

INCIDENCE DIMENSION AND 2-PACKING NUMBER IN GRAPHS

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Abstract. Let $G = (V, E)$ be a graph. A set of vertices A is an incidence generator for G if for any two distinct edges $e, f \in E(G)$ there exists a vertex from A which is an endpoint of either e or f . The smallest cardinality of an incidence generator for G is called the incidence dimension and is denoted by $\dim_I(G)$. A set of vertices $P \subseteq V(G)$ is a 2-packing of G if the distance in G between any pair of distinct vertices from P is larger than two. The largest cardinality of a 2-packing of G is the packing number of G and is denoted by $\rho(G)$. In this article, the incidence dimension is introduced and studied. The given results show a close relationship between $\dim_I(G)$ and $\rho(G)$. We first note that the complement of any 2-packing in graph G is an incidence generator for G , and further show that either $\dim_I(G) = |V(G)| - \rho(G)$ or $\dim_I(G) = |V(G)| - \rho(G) - 1$ for any graph G . In addition, we present some bounds for $\dim_I(G)$ and prove that the problem of determining the incidence dimension of a graph is NP-hard.

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1. INTRODUCTION

The famous Gallai's theorem states that

$$\alpha(G) + \beta(G) = n,$$

where G is a graph on n vertices, $\beta(G)$ is the vertex covering number of G and $\alpha(G)$ is the independence number of G . Its beauty does not only lie in the numbers. Any vertex cover set S of cardinality $\beta(G)$ determines an independent set I of cardinality $\alpha(G)$, which is the complement of S . With such elegance, the hunt for analog results is always open.

One possibility arises from observing an independent set from a different (but equivalent) perspective. It comes from the notion of k -packings in graphs. A set $P \subseteq V(G)$ is a k -packing set (or k -packing for short) of G if the distance between any pair of distinct vertices from P is greater than k . The k -packing number of G is the maximum cardinality of any k -packing of G and is denoted by $\rho_k(G)$. Clearly, 1-packings represent independent

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sets of G , and maximum 1-packings are maximum independent sets of G . A $\rho_k(G)$ -set is a packing of cardinality $\rho_k(G)$. Since we are interested only in 2-packings, from now on, we will use the terminology $\rho(G)$, instead of $\rho_2(G)$, and packing, instead of 2-packing. A natural question concerns finding an analogy to Gallai's theorem for maximum k -packings for $k \geq 2$. We deal with this problem for $k = 2$.

For a longer period of time packings were considered as a natural lower bound for the domination number of graphs (for definitions, terminology and more information on domination in graphs we suggest the books [12, 13]). One of the first results (and indeed very remarkable) of that type is from Meir and Moon [24], who have shown that $\rho(T) = \gamma(T)$ for every tree T (using different notation). Efficient closed domination graphs represent a class of graphs with $\rho(G) = \gamma(G)$, where both maximum 2-packing sets and minimum dominating sets coincide. In such a case, we call a minimum dominating set a 1-perfect code. The study of perfect codes in graphs was initiated by Biggs [1]. Later it was intensively studied. We recommend [19] for further information and references.

In the last decade, the packing number became more interesting, not only in connection with the domination number. For instance, the relationship between the packing number and maximal packings of minimum cardinality, called also the lower packing number, is investigated in [27]. A connection (in the form of an upper bound) between the packing number and the double domination is presented in [25]. Graphs for which their packing number equals the packing number of their complement are described in [4]. Henning *et al.* [14] shows that the domination number is bounded from above by the packing number multiplied with the maximum degree of a graph.

A generalization of packing, called k -limited packing, is presented in [8]. Here every vertex can have at most k neighbors in a k -limited packing set S . To achieve some bounds, a probabilistic approach to k -limited packings was introduced in [7]. A further generalization of it is shown in [3]. It brings a dynamic approach concerning the vertices of G , where different vertices can have a different number of neighbors in a generalized limited packing. As shown in [3], the problem of computing the packing number of graphs is NP-hard but polynomially solvable for P_4 -tidy graphs.

Our goal is to continue finding several contributions on packings in connection with other topics in graphs. In recent years, this has attracted several researchers' attention who also tried to find the connections with the parameters close to metric dimension. The metric dimension of graph G (introduced first in [11, 28]) and its large number of variants are due to their properties of uniquely recognizing (identifying or determining) vertices or edges of G nowadays largely studied. Some of the most recent variants deal with uniquely identifying the edges of a considered graph (which is in some way one of the ideas that motivated this contribution). The first works on these recent topics are [17, 18]. Another metric parameter that can be taken as a predecessor of the present study concerns identification of vertices throughout neighborhoods (see [15]). We next describe all of these related concepts.

Given graph G , a set S of vertices of G is an *adjacency generator*¹ for G if for any two different vertices $u, v \in V(G) - S$, there exists $x \in S$ such that $|N(x) \cap \{u, v\}| = 1$. An adjacency generator of minimum cardinality is called an *adjacency basis* for G . Its cardinality is the *adjacency dimension* of G and is denoted by $\dim_A(G)$. These concepts were first introduced in [15] as a tool while studying some metric properties of the lexicographic product of graphs. More results on the adjacency dimension of graphs can be found in [5, 6].

Given a vertex $v \in V(G)$ and an edge $e = uw \in E(G)$, the distance between the vertex v and the edge e is defined as $d(e, v) = \min\{d(u, v), d(w, v)\}$. A vertex $w \in V(G)$ *distinguishes* two edges $e_1, e_2 \in E(G)$ if $d(w, e_1) \neq d(w, e_2)$. A nonempty set $S \subset V(G)$ is an *edge metric generator* for G if any two edges of G are distinguished by some vertex of S . An edge metric generator with the smallest possible cardinality is called an *edge metric basis* for G . Its cardinality is the *edge metric dimension* and is denoted by $\text{edim}(G)$. This concept was introduced in [18]. Some other studies on the edge metric dimension of graphs appeared in [10, 20–22, 26, 30]. Moreover, for some extra information on these parameters, and several other works on the metric dimension of

¹In fact, these sets were called adjacency resolving sets in [15], where the concept was first described.

graphs and its variants, in particular for the ones identifying edges of graphs, we suggest the two recent surveys [23, 29].

As a kind of a mix with respect to the last two parameters, we introduce the concept of incidence dimension in graphs which arises from the two concepts above in some natural way of research evolution. We formally define it in the next section. The definition is based on the existence of some properties of the complement of a packing set. Then we present a connection between the incidence dimension and packing number of graphs. A section about the complexity of the incidence dimension follows. We conclude this work with some additional information about the incidence dimension.

We consider only finite, undirected and simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a fixed $v \in V(G)$, set $\{u \in V(G) : uv \in E(G)\}$ represents the *open neighborhood* of v and is denoted by $N(v)$. The *degree* of v is $d(v) = |N(v)|$. The *closed neighborhood* of $v \in V(G)$ is $N[v] = N(v) \cup \{v\}$. The *distance* $d(u, v)$ between any two vertices u and v is the minimum number of edges on a path between them. Given a set of vertices S of G , we use $G - S$ to denote the graph obtained from G by removing all the vertices of S and the edges incident with them. If $S = \{v\}$ for some vertex v , then we simply write $G - v$. The subgraph of G induced by $D \subset V(G)$ will be denoted by $G[D]$.

2. DEFINING THE INCIDENCE DIMENSION AND ITS CONNECTION WITH PACKING NUMBER

As mentioned in the introduction, we are interested in some properties of the complement of the packing sets of graph G . The following result provides a motivation for the definition of the incidence dimension.

Proposition 2.1. *If a set $X \subseteq V(G)$ is a packing set of graph G , then the set $S = V(G) \setminus X$ is a vertex cover of G and for any two different edges e and f there exists $x \in S$ such that either x is incident with e or x is incident with f .*

Proof. First, let $e = uv$ be an arbitrary edge of G . If $\{u, v\} \cap S = \emptyset$, then $u, v \in X$. Since $d(u, v) = 1$, this yields a contradiction with X being a packing set of G . Thus, S is a vertex cover. (This also follows from the fact that every packing is also a 1-packing and, with this, an independent set. Since the complements of independent sets are vertex covers, the result follows.)

Take now two different arbitrary edges $e, f \in E(G)$. If $e = uv$ and $f = ab$ are not incident, then they are distinguished by one endpoint of e which exists in S because S is a vertex cover. Otherwise they are incident in one vertex, say in $u = a$. If $\{v, b\} \cap S = \emptyset$, then $v, b \in X$. This is a contradiction with X being a packing of G as $d(b, v) \leq 2$. So at least one of v or b , say v , is in S and v is the desired vertex. \square

With the second property of the above proposition, we are able to define the incidence dimension as follows. For this, note that it is meaningful to demand a minimum cardinality set with this mentioned property in order to retain the analogy with the relationship between independence number and vertex cover number.

Definition 2.2. Given two edges $e, f \in E(G)$ and a vertex $x \in V(G)$, we say that x (*incidently*) *resolves* or *distinguishes* the pair e, f if either $x \in e$ or $x \in f$ (exactly one of these two edges is incident with x). A set S of vertices of G is an *incidence generator* for G if for any two different edges $e, f \in E(G)$ there exists a vertex $x \in S$ such that x incidently resolves the pair e, f . An incidence generator of minimum cardinality is called an *incidence basis* for G , and its cardinality, denoted by $\dim_I(G)$, is the *incidence dimension* of G .

From this, we can immediately see that we cannot expect such an elegant result as in the case of Gallai's theorem. Indeed, we can see that for K_2 , an empty set is an incidence generator of minimum cardinality since there exists only one edge in K_2 , and we have $\dim_I(K_2) = 0$. Clearly, this can be extended to any graph with only one edge. However, as soon as there are two edges in G , we have $\dim_I(K_2) > 0$.

The next observation is that if S is an incidence generator for G , then there exists at most one edge with both endpoints outside of S . Namely, two such edges would not be incidently resolved by S , a contradiction with S being an incidence generator for G .

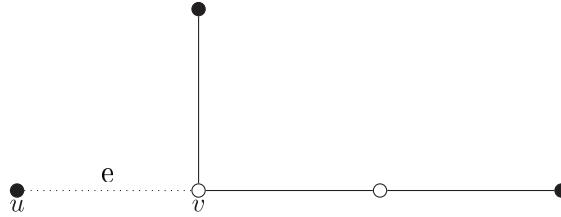


FIGURE 1. Graph G and edge e for which it holds that $\rho(G - e) > |P_e(G)|$.

Before we state a deeper connection between the incidence dimension and the packing number of a graph, we need some additional terminology.

Definition 2.3. Let $e = uv$ be an edge of graph G . The e -critical packing of $G - e$, denoted by $P_e(G)$, is a maximum packing of the graph $G - e$ with the following property.

$$\text{If } |\{u, v\} \cap P_e(G)| < 2, \text{ then } P_e(G) \text{ is a packing of } G. \quad (2.1)$$

Notice that both u and v can be in $P_e(G)$, and (2.1) is trivially fulfilled. Clearly, $P_e(G)$ is not a packing of G in such a case. However, by removing either u or v from $P_e(G)$, we obtain a packing of G . If $u, v \notin P_e(G)$, the set $P_e(G)$ is also a packing for G . If exactly one endpoint of e , say u , is inside $P_e(G)$, then $N_G(v) \cap P_e(G) = \{u\}$ or else $P_e(G)$ is not a packing of G contradicting (2.1). Therefore we have

$$\rho(G) \leq |P_e(G)| \leq \rho(G) + 1. \quad (2.2)$$

Figure 1 shows an example of graph G , which for the drawn dashed edge has an e -critical packing smaller than $\rho(G - e)$. Black vertices represent the unique maximum packing of $G - e$, which does not fulfill (2.1) and is therefore not e -critical. Hence, the cardinality of every e -critical packing is two.

Theorem 2.4. *If G is a graph of order n and k is an integer defined as $k = \max_{e \in E(G)} |P_e(G)|$, then $\dim_I(G) = n - k$. An incidence basis for the graph G is any set $S = V(G) \setminus P_e(G)$ for which it holds that $k = |P_e(G)|$.*

Proof. Let $e = uv$ and $P = P_e(G)$ such that $k = |P_e(G)|$. We want to prove that $S = V(G) \setminus P$ is an incidence generator for G .

If $|\{u, v\} \cap P| < 2$, then P is also a packing of G by (2.1). Due to Proposition 2.1, S is an incidence generator for G . Otherwise $u, v \in P$. Hence, P is a packing of the graph $G - e$. Due to Proposition 2.1, S is an incidence generator for $G - e$ with the property that every edge of $G - e$ has at least one endpoint in S . Since S is an incidence generator for $G - e$, we have to consider only the pairs of edges where one edge is e . Take edge e and an arbitrary edge $f \neq e$ of G . Clearly, the edges e and f are distinguished by the endpoint of f that is in S . It follows that S is an incidence generator for G and $\dim_I(G) \leq n - k$.

Now suppose that there exists an incidence generator S' for G with cardinality $|S'| = d' < n - k$. Since S' is an incidence generator for G , it follows that there exists at most one edge in G induced by $P'_e(G) = V(G) \setminus S'$. Suppose that such an edge $e = uv$ exists. First, notice that in $G - e$ there is no edge between two vertices of $P'_e(G)$. Since S' is an incidence generator for G , there do not exist two arbitrary vertices $x, y \in P'_e(G)$ such that $d_{G-e}(x, y) = 2$. Thus, it follows that $P'_e(G)$ is a packing for $G - e$ and both $u, v \in P'_e(G)$. Since the cardinality of $P'_e(G)$ is $n - d' > k$, we obtain a contradiction with the maximality of $P_e(G)$. If there is no edge in the graph induced by $P'_e(G)$, then $P'_e(G)$ is a packing of G by (2.1). Every packing of G is also a packing of $G - e$ with property (2.1) for an arbitrary edge e . Again, this is a contradiction with the maximality of $P_e(G)$. Therefore, we deduce that there is no incidence generator with cardinality less than $n - k$. \square

A direct consequence of Theorem 2.4, together with (2.2), follows.

Corollary 2.5. *For every graph G of order n it holds that $n - \rho(G) - 1 \leq \dim_I(G) \leq n - \rho(G)$.*

This yields a natural partition of graphs into two classes, those whose incidence dimension equals $|V(G)| - \rho(G) - 1$ and those for which $\dim_I(G) = |V(G)| - \rho(G)$. To show that graph G belongs to the first class, we need to find an edge e such that $|P_e(G)| = \rho(G) + 1$. For the second class, we need to show that for each edge e , we have $|P_e(G)| = \rho(G)$.

We next derive exact results for the incidence dimension for some graph families, and we first consider a class of graphs called edge-triangular. A graph is called *edge-triangular* if every edge of a graph is in at least one 3-cycle.

Proposition 2.6. *Let S be any incidence generator for graph G . Then the graph G is edge-triangular if and only if for every $e = uv \in E(G)$ it holds that $|\{u, v\} \cap S| > 0$.*

Proof. Let G be an edge-triangular graph. Suppose there exists an edge $e = uv$ and an incidence generator S for G such that $|\{u, v\} \cap S| = 0$. Since G is an edge-triangular graph, there exists a vertex w such that $uvwu$ is a triangle. Note that the vertex w has to be in S because e and uw have to be distinguished by at least one endpoint. However, the edges uw and vw are not distinguished by any vertex from S . This is a contradiction with S being an incidence generator.

Conversely, suppose that the graph G is not edge-triangular. Thus, there exists an edge $e = uv$ that is not a part of a triangle. Consequently, the set $S = V(G) \setminus \{u, v\}$ is an incidence generator for G , which means $|\{u, v\} \cap S| = 0$. \square

Proposition 2.6 implies the following corollary.

Corollary 2.7. *Let G be an edge-triangular graph. The set S is an incidence generator for G if and only if $V(G) \setminus S$ is a packing of G .*

Proof. Suppose that S is an incidence generator. Due to Proposition 2.6 there are no two vertices at distance 1 in $V(G) \setminus S$. Since S is an incidence generator, there are also not two vertices at distance 2 in $V \setminus S$. It follows that $V(G) \setminus S$ is a packing of G .

By Proposition 2.1, the converse holds for any graph G . \square

Proposition 2.8. *Let n, r and t be integers.*

- (i) *If $n \geq 3$, then $\dim_I(K_n) = n - 1$.*
- (ii) *If $n \geq 3$, then $\dim_I(P_n) = \left\lfloor \frac{2(n-1)}{3} \right\rfloor$.*
- (iii) *If $n \geq 4$, then $\dim_I(C_n) = \left\lfloor \frac{2n}{3} \right\rfloor$.*
- (iv) *If $r, t \geq 1$, then $\dim_I(K_{r,t}) = r + t - 2$.*

Proof. (i) Clearly, K_n is an edge-triangular graph. So, the result follows from Corollary 2.7, and the fact that any set containing all but one vertex of any graph G is an incidence generator for G . Such an incident generator is minimum because $V(K_n) - \{x, y\}$ is not an incident generator for different $x, y \in V(K_n)$ as edges xw and yw are not distinguished by $V(K_n) - \{x, y\}$ for any $w \in V(K_n) - \{x, y\}$.

(ii) Let $V(P_n) = \{v_0, v_1, \dots, v_{n-1}\}$ such that $v_i v_{i+1} \in E(P_n)$ for every $i \in \{0, \dots, n-2\}$. Consider the set $S' = \{v_i : i \geq 2 \text{ and } i \equiv 0 \text{ or } i \equiv 2 \pmod{3}\}$. Note that any two edges of P_n are incidently resolved by S' , and so $\dim_I(P_n) \leq |S'| = \left\lfloor \frac{2(n-1)}{3} \right\rfloor$.

On the other hand, let S be an incidence basis for P_n . There could be at most one edge which is not incident to any vertex of S . Also, if $v_i \in S$ and $i \neq 0, n-1$, then $v_{i-1} \in S$ or $v_{i+1} \in S$ or both. Thus, at least two of any

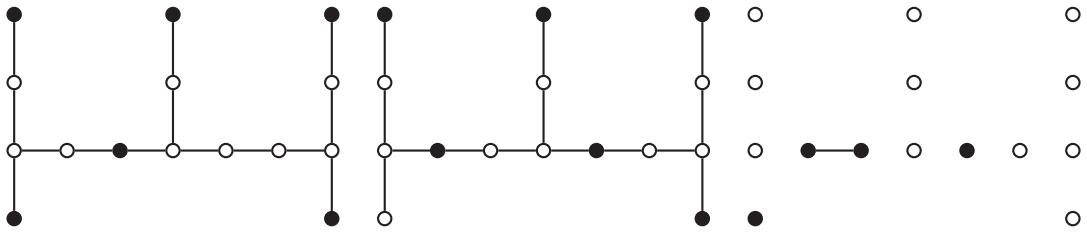


FIGURE 2. An example showing that the converse implication of Theorem 2.9 does not hold in general.

three consecutive vertices v_i, v_{i+1}, v_{i+2} are in S . According to these facts, $\dim_I(P_n) = |S| \geq \left\lfloor \frac{2(n-1)}{3} \right\rfloor$, which completes the proof of (ii).

(iii) Let $e = uv$ be any edge of C_n . Clearly, $C_n - e \cong P_n$. It is well known, see [24], that $\rho(T) = \gamma(T)$ for every tree T and we have $\rho(P_n) = \lceil \frac{n}{3} \rceil$. Moreover, there always exists a $\rho(P_n)$ -set P such that $u, v \in P$. Hence, $|P_e(C_n)| = \lceil \frac{n}{3} \rceil$. On the other hand, $\rho(C_n) = \lfloor \frac{n}{3} \rfloor$. If $n \neq 3k$, then $|P_e(C_n)| = \rho(C_n) + 1$, and by Theorem 2.4, we have $\dim_I(C_n) = n - \lceil \frac{n}{3} \rceil = \lfloor \frac{2n}{3} \rfloor$. For $n = 3k$ we have $|P_e(C_n)| = \rho(C_n) = k$ for every edge e and, again by Theorem 2.4, we have $\dim_I(C_n) = n - \rho(C_n) = 3k - k = \frac{2n}{3} = \lfloor \frac{2n}{3} \rfloor$. So, we are done with the proof of (iii).

(iv) Clearly $P_e(K_{r,t}) = \{u, v\}$ for any edge $e = uv$ of $K_{r,t}$, while $\rho(K_{r,t}) = 1$. By Corollary 2.5 and a comment after it we have $\dim_I(K_{r,t}) = r + t - 2$, because all edges are symmetric to each other. \square

Recall that $A \triangle B$ denotes the symmetric difference of the sets A and B .

Theorem 2.9. Let G be a graph. If $\dim_I(G) = |V(G)| - \rho(G) - 1$, then $G[P_1 \triangle P_2]$ is not an empty graph for some maximum packings P_1 and P_2 of G .

Proof. Let G be a graph and let S be an incidence basis of G such that $\dim_I(G) = |V(G)| - \rho(G) - 1$. Let $P = V(G) - S$. Since $|P| = \rho(G) + 1$, P is not a packing for G . So, there exist two different vertices $u, v \in P$, such that $1 \leq d(u, v) \leq 2$. If $d(u, v) = 2$, then the edges uw and wv , where w is a common neighbour of u and v , are not resolved by S , a contradiction. Thus $d(u, v) = 1$. Let $P_1 = V(G) - (S \cup \{u\})$ and $P_2 = V(G) - (S \cup \{v\})$. The cardinality of both packings is maximum possible, since $|P_1| = |P_2| = |V(G)| - ((|V(G)| - \rho(G) - 1) + 1) = \rho(G)$. Since u and v are adjacent it follows that $G[P_1 \Delta P_2]$ is not an empty graph and the proof is completed. \square

The converse implication of Theorem 2.9 does not hold in general, as we can see from the example in Figure 2. Two different packings of G are presented by black vertices on the left and in the middle of Figure 2. On the right side, there is a picture of $G[P_1 \triangle P_2]$, which is clearly not an empty graph. However, the incidence dimension of G is not equal to $|V(G)| - \rho(G) - 1$ because one can check that $P_e(G) = \rho(G)$ for every edge $e \in E(G)$ as mentioned in a comment after Corollary 2.5.

We next provide an exact result for the class of graphs with the unique maximum packing. Two characterizations of trees with the unique maximum packing were presented recently in [2].

Theorem 2.10. *If G is a graph with the unique maximum packing P , then $\dim_I(G) = |V(G)| - \rho(G)$.*

Proof. Let G be a graph and P its unique maximum packing. To prove that $\dim_I(G) = |V(G)| - \rho(G)$, we will use the contraposition of Theorem 2.9: *If $G[P_1 \Delta P_2]$ is an empty graph, then $\dim_I(G) \neq |V(G)| - \rho(G) - 1$.* It follows that $P_1 = P_2 = P$, because P is a unique maximum packing of T . So, $G[P_1 \Delta P_2]$ is an empty graph for any maximum packings P_1 and P_2 and $\dim_I(G) \neq |V(G)| - \rho(G) - 1$. By Corollary 2.5 we conclude that $\dim_I(G) = |V(G)| - \rho(G)$. \square

3. COMPLEXITY OF THE PROBLEM

In this section, we consider the computational complexity of computing the incidence dimension of a graph. We show that the problem of finding the incidence dimension of an arbitrary graph is NP-hard. For this, we strongly rely on edge-triangular graphs. We consider the following decision problem.

INCIDENCE DIMENSION PROBLEM (IDIM problem for short)
 INSTANCE: Graph G of order $n \geq 3$ and an integer $1 \leq r \leq n - 1$.
 QUESTION: Is $\dim_I(G) \leq r$?

To study the complexity of IDIM problem, we make a reduction from 3-SAT problem, which is one of the most classical NP-complete problems known in the literature. For more information on 3-SAT problem and reducibility of NP-complete problems in general, we suggest [9].

Theorem 3.1. *IDIM problem is NP-complete.*

Proof. For a set of vertices S guessed for the problem by a nondeterministic algorithm, one needs to iterate through all pairs of edges and check that every pair is incidently resolved by a vertex from S . This can be done in polynomial time, and therefore IDIM problem is in NP. We make a polynomial transformation of 3-SAT problem to IDIM problem in the following way. Consider an arbitrary instance of 3-SAT problem, *i.e.*, a finite set $U = \{u_1, \dots, u_n\}$ of Boolean variables and a collection $C = \{c_1, \dots, c_m\}$ of clauses, each containing three Boolean variables from U . We will construct graph $G = (V, E)$ and set a positive integer $r \leq |V| - 1$, such that $\dim_I(G) \leq r$ if and only if C is satisfiable. The construction will be made up of several gadgets and edges between them.

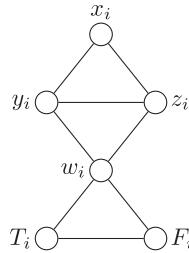
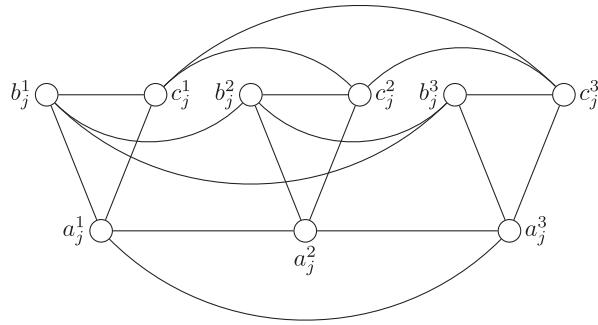
For each variable $u_i \in U$, $1 \leq i \leq n$, we construct a truth-setting gadget $X_i = (V_i, E_i)$, with $V_i = \{x_i, y_i, z_i, w_i, T_i, F_i\}$ and $E_i = \{x_iy_i, x_iz_i, y_iz_i, y_iw_i, z_iw_i, w_iT_i, w_iF_i, T_iF_i\}$, see Figure 3. Each truth-setting gadget is connected to the rest of the graph only through nodes T_i and F_i , which are representing values TRUE and FALSE, respectively.

Claim. Let u_i be an arbitrary variable in U . Any incidence generator S must contain at least four vertices from its truth-setting gadget. Moreover, if there are exactly four vertices from a truth-setting gadget in S , then $y_i, w_i, z_i \in S$ and $x_i \notin S$.

Proof. Towards contradiction suppose that there exists an incidence generator S with less than four vertices from the truth-setting gadget corresponding to u_i . It follows that there exists a set of three vertices $W_i = \{v_1, v_2, v_3\} \subset V_i$ that are not in S . Make a partition of V_i into two sets $V_i = \{x_i, y_i, z_i\} \cup \{w_i, T_i, F_i\}$. There are at least two vertices from W_i in one of the partition sets. Since each partition set forms a triangle, it follows that there is an edge lying outside S , a contradiction with Proposition 2.6.

Suppose now that exactly four vertices from a truth-setting gadget are in S . If $w_i \notin S$, then $T_i, F_i, y_i, z_i \in S$ because otherwise we have a contradiction with Proposition 2.6. But then $x_i \notin S$ and edges w_iz_i and x_iz_i are not distinguished by S , a contradiction. Hence, $w_i \in S$. If $y_i \notin S$ (resp. $z_i \notin S$), then $x_i \in S$ and $z_i \in S$ (resp. $y_i \in S$) to fulfill Proposition 2.6. Clearly, exactly one of T_i and F_i is in S . If $T_i \in S$, then edges F_iw_i and y_iw_i (resp. z_iw_i) are not distinguished by S , a contradiction. Similar, if $F_i \in S$, then edges T_iw_i and y_iw_i (resp. z_iw_i) are not distinguished by S , a contradiction again. Thus, $y_i, z_i \in S$. If in addition $x_i \in S$, then for the triangle $w_iT_iF_iw_i$ we have a contradiction with Proposition 2.6. Therefore, $x_i \notin S$. \square

For each clause $c_j = y_j^1 \vee y_j^2 \vee y_j^3$, $1 \leq j \leq m_j$, where y_j^k is a literal in the clause c_j , we construct a satisfaction testing gadget $Y_j = (V'_j, E'_j)$, with $V'_j = \{a_j^1, b_j^1, c_j^1, a_j^2, b_j^2, c_j^2, a_j^3, b_j^3, c_j^3\}$ and $E'_j = \{a_j^1b_j^1, a_j^1c_j^1, b_j^1c_j^1, a_j^2b_j^2, a_j^2c_j^2, b_j^2c_j^2, a_j^3b_j^3, a_j^3c_j^3, b_j^3c_j^3, a_j^1a_j^2, a_j^2a_j^3, a_j^3a_j^1, b_j^1b_j^2, b_j^2b_j^3, b_j^3b_j^1, c_j^1c_j^2, c_j^2c_j^3, c_j^3c_j^1\}$ (see Fig. 4). Notice that the satisfaction testing gadget is isomorphic to the Cartesian product $C_3 \square C_3$.

FIGURE 3. The truth-setting gadget for variable u_i .FIGURE 4. The satisfaction testing gadget for clause c_j .

Claim. If c_j is an arbitrary clause in C and $Y_j = (V'_j, E'_j)$ its satisfaction testing gadget, then any incidence generator must contain at least 8 vertices from V'_j .

Proof. Towards a contradiction, suppose there exists an incidence generator S with less than 8 vertices from the satisfaction testing gadget corresponding to c_j . It follows that there exist two vertices $x, y \in V'_j$ that are not in S . Since the diameter of $C_3 \square C_3$ is 2, the vertices x and y are either at a distance one or two. Every edge of Y_j is a part of some triangle, and so x and y cannot be at a distance one, due to Proposition 2.6. Thus, there is a vertex z such that the edges xz and yz exist. But those two edges are not resolved by any endpoint, a contradiction. \square

We also add some edges to connect the truth-setting gadgets with corresponding satisfaction testing gadgets. If a variable u_i occurs as a literal y_j^k in a clause $c_j = y_j^1 \vee y_j^2 \vee y_j^3$, then we add the following edges. If y_j^k is a positive literal, then we add the edges $F_i b_j^k$ and $F_i c_j^k$. If a variable y_j^k is a negative literal in a clause c_j , then we add the edges $T_i b_j^k$ and $T_i c_j^k$. For each clause $c_j \in C$, denote those six added edges with E''_j . We call them *communication* edges. Figure 5 shows the edges that were added corresponding to the clause $c_j = (\bar{u}_1 \vee \bar{u}_2 \vee u_3)$, where \bar{u}_1 and \bar{u}_2 represent the negative literal corresponding to the variables u_1 and u_2 , respectively.

The construction of IDIM instance is then completed by setting $r = 4n + 8m$ and $G = (V, E)$, where

$$V = \left(\bigcup_{i=1}^n V_i \right) \cup \left(\bigcup_{j=1}^m V'_j \right)$$

and

$$E = \left(\bigcup_{i=1}^n E_i \right) \cup \left(\bigcup_{j=1}^m E'_j \right) \cup \left(\bigcup_{j=1}^m E''_j \right)$$

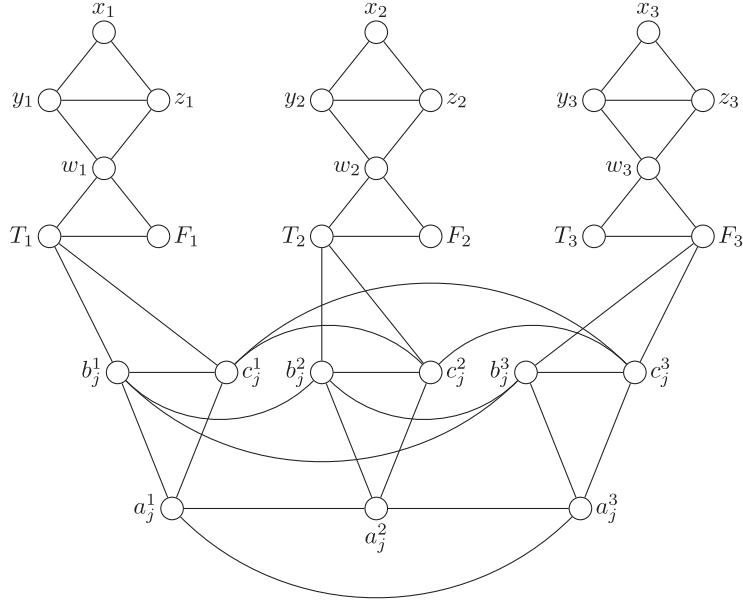


FIGURE 5. The subgraph associated to the clause $c_j = (\bar{u_1} \vee \bar{u_2} \vee u_3)$.

One can execute the described construction in polynomial time. Notice also that G is edge-triangular.

If we show that C is satisfiable if and only if G has an incidence dimension less or equal than r , the proof of NP-completeness is completed. From Claims 3 and 3 we get the following corollary.

Corollary 3.2. *The incidence dimension of G constructed above is at least $r = 4n + 8m$.*

The following lemmas together with Corollary 3.2 complete the proof for IDIM problem being NP-complete.

Lemma 3.3. *If C is satisfiable, then $\dim_I(G) = r$.*

Proof. We construct an incidence generator S of size r based on a truth assignment of elements from the set U that satisfies the collection of clauses C . Let $t : U \rightarrow \{\text{TRUE}, \text{FALSE}\}$ be a truth assignment that satisfies the collection of clauses C . For each clause $c_j = y_j^1 \vee y_j^2 \vee y_j^3$, from C , put into S the vertices b_j^k, c_j^k for $k \in \{1, 2, 3\}$. Since the collection of clauses C is satisfiable, there exists a literal y_j^k , $k \in \{1, 2, 3\}$, that satisfies c_j . Fix one such k and put into S the other two vertices a_j^ℓ for $\ell \in \{1, 2, 3\} \setminus \{k\}$.

For each Boolean variable $u_i \in U$, put into S the vertices $\{y_i, z_i, w_i\}$. Also add to the set S , the vertex F_i if $t(u_i) = \text{TRUE}$, or the vertex T_i if $t(u_i) = \text{FALSE}$. The cardinality of the constructed set S is clearly $r = 4n + 8m$.

We now take a look at the set $X = V(G) \setminus S$. For each $u_i \in U$ there are x_i and exactly one of the vertices from the set $\{T_i, F_i\}$ in X . The distance between these two vertices is three. For each $c_j \in C$ exactly one of the vertices a_j^1, a_j^2, a_j^3 is in X . The vertex that is in X corresponds to the variable that satisfies c_j . It follows that this vertex is at a distance three or more from all the other vertices in X . All other possible pairs of vertices in X are also at a distance greater or equal to three. It follows that X is a packing of G . From Corollary 2.7 follows that S is an incidence generator for G since G is edge-triangular. Clearly, $|S| = r$ and we are done by Corollary 3.2. \square

Lemma 3.4. *If $\dim_I(G) = r$, then the collection of clauses C is satisfiable.*

Proof. Let S be an arbitrary incidence generator for G of cardinality r . The set S must contain at least eight vertices from each satisfaction testing gadget and at least four vertices from each truth-setting gadget due to Claims 3 and 3. Since $|S| = r = 8m + 4n$, it follows that in S there are exactly four vertices from each truth-setting component and exactly eight vertices from each satisfaction testing component. Since the graph G is edge-triangular, and together with Corollary 2.7, it follows that $X = V(G) \setminus S$ is a packing for G . Moreover, for each $i \in \{1, \dots, n\}$ it holds that $x_i \in X$ and exactly one of the vertices T_i, F_i is in X , by Claim 3. For each $j \in \{1, \dots, m\}$ exactly one of the vertices a_j^1, a_j^2, a_j^3 is in X because b_j^i and c_j^i , $i \in \{1, 2, 3\}$, are in a common triangle with either T_ℓ or F_ℓ where u_ℓ belongs to the clause c_j .

We now define a function that satisfies all clauses from C . For an arbitrary $i \in \{1, \dots, n\}$, let $v_i \in \{T_i, F_i\} \cap X$. Let $t : U \rightarrow \{\text{TRUE}, \text{FALSE}\}$ be as follows:

$$t(u_i) = \begin{cases} \text{TRUE}, & v_i = T_i, \\ \text{FALSE}, & v_i = F_i. \end{cases}$$

We need to show that t is a satisfying truth assignment for C . Let $c_j = y_j^1 \vee y_j^2 \vee y_j^3 \in C$ be an arbitrary clause and denote the corresponding boolean variables with $u_{j_1}, u_{j_2}, u_{j_3}$, respectively. To show that at least one of its literals has value **TRUE**, take the vertex from V'_j that belongs to X . There is exactly one of the vertices a_j^1, a_j^2, a_j^3 in X . Let a_j^k , $k \in \{1, 2, 3\}$, be the vertex that is in X . The communication edges are added in such a way that a_j^k can be in X (packing set) only if u_{j_k} occurs in c_j as:

- a positive literal and $v_{j_k} = T_{j_k}$;
- a negative literal and $v_{j_k} = F_{j_k}$.

In both cases, c_j is satisfied by the literal corresponding to the variable u_{j_k} . It finally follows that C is satisfiable, which completes the proof of this lemma. \square

Lemmas 3.3 and 3.4 show that the above construction is a polynomial transformation from 3-SAT to IDIM problem. Therefore, the proof of Theorem 3.1 is completed. \square

The proof of Theorem 3.1 yields the following result.

Corollary 3.5. *The problem of finding the incidence dimension of a graph is NP-hard.*

4. SOME FINAL REMARKS ON $\dim_I(G)$

Given any graph G , it is easy to see that the set $V(G)$ minus one arbitrary vertex is an incident generator for G . On the other hand, given an incidence basis for G , for all but probably one edge in $E(G)$, at least one of its endpoints belongs to S . Moreover, for any three edges incident with the same vertex, at least two different endpoints of two different edges must be in S too. In consequence, the following bounds are easy to deduce.

Remark 4.1. If G is a connected graph of order n with at least two edges, then

$$\left\lfloor \frac{n}{2} \right\rfloor \leq \dim_I(G) \leq n - 1.$$

The lower bound of Remark 4.1 is achieved for a path P_n , $n \in \{3, 4, 5, 6, 8\}$, a cycle C_4 , a star $K_{1,3}$, and some graphs obtained by attaching a pendant vertex or an edge to some vertices of previously mentioned examples. While it is not clear if this list is complete, we can entirely describe all graphs achieving the upper bound of Remark 4.1.

Proposition 4.2. *Let G be a connected graph of order n with at least two edges. Then $\dim_I(G) = n - 1$ if and only if any two vertices of G have a common neighbor.*

Proof. If there are two different vertices $x, y \in V(G)$ such that they do not have a common neighbor, then it is not difficult to see that the set $V(G) - \{x, y\}$ is an incidence generator for G . Thus, to have an incidence generator of order $n - 1$, it is required that any two different vertices of G have a common neighbor and vice versa. \square

Now, concerning the bounds of Remark 4.1, we next study the existence of graphs G of order n and incidence dimension r for any r, n such that $\lfloor \frac{n}{2} \rfloor \leq r \leq n - 1$.

Proposition 4.3. *For any integers r, n with $2 \leq \lfloor \frac{n}{2} \rfloor \leq r \leq n - 1$ there exists graph G of order n such that $\dim_I(G) = r$.*

Proof. If $r = 2$, then $n \in \{4, 5\}$. In such a case paths P_4 and P_5 , respectively, satisfy the requirements. Hence, from now on we may assume $r \geq 3$.

Consider n is odd and $r = \lfloor \frac{n}{2} \rfloor < n - 1$. Let $G_{r,n}$ be the graph obtained as follows.

- We begin with a complete graph K_r with vertex set $V = \{u_1, \dots, u_r\}$.
- Add $r + 1$ vertices w, v_1, \dots, v_r .
- Add the edges $u_i v_i$ for every $i \in \{1, \dots, r\}$ and the edge wv_1 .

Clearly, $G_{r,n}$ has order $2r + 1 = n$. It is not difficult to see that V is an incidence generator for $G_{r,n}$ and so, $\dim_I(G_{r,n}) \leq r$. Now, suppose $\dim_I(G_{r,n}) < r$ and let S be an incidence basis for $G_{r,n}$. That means that there is at least one vertex $u_j \in V$, such that $u_j \notin S$. If there exists some other vertex $u_i \in V$, $i \neq j$, such that $u_i \notin S$, then there are two edges $u_j u_k$, $u_i u_k$, with $k \neq i, j$ (since $r \geq 3$), such that they are not incidently resolved by S , a contradiction. Thus, $V - \{u_j\} \subseteq S$, which means $|S| = r - 1$ and $S = V - \{u_j\}$. But, in such a case, the edges wv_1 and $u_j v_j$ are not incidently resolved by S , which is a contradiction again. As a consequence, $\dim_I(G_{r,n}) = r$.

We next consider (n is even and $r = \lfloor \frac{n}{2} \rfloor < n - 1$) or $\lfloor \frac{n}{2} \rfloor < r < n - 1$. Let $G'_{r,n}$ be the graph obtained as follows.

- We begin with a complete graph K_r with vertex set $V = \{u_1, \dots, u_r\}$.
- Add $n - r$ vertices v_1, \dots, v_{n-r} .
- Add the edges $u_i v_i$ for every $i \in \{1, \dots, n - r - 1\}$.
- Add the edges $v_{n-r} u_i$ for every $i \in \{n - r, \dots, r\}$.
- Add the edge $v_1 v_2$ (notice that such two vertices always exist because $r < n - 1$).

Clearly the order of $G'_{r,n}$ is n and we can easily notice that V is an incidence generator for $G'_{r,n}$ and so, $\dim_I(G'_{r,n}) \leq r$. Hence, suppose $\dim_I(G'_{r,n}) < r$ and let S' be an incidence basis for $G'_{r,n}$. In consequence, there is at least one vertex $u_j \in V$ such that $u_j \notin S'$. A similar procedure as earlier leads to the fact that $\dim_I(G'_{r,n}) = r - 1$ and that $S' = V - \{u_j\}$. However, in this case there is an edge $u_j v_l$ for some $l \in \{1, \dots, r\}$ such that the edges $v_1 v_2$ and $u_j v_l$ are not incidently resolved by S' , a contradiction. Therefore, $\dim_I(G'_{r,n}) = r$.

We finally consider the situation $r = n - 1$, which is straightforward to realize by just taking the complete graph K_n , which completes the proof. \square

It is natural to think that the incidence dimension is related to (edge or adjacency) dimension of graphs. Accordingly, we conclude this work by comparing $\dim_I(G)$ with $\dim_e(G)$ and $\dim_A(G)$.

Proposition 4.4. *For any graph G without isolated vertices, $\dim_I(G) \geq \max\{\dim_A(G), \dim_e(G)\}$.*

Proof. Let S be an incidence basis for G . Consider two different vertices $x, y \in V(G) - S$. If $N(x) \cap S = \emptyset$ and $N(y) \cap S = \emptyset$, then since G has no isolated vertices, there are at least two edges xx' and yy' such that $x', y' \notin S$. Thus, xx' and yy' are not incidently resolved by any vertex of S , which is a contradiction. So $N(x) \cap S \neq \emptyset$ or $N(y) \cap S \neq \emptyset$. Now, suppose $N(x) \cap S = N(y) \cap S$. Hence, there exists a vertex $w \in S$ such that the edges xw

and yw are not incidently resolved by any vertex of S , a contradiction again. Thus, $N(x) \cap S \neq N(y) \cap S$ and, as a consequence, S is an adjacency generator for G and $\dim_I(G) \geq \dim_A(G)$.

Now, since any two edges e_1, e_2 are incident to at least two different vertices x, y , and at least one of x, y must be in S , it is clear that the edges e_1, e_2 are distinguished by x or y . So, S is also an edge metric generator for G , and $\dim_I(G) \geq \dim_e(G)$, which completes the proof. \square

From [15] we know that $\dim_A(K_{r,t}) = r + t - 2$. Also, from [18], we have $\dim_e(K_{r,t}) = r + t - 2$. Now, from Proposition 2.8 (iv), we observe that the bound of Proposition 4.4 is tight. In such a case, we have equality $\dim_I(K_{r,t}) = \dim_A(K_{r,t}) = \dim_e(K_{r,t})$. An interesting problem is then to characterize the families of graphs for which the bound of Proposition 4.4 is achieved, and moreover, finding whether the situations $\dim_I(K_{r,t}) = \dim_A(K_{r,t}) \neq \dim_e(K_{r,t})$, $\dim_I(K_{r,t}) = \dim_e(K_{r,t}) \neq \dim_A(K_{r,t})$ or $\dim_I(K_{r,t}) = \dim_A(K_{r,t}) = \dim_e(K_{r,t})$ happen.

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