x, C_1) + \cdots + d(x, C_n)]. \tag{3.1}$$

The above inequality is very well studied in literature and it is known under various names: bounded linear regularity [7], metric regularity [17], local linear regularity [30], linear coherence [33], subtransversality [18] (more details can be found in [7,16–18,22,24–26,28–33] and in the references therein). Note that, several authors have established the conditions ensuring this inequality. Among them, we may cite, Proposition 1 of [25] in the setting of normed linear spaces, Theorem 5.1 of [16], Proposition 3.1 of [22] and Theorem 3.1(ii) of [24] in the setting of general Banach spaces, Theorem 1 of [26] in the setting of Euclidean spaces, Theorem 6.44 of [33] and Theorem 3.1 and Corollary 3.2 of [31] in the case of Asplund spaces, and Corollary 4.2 of [30] in the setting of Reflexive spaces.

We can now state necessary conditions for the problem (SP_1).

Theorem 3.1. *Let $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$ with $\bar{x} \in C$. Assume that (3.1) holds for $C_1 = (C \times Y)$ and $C_2 = \mathrm{gr}(F)$. If $(\bar{x}, \bar{y})$ is a local sharp minimizer of order $\gamma = 1$ in the weak sense for* (SP_1) *then there exist $c > 0$, $x_1^*, x_2^* \in X^* \times X^*$ and $y^* \in (-K_Y)^\circ \backslash \{0_{Y^*}\}$ such that*

$$(-x_1^* - x_2^*, -y^*) \in \partial d((\bar{x}, \bar{y}), \mathrm{gr}(F)) \quad \text{with} \quad \|x_1^*\|_{X^*} \leq c \quad \text{and} \quad x_2^* \in N(\bar{x}, C),$$

where $N(\bar{x}, C)$ is the Clarke normal cone of C at $\bar{x}$.

Proof. By assumptions, Proposition 2.10 gives that, $(\bar{x}, \bar{y})$ is a local minimal solution for the unconstrained scalar problem

$$\begin{cases} \text{Minimize} \;\; \phi(x,y) + \alpha\bar{c}[d((x,y), \mathrm{gr}(F)) + d(x, C)] \\ \text{subject to} \quad (x,y) \in X \times Y, \end{cases}$$

where $\phi(x, y) = \Delta_{-K_Y}(y - \bar{y}) + c\|x - \bar{x}\|$. Since ϕ and the distance function are Lipschitzians we obtain that

$$\begin{aligned} (0,0) &\in \partial\big[\phi + \alpha\bar{c}d(\cdot\,, \mathrm{gr}(F)) + \alpha\bar{c}d(\cdot\,, C)\big](\bar{x}, \bar{y}), \\ &\subset \partial(c\|\cdot - \bar{x}\|)(\bar{x}) \times \partial\Delta_{-K_Y}(0) + \alpha\bar{c}\partial d(\cdot\,, \mathrm{gr}(F))(\bar{x}, \bar{y}) + \alpha\bar{c}\partial d(\cdot\,, C)(\bar{x}) \times \{0\}, \\ &\subset c\overline{\mathbb{B}}_{X^*} \times (-K_Y)^\circ + \alpha\bar{c}\partial d((\bar{x}, \bar{y}), \mathrm{gr}(F)) + N(\bar{x}, C) \times \{0\}, \end{aligned}$$

because $\partial\Delta_{-K_Y}(0) \subset (-K_Y)^\circ$. Thus, there exist $x_1^* \in X^*$ with $\|x_1^*\|_{X^*} \leq c$, $x_2^* \in N(\bar{x}, C)$ and $y^* \in (-K_Y)^\circ$ such that

$$(-x_1^* - x_2^*, -y^*) \in \partial d((\bar{x}, \bar{y}), \mathrm{gr}(F)).$$

Proposition 2.1(vii) gives that $y^* \neq 0_{Y^*}$ and the proof is complete. □

Next, we establish implicit Fritz-John multipliers for (SP_2).

Theorem 3.2. *Let $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$ with $\bar{x} \in C$ and $\bar{z} \in G(\bar{x}) \cap (-K_Z)$. Assume that (3.1) holds for $C_1 = (C \times Y \times Z)$ and $C_2 = \mathrm{gr}(F, G)$. If $(\bar{x}, \bar{y})$ is a local sharp minimizer of order $\gamma = 1$ in the weak sense for* (SP_2) *then there exist $c > 0$, $(x_1^*, x_2^*) \in X^* \times N(\bar{x}, C)$, $y^* \in (-K_Y)^\circ \backslash \{0_{Y^*}\}$, $z^* \in (-K_Z)^\circ \backslash \{0_{Z^*}\}$ and $(\alpha_1, \alpha_2) \in (\mathbb{R}_+ \times \mathbb{R}_+) \backslash \{0, 0\}$ such that*

(i) $\|x_1^*\|_{X^*} \leq c$,
(ii) $\langle z^*, \bar{z}\rangle = -d(\bar{z}, (Z \backslash -K_Z))$,
(iii) $(-x_2^* - \alpha_1 x_1^*, -\alpha_1 y^*, -\alpha_2 z^*) \in \partial d((\bar{x}, \bar{y}, \bar{z}), \mathrm{gr}(F, G))$.

Proof. By Proposition 2.11 together with the assumption (3.1) we obtain that

$$\begin{aligned} (0,0,0) &\in \partial[g + \alpha\bar{c}d(\cdot\,, \mathrm{gr}(F,G)) + \alpha\bar{c}d(\cdot, C)](\bar{x}, \bar{y}, \bar{z}) \\ &\subset \mathrm{co}([\partial(c\|\cdot - \bar{x}\|)(\bar{x}) \times \partial\Delta_{-K_Y}(0)] \cup \partial\Delta_{-K_Z}(\bar{z})) + \alpha\bar{c}\partial d((\bar{x}, \bar{y}, \bar{z}), \mathrm{gr}(F,G)) + \alpha\bar{c}\partial d(\bar{x}, C) \times \{(0,0)\}, \end{aligned}$$

where co is the convex hull. Hence there are $x_2^* \in N(\bar{x}, C)$ and $(\alpha_1, \alpha_2) \in (\mathbb{R}_+ \times \mathbb{R}_+) \backslash \{(0,0)\}$ such that

$$(-x_2^*, 0, 0) \in \alpha_1[\partial(c\|\cdot - \bar{x}\|)(\bar{x}) \times \partial\Delta_{-K_Y}(0)] \times \{0\} + \{(0,0)\} \times \alpha_2\partial\Delta_{-K_Z}(\bar{z}) + \alpha\bar{c}\partial d((\bar{x}, \bar{y}, \bar{z}), \mathrm{gr}(F,G)).$$

By using similar arguments as in the proof of Theorem 3.1, there exist $x_1^* \in X^*$ with $\|x_1^*\|_{X^*} \leq c$, $y^* \in (-K_Y)^\circ \backslash \{0_{Y^*}\}$ and $z^* \in \partial\Delta_{-K_Z}(\bar{z})$ such that

$$(-x_2^* - \alpha_1 x_1^*, -\alpha_1 y^*, -\alpha_2 z^*) \in \partial d((\bar{x}, \bar{y}, \bar{z}), \mathrm{gr}(F, G)).$$

On the other hand one has

$$\langle z^*, z - \bar{z}\rangle \leq \Delta_{-K_Z}(z) - \Delta_{-K_Z}(\bar{z}), \qquad \text{for all } z \in Z. \tag{3.2}$$

Taking in (3.2), $z = 2\bar{z}$ and $z = \frac{1}{2}\bar{z}$, it follows that

$$\langle z^*, \bar{z}\rangle = \Delta_{-K_Z}(\bar{z}) = -d(\bar{z}, (Z\setminus -K_Z)). \tag{3.3}$$

Combining (3.2) and (3.3) we get that

$$\langle z^*, z\rangle \leq 0 \quad \text{for all } z \in -K_Z,$$

that is $z^* \in (-K_Z)^\circ$. Proposition 2.1(vii) implies that $z^* \neq 0_{Z^*}$ and the proof is complete. □

Remark 3.3. The assumption (3.1) in the above theorems is always true in the case of unconstrained set-valued optimization problems. Besides, the results obtained here are different than those of Amahroq and Taa [2]. Indeed, x_1^* is not necessarily zero as in Amahroq and Taa [2]. Comparing with the results of Durea and Strugariu [13], in the above theorems the graph of F (resp, the set-valued data F) is not necessarily locally closed (resp. locally Lipschitz-like) and X, Y are Banach spaces. The Lipschitz-like condition in Durea and Strugariu [13] is replaced here by (3.1). Obviously, the Asplund space condition in Durea and Strugariu [13] is more restrictive, but their conclusion uses the Mordukhovich normal cone which is smaller than the Clarke normal cone considered above.

4. Sharp Karush–Kuhn–Tucker multipliers (for $\gamma = 1$)

In order to establish necessary optimality conditions in terms of Karush–Kuhn–Tucker multipliers for the constrained problem (SP_2) we shall use the following metrical regularity condition.

Definition 4.1. A set-valued map G is said to be metrically regular at $(\bar{x}, \bar{z}) \in \mathrm{gr}(G)$ with $\bar{z} \in (-K_Z)$ relative to $-K_Z$ if there exist $\bar{k} > 0$ and neighborhoods V and W of $\bar{x}$ and $\bar{z}$ respectively such that

$$d(x, G^-(-K_Z)) \leq \bar{k} d(z, (-K_Z)),$$

for all $x \in V$ and $z \in W \cap G(x)$.

This regularity condition is well known in the literatures when G is a single valued mapping (see [23] and the references therein) and in the above general form it has been considered in Amahroq and Thibault [3] and studied in Amahroq *et al.* [4]. For verifiable conditions ensuring this condition by virtue of the set-valued map M, see Amahroq *et al.* [4].

Now we state sharp Karush–Kuhn–Tucker multipliers of order $\gamma = 1$ for the constrained problem (SP_2).

Theorem 4.2. *Let $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$ with $\bar{x} \in C$ and $\bar{z} \in G(\bar{x}) \cap (-K_Z)$. Assume that (3.1) holds for $C_1 = G^-(-K_Z) \times Y$, $C_2 = C \times Y$ and $C_3 = \mathrm{gr}(F)$, and G is metrically regular at $(\bar{x}, \bar{z})$ relative to $-K_Z$. If $(\bar{x}, \bar{y})$ is a local sharp minimizer of order $\gamma = 1$ in the weak sense for (SP_2) then there exist $c > 0$, $x^* \in X^*$, $y^* \in (-K_Y)^\circ \setminus \{0_{Y^*}\}$, $z^* \in (-K_Z)^\circ$ and $r > 0$ such that*

(i) $\|x^*\|_{X^*} \leq c$ *and* $\langle z^*, \bar{z}\rangle = 0$,
(ii) $\langle y^*, y\rangle + \langle z^*, z\rangle \leq 0$, for all $y \in (-K_Y)$ and $z \in (-K_Z)$,
(iii) $(-x^*, -y^*, -z^*) \in \partial d((\bar{x}, \bar{y}), \mathrm{gr}(F)) \times \{0\} + N(\bar{x}, C) \times \{(0, 0)\} + r\partial h(\bar{x}, \bar{y}, \bar{z})$,

where $h(x, y, z) = d((x, z), \mathrm{gr}(G))$.

Proof. Applying again the Proposition 2.10 with the feasible set $G^-(-K_Z) \cap C$, so $(\bar{x}, \bar{y})$ is a local solution of the unconstrained scalar problem (S_1) and by the metric regularity assumption together with (3.1), we obtain that $(\bar{x}, \bar{y}, \bar{z})$ is a local minimizer of the problem

$$\begin{cases} \text{Minimize } \ \phi(x, y) + \alpha\bar{c}\big[d((x, y), \mathrm{gr}(F)) + d(x, C) + \bar{k} d(z, -K_Z)\big] \\ \text{subject to } \ \ x \in V, \ y \in Y \text{ and } z \in W \cap G(x), \end{cases}$$

where V, W and $\bar{k}$ are given by Definition 4.1. Applying the Clarke penalization ([11], Prop. 2.4.3), we get that $(\bar{x}, \bar{y}, \bar{z})$ is a local minimizer of the unconstrained problem

$$\begin{cases} \text{Minimize} \quad \phi(x, y) + \alpha\bar{c}\big[d((x, y), \mathrm{gr}(F)) + d(x, C) + \bar{k}d(z, -K_Z)\big] + rd((x, z), \mathrm{gr}(G)) \\ \text{subject to} \quad (x, y, z) \in X \times Y \times Z, \end{cases}$$

where $r = \bar{c}(1 + 2\alpha + \alpha\bar{k})$. Hence

$$(0, 0, 0) \in \partial\phi(\bar{x}, \bar{y}) \times \{0\} + \alpha\bar{c}\partial d((\bar{x}, \bar{y}), \mathrm{gr}(F)) \times \{0\} + N(\bar{x}, C) \times \{(0, 0)\} + \{(0, 0)\} \times N(\bar{z}, -K_Z) + r\partial h(\bar{x}, \bar{y}, \bar{z}).$$

Then there exist $(x^*, y^*) \in \partial\phi(\bar{x}, \bar{y})$ and $z^* \in N(\bar{z}, -K_Z)$ such that

$$(-x^*, -y^*, -z^*) \in \partial d((\bar{x}, \bar{y}), \mathrm{gr}(F)) \times \{0\} + N(\bar{x}, C) \times \{(0, 0)\} + r\partial d((\bar{x}, \bar{z}), \mathrm{gr}(G)).$$

By using similar arguments as in the proof of Theorem 3.1, we obtain that $\|x^*\|_{X^*} \leq c$ and the fact that $y^* \in (-K_Y)^\circ \backslash \{0_{Y^*}\}$.

On the other hand, $z^* \in N(\bar{z}, -K_Z)$ implies that

$$\langle z^*, z - \bar{z}\rangle \leq 0, \qquad \text{for all } z \in (-K_Z), \tag{4.1}$$

and therefore (by taking $z = 2\bar{z}$ and $z = \frac{1}{2}\bar{z}$ in the previous inequality)

$$\langle z^*, \bar{z}\rangle = 0. \tag{4.2}$$

Combining (4.1) and (4.2) we get that

$$\langle z^*, z\rangle \leq 0, \quad \text{for all } z \in (-K_Z),$$

that is $z^* \in (-K_Z)^\circ$. Whence (ii) holds from the fact that $y^* \in (-K_Y)^\circ$ and $z^* \in (-K_Z)^\circ$. □

5. Necessary conditions for sharp minima of higher order in the weak sense

The following results provide necessary conditions for sharp minimizers of order $\gamma \geq 2$ in the weak sense where γ is integer.

Theorem 5.1. *Let $\gamma \geq 2$ (γ integer), $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$ with $\bar{x} \in C$. Assume that (3.1) holds for $C_1 = (C \times Y)$ and $C_2 = \mathrm{gr}(F)$. If $(\bar{x}, \bar{y})$ is a local sharp minimizer of order γ in the weak sense for (SP_1) then there exist $c, \bar{c}, \alpha > 0$ such that*

$$(0, 0) \in \partial[\Delta_{-K_Y}(\cdot - \bar{y}) + c\| \cdot - \bar{x}\|^\gamma + \alpha\bar{c}d(\cdot\ , \mathrm{gr}(F))](\bar{x}, \bar{y}) + N(\bar{x}, C) \times \{0\}.$$

Proof. It is enough to apply Proposition 2.10 together with the assumption (3.1) to get the proof. □

Theorem 5.2. *Let $\gamma \geq 2$ (γ integer), $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$ with $\bar{x} \in C$ and $\bar{z} \in G(\bar{x}) \cap (-K_Z)$. Assume that (3.1) holds for $C_1 = (C \times Y \times Z)$ and $C_2 = \mathrm{gr}(F, G)$. If $(\bar{x}, \bar{y})$ is a local sharp minimizer of order γ in the weak sense for (SP_2) then there exist $c, \bar{c}, \alpha > 0$ and $x^* \in N(\bar{x}, C)$ such that*

$$(-x^*, 0, 0) \in \partial[\max(\Delta_{-K_Y}(\cdot - \bar{y}) + c\| \cdot - \bar{x}\|^\gamma, \Delta_{-K_Z}) + \alpha\bar{c}d(\cdot\ , \mathrm{gr}(F, G))](\bar{x}, \bar{y}, \bar{z}).$$

Proof. The proof follows directly from Proposition 2.11 together with the assumption (3.1). □

Theorem 5.3. *Let $\gamma \geq 2$ (γ integer), $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$ and $\bar{z} \in G(\bar{x}) \cap (-K_Z)$. Assume that (3.1) holds for $C_1 = G^{-}(-K_Z) \times Y$, $C_2 = C \times Y$ and $C_3 = \mathrm{gr}(F)$, and G is metrically regular at $(\bar{x}, \bar{z})$ relative to $-K_Z$. If $(\bar{x}, \bar{y})$ is a local sharp minimizer of order γ in the weak sense for* (SP_2) *then there exist $x^* \in N(C, \bar{x})$, $z^* \in (-K_Z)^\circ$ and $\alpha, r, c, \bar{c} > 0$ such that*

$$(-x^*, 0, -z^*) \in \partial(\Delta_{-K_Y}(\cdot - \bar{y}) + c\| \cdot - \bar{x}\|^\gamma + \alpha\bar{c}d(\cdot, \mathrm{gr}(F)))(\bar{x}, \bar{y}) + r\partial h(\bar{x}, \bar{y}, \bar{z}),$$

where $h(x, y, z) = d((x, z), \mathrm{gr}(G))$.

Proof. By using the same reasoning as in the proof of Theorem 4.2 we obtain that

$$(0, 0, 0) \in \partial\big[\phi + \alpha\bar{c}\big[d(\cdot, \mathrm{gr}(F)) + d(\cdot, C) + \bar{k}d(\cdot, -K_Z)\big] + rd(\cdot, \mathrm{gr}(G))\big](\bar{x}, \bar{y}, \bar{z}),$$

where $\phi(x, y) = \Delta_{-K_Y}(y - \bar{y}) + c\|x - \bar{x}\|^\gamma$. Thus the proof is completed. □

Remark 5.4. The conclusions in Theorems 5.1–5.3 are presented in terms of Fermat rules, because if we apply the subdifferential of the sum we will lose the dependence with γ.

6. Necessary conditions for sharp minima in the strong sense

It is worth noting that Definitions 2.2 and 2.3 still work when $\mathrm{int}(K_Y) = \emptyset$. Therefore, some discussion on necessary optimality conditions for sharp minima in the strong sense will be of interest, especially when the interior of the ordering cone K_Y is empty.

In this section, the interior of K_Y is not necessarily nonempty. Let us start with the following scalarization results which will be crucial steps in the sequel.

Proposition 6.1. *Let $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$ with $\bar{x} \in C$ and $\gamma \in \mathbb{N}^*$. If $(\bar{x}, \bar{y})$ is a local sharp minimizer of order γ in the strong sense for* (SP_1) *then there exists $c > 0$ such that $(\bar{x}, \bar{y})$ is a local minimal solution for the unconstrained scalar problem*

$$\begin{cases} \text{Minimize} & \phi(x, y) + \bar{c}d((x, y), \mathrm{gr}(F) \cap (C \times Y)) \\ \text{subject to} & (x, y) \in X \times Y, \end{cases}$$

where $\bar{c} = \max(1, c\gamma)$.

Proof. By assumption there exist $c > 0$ and a neighborhood U of $\bar{x}$ such that, for all $x \in U \cap C$ and $y \in F(x)$ one has

$$y - \bar{y} + c\|x - \bar{x}\|^\gamma \overline{\mathbb{B}}_Y \subset (Y \backslash (-K_Y)) \cup \{0\}, \tag{6.1}$$

whence, for all $b \in \overline{\mathbb{B}}_Y$

$$y - \bar{y} + c\|x - \bar{x}\|^\gamma b \in Y \backslash (-K_Y) \qquad \text{or} \qquad y - \bar{y} + c\|x - \bar{x}\|^\gamma b = 0, \tag{6.2}$$

from the definition of Δ_{-K_Y} together with Proposition 2.1(vi), we get that

$$\Delta_{-K_Y}(y - \bar{y} + c\|x - \bar{x}\|^\gamma b) \geq 0, \qquad \text{or} \qquad \Delta_{-K_Y}(y - \bar{y} + c\|x - \bar{x}\|^\gamma b) = 0.$$

Thus

$$\Delta_{-K_Y}(y - \bar{y} + c\|x - \bar{x}\|^\gamma b) \geq 0, \qquad \forall b \in \overline{\mathbb{B}}_Y.$$

The rest of the proof is practically the same as that of Proposition 2.10. □

Proposition 6.2. *Let $(\bar{x},\bar{y})\in \mathrm{gr}(F)$ with $\bar{x}\in C$ and $\gamma\in\mathbb{N}^*$. If $(\bar{x},\bar{y})$ is a local sharp minimizer of order γ in the strong sense for* (SP_2) *then there exists $c>0$ such that for all $\bar{z}\in G(\bar{x})\cap(-K_Z)$, $(\bar{x},\bar{y},\bar{z})$ is a local minimal solution of the unconstrained scalar problem*

$$\begin{cases}\text{Minimize} \quad g(x,y,z)+\bar{c}d((x,y,z),\mathrm{gr}(F,G)\cap(C\times Y\times Z))\\ \text{subject to} \quad (x,y,z)\in X\times Y\times Z,\end{cases}$$

where g, $\bar{c}$ and (F,G) are defined in Proposition 2.11.

Proof. In the same manner as the proof of Proposition 2.11, we show that $(\bar{x},\bar{y},\bar{z})$ is a local minimal solution of the problem

$$\begin{cases}\text{Minimize} \quad g(x,y,z)\\ \text{subject to} \quad (x,y,z)\in \mathrm{gr}(F,G)\cap(C\times Y\times Z).\end{cases}$$

Indeed. Suppose the contrary. Then there exists a sequence $(x_n,y_n,z_n)\in \mathrm{gr}(F,G)\cap(C\times Y\times Z)$ such that for all $b\in\overline{\mathbb{B}}_Y$ and $n\in\mathbb{N}$ one has

$$\Delta_{-K_Y}(y_n-\bar{y}+c\|x_n-\bar{x}\|^{\gamma}b)<0 \quad \text{and} \quad z_n\in G(x_n)\cap(-K_Z),$$

it follows from the definition of Δ_{-K_Y} together with Proposition 2.1(vi) that

$$y_n-\bar{y}+c\|x_n-\bar{x}\|^{\gamma}b\in(-K_Y)\backslash\{0\}, \quad y_n\in F(x_n) \quad \text{and} \quad z_n\in G(x_n)\cap(-K_Z).$$

This contradicts the fact that $(\bar{x},\bar{y})$ is a local sharp minimizer of order γ in the strong sense for (SP_2). The rest of the proof is the same as that of Proposition 2.11. $\square$

The following corollaries can be proved in the same manner as the proofs of Theorems 3.1, 3.2, 4.2, 5.1, 5.2, and 5.3 respectively, based on Propositions 6.1 and 6.2.

Corollary 6.3. *Under the setting of Theorem 3.1, if $(\bar{x},\bar{y})$ is a local sharp minimizer of order $\gamma=1$ in the strong sense for* (SP_1) *then there exist $c>0$, $x_1^*,x_2^*\in X^*\times X^*$ and $y^*\in(-K_Y)^{\circ}\backslash\{0_{Y^*}\}$ such that*

$$(-x_1^*-x_2^*,-y^*)\in\partial d((\bar{x},\bar{y}),\mathrm{gr}(F)) \quad \text{with} \quad \|x_1^*\|_{X^*}\le c \quad \text{and} \quad x_2^*\in N(\bar{x},C),$$

where $N(\bar{x},C)$ is the Clarke normal cone of C at $\bar{x}$.

Corollary 6.4. *Under the setting of Theorem 3.2, if $(\bar{x},\bar{y})$ is a local sharp minimizer of order $\gamma=1$ in the strong sense for* (SP_2) *then there exist $c>0$, $(x_1^*,x_2^*)\in X^*\times N(\bar{x},C)$, $y^*\in(-K_Y)^{\circ}\backslash\{0_{Y^*}\}$, $z^*\in(-K_Z)^{\circ}\backslash\{0_{Z^*}\}$ and $(\alpha_1,\alpha_2)\in(\mathbb{R}_+\times\mathbb{R}_+)\backslash\{0,0\}$ such that*

(i) $\|x_1^*\|_{X^*}\le c$,
(ii) $\langle z^*,\bar{z}\rangle=-d(\bar{z},(Z\backslash-K_Z))$,
(iii) $(-x_2^*-\alpha_1x_1^*,-\alpha_1y^*,-\alpha_2z^*)\in\partial d((\bar{x},\bar{y},\bar{z}),\mathrm{gr}(F,G))$.

Corollary 6.5. *Under the setting of Theorem 4.2, if $(\bar{x},\bar{y})$ is a local sharp minimizer of order $\gamma=1$ in the strong sense for* (SP_2) *then there exist $c>0$, $x^*\in X^*$, $y^*\in(-K_Y)^{\circ}\backslash\{0_{Y^*}\}$, $z^*\in(-K_Z)^{\circ}$ and $r>0$ such that*

(i) $\|x^*\|_{X^*}\le c$ *and* $\langle z^*,\bar{z}\rangle=0$,
(ii) $\langle y^*,y\rangle+\langle z^*,z\rangle\le 0$, for all $y\in(-K_Y)$ and $z\in(-K_Z)$,
(iii) $(-x^*,-y^*,-z^*)\in\partial d((\bar{x},\bar{y}),\mathrm{gr}(F))\times\{0\}+N(\bar{x},C)\times\{(0,0)\}+r\partial h(\bar{x},\bar{y},\bar{z})$,

where $h(x,y,z)=d((x,z),\mathrm{gr}(G))$.

Corollary 6.6. *Under the setting of Theorem 5.1, if $(\bar{x},\bar{y})$ is a local sharp minimizer of order $\gamma\ge 2$ (γ integer) in the strong sense for* (SP_1) *then there exist $c,\bar{c},\alpha>0$ such that*

$$(0,0)\in\partial\big[\Delta_{-K_Y}(\cdot-\bar{y})+c\|\cdot-\bar{x}\|^{\gamma}+\alpha\bar{c}d(\cdot\ ,\mathrm{gr}(F))\big](\bar{x},\bar{y})+N(\bar{x},C)\times\{0\}.$$

Corollary 6.7. *Under the setting of Theorem 5.2, if $(\bar{x},\bar{y})$ is a local sharp minimizer of order $\gamma \geq 2$ (γ integer) in the strong sense for* (SP_2) *then there exist $c,\bar{c},\alpha > 0$ and $x^* \in N(\bar{x},C)$ such that*

$$(-x^*,0,0) \in \partial[\max(\Delta_{-K_Y}(\cdot-\bar{y}) + c\|\cdot-\bar{x}\|^\gamma, \Delta_{-K_Z}) + \alpha\bar{c}d(\cdot\ ,\mathrm{gr}(F,G))](\bar{x},\bar{y},\bar{z}).$$

Corollary 6.8. *Under the setting of Theorem 5.3, if $(\bar{x},\bar{y})$ is a local sharp minimizer of order $\gamma \geq 2$ (γ integer) in the strong sense for* (SP_2) *then there exist $x^* \in N(C,\bar{x})$, $z^* \in (-K_Z)^\circ$ and $\alpha, r, c, \bar{c} > 0$ such that*

$$(-x^*,0,-z^*) \in \partial(\Delta_{-K_Y}(\cdot-\bar{y}) + c\|\cdot-\bar{x}\|^\gamma + \alpha\bar{c}d(\cdot,\mathrm{gr}(F)))(\bar{x},\bar{y}) + r\partial h(\bar{x},\bar{y},\bar{z}),$$

where $h(x,y,z) = d((x,z),\mathrm{gr}(G))$.

7. Sufficient conditions

Let S be a nonempty subset of X and let $\bar{x} \in S$. The radial cone $R(S,\bar{x})$ of S at $\bar{x}$ is the subset of X defined by

$$R(S,\bar{x}) := \mathrm{cl}(\mathbb{R}_+(S-\bar{x})),$$

where cl denotes the closure. It is obvious to see that for all $(\bar{x},\bar{y}) \in \mathrm{gr}(F)$

$$[R(\mathrm{gr}(F),(\bar{x},\bar{y}))]^\circ = \{(x^*,y^*) \in X^* \times Y^* \mid \langle (x^*,y^*),(x,y)-(\bar{x},\bar{y})\rangle \leq 0, \quad \forall (x,y) \in \mathrm{gr}(F)\}.$$

Note that, the Clarke normal cone $N(S,\bar{x})$ reduces to the normal cone of convex analysis when S is convex, *i.e.*,

$$N(S,\bar{x}) = [R(S,\bar{x})]^\circ = \{x^* \in X^* \mid \langle x^*, x-\bar{x}\rangle \leq 0, \quad \forall x \in S\}.$$

We begin with the following theorem that provides sufficient optimality conditions for sharp minima in the weak sense for (SP_2) without any convexity assumption on the set-valued objective mapping.

Theorem 7.1. *Let $\gamma > 0$, $(\bar{x},\bar{y}) \in \mathrm{gr}(F)$ and $\bar{z} \in G(\bar{x}) \cap (-K_Z)$. Assume that there exist $y^* \in (-K_Y)^\circ\backslash\{0\}$ and $z^* \in (-K_Z)^\circ$ such that*

$$(0,-y^*,-z^*) \in \left[R\big(M_{(\mathrm{gr}(F),\gamma)} \cap (C\times Y),(\bar{x},\bar{y})\big)\right]^\circ \times \{0_{Z^*}\} + [R(C,\bar{x})]^\circ \times \{(0_{Y^*},0_{Z^*})\} + A \circ [R(\mathrm{gr}(G),(\bar{x},\bar{z}))]^\circ, \tag{7.1}$$

where $M_{(\mathrm{gr}(F),\gamma)} = \{(x, y+\|x-\bar{x}\|^\gamma b) \mid (x,y) \in \mathrm{gr}(F),\ b \in \overline{\mathbb{B}}_Y\}$ and A is defined from $X^* \times Z^*$ to $X^* \times Y^* \times Z^*$ by $A(x^*,z^*) = (x^*,0_{Y^*},z^*)$. Then $(\bar{x},\bar{y})$ is a sharp minimizer of order γ in the weak sense for the problem (SP_2).

Proof. Reasoning ad absurdum, suppose that $(\bar{x},\bar{y})$ is not a sharp minimizer of order γ in the weak sense for the problem (SP_2). Then, there exist $x_1 \in C$, $y_1 \in F(x_1)$, $z_1 \in G(x_1) \cap (-K_Z)$ and $b_1 \in \overline{\mathbb{B}}_Y$ such that

$$\bar{y} - y_1 + \|x_1-\bar{x}\|^\gamma b_1 \in \mathrm{int}(K_Y).$$

Since $y^* \in (-K_Y)^\circ\backslash\{0_{Y^*}\}$, we may choose $y_0 \in Y\backslash\{0\}$ such that

$$\langle -y^*, y_0\rangle < 0.$$

As the sequence $\bar{y} - y_1 + \|x_1-\bar{x}\|^\gamma b_1 - (n+1)^{-1}y_0$ converges to $\bar{y} - y_1 + \|x_1-\bar{x}\|^\gamma b_1$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\bar{y} - y_1 + \|x_1-\bar{x}\|^\gamma b_1 - (n+1)^{-1}y_0 \in \mathrm{int}(K_Y),$$

using the fact that $y^* \in (-K_Y)^\circ$, one has

$$\left\langle y^*, \bar{y} - y_1 + \|x_1-\bar{x}\|^\gamma b_1 - (n+1)^{-1}y_0\right\rangle \geq 0,$$

that is

$$\langle y^*, \bar{y} - y_1 + \|x_1 - \bar{x}\|^\gamma b_1\rangle \geq (n+1)^{-1}\langle y^*, y_0\rangle > 0,$$

then

$$\langle y^*, \bar{y} - y_1 + \|x_1 - \bar{x}\|^\gamma b_1\rangle > 0. \tag{7.2}$$

From (7.1), there exist $x^* \in [R(C,\bar{x})]^\circ$, $(u_1^*, v^*) \in [R(M_{(\mathrm{gr}(F),\gamma)} \cap (C \times Y), (\bar{x},\bar{y}))]^\circ$ and $(u_2^*, w^*) \in [R(\mathrm{gr}(G), (\bar{x},\bar{z}))]^\circ$ such that

$$(-x^*, -y^*, -z^*) = (u_1^* + u_2^*, v^*, w^*). \tag{7.3}$$

Since $(x_1, y_1 - \|x_1 - \bar{x}\|^\gamma b_1) \in M_{(\mathrm{gr}(F),\gamma)} \cap (C \times Y)$ and $(x_1, z_1) \in \mathrm{gr}(G)$ one has

$$\langle u_1^*, x_1 - \bar{x}\rangle + \langle v^*, y_1 - \bar{y} - \|x_1 - \bar{x}\|^\gamma b_1\rangle \leq 0, \tag{7.4}$$

and

$$\langle u_2^*, x_1 - \bar{x}\rangle + \langle w^*, z_1 - \bar{z}\rangle \leq 0. \tag{7.5}$$

By adding the above relations (7.4) and (7.5) we get that

$$\langle -x^*, x_1 - \bar{x}\rangle + \langle -y^*, y_1 - \bar{y} - \|x_1 - \bar{x}\|^\gamma b_1\rangle + \langle -z^*, z_1 - \bar{z}\rangle \leq 0,$$

since $x^* \in [R(C,\bar{x})]^\circ$ and $z^* \in N(\bar{z}, -K_Z)$ we obtain that

$$\langle y^*, \bar{y} - y_1 + \|x_1 - \bar{x}\|^\gamma b_1\rangle \leq \langle x^*, x_1 - \bar{x}\rangle + \langle z^*, z_1 - \bar{z}\rangle \leq 0,$$

which is a contradiction with (7.2). □

The following theorem is a direct consequence of Theorem 7.1.

Theorem 7.2. *Let $\gamma > 0$ and $(\bar{x},\bar{y}) \in \mathrm{gr}(F)$. Assume that there exists $y^* \in (-K_Y)^\circ \backslash \{0\}$ such that*

$$(0, -y^*) \in \left[R\big(M_{(\mathrm{gr}(F),\gamma)} \cap (C \times Y), (\bar{x},\bar{y})\big)\right]^\circ + [R(C,\bar{x})]^\circ \times \{0_{Y^*}\}.$$

Then $(\bar{x},\bar{y})$ is a sharp minimizer of order γ in the weak sense for the problem (SP$_1$).

Proof. In Theorem 7.1 let us take $G : X \rightrightarrows Z$ with $\mathrm{gr}(G) = X \times Z$, of course, the hypotheses of the theorem hold, so the conclusion follows. □

In the next results, we show that necessary conditions in Sections 3–5 may be sufficient conditions under some convexity assumptions. For that purpose, we recall the following notion of γ-strong convexity of a set-valued map that introduced in Definition 2.9 of [5].

Definition 7.3. Let $\gamma > 0$. It is said that F is γ-strongly convex set-valued mapping with a constant $c > 0$ if there exists a function $g : [0,1] \to \mathbb{R}^+$ with

$$\lim_{\theta \to 0} \frac{g(\theta)}{\theta} = 1 \quad \text{and} \quad g(0) = g(1) = 0,$$

such that for all $x_1, x_2 \in X$ and $\theta \in [0,1]$

$$\theta F(x_1) + (1-\theta)F(x_2) + cg(\theta)\|x_1 - x_2\|^\gamma \overline{\mathbb{B}}_Y \subset F(\theta x_1 + (1-\theta)x_2).$$

Remark 7.4. It is obvious that, F is γ-strongly convex set-valued mapping implies that F is convex, *i.e.*, for all $x_1, x_2 \in X$ and $\theta \in [0,1]$ one has

$$\theta F(x_1) + (1-\theta)F(x_2) \subset F(\theta x_1 + (1-\theta)x_2),$$

or equivalently, $\mathrm{gr}(F)$ is a convex set.

The following result is a direct consequence of Theorem 2.12 from [5].

Lemma 7.5. *Let $\gamma > 0$. Assume that F is γ-strongly convex set-valued mapping with a constant $c > 0$ and $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$. Then for all $x \in X$, $y \in F(x)$ and $b \in \overline{\mathbb{B}}_Y$ one has*

$$(x - \bar{x}, y - \bar{y} - c\|x - \bar{x}\|^{\gamma} b) \in R(\mathrm{gr}(F), (\bar{x}, \bar{y})).$$

Now, we provide sufficient conditions for sharp minima for (SP_2) under strong convexity assumption on the set-valued objective mapping.

Theorem 7.6. *Let $\gamma > 0$. Assume that F is γ-strongly convex set-valued mapping with a constant $c > 0$, G is convex, $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$, $\bar{z} \in G(\bar{x}) \cap (-K_Z)$ and C is a convex set. Assume further that there exist $y^* \in (-K_Y)^{\circ} \backslash \{0\}$ and $z^* \in (-K_Z)^{\circ}$ such that*

$$(0, -y^*, -z^*) \in N(\mathrm{gr}(F), (\bar{x}, \bar{y})) \times \{0_{Z^*}\} + N(C, \bar{x}) \times \{(0_{Y^*}, 0_{Z^*})\} + A \circ [N(\mathrm{gr}(G), (\bar{x}, \bar{z}))], \tag{7.6}$$

where A is defined in Theorem 7.1. Then $(\bar{x}, \bar{y})$ is a sharp minimizer of order γ in the weak sense for the problem (SP_2).

Proof. By applying Lemma 7.5, we obtain that

$$(x - \bar{x}, y - \bar{y} - c\|x - \bar{x}\|^{\gamma} b) \in R(\mathrm{gr}(F), (\bar{x}, \bar{y})), \quad \text{for all } (x, y) \in \mathrm{gr}(F) \cap (C \times Y) \text{ and } b \in \overline{\mathbb{B}}_Y.$$

Since $\mathrm{gr}(F)$ is a convex set then

$$N(\mathrm{gr}(F), (\bar{x}, \bar{y})) = [R(\mathrm{gr}(F), (\bar{x}, \bar{y}))]^{\circ}.$$

From (7.6), there exist $(u_1^*, v^*) \in N(\mathrm{gr}(F), (\bar{x}, \bar{y}))$, $x^* \in N(C, \bar{x})$ and $(u_2^*, w^*) \in N(\mathrm{gr}(G), (\bar{x}, \bar{z}))$ such that

$$(0, -y^*, -z^*) = (u_1^*, v^*, 0) + (x^*, 0, 0) + (u_2^*, 0, w^*). \tag{7.7}$$

Based on these observations, one has

$$\langle (u_1^*, v^*), (x, y - c\|x - \bar{x}\|^{\gamma} b) - (\bar{x}, \bar{y}) \rangle \leq 0, \quad \text{for all } (x, y) \in \mathrm{gr}(F) \cap (C \times Y) \text{ and } b \in \overline{\mathbb{B}}_Y,$$

or equivalently,

$$(u_1^*, v^*) \in [R(M_{(\mathrm{gr}(F), \gamma)} \cap (C \times Y), (\bar{x}, \bar{y}))]^{\circ}.$$

By (7.7) together with the convexity of $\mathrm{gr}(G)$ and C, we get that

$$(0, -y^*, -z^*) \in [R(M_{(\mathrm{gr}(F), \gamma)} \cap (C \times Y), (\bar{x}, \bar{y}))]^{\circ} \times \{0_{Z^*}\} + [R(C, \bar{x})]^{\circ} \times \{(0_{Y^*}, 0_{Z^*})\} + A \circ [R(\mathrm{gr}(G), (\bar{x}, \bar{z}))]^{\circ}.$$

Theorem 7.1 completes the proof. □

The following theorem is a direct consequence of Theorem 7.6.

Theorem 7.7. *Let $\gamma > 0$. Suppose that F is γ-strongly convex set-valued mapping with a constant $c > 0$, $(\bar{x}, \bar{y}) \in \mathrm{gr}(F)$ and C is a convex set. Assume that there exists $y^* \in (-K_Y)^{\circ} \backslash \{0\}$ such that*

$$(0, -y^*) \in N(\mathrm{gr}(F), (\bar{x}, \bar{y})) + N(C, \bar{x}) \times \{0_{Y^*}\}.$$

Then $(\bar{x}, \bar{y})$ is a sharp minimizer of order γ in the weak sense for the problem (SP_1).

Proof. It suffices to apply Theorem 7.6 for $\mathrm{gr}(G) = X \times Z$. □

8. Conclusions

This paper studies three notions of weak sharp minima in set-valued optimization problems and provides some links between them. A natural extension is chosen to establish necessary optimality conditions for constrained set-valued optimization problems in terms of Lagrange multiplier rules, mainly in terms of Clarke differentiation objects (subdifferentials and normal cones). Under suitable assumptions, necessary optimality conditions become sufficient optimality conditions. Besides, some sufficient optimality conditions are derived without any convexity assumption on the set-valued objective mapping.

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