COMPUTING SOME ROLE ASSIGNMENTS OF CARTESIAN PRODUCT OF GRAPHS

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Abstract. In social networks, a role assignment is such that individuals play the same role, if they relate in the same way to other individuals playing counterpart roles. When a smaller graph models the social roles in a network, this gives rise to the decision problem called r-Role Assignment whether it exists such an assignment of r distinct roles to the vertices of the graph. This problem is known to be NP-complete for any fixed $r \geq 2$. The Cartesian product of graphs is one of the most studied operation on graphs and has numerous applications in diverse areas, such as Mathematics, Computer Science, Chemistry and Biology. In this paper, we determine the computational complexity of r-Role Assignment restricted to Cartesian product of graphs, for $r = 2, 3$. In fact, we show that the Cartesian product of graphs is always 2-role assignable, however the problem of 3-Role Assignment is still NP-complete for this class.

1. Introduction

Graphs have been used for centuries as a modeling tool in which vertices typically represent objects and edges the relationship between them. The role assignments in graphs emerged from the concept of *covering* introduced by Augluin in 1980 [1] and is a tool for networks of processors. Later on, based on graph models for social networks, Everett and Borgatti [8] formalized this idea under the name of *role coloring*. In this sense, *r-role assignment* of a simple graph G is an assignment of r distinct roles to the vertices of G which obeys the rule that two vertices have the same role, if their neighbors have the same set of roles. Furthermore, such an assignment defines a *role graph*, where vertices are the r distinct roles and there is an edge between two roles whenever there are two related vertices in the graph G that correspond to those roles. Observe that, the role graph is a graph that has no multiple edges, but allows loops since two neighbours in G can have the same role. Specifically, if the role graph has all possible relationships except loops, role assignment correspond to *fall coloring*, introduced by Dunbar *et al.* [7] in 2000.

We define the problem r-Role Assignment as follows:

> *r*-ROLE ASSIGNMENT
> **Instance:** A simple graph G.
> **Question:** Does G admits a *r*-role assignment?

Various studies of role assignment point out applications. For example, in [16] the authors show that any network represented by a graph, with minimum degree bounded by a suitable bound that depends on r, has a *r*-role assignment. Others applications may be found in social networks [8,17] and distributed computing [3,4].

In 2005, Fiala and Paulusma [9] showed that *r*-ROLE ASSIGNMENT is NP-complete for fixed $r \geq 3$. Prior to this work, in 2001, Roberts and Sheng [17] proved that 2-ROLE ASSIGNMENT is NP-complete. However, the problem of *r*-ROLE ASSIGNMENT can be solved in polynomial time for trees [10] and for proper interval graphs [13]. For chordal and split graphs, the following dichotomy for the complexity of *r*-ROLE ASSIGNMENT arises. While for chordal graph, the problem is solvable in linear time for $r = 2$ and NP-complete for $r \geq 3$ [19]; for split graphs, the problem is trivial, with true answer, for $r = 2$, solvable in polynomial time for $r = 3$, and NP-complete for any fixed $r \geq 4$ [6]. Some properties of role assignment has also been studied under some graph operations, such as Cartesian product [21]. For complementary prism, which arised of complementary product which is a generalization of Cartesian product, the 2-ROLE ASSIGNMENT has answer true except for the complementary prism of a path with three vertices [2].

In this work, we study the problem of *r*-ROLE ASSIGNMENT of Cartesian product in the computational complexity point of view. Cartesian product of graphs is one of the most studied operation on graphs. Introduced by Sabidussi [18] in 1959, it has been applied in many areas since then, for example in space structures [14] and interconnection networks [5]. The study of networks is a clear connection between Cartesian product and role assignment. Whereas role assignment allows to study a network through a smaller graph, Cartesian product is often used for modeling one. In that sense, the term Product Networks [20] has raised in the literature to denominated Cartesian product.

Returning to the specific role assignment that correspond to fall coloring, Laskar and Lyle [15] showed that the problem for 3 colors (or 3 roles) is NP-complete even when restricted to the class of bipartite graphs. They also construct fall colourable graphs using Cartesian product.

This paper is organized as follows. In Section 2, we set notations and terminology. Concerning the problem of *r*-ROLE ASSIGNMENT restricted to Cartesian product, we show in Section 3 that it is trivial, with true answer, for $r = 2$. In Section 4, we prove that it is NP-complete for $r = 3$ and in Section 5, we conclude by conjecturing that it is also NP-complete for $r \geq 4$.

2. PRELIMINARIES

In this section, we state the graph terminology and notations used in this paper. All graphs considered are undirected, finite, non-trivial and have no multiple edges. A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. The vertices u and v are *adjacent* or *neighbors* if they are joined by an edge e, also denoted by uv. In this case, u and v are *incident* to e and e is *incident* to u and v. A *loop* is an edge incident to only one vertex. The *neighborhood* of a vertex v, denoted by $N_G(v)$, is the set of all neighbors of v in G. A *simple graph* is a graph without loops. In a simple graph G, the *degree* of a vertex v is the cardinality of $N_G(v)$. A vertex of degree zero is *isolated*. The neighborhood of a subset V of $V(G)$, denoted as $N_G(V)$, is the union of the neighborhoods of the vertices of V.

A *path* is a sequence of distinct vertices with an edge between each pair of consecutive vertices. For $n \geq 2$, we denote a path on n vertices by P_n or by the sequence of vertices $v_1, \ldots, v_n$. For $n \geq 3$, a *cycle graph*, denoted by C_n, is a connected simple graph on n vertices all having a degree 2.

Given a simple graph G and a graph R, possibly with loops, we set $1, \ldots, r$ the vertices of R, also called *roles*. A *R-role assignment* of G is a surjective vertex mapping $p : V(G) \to V(R)$ such that $p(N_G(v)) = N_R(p(v))$ for all $v \in V(G)$. A graph G has a *r-role assignment* if it admits a R-role assignment for some graph R, called the *role graph*, with $|V(R)| = r$. From now on, all graphs (except maybe the role graph) are simple.

For Cartesian product, we follow the terminology of Hammack *et al.* [12]. Let G and H be two graphs. The *Cartesian product* of G and H is a graph, denoted as $G \square H$, whose vertex set is $V(G) \times V(H)$ and two vertices (u, v) and (x, y) are adjacent precisely if $u = x$ and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$.

Observe that if H is a *trivial graph*, then $G \square H \simeq G$ and the complexity of the problem r-ROLE ASSIGNMENT is already known. This is why we always consider G and H to be non-trivial graphs.

3. CARTESIAN PRODUCT WITH 2-ROLE ASSIGNMENT

In this section, we show that the Cartesian product of any two non-trivial graphs G and H has a 2-role assignment.

Theorem 3.1. *Let G and H be non-trivial graphs. Then $G \square H$ has a 2-role assignment.*

Proof. If $G \square H$ is disconnected, then according to Roberts and Sheng [17], $G \square H$ has a 2-role assignment. In fact, since G and H are non-trivial graphs, $G \square H$ has no isolated vertex. Therefore, by assigning role 1 to every vertex of one connected component and role 2 to the other vertices, we obtain a 2-role assignment, where the role graph is the disjoint union of two vertices, both with loops. Then, suppose that $G \square H$ is connected, that is, both G and H are connected. Since H is connected, there is a vertex ordering of $V(H)$, denoted by $v_1, \ldots, v_n$, with $|V(H)| = n$ such that $v_{i+1} \in N_H(\{v_1, \ldots, v_i\})$ for $i = 1, \ldots, n-1$. We first define recursively $\ell : V(H) \to \{1, 2\}$ by:

$$\ell(v_1) = 1.$$
$$\ell(v_i) = \ell(v_j) \bmod 2 + 1, \text{ where } j = \min\{k \mid v_i v_k \in E(H)\}, \quad \text{for } i = 2, \ldots, n.$$

We note that in the definition of $\ell(v_i)$, we have that $j \leq i - 1$. We use ℓ to define a 2-role assignment p of $G \square H$ by $p(u, v) = \ell(v)$ for all $(u, v) \in V(G \square H)$. We show that p is a R-role assignment, where R is the path P_2 with loops on both roles. For this purpose, we will see that $p(N_{G \square H}((u, v_i))) = \{1, 2\}$ for all $u \in V(G)$, $i = 1, \ldots, n$.

Let $u \in V(G)$. As G is a non-trivial connected graph, there exists $x \in N_G(u)$. We note that, for all $i = 1, \ldots, n$, (u, v_i) is a neighbor of (x, v_i) in $G \square H$ and $p((u, v_i)) = \ell(v_i) = p((x, v_i))$. Therefore, all vertices of $G \square H$ have a neighbor with the same role.

It remains to see that there is a neighbor of (u, v_i) with different role. For $i = 1$, $p((u, v_1)) = 1$. Since $v_2 \in N_H(v_1)$, $\ell(v_2) = 2$ and we obtain (u, v_2), a neighbor of (u, v_1), of role 2. For $i \geq 2$, there exists $j \in \{1, \ldots, i-1\}$, such that $v_i v_j \in E(H)$ and $\ell(v_i) = \ell(v_j) \bmod 2 + 1$. Clearly, $\ell(v_i) \neq \ell(v_j)$. Hence, (u, v_j) is a neighbor of (u, v_i) in $G \square H$ such that $p((u, v_i)) \neq p((u, v_j))$. $\qquad\square$

4. CARTESIAN PRODUCT WITH 3-ROLE ASSIGNMENT

Inspired by the constructions presented by van 't Hof *et al.* [19] and Dourado [6], we design a new construction to prove that the decision problem related to finding a 3-role assignment for Cartesian product of two graphs remains NP-complete. For this aim, we introduce the NP-complete problem of HYPERGRAPH 2-COLORING, also known as SET SPLITTING, see [11]. A *hypergraph* $\mathcal{H}$ is a pair $\mathcal{H} = (\mathcal{V}(\mathcal{H}), \mathcal{S}(\mathcal{H}))$, where $\mathcal{V}(\mathcal{H})$ is a set of vertices, and $\mathcal{S}(\mathcal{H})$ is a set of non-empty subsets of $\mathcal{V}(\mathcal{H})$ called *hyperedges*. We consider hyperedges with at least two elements.

A surjective mapping $c : \mathcal{V}(\mathcal{H}) \to \{1, 2\}$ is a k-COLORING, if every hyperedge in $\mathcal{S}(\mathcal{H})$ contains at least two vertices u and v with $c(u) \neq c(v)$. The HYPERGRAPH 2-COLORING problem asks whether a given hypergraph has a 2-coloring.

Given a hypergraph $\mathcal{H}$, we require to construct a Cartesian product of two graphs that will serve as an instance of r-ROLE ASSIGNMENT. For this, we will consider $G(\mathcal{H}) \square P_2$. The construction of $G(\mathcal{H})$, defined in the sequel, is based on the incidence graph. The *incidence graph* $I = (V(I), E(I))$ of a hypergraph $\mathcal{H}$ is a bipartite graph whose vertex set is $V(I) = \mathcal{V}(\mathcal{H}) \cup \mathcal{S}(\mathcal{H})$, and edge set $E(I) = \{vS \mid v \in \mathcal{V}(\mathcal{H}), S \in \mathcal{S}(\mathcal{H}) \text{ with } v \in S\}$.
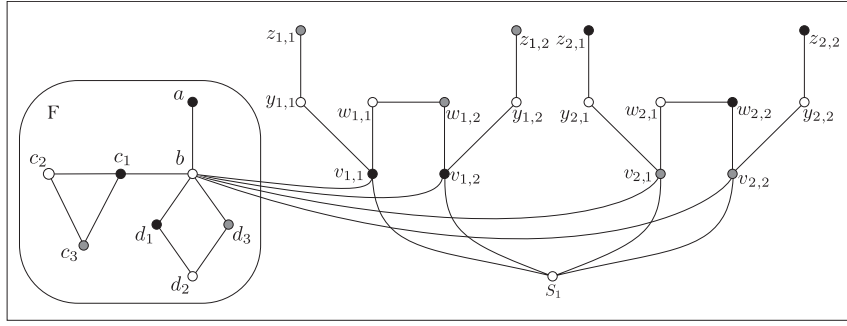
FIGURE 1. Part of a 3-role assignment, with $\mathcal{V}(\mathcal{H}) = \{v_1, v_2\}$ and $\mathcal{S}(\mathcal{H}) = \{S_1\}$, where $S_1 = \{v_1, v_2\}$, of $G(\mathcal{H}) \square P_2$ $(V(G(\mathcal{H}))'$ is not shown).

Construction 1. Given a hypergraph $\mathcal{H}$ with $\mathcal{V}(\mathcal{H}) = \{v_1, \ldots, v_n\}$, $n \geq 2$ and $\mathcal{S}(\mathcal{H}) = \{S_1, \ldots, S_m\}$, $m \geq 1$. We construct a graph $G(\mathcal{H})$, arising from the incidence graph I of $\mathcal{H}$:

– For $i = 1, \ldots, n$, we remove v_i from $V(I)$, as well as its incident edges from $E(I)$ and replace it by a copy of a path on eight vertices denoted by $z_{i,1} y_{i,1} v_{i,1} w_{i,1} w_{i,2} v_{i,2} y_{i,2} z_{i,2}$. We add two edges $v_{i,1} S_j, v_{i,2} S_j$, for every edge $v_i S_j \in E(I)$;
– We add the subgraph F, see Figure 1, and the edges $bv_{i,1}, bv_{i,2}$, for every $i \in \{1, \ldots, n\}$.

As we consider $G(\mathcal{H}) \square P_2$, we use the following simplified notations. Considering that $V(P_2) = \{u_1, u_2\}$, we identify x with the vertex (x, u_1) and denote by x' the vertex (x, u_2) for any $x \in V(G(\mathcal{H}))$. We say that x' is the *corresponding vertex* of x. Let $X \subseteq V(G(\mathcal{H}))$, we identify X with $\{(x, v_1) \mid x \in X\}$ and denote by X' the set $\{(x, v_2) \mid x \in X\}$. For short, we denote $N(x) = N_{G(\mathcal{H}) \square P_2}(x, v_1)$ and $N(x') = N_{G(\mathcal{H}) \square P_2}(x, v_2)$ for any $x \in V(G(\mathcal{H}))$.

Let R_3 be the graph with vertex set $\{1, 2, 3\}$ arising from the cycle C_3 by the addition of a loop on role 2. Theorem 4.1 presents a R_3-role assignment for $G(\mathcal{H}) \square P_2$ when a 2-coloring of $\mathcal{H}$ is known.

Theorem 4.1. *Let $\mathcal{H}$ be a hypergraph and $G(\mathcal{H})$ the graph obtained from Construction 1. If $\mathcal{H}$ has a 2-coloring, then $G(\mathcal{H}) \square P_2$ has a R_3-role assignment.*

Proof. Let $c : V(\mathcal{H}) \to \{1, 3\}$ be a 2-coloring of $\mathcal{H}$. To simplifies the definition of a role assignment, we introduce the following subsets of $V(G(\mathcal{H}))$.

$$A_1 = \{v_{i,1}, v_{i,2} \mid v_i \in \mathcal{V}(\mathcal{H}), c(v_i) = 1\} \cup \{z_{i,1}, z_{i,2}, w_{i,2} \mid v_i \in \mathcal{V}(\mathcal{H}), c(v_i) = 3\} \cup \{a, c_1, d_1\}.$$
$$A_2 = \{y_{i,1}, y_{i,2}, w_{i,1} \mid v_i \in \mathcal{V}(\mathcal{H})\} \cup \mathcal{S}(\mathcal{H}) \cup \{b, c_2, d_2\}.$$
$$A_3 = \{v_{i,1}, v_{i,2} \mid v_i \in \mathcal{V}(\mathcal{H}), c(v_i) = 3\} \cup \{z_{i,1}, z_{i,2}, w_{i,2} \mid v_i \in \mathcal{V}(\mathcal{H}), c(v_i) = 1\} \cup \{c_3, d_3\}.$$

We define a role assignment $p : V(G(\mathcal{H}) \square P_2) \to \{1, 2, 3\}$ as follows. For every $x \in V(G(\mathcal{H}))$, $p(x) = i$, if $x \in A_i$ and:

$$p(x') = \begin{cases} 1, & \text{if } x \in A_3; \\ 2, & \text{if } x \in A_2; \\ 3, & \text{if } x \in A_1. \end{cases}$$

See an example in Figure 1, in which the vertices in black, white and gray receive roles 1, 2 and 3, respectively. It is easy to see that p is a R_3-role assignment of $G(\mathcal{H}) \square P_2$. $\qquad\square$

Observe that, if the graph G is connected, then the role graph R of any role assignment of G is also connected. We show in the following lemma, that R_3 is the unique role graph for a 3-role assignment of $G(\mathcal{H}) \square P_2$.

Lemma 4.2. *Let $\mathcal{H}$ be a hypergraph and $G(\mathcal{H})$ the graph obtained from Construction 1. If $G(\mathcal{H}) \square P_2$ has a R-role assignment with $|V(R)| = 3$, then $R \simeq R_3$.*

Proof. Let $p : V(G(\mathcal{H}) \square P_2) \to V(R)$ be a 3-role assignment for $G(\mathcal{H}) \square P_2$ with $V(R) = \{1, 2, 3\}$. Since $G(\mathcal{H}) \square P_2$ is connected, so R is. Hence, we may assume that $1, 3 \in N_R(2)$. We show that $R \simeq R_3$.

First, suppose that $p(b), p(b') \in \{1, 3\}$. We note that if $p(a) \in \{1, 3\}$, then $p(a') = 2$. Therefore, by symmetry, we may assume that $p(a) = 2$. Hence, $\{p(a'), p(b)\} = \{1, 3\}$ and $N_R(2) = \{1, 3\}$. Without lost of generality, we suppose that $p(a') = 1$ and $p(b) = 3$. If $p(b') = 1$, then $3 \in N_R(1)$, but $N_R(1) = p(N(a')) = \{1, 2\}$, a contradiction. Hence, $p(b') = 3$. Since, $N_R(1) = \{2, 3\}$, $N_R(2) = \{1, 3\}$ and $N_R(3) = \{1, 2, 3\}$, we have that $R \simeq R_3$.

From now on, $p(b) = 2$ or $p(b') = 2$. By symmetry, suppose that $p(b) = 2$. Its important remember that $p(a) \neq 2$, otherwise since the degree of a in $G(\mathcal{H}) \square P_2$ is 2, a does not have enough neighbors to assign the roles $1, 2, 3$. Up to analyze each possibility of roles for a and a', we divide the proof in the following cases:

Case 1. $p(a) = 1$ and $p(a') = 1$,
Case 2. $p(a) = 1$ and $p(a') = 2$ and
Case 3. $p(a) = 1$ and $p(a') = 3$.

We remark that the case $p(a) = 3$ and $p(a') = 1$ is similar to Case 3, given that $p(b') = 2$. On the other hand, the case $p(a) = 3$ and $p(a') = 2$ is equivalent to the Case 2 by interchanging the roles 1 and 3. The same occurs in the case that $p(a) = 3$ and $p(a') = 3$, that is equivalent to Case 1. In Case 3, we obtain $p(b) = p(b') = 2$ and $R = R_3$ as desired. We show contradictions in Cases 1 and 2.

Case 1. $p(a) = 1$ and $p(a') = 1$.

We have that $p(b) = p(b') = 2$, $N_R(1) = \{1, 2\}$ and $N_R(2) = \{1, 2, 3\}$. Two possibilities exist depending if there is a loop on role 3 or not.

First, consider that there is a loop on role 3. Observe that R is the path P_3 with loops on all roles. We show contradictions in the roles of c_1, c_2, c_3 and their corresponding vertices. We consider the possible roles for c_1.

If $p(c_1) = 1$, since there is a loop on role 1, we have two possibilities: $1 \in p(\{c_2, c_3\})$ or $p(c_1') = 1$. If $1 \in p(\{c_2, c_3\})$, we may assume that $p(c_2) = 1$. Since $p(N(c_3)) = \{1, p(c_3')\}$ and $|N_R(2)| = 3$, we have that $p(c_3) = 1$ and $p(c_2') = 2$. Hence, every neighbor of c_2' is adjacent to a vertex with role 1, a contradiction since $3 \in N_R(2)$, but $3 \notin N_R(1)$.

Notice that, with similar arguments used for $p(c_1) = 1$, we may find a contradiction for $p(c_1) = 3$. Thus, remains to consider $p(c_1) = 2$. By symmetry, we may assume that $p(c_1') = 2$. Since $N_R(2) = \{1, 2, 3\}$, we have $\{p(c_2), p(c_3)\} = \{1, 3\}$, a contradiction, because $1 \notin N_R(3)$.

Next, consider that there is no loop on role 3. Observe that R is the path P_3 with loop on roles 1 and 2. We show contradictions in the roles of d_1, d_2, d_3 and their corresponding vertices. We consider the possible roles for d_1.

We begin by role 3, that is $p(d_1) = 3$. Since $N_R(3) = \{2\}$, we have that $p(d_2) = 2$ and $p(d_1') = 2$. Looking at the roles of the neighborhood of d_1', we obtain that $p(d_2') = 1$. On the other hand, d_2 must have a neighbor of role 2, which must be d_3. But, $p(N(d_3)) = \{2, p(d_3')\}$ and $|N_R(2)| = 3$, a contradiction. Thus, we may assume that $3 \notin p(\{d_1, d_3, d_1', d_3'\})$.

If $p(d_1) = 2$, since $p(d_1') \neq 3$, we get that $p(d_2) = 3$ and $p(d_1') = 1$. Recall that $N_R(3) = \{2\}$, then $p(d_2') = 2$. This implies that $p(N(d_1')) = \{2\}$, a contradiction, since $N_R(1) = \{1, 2\}$. We conclude that $p(\{d_1, d_3, d_1', d_3'\}) = \{1\}$. Considering that $p(N(d_2)) = \{1, p(d_2')\}$, $N_R(1) = \{1, 2\}$ and $|N_R(2)| = 3$, we obtain that $p(d_2) = 1$. Similarly, $p(d_2') = 1$, a contradiction.

Case 2. $p(a) = 1$ and $p(a') = 2$.

Recall that $p(b) = 2$. We have that $p(b') = 3$, $N_R(1) = \{2\}$ and $N_R(2) = \{1, 3\}$. Observe that R is the path P_3 with a possible loop on role 3. We show contradictions in the roles of c_1, c_2, c_3 and their corresponding vertices. We consider the possible roles for c_1. Recall that, since there is no loop on role 2, $p(c_1) \neq 2$.

If $p(c_1) = 1$, as $N_R(1) = \{2\}$, then $p(c_2) = p(c_3) = 2$, a contradiction to the fact that there is no loop on role 2. Thus, $p(c_1) = 3$. Given that $N_R(3) = \{2,3\}$ and $2 \notin N_R(2)$, we may suppose that $p(c_2) = 3$. We know that $p(c_3) \in \{2,3\}$. If $p(c_3) = 2$, then $p(c_3') = 1$, and we conclude that $p(c_1') = p(c_2') = 2$, a contradiction. Hence, $p(c_3) = 3$ and this leads to $p(c_2') = p(c_3') = 2$, a contradiction.   □

Next, we show that one can obtain a 2-coloring for $\mathcal{H}$ when a 3-role assignment for $G(\mathcal{H})\square P_2$ is known.

Theorem 4.3. *Let $\mathcal{H}$ be a hypergraph and $G(\mathcal{H})$ the graph obtained from Construction 1. If $G(\mathcal{H})\square P_2$ has a 3-role assignment, then $\mathcal{H}$ has a 2-coloring.*

Proof. Let p be a R-role assignment for $G(\mathcal{H})\square P_2$, with $|V(R)| = 3$. By Lemma 4.2, we may assume that $R = R_3$. Observe that $|N_R(2)| = 3$, this implies that the vertices of degree 2, in $G(\mathcal{H})\square P_2$, cannot have role 2. Let $x \in V(G(\mathcal{H}))\square P_2$, such that $N(x) = \{x', y\}$ for some $y \in V(G(\mathcal{H}))$. Thus, x and x' cannot have role 2, that is $p(x), p(x') \in \{1,3\}$. Since, $2 \in N_R(1) \cap N_R(3)$ and $p(x) \neq p(x')$, we obtain that $p(y) = p(y') = 2$.

Let $i \in \{1,\ldots,n\}$. By the above, $p(\{y_{i,1}, y_{i,2}, y_{i,1}', y_{i,2}'\}) = \{2\}$. Since $N_R(2) = \{1,2,3\}$, we obtain that $p(v_{i,1}), p(v_{i,2}), p(v_{i,1}'), p(v_{i,2}') \in \{1,3\}$. First, we show that $p(v_{i,1}) = p(v_{i,2})$. Suppose, by contradiction, $p(v_{i,1}) = 1$ and $p(v_{i,2}) = 3$. Recall that there is no loop on roles 1 and 3, therefore $p(v_{i,1}') = 3$ and $p(v_{i,2}') = 1$. If $p(w_{i,1}) \neq 2$, then $2 \in \{p(w_{i,2}), p(w_{i,1}')\}$. Thus, by symmetry, we may assume that $p(w_{i,1}) = 2$.

Notice that $|N(w_{i,1})| = 3$ and $p(v_{i,1}) = 1$, hence $p(\{w_{i,2}, w_{i,1}'\}) = \{2,3\}$. Since $p(w_{i,2}) \in N_R(2) \cap N_R(3) = \{1,2\}$, we get $p(w_{i,2}) = 2$ and $p(w_{i,1}') = 3$, a contradiction to the fact that there is no loop on role 3. We conclude by symmetry that $p(v_{i,1}) = p(v_{i,2})$ and $p(v_{i,1}') = p(v_{i,2}')$, for $i = 1,\ldots,n$.

Now, let $j \in \{1,\ldots,m\}$. We show that $p(S_j) = p(S_j') = 2$. Suppose by contradiction that $p(S_j) \neq 2$. Since $p(N_{G(\mathcal{H})}(S_j)) \subseteq \{1,3\}$, we have that $p(S_j') = 2$, which leads to a contradiction, given that $p(N(S_j')) \subseteq \{1,3\}$. Hence, we obtain that $p(S_j) = p(S_j') = 2$, for $j = 1,\ldots,m$.

Finally, we define a mapping $c : V(\mathcal{H}) \to \{1,3\}$ given by $c(v_i) = p(v_{i,1})$. We show that c is a 2-coloring for $\mathcal{H}$. Let $S_j \in \mathcal{S}(\mathcal{H})$. Recall that S_j represents a vertex in $V(G(\mathcal{H}))$, and consequently in $V(G(\mathcal{H})\square P_2)$. We have that $N(S_j) = \{v_{i,1}, v_{i,2} \mid v_i \in S_j \text{ (in } \mathcal{H})\} \cup \{S_j'\}$. Since $p(S_j) = p(S_j') = 2$, for $j = 1,\ldots,m$, $p(v_{i,1}) = p(v_{i,2}) \in \{1,3\}$, for $i = 1,\ldots,n$, and p is a R_3-role assignment, there exist $s,t \in \{1,\ldots,n\}$, such that $v_{s,1}, v_{t,1} \in N(S_j)$, $p(v_{s,1}) = 1$, and $p(v_{t,1}) = 3$. This implies that $c(v_s) = 1$ and $c(v_t) = 3$ with $v_s, v_t \in S_j$ (in $\mathcal{H}$), which defines a 2-coloring for $\mathcal{H}$.   □

The above results imply directly in the NP-completeness of 3-ROLE ASSIGNMENT.

Theorem 4.4. *The problem* 3-ROLE ASSIGNMENT *is* NP-*complete even when restricted to the Cartesian product of two non-trivial graphs.*

Proof. The problem is clearly in NP (*cf.* [17]). To show the NP-hardness, we use a reduction from the NP-complete problem HYPERGRAPH 2-COLORING [11]. Given a hypergraph $\mathcal{H}$, we construct the graph $G(\mathcal{H})$ according to Construction 1, which is used to compute $G(\mathcal{H})\square P_2$. It is easy to see that $G(\mathcal{H})\square P_2$ may be obtained in polynomial time. By Theorems 4.1 and 4.3, we obtain that $\mathcal{H}$ has 2-coloring if and only if $G(\mathcal{H})\square P_2$ has a 3-role assignment, and the proof is complete.   □

5. CONCLUSION

In this paper, we have shown that the problem r-ROLE ASSIGNMENT restricted to Cartesian product is trivial, with true answer, for $r = 2$ and NP-complete for $r = 3$. As we have seen in Introduction, the known literature results have reported that when r-ROLE ASSIGNMENT is NP-complete for some r, the problem remains NP-complete for any integer greater than r. Then, the following conjecture rise naturally:

Conjecture. *For $r \geq 4$, the problem r-ROLE ASSIGNMENT is* NP-*complete even when restricted to the Cartesian product of two non-trivial graphs.*

As future works, we suggest to determine the computational complexity of r-ROLE ASSIGNMENT restricted to other graph products.

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