RAIRO-Oper. Res. 56 (2022) 1–22
https://doi.org/10.1051/ro/2021183

RAIRO Operations Research
www.rairo-ro.org

EQUILIBRIUM REINSURANCE-INVESTMENT STRATEGY WITH A COMMON SHOCK UNDER TWO KINDS OF PREMIUM PRINCIPLES

Junna Bi, Danping Li* and Nan Zhang

Abstract. This paper investigates the optimal mean-variance reinsurance-investment problem for an insurer with a common shock dependence under two kinds of popular premium principles: the variance premium principle and the expected value premium principle. We formulate the optimization problem within a game theoretic framework and derive the closed-form expressions of the equilibrium reinsurance-investment strategy and equilibrium value function under the two different premium principles by solving the extended Hamilton–Jacobi–Bellman system of equations. We find that under the variance premium principle, the proportional reinsurance is the optimal reinsurance strategy for the optimal reinsurance-investment problem with a common shock, while under the expected value premium principle, the excess-of-loss reinsurance is the optimal reinsurance strategy. In addition, we illustrate the equilibrium reinsurance-investment strategy by numerical examples and discuss the impacts of model parameters on the equilibrium strategy.

Mathematics Subject Classification. 90C39, 91B30, 91G80.

Received June 15, 2021. Accepted December 14, 2021.

1. Introduction

The study of an insurer's optimal reinsurance-investment problem has attracted a lot of attention in the literature of actuarial science in the past few years. The optimization criteria which are commonly used in these optimal reinsurance-investment problems include maximizing the expected utility of the terminal wealth of an insurer (see *e.g.*, [10, 12, 18, 20, 21, 26, 27]), minimizing the ruin probability of an insurer (see *e.g.*, [10, 22, 23]), and the mean-variance criterion (see *e.g.*, [3–5, 16]). In this paper, we take the mean-variance criterion as the optimization criterion.

For the optimal reinsurance-investment problem of an insurer with more lines of insurance business, we have to consider the dependence among the lines of the insurance business. For example, a hurricane can lead to different kinds of insurance claims such as death claims, household claims, and so on. The common shock risk model is often used to describe the dependence between different classes of an insurance business. In this model, there is a common shock affecting the claim numbers of all classes in addition to their underlying risks. Yuen *et al.* [30] and Liang and Yuen [17] assumed that the claim number processes were correlated through a common shock, and studied the optimal proportional reinsurance strategy. More researches about dependent insurance

Keywords. Optimal reinsurance-investment, common shock, premium principles, mean-variance criterion, equilibrium strategy.

School of Statistics, KLATASDS-MOE, East China Normal University, Shanghai 200062, P.R. China.
*Corresponding author: dpli@fem.ecnu.edu.cn

risks can be found in Yuen *et al.* [29], Bai *et al.* [2], Bi *et al.* [5], Zhang and Liang [34], Liang *et al.* [18], and references therein.

It is well-known that the optimal reinsurance-investment problems under the mean-variance criterion in a continuous time framework are time inconsistent due to the fact that variance lacks of the iterated-expectation property. So the dynamic programming cannot be directly applied to such problems. Actually, the optimal strategies in most of the mean-variance works mentioned above are called pre-committed strategies. A pre-committed strategy means that if the decision makers can commit themselves at the initial time, they can choose a strategy that is optimal from the point of view at the initial time, and then constrain themselves to abide by it in the future, although the strategy is no longer optimal for the future time. Considering a real-world example, if an investor adopts the pre-committed strategy, it is optimal only when sitting at the initial time but no longer optimal in the remaining time interval, which is time-inconsistent, while this investor hopes to choose a strategy which is optimal at every remaining time, *i.e.*, she/he prefers the time-consistent strategy. For this reason, more and more scholars have become to study the time-consistent strategies for the dynamic mean-variance problems. Björk and Murgoci [6] developed a general theory for time inconsistent stochastic control problems, and derived an extended Hamilton–Jacobi–Bellman (HJB) system of equations that can be used to solve for the equilibrium strategy, which is time-consistent. From then on, many works have addressed the equilibrium strategies of dynamic mean-variance problems within the game theoretic framework, including Zeng and Li [31], Li and Li [15], Björk *et al.* [8], Björk and Murgoci [7], Wu and Zeng [25], Björk *et al.* [9], Wei and Wang [24], and Liu and Chen [19].

A reinsurance policy consists of one risk sharing function which determines how the insurer and the reinsurer share risk, and one premium share function which indicates how premium is diverted between the insurer and the reinsurer. The optimal risk sharing functions have different forms under different reinsurance premium principles. Kaluszka [14] studied the optimal reinsurance under premium principles based on the mean and variance of the reinsurer's share of the total claim amount for both global reinsurance and local reinsurance. Hipp and Taksar [13] solved the problem of ruin probability minimization and they found that the proportional reinsurance was optimal under the variance premium principle. Bai *et al.* [2] showed that an excess-of-loss reinsurance policy was the optimal form that minimized the ruin probability under the expected value premium principle. Zeng and Luo [32] modeled reinsurance as a cooperation game, then Pareto-optimal policies were studied and classified as either excess-of-loss or proportional reinsurance based on choices of premium share functions. Zhang *et al.* [33] analyzed the optimal reinsurance strategy for insurers with a generalized mean-variance premium principle. They derived the form of optimal reinsurance under the criteria of maximizing the expected utility function of terminal wealth and minimizing the probability of ruin. Li *et al.* [16] studied an insurer's reinsurance problem under a mean-variance criterion. They showed that excess-of-loss was the equilibrium reinsurance strategy under a spectrally negative Lévy insurance model when the reinsurance premium was computed according to the expected value premium principle. These papers motivate us to consider the optimal reinsurance forms in an optimal reinsurance-investment problem with a common shock under different premium principles.

In this paper, we investigate the equilibrium reinsurance-investment strategy with common shock dependence under two popular reinsurance premium principles in practice: the variance premium principle and the expected value premium principle. The expected value premium principle is commonly used in life insurance which has stable and smooth claim frequency and claim sizes. The variance premium principle is extensively used in property insurance. The variance principle permits the company to take the fluctuations (variance) of claims into consideration when pricing insurance contracts. We assume that the insurer has two dependent classes of insurance businesses, which are subject to a common shock. The common shock can be interpreted as a claim that affects two lines of businesses at the same time, such as motor and life insurances, or as a natural disaster that causes different kinds of claims. The insurer can invest in a financial market consisting of a risk-free asset and a risky asset whose price process follows a geometric Brownian motion. Under the mean-variance criterion, we formulate the optimal reinsurance-investment problem within a game theoretic framework. We firstly prove that the optimal reinsurance contract is a proportional reinsurance under the variance premium principle, and the optimal reinsurance contract is an excess-of-loss reinsurance under the expected value premium principle.

Then by using the technique of stochastic control theory and solving the corresponding extended Hamilton–Jacobi–Bellman (HJB) system of equations, we derive the closed-form expressions of the equilibrium reinsurance-investment strategy and the equilibrium value function. We further use numerical examples to show the impacts of model parameters on the equilibrium reinsurance-investment strategies under two kinds of premium principles.

The main contributions of this paper are as follows: First, we incorporate a common shock into the insurer's reserve model and then consider the time-inconsistent reinsurance-investment problem under two reinsurance premium principles. Our model extends the results in Zeng and Luo [32] and Zhang *et al.* [33], where they did not consider common shock dependence and time-consistent strategies. Second, for mathematical simplicity, many papers use the proportional reinsurance directly, but we use the general reinsurance form and prove that the optimal reinsurance strategies have different forms under different reinsurance premium principles. We find that the optimal reinsurance contract is a proportional reinsurance under the variance premium principle, and the optimal reinsurance contract is an excess-of-loss reinsurance under the expected value premium principle. Third, we find that under the variance premium principle, the optimal proportions for the first class and the second class of claims are the same. However, under the expected value premium principle, the optimal reinsurance strategies for the first class and the second class of claims are different.

This paper is organized as follows. Section 2 presents the assumptions and problem formulation. Section 3 studies a mean-variance reinsurance-investment problem under variance premium principle, and derives the explicit solution to this problem by solving the extended HJB system of equations. Section 4 solves the mean-variance reinsurance-investment problem under the expected value premium principle. Section 5 uses numerical examples to illustrate the derived equilibrium reinsurance-investment strategies under two kinds of premium principles. Section 6 concludes this paper.

2. The model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a filtered probability space satisfying the usual conditions, *i.e.*, $\mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is right continuous and complete with respect to $\mathcal{P}$. $\mathcal{F}_t$ stands for the information of the market available until time t, which is generated by the standard Brownian motions $B_1(t)$, $B_2(t)$. That is, $\mathcal{F}_t$ is the accumulated information up to time t. $T > 0$ is a fixed time horizon. In addition, we assume that there is no transaction cost and tax in the financial market or the insurance market, and trading takes place continuously.

The reserve process $\left\{\bar{X}(t)\right\}_{t\geq 0}$ of the insurer is modeled by

$$\bar{X}(t) = \bar{X}_0 + c_0 t - \left(\sum_{i=1}^{N_1(t)+N(t)} Z_{1i} + \sum_{i=1}^{N_2(t)+N(t)} Z_{2i}\right). \tag{2.1}$$

Here $\bar{X}_0$ is the deterministic initial reserve of the insurer and the constant c_0 is the premium rate. $\{N_1(t)\}_{t\geq 0}$, $\{N_2(t)\}_{t\geq 0}$ and $\{N(t)\}_{t\geq 0}$ are three independent Poisson processes with intensity parameters $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda > 0$, respectively. The counting processes $N_1(t)+N(t)$ and $N_2(t)+N(t)$ represent the numbers of claims during the time interval $[0,t]$ for the first class and second class, respectively. Z_{1i} is the size of the ith claim for the first class and $\{Z_{1i}, i \geq 1\}$ is assumed to be an i.i.d. sequence with common distribution $F_{Z_1}(\cdot)$, $\mathbb{E}(Z_{1i}) = \mu_{1Z_1} > 0$ and $\mathbb{E}\left(Z_{1i}^2\right) = \mu_{2Z_1} > 0$. Z_{2i} is the size of the ith claim for the second class and $\{Z_{2i}, i \geq 1\}$ is assumed to be an i.i.d. sequence with common distribution $F_{Z_2}(\cdot)$, $\mathbb{E}(Z_{2i}) = \mu_{1Z_2} > 0$ and $\mathbb{E}(Z_{2i}^2) = \mu_{2Z_2} > 0$. Thus the compound Poisson processes $S_1(t) := \sum_{i=1}^{N_1(t)+N(t)} Z_{1i}$ and $S_2(t) := \sum_{i=1}^{N_2(t)+N(t)} Z_{2i}$ represent the cumulative amounts of claims for the first class and the second class in time interval $[0,t]$, respectively. $\{N_1(t)\}_{t\geq 0}$, $\{N_2(t)\}_{t\geq 0}$, $\{N(t)\}_{t\geq 0}$, $\{Z_{1i}, i \geq 1\}$, and $\{Z_{2i}, i \geq 1\}$ are mutually independent. It is obvious that the dependence of the two classes of businesses is due to a common shock governed by the counting process $\{N(t)\}_{t\geq 0}$. This model has been studied extensively in the literature; see *e.g.*, Yuen *et al.* [28], Yuen *et al.* [29], Yuen *et al.* [30] and Liang and Yuen [17].

Moreover, we allow the insurance company applies reinsurance strategies $g_1(t, Z_1)$ and $g_2(t, Z_2)$ for the first and second classes of claims. $g_1(t, Z_1)$ and $g_2(t, Z_2)$ are increasing risk share functions for the two variables

Z_1 and Z_2, with $0 \leq g_1(t, Z_1) \leq Z_1$ and $0 \leq g_2(t, Z_2) \leq Z_2$. $g_1(t, Z_1)$ and $g_2(t, Z_2)$ are the parts of each random claim retained by the insurance company while the rest $Z_1 - g_1(t, Z_1)$ and $Z_2 - g_2(t, Z_2)$ are ceded to the reinsurer. Let the reinsurance premium rate at time t be $\delta(g_1, g_2)$. In Sections 3 and 4, we consider the variance premium principle and the expected value premium principle respectively. Let $\left\{\widetilde{X}^{g_1,g_2}(t)\right\}_{t\geq 0}$ denote the associated surplus process, *i.e.*, $\widetilde{X}^{g_1,g_2}(t)$ is the wealth of the insurer at time t under the strategy $(g_1(\cdot,\cdot), g_2(\cdot,\cdot))$. This process then evolves as

$$d\widetilde{X}^{g_1,g_2}(t) = [c_0 - \delta(g_1, g_2)]dt - d\sum_{i=1}^{N_1(t)+N(t)} g_1(t, Z_{1i}) - d\sum_{i=1}^{N_2(t)+N(t)} g_2(t, Z_{2i}). \tag{2.2}$$

Due to the jumps in the reserve process $\left\{\widetilde{X}(t)\right\}_{t\geq 0}$, it is not feasible to derive the optimal investment-reinsurance strategy in this paper explicitly. As most studies on the optimization problem (see *e.g.*, [1,2,10,17,28], and so on), we can consider the problem under the diffusion approximation of the reserve process $\left\{\widetilde{X}(t)\right\}_{t\geq 0}$. According to Grandell [11] (pages 15–17), we consider the diffusion approximations, *i.e.*, approximating the reserve process (2.2) by a Brownian motion with drift. Mathematically, such approximations are based on the theory of weak convergence of probability measures. One way to express this diffusion approximation is, if the classical risk model is regarded to be "large deviation", the diffusion model is related to "the central limit theorem". By the similar calculations as those in Bai *et al.* [2], we have the diffusion approximation for our two-dimensional reserve process (2.2) in the following form

$$\begin{aligned} d\widehat{X}^{g_1,g_2}(t) &= [c_0 - \delta(g_1, g_2) - (\lambda+\lambda_1)\mathbb{E}g_1(t, Z_1) - (\lambda+\lambda_2)\mathbb{E}g_2(t, Z_2)]dt \\ &\quad + \sqrt{(\lambda+\lambda_1)\mathbb{E}[g_1(t, Z_1)^2] + (\lambda+\lambda_2)\mathbb{E}[g_2(t, Z_2)^2] + 2\lambda\mathbb{E}g_1(t, Z_1)\mathbb{E}g_2(t, Z_2)}\,dB_1(t), \end{aligned} \tag{2.3}$$

where $B_1(t)$ is a standard Brownian motion.

Suppose that the insurer is allowed to invest all of his/her wealth in a financial market consisting of one risk-free asset (the money market instrument or the bond) and one risky asset (stock). We consider the financial market where the two assets are traded continuously on a finite time horizon $[0, T]$.

The price process of the risk-free asset is given by

$$dP_0(t) = r_0(t)P_0(t)dt,\ t \in [0, T],$$

where the deterministic function $r_0(t)(> 0)$ is the interest rate of the risk-free asset.

The price of the risky asset is modeled by the following stochastic differential equation

$$dP_1(t) = P_1(t)[r_1(t)dt + \sigma(t)dB_2(t)],\ t \in [0, T],$$

where $r_1(t)(> r_0(t))$ is the appreciation rate and $\sigma(t)$ is the volatility coefficient of the risky asset. $B_2(t)$ is a standard $\{\mathcal{F}_t\}_{t\geq 0}$-adapted Brownian motion which is independent with $B_1(t)$. We assume that $r_0(t)$, $r_1(t)$ and $\sigma(t)$ are deterministic, Borel-measurable and bounded on $[0, T]$.

Let $X(t)$ denote the insurer's wealth at time t and $u(t)$ denote the total market value of the insurer's wealth in the risky asset. Then $X(t) - u(t)$ is the value of the insurer's wealth in the risk-free asset. A restriction we will consider in this paper is the prohibition of short-selling of the stock, *i.e.*, $u(t) \geq 0$. But the market value of the insurer's wealth in the risk-free asset is not constrained.

We call $\pi(t) := (g_1(t,\cdot), g_2(t,\cdot), u(t))'$ an admissible strategy if $\pi(t)$ is $\mathcal{F}_t$-predictable and satisfies $0 \leq g_1(t, Z_1) \leq Z_1$, $0 \leq g_2(t, Z_2) \leq Z_2$, $u(t) \geq 0$, $\mathbb{E}\left[\int_0^t g_1(s, Z_1)^2 \mathrm{d}s\right] < \infty$, $\mathbb{E}\left[\int_0^t g_2(s, Z_2)^2 \mathrm{d}s\right] < \infty$, and $\mathbb{E}[\int_0^t u(s)^2 \mathrm{d}s] < \infty$ for all $t \geq 0$.

We denote the set of all admissible strategies by Π. Let $X^\pi(t)$ denote the wealth process when the strategy $\pi(\cdot)$ is applied. Then the reserve process $X^\pi(t)$ is given by

$$\begin{aligned}dX^\pi(t) = \Big\{&r_0(t)X^\pi(t) + [r_1(t) - r_0(t)]u(t) + c_0 - \delta(g_1, g_2) - (\lambda+\lambda_1)\mathbb{E}g_1(t, Z_1)\\&- (\lambda+\lambda_2)\mathbb{E}g_2(t, Z_2)\Big\}dt + \sigma(t)u(t)dB_2(t)\\&+ \sqrt{(\lambda+\lambda_1)\mathbb{E}[g_1(t, Z_1)^2] + (\lambda+\lambda_2)\mathbb{E}[g_2(t, Z_2)^2] + 2\lambda\mathbb{E}g_1(t, Z_1)\mathbb{E}g_2(t, Z_2)}dB_1(t).\end{aligned}\tag{2.4}$$

We will formulate the problem within a game theoretic framework, which is developed by Björk and Murgoci [6]. The optimization problem for the insurer with the objective function which we want to maximize is given by

$$\begin{aligned}J(t, x, \pi) &= \mathbb{E}_{t,x}[X^\pi(T)|\mathcal{F}_t] - \frac{\gamma}{2}\mathbb{V}ar_{t,x}[X^\pi(T)|\mathcal{F}_t]\\&= \mathbb{E}_{t,x}[X^\pi(T)|\mathcal{F}_t] - \frac{\gamma}{2}\Big\{\mathbb{E}_{t,x}\Big[(X^\pi(T))^2|\mathcal{F}_t\Big] - [\mathbb{E}_{t,x}(X^\pi(T)|\mathcal{F}_t)]^2\Big\},\\&= \mathbb{E}_{t,x}[F(X^\pi(T))|\mathcal{F}_t] + G(\mathbb{E}_{t,x}(X^\pi(T)|\mathcal{F}_t)),\end{aligned}\tag{2.5}$$

where

$$F(y) = y - \frac{\gamma}{2}y^2,\ G(y) = \frac{\gamma}{2}y^2,$$

$\mathbb{E}_{t,x}[\cdot|\mathcal{F}_t]$ and $\mathbb{V}ar_{t,x}[\cdot|\mathcal{F}_t]$ are the conditional expectation and variance given filtration $\mathcal{F}_t$ with $X^\pi(t) = x$ for $(t, x) \in [0, T] \times \mathbb{R}$, respectively, and γ is the risk aversion coefficient of the insurer.

Since the objective function $J(t, x, \pi)$ in (2.5) involves with $\mathbb{V}ar_{t,x}[\cdot]$, that is, $J(t, x, \pi)$ is a nonlinear function of the expected value $\mathbb{E}_{t,x}[\cdot]$, so the optimization problem is time-inconsistent. We solve this time-inconsistent problem within a game theoretic framework and look for Nash subgame perfect equilibrium solutions. Now we recall the following definition of an equilibrium control and equilibrium value function, which is taken from Björk and Murgoci [6].

Definition 2.1. Given a control law π^*, which can be informally viewed as a candidate equilibrium law, choose a fixed $\pi \in \Pi$, a fixed real number $l > 0$ and a fixed arbitrarily chosen initial point $(t, y) \in [0, T] \times \mathbb{R}$, and construct a control law π_l by

$$\pi_l(s, y) = \begin{cases}\pi(s, y), & t \le s < t + l,\ \ y \in \mathbb{R},\\ \pi^*(s, y), & t + l \le s \le T,\ y \in \mathbb{R}.\end{cases}$$

If

$$\lim_{l\to 0}\inf \frac{J(t, x, \pi^*) - J(t, x, \pi_l)}{l} \ge 0$$

for all $\pi \in \Pi$ and $(t, x) \in [0, T] \times \mathbb{R}$, we say that π^* is an equilibrium control law. The equilibrium value function is defined by

$$W(t, x) = J(t, x, \pi^*).$$

Based on the definition above, the equilibrium strategy is time-consistent, the equilibrium strategy is thus the optimal time-consistent strategy. Our objective is to find an equilibrium strategy π^* and the corresponding equilibrium value function.

Let $C^{1,2}([0, T] \times \mathbb{R})$ denote the space of the bivariate functions $\phi(t, x)$ such that $\phi(t, x)$ and its derivatives $\phi_t(t, x)$, $\phi_x(t, x)$, $\phi_{xx}(t, x)$ are continuous on $[0, T] \times \mathbb{R}$. For any function $\phi(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$ and any fixed $\pi \in \Pi$, the usual infinitesimal generator $\mathcal{A}^\pi$ for process (2.4) is defined by

$$\begin{aligned}\mathcal{A}^\pi\phi(t,x) = \phi_t + \Big\{ & r_0(t)x + [r_1(t) - r_0(t)]u(t) + c_0 - \delta(g_1,g_2) \\ & - (\lambda+\lambda_1)\mathbb{E}g_1(t,Z_1) - (\lambda+\lambda_2)\mathbb{E}g_2(t,Z_2)\Big\}\phi_x + \frac{1}{2}\Big\{(\lambda+\lambda_1)\mathbb{E}\big[g_1(t,Z_1)^2\big] \\ & + (\lambda+\lambda_2)\mathbb{E}\big[g_2(t,Z_2)^2\big] + 2\lambda\mathbb{E}g_1(t,Z_1)\mathbb{E}g_2(t,Z_2) + \sigma^2(t)u^2(t)\Big\}\phi_{xx}. \end{aligned} \tag{2.6}$$

Next we give the extended HJB system of equations and the corresponding Verification Theorem in the following theorem.

Theorem 2.2 (Verification Theorem)**.** *For the Nash equilibrium problem, if there exist functions $V(t,x)$ and $g(t,x)$ satisfying the following conditions: $\forall(t,x) \in [0,T]\times\mathbb{R}$ and $y \in \mathbb{R}$,*

$$\begin{cases} \sup\limits_{\pi\in\Pi}\{\mathcal{A}^\pi V(t,x) - \mathcal{A}^\pi(G\diamond g)(t,x) + \mathcal{H}^\pi g(t,x)\} = 0,\ \ 0\le t\le T, \\ \mathcal{A}^{\pi^*}g(t,x) = 0,\ \ 0\le t\le T, \\ V(T,x) = x, \\ g(T,x) = x, \end{cases} \tag{2.7}$$

and

$$\pi^* = \arg\sup_{\pi\in\Pi}\{\mathcal{A}^\pi V(t,x) - \mathcal{A}^\pi(G\diamond g)(t,x) + \mathcal{H}^\pi g(t,x)\},$$

then $W(t,x) = V(t,x)$, i.e., $V(t,x)$ is the equilibrium value function, π^* is the equilibrium reinsurance-investment strategy, and $g(t,x)$ has the following probabilistic interpretation:

$$g(t,x) = \mathbb{E}_{t,x}\Big[X^{\pi^*}(T)\Big], \tag{2.8}$$

where $G\diamond g$ and $\mathcal{H}^\pi g$ are defined as follows

$$\begin{cases} G\diamond g(t,x) = G(g(t,x)), \\ \mathcal{H}^\pi g(t,x) = G_y(g(t,x))\times\mathcal{A}^\pi g(t,x), \\ G_y(y) = \dfrac{dG}{dy}(y). \end{cases}$$

Equation (2.7) is also called the extended HJB system of equations.

Proof. The derivation of the extended HJB system of equations (2.7) and the proof of the Verification Theorem can be obtained by using the standard arguments similar to those used in Section 4 of Björk and Murgoci [6]. So we omit the details here. ☐

3. Equilibrium strategy under variance premium principle

In this section, we assume that the reinsurance premium is calculated according to the variance premium principle. That is,

$$\begin{aligned}\delta(g_1,g_2) = & (\lambda+\lambda_1)(\mu_{1Z_1} - \mathbb{E}g_1(t,Z_1)) + (\lambda+\lambda_2)(\mu_{1Z_2} - \mathbb{E}g_2(t,Z_2)) \\ & + \Lambda\big\{(\lambda+\lambda_1)\mathbb{E}[Z_1 - g_1(t,Z_1)]^2 + (\lambda+\lambda_2)\mathbb{E}[Z_2 - g_2(t,Z_2)]^2 \\ & + 2\lambda(\mu_{1Z_1} - \mathbb{E}g_1(t,Z_1))(\mu_{1Z_2} - \mathbb{E}g_2(t,Z_2))\big\}, \end{aligned} \tag{3.1}$$

here Λ is the reinsurer's safety loading.

Then the reserve process $X^\pi(t)$ given by (2.4) becomes

$$
\begin{aligned}
dX^\pi(t) = \bigg\{ & r_0(t)X^\pi(t) + [r_1(t) - r_0(t)]u(t) + c_0 - (\lambda+\lambda_1)\mu_{1Z_1} - (\lambda+\lambda_2)\mu_{1Z_2} \\
& - \Lambda\big[(\lambda+\lambda_1)\mathbb{E}(Z_1 - g_1(t,Z_1))^2 + (\lambda+\lambda_2)\mathbb{E}(Z_2 - g_2(t,Z_2))^2 \\
& + 2\lambda(\mu_{1Z_1} - \mathbb{E}g_1(t,Z_1))(\mu_{1Z_2} - \mathbb{E}g_2(t,Z_2))\big]\bigg\} dt + \sigma(t)u(t)dB_2(t) \\
& + \sqrt{(\lambda+\lambda_1)\mathbb{E}[g_1(t,Z_1)^2] + (\lambda+\lambda_2)\mathbb{E}[g_2(t,Z_2)^2] + 2\lambda\mathbb{E}g_1(t,X)\mathbb{E}g_2(t,Z_2)}\, dB_1(t),
\end{aligned} \tag{3.2}
$$

and the usual infinitesimal generator $\mathcal{A}$ for process (3.2) becomes

$$
\begin{aligned}
\mathcal{A}^\pi\phi(t,x) = \phi_t + \bigg\{ & r_0 x + (r_1 - r_0)u + c_0 - (\lambda+\lambda_1)\mu_{1Z_1} - (\lambda+\lambda_2)\mu_{1Z_2} \\
& - \Lambda\big[(\lambda+\lambda_1)\mathbb{E}(Z_1 - g_1(t,Z_1))^2 + (\lambda+\lambda_2)\mathbb{E}(Z_2 - g_2(t,Z_2))^2 \\
& + 2\lambda(\mu_{1Z_1} - \mathbb{E}g_1(t,Z_1))(\mu_{1Z_2} - \mathbb{E}g_2(t,Z_2))\big]\bigg\}\phi_x + \frac{1}{2}\bigg\{(\lambda+\lambda_1)\mathbb{E}\big[g_1(t,Z_1)^2\big] \\
& + (\lambda+\lambda_2)\mathbb{E}\big[g_2(t,Z_2)^2\big] + 2\lambda\mathbb{E}g_1(t,Z_1)\mathbb{E}g_2(t,Z_2) + \sigma^2u^2\bigg\}\phi_{xx}.
\end{aligned} \tag{3.3}
$$

After a number of elementary calculations, we obtain the following result about the extended HJB system of equations.

Proposition 3.1. *The extended HJB system of equations (2.7) can be simplified as follows:*

$$
\begin{aligned}
V_t + \sup_{\pi\in\Pi}\bigg\{ & \Big[r_0x + (r_1 - r_0)u + c_0 - (\lambda+\lambda_1)\mu_{1Z_1} - (\lambda+\lambda_2)\mu_{1Z_2}\Big]V_x \\
& - \Lambda V_x\Big[(\lambda+\lambda_1)\mathbb{E}(Z_1 - g_1(t,Z_1))^2 + (\lambda+\lambda_2)\mathbb{E}(Z_2 - g_2(t,Z_2))^2 + 2\lambda(\mu_{1Z_1} \\
& - \mathbb{E}g_1(t,Z_1))(\mu_{1Z_2} - \mathbb{E}g_2(t,Z_2))\Big] + \frac{1}{2}(V_{xx} - G_{yy}g_x^2)\Big[(\lambda+\lambda_1)\mathbb{E}\big[g_1(t,Z_1)^2\big] \\
& + (\lambda+\lambda_2)\mathbb{E}\big[g_2(t,Z_2)^2\big] + 2\lambda\mathbb{E}g_1(t,Z_1)\mathbb{E}g_2(t,Z_2) + \sigma^2u^2\Big]\bigg\} = 0,
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
g_t + \bigg\{ & r_0x + (r_1 - r_0)u^* + c_0 - (\lambda+\lambda_1)\mu_{1Z_1} - (\lambda+\lambda_2)\mu_{1Z_2} \\
& - \Lambda\big[(\lambda+\lambda_1)\mathbb{E}(Z_1 - g_1^*(t,Z_1))^2 + (\lambda+\lambda_2)\mathbb{E}(Z_2 - g_2^*(t,Z_2))^2 \\
& + 2\lambda(\mu_{1Z_1} - \mathbb{E}g_1^*(t,Z_1))(\mu_{1Z_2} - \mathbb{E}g_2^*(t,Z_2))\big]\bigg\}g_x + \frac{1}{2}\bigg\{(\lambda+\lambda_1)\mathbb{E}[g_1^*(t,Z_1)^2] \\
& + (\lambda+\lambda_2)\mathbb{E}\big[g_2^*(t,Z_2)^2\big] + 2\lambda\mathbb{E}g_1^*(t,Z_1)\mathbb{E}g_2^*(t,Z_2) + \sigma^2(u^*)^2\bigg\}g_{xx} = 0.
\end{aligned} \tag{3.5}
$$

Proof. Recall that the reserve process and the infinitesimal generator are given in (3.2) and (3.3), respectively, then we have

$$
\begin{aligned}
\mathcal{A}^\pi V(t,x) = V_t + \bigg\{ & r_0x + (r_1 - r_0)u + c_0 - (\lambda+\lambda_1)\mu_{1Z_1} - (\lambda+\lambda_2)\mu_{1Z_2} \\
& - \Lambda\big[(\lambda+\lambda_1)\mathbb{E}(Z_1 - g_1(t,Z_1))^2 + (\lambda+\lambda_2)\mathbb{E}(Z_2 - g_2(t,Z_2))^2 \\
& + 2\lambda(\mu_{1Z_1} - \mathbb{E}g_1(t,Z_1))(\mu_{1Z_2} - \mathbb{E}g_2(t,Z_2))\big]\bigg\}V_x + \frac{1}{2}\bigg\{(\lambda+\lambda_1)\mathbb{E}\big[g_1(t,Z_1)^2\big] \\
& + (\lambda+\lambda_2)\mathbb{E}\big[g_2(t,Z_2)^2\big] + 2\lambda\mathbb{E}g_1(t,Z_1)\mathbb{E}g_2(t,Z_2) + \sigma^2u^2\bigg\}V_{xx},
\end{aligned} \tag{3.6}
$$

$$\begin{aligned}
\mathcal{A}^{\pi}(G\diamond g)(t,x) &= \mathcal{A}^{\pi}G(x,g(t,x))\\
&= G_y g_t + \Big\{ r_0 x + (r_1-r_0)u + c_0 - (\lambda+\lambda_1)\mu_{1Z_1} - (\lambda+\lambda_2)\mu_{1Z_2}\\
&\quad - \Lambda\big[(\lambda+\lambda_1)\mathbb{E}(Z_1-g_1(t,Z_1))^2 + (\lambda+\lambda_2)\mathbb{E}(Z_2-g_2(t,Z_2))^2\\
&\quad + 2\lambda(\mu_{1Z_1}-\mathbb{E}g_1(t,Z_1))(\mu_{1Z_2}-\mathbb{E}g_2(t,Z_2))\big]\Big\}(G_x+G_y g_x)\\
&\quad + \frac{1}{2}\Big\{(\lambda+\lambda_1)\mathbb{E}\big[g_1(t,Z_1)^2\big] + (\lambda+\lambda_2)\mathbb{E}\big[g_2(t,Z_2)^2\big] + 2\lambda\mathbb{E}g_1(t,Z_1)\mathbb{E}g_2(t,Z_2)\\
&\quad + \sigma^2 u^2\Big\}\big[G_{xx} + G_{yy}g_x^2 + G_y g_{xx} + 2G_{xy}g_x\big],
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\mathcal{H}^{\pi}g(t,x) &= G_y(x,g(t,x)) \times \mathcal{A}^{\pi}g(t,x)\\
&= G_y g_t + \Big\{ r_0 x + (r_1-r_0)u + c_0 - (\lambda+\lambda_1)\mu_{1Z_1} - (\lambda+\lambda_2)\mu_{1Z_2}\\
&\quad - \Lambda\big[(\lambda+\lambda_1)\mathbb{E}(Z_1-g_1(t,Z_1))^2 + (\lambda+\lambda_2)\mathbb{E}(Z_2-g_2(t,Z_2))^2\\
&\quad + 2\lambda(\mu_{1Z_1}-\mathbb{E}g_1(t,Z_1))(\mu_{1Z_2}-\mathbb{E}g_2(t,Z_2))\big]\Big\}G_y g_x + \frac{1}{2}\Big\{(\lambda+\lambda_1)\mathbb{E}\big[g_1(t,Z_1)^2\big]\\
&\quad + (\lambda+\lambda_2)\mathbb{E}\big[g_2(t,Z_2)^2\big] + 2\lambda\mathbb{E}g_1(t,Z_1)\mathbb{E}g_2(t,Z_2) + \sigma^2u^2\Big\}G_y g_{xx}.
\end{aligned} \tag{3.8}$$

Insert (3.6), (3.7) and (3.8) into the first equation of (2.7), and recall that $G(y) = \frac{\gamma}{2}y^2$, we have (3.4) and (3.5). □

In the following, we will focus on finding the equilibrium reinsurance-investment strategy and the corresponding value function.

Theorem 3.2. *If the value function satisfies $V_x \geq 0$, $V_{xx} \leq 0$, then the optimal risk sharing functions are in the forms of $g_{1p}(t,Z_1) = p_1(t)Z_1,\ 0 \leq p_1(t) \leq 1$, $g_{2p}(t,Z_2) = p_2(t)Z_2,\ 0 \leq p_2(t) \leq 1$ under the variance premium principle. In other words, the optimal risk sharing functions are proportional.*

Proof. Let $g_1(\cdot,\cdot)$ be an arbitrary risk sharing function and $g_{1p}(\cdot,\cdot)$ be a proportional risk sharing function

$$g_{1p}(t,Z_1) = p_1(t)Z_1,\ 0 \leq p_1(t) \leq 1.$$

Then with Cauchy–Schwartz inequality, we have

$$\begin{aligned}
\mathbb{E}\big[(Z_1-g_1(t,Z_1))^2\big] &= \mathbb{E}\big[Z_1^2\big] - 2\mathbb{E}[Z_1 g_1(t,Z_1)] + \mathbb{E}\big[g_1(t,Z_1)^2\big]\\
&\geq \mathbb{E}\big[Z_1^2\big] - 2\sqrt{\mathbb{E}[Z_1^2]}\sqrt{\mathbb{E}[g_1(t,Z_1)^2]} + \mathbb{E}\big[g_1(t,Z_1)^2\big].
\end{aligned}$$

The inequality above becomes equality when the relationship between $g_1(t,Z_1)$ and Z_1 are proportional. That is

$$\begin{aligned}
\mathbb{E}\big[Z_1^2\big] - 2\sqrt{\mathbb{E}[Z_1^2]}\sqrt{\mathbb{E}[g_{1p}(t,Z_1)^2]} + \mathbb{E}\big[g_{1p}(t,Z_1)^2\big] &= \mathbb{E}\big[Z_1^2\big] - 2p_1(t)\mathbb{E}\big[Z_1^2\big] + p_1(t)^2\mathbb{E}\big[(Z_1)^2\big]\\
&= \mathbb{E}\big[(Z_1-g_{1p}(t,Z_1))^2\big].
\end{aligned}$$

Then we have

$$\mathbb{E}\big[(Z_1-g_1(t,Z_1))^2\big] \geq \mathbb{E}\big[(Z_1-g_{1p}(t,Z_1))^2\big].$$

Similarly, we have

$$\begin{aligned}\mathbb{E}\big[g_1(t,Z_1)^2\big] &= \mathbb{E}\big\{[Z_1-(Z_1-g_1(t,Z_1))]^2\big\}\\&= \mathbb{E}\big[Z_1^2\big]-2\mathbb{E}[Z_1(Z_1-g_1(t,Z_1))]+\mathbb{E}\big[(Z_1-g_1(t,Z_1))^2\big]\\&\geq \mathbb{E}\big[Z_1^2\big]-2\sqrt{\mathbb{E}[Z_1^2]}\sqrt{\mathbb{E}[(Z_1-g_1(t,Z_1))^2]}+\mathbb{E}\big[(Z_1-g_1(t,Z_1))^2\big].\end{aligned}$$

The inequality becomes equality when the relationship between $g_1(t,Z_1)$ and Z_1 are proportional. That is

$$\begin{aligned}&\mathbb{E}\big[Z_1^2\big]-2\sqrt{\mathbb{E}[Z_1^2]}\sqrt{\mathbb{E}[(Z_1-g_{1p}(t,Z_1))^2]}+\mathbb{E}\big[(Z_1-g_{1p}(t,Z_1))^2\big]\\&= \mathbb{E}\big[Z_1^2\big]-2(1-p_1(t))\mathbb{E}\big[Z_1^2\big]+(1-p_1(t))^2\mathbb{E}\big[Z_1^2\big]\\&= \mathbb{E}\big[g_{1p}(t,Z_1)^2\big].\end{aligned}$$

Then we have

$$\mathbb{E}\big[g_1(t,Z_1)^2\big]\geq \mathbb{E}\big[g_{1p}(t,Z_1)^2\big].$$

By similar calculations, we have

$$\mathbb{E}\big[(Z_2-g_2(t,Z_2))^2\big]\geq \mathbb{E}\big[(Z_2-g_{2p}(t,Z_2))^2\big],$$

$$\mathbb{E}\big[g_2(t,Z_2)^2\big]\geq \mathbb{E}\big[g_{2p}(t,Z_2)^2\big],$$

for an arbitrary risk sharing function $g_2(\cdot)$ and a proportional risk sharing function

$$g_{2p}(t,Z_2)=p_2(t)Z_2,\ \ 0\leq p_2(t)\leq 1.$$

Note that $\Lambda>0$, $V_x\geq 0$, $V_{xx}\leq 0$, $G_{yy}=\gamma>0$, $g_x^2\geq 0$, then we get

$$\begin{aligned}\mathcal{L}^{g_1,g_2}V := &\Big[r_0x+(r_1-r_0)u+c_0-(\lambda+\lambda_1)\mu_{1Z_1}-(\lambda+\lambda_2)\mu_{1Z_2}\\&-\Lambda\big[(\lambda+\lambda_1)\mathbb{E}(Z_1-g_1(t,Z_1))^2+(\lambda+\lambda_2)\mathbb{E}(Z_2-g_2(t,Z_2))^2\\&+2\lambda(\mu_{1Z_1}-\mathbb{E}g_1(t,Z_1))(\mu_{1Z_2}-\mathbb{E}g_2(t,Z_2))\big]\Big]V_x+\frac{1}{2}V_{xx}\Big[(\lambda+\lambda_1)\mathbb{E}\big[g_1(t,Z_1)^2\big]\\&+(\lambda+\lambda_2)\mathbb{E}\big[g_2(t,Z_2)^2\big]+2\lambda\mathbb{E}g_1(t,Z_1)\mathbb{E}g_2(t,Z_2)+\sigma^2u^2\Big]\\&-\frac{1}{2}G_{yy}g_x^2\Big[(\lambda+\lambda_1)\mathbb{E}\big[g_1(t,Z_1)^2\big]+(\lambda+\lambda_2)\mathbb{E}\big[g_2(t,Z_2)^2\big]+2\lambda\mathbb{E}g_1(t,Z_1)\mathbb{E}g_2(t,Z_2)\\&+\sigma^2u^2\Big]\leq \mathcal{L}^{g_{1p},g_{2p}}V.\end{aligned}$$

In other words, the optimal risk sharing functions are proportional. □

Next we just consider the proportional reinsurance strategy, that is $g_1(t,Z_1)=p_1(t)Z_1$ and $g_2(t,Z_2)=p_2(t)Z_2$, and give the equilibrium reinsurance-investment strategy and the corresponding equilibrium value function under the variance premium principle in the following theorem.

Theorem 3.3. *The equilibrium reinsurance-investment strategy $\pi^*=(g_1^*,g_2^*,u^*)$ for problem (2.5) with (3.2) under variance premium principle (3.1) is*

$$g_1^*(t,Z_1)=\frac{2\Lambda}{2\Lambda+\gamma e^{r_0(T-t)}}Z_1, \tag{3.9}$$

$$g_2^*(t, Z_2) = \frac{2\Lambda}{2\Lambda + \gamma e^{r_0(T-t)}} Z_2, \tag{3.10}$$

$$u^*(t) = \frac{r_1 - r_0}{\sigma^2} \frac{1}{\gamma e^{r_0(T-t)}}. \tag{3.11}$$

The corresponding equilibrium value function of the extended HJB system of equations (2.7) is

$$V(t, x) = A(t)x + B(t),$$
$$g(t, x) = a(t)x + b(t),$$

with

$$A(t) = a(t) = e^{r_0(T-t)}, \tag{3.12}$$

$$B(t) = \int_t^T \left[e^{r_0(T-\tau)}(c_0 - a_1 - a_2) - \frac{\Lambda\gamma e^{2r_0(T-\tau)}}{2\Lambda + \gamma e^{r_0(T-\tau)}} \left(\sigma_1^2 + \sigma_2^2 + 2\lambda\mu_{1Z_1}\mu_{1Z_2}\right) + \frac{(r_1 - r_0)^2}{2\sigma^2\gamma} \right] \mathrm{d}\tau, \tag{3.13}$$

$$b(t) = \int_t^T \left[e^{r_0(T-\tau)}(c_0 - a_1 - a_2) - \Lambda e^{r_0(T-\tau)} \left(\frac{\gamma e^{r_0(T-\tau)}}{2\Lambda + \gamma e^{r_0(T-\tau)}} \right)^2 \cdot \left(\sigma_1^2 + \sigma_2^2 + 2\lambda\mu_{1Z_1}\mu_{1Z_2}\right) + \frac{(r_1 - r_0)^2}{\sigma^2\gamma} \right] \mathrm{d}\tau. \tag{3.14}$$

Proof. Set $a_1 := (\lambda_1 + \lambda)\mu_{1Z_1}$, $a_2 := (\lambda_2 + \lambda)\mu_{1Z_2}$, $\sigma_1 := \sqrt{(\lambda_1 + \lambda)\mu_{2Z_1}}$ and $\sigma_2 := \sqrt{(\lambda_2 + \lambda)\mu_{2Z_2}}$. For the proportional reinsurance strategy, *i.e.*, $g_1(t, Z_1) = p_1(t)Z_1$ and $g_2(t, Z_2) = p_2(t)Z_2$, (3.4) becomes

$$\begin{aligned} V_t + \sup_{\pi\in\Pi} \Big\{ & \big[r_0 x + (r_1 - r_0)u + c_0 - a_1 - a_2 \\ & - \Lambda\big((1-p_1)^2\sigma_1^2 + (1-p_2)^2\sigma_2^2 + 2(1-p_1)(1-p_2)\lambda\mu_{1Z_1}\mu_{1Z_2}\big)\big] V_x \\ & + \frac{1}{2} V_{xx}\big(\sigma_1^2 p_1^2 + \sigma_2^2 p_2^2 + 2p_1 p_2\lambda\mu_{1Z_1}\mu_{1Z_2} + \sigma^2 u^2\big) \\ & - \frac{1}{2} G_{yy} g_x^2 \big(\sigma_1^2 p_1^2 + \sigma_2^2 p_2^2 + 2p_1 p_2 \lambda\mu_{1Z_1}\mu_{1Z_2} + \sigma^2 u^2\big) \Big\} = 0. \end{aligned} \tag{3.15}$$

Set

$$f(u) := \frac{1}{2}\big(V_{xx} - G_{yy} g_x^2\big)\sigma^2 u^2 + V_x(r_1 - r_0)u. \tag{3.16}$$

The maximizer of (3.16) is

$$u^* = -\frac{V_x(r_1 - r_0)}{2 \times \frac{1}{2}(V_{xx} - G_{yy} g_x^2)\sigma^2} = \frac{r_1 - r_0}{\sigma^2} \frac{V_x}{\gamma g_x^2 - V_{xx}}, \tag{3.17}$$

in which we use the fact $G(y) = \frac{\gamma}{2} y^2$.

Next we look for the equilibrium reinsurance strategy and the corresponding equilibrium value function. Let

$$\begin{aligned} H(p_1, p_2) := & -\Lambda V_x \big[(1-p_1)^2\sigma_1^2 + (1-p_2)^2\sigma_2^2 + 2(1-p_1)(1-p_2)\lambda\mu_{1Z_1}\mu_{1Z_2}\big] \\ & + \frac{1}{2}(V_{xx} - \gamma g_x^2)(\sigma_1^2 p_1^2 + \sigma_2^2 p_2^2 + 2p_1 p_2 \lambda\mu_{1Z_1}\mu_{1Z_2}) \\ = & \left(-\Lambda V_x + \frac{1}{2} V_{xx} - \frac{\gamma}{2} g_x^2\right)\sigma_1^2 p_1^2 + \left(-\Lambda V_x + \frac{1}{2} V_{xx} - \frac{\gamma}{2} g_x^2\right)\sigma_2^2 p_2^2 \\ & + 2\Lambda V_x\big(\sigma_1^2 + \lambda\mu_{1Z_1}\mu_{1Z_2}\big)p_1 + 2\Lambda V_x\big(\sigma_2^2 + \lambda\mu_{1Z_1}\mu_{1Z_2}\big)p_2 \\ & + \big(-2\Lambda V_x + V_{xx} - \gamma g_x^2\big)\lambda\mu_{1Z_1}\mu_{1Z_2} p_1 p_2 \end{aligned}$$

$$- \Lambda V_x\big(\sigma_1^2+\sigma_2^2+2\lambda\mu_{1Z_1}\mu_{1Z_2}\big).$$

Then we have

$$\begin{cases} \dfrac{\partial H(p_1,p_2)}{\partial p_1} = \sigma_1^2\big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)p_1+2\Lambda V_x\big(\sigma_1^2+\lambda\mu_{1Z_1}\mu_{1Z_2}\big) \\ \qquad\qquad + \big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)\lambda\mu_{1Z_1}\mu_{1Z_2}p_2, \\ \dfrac{\partial H(p_1,p_2)}{\partial p_2} = \sigma_2^2\big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)p_2+2\Lambda V_x\big(\sigma_2^2+\lambda\mu_{1Z_1}\mu_{1Z_2}\big) \\ \qquad\qquad + \big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)\lambda\mu_{1Z_1}\mu_{1Z_2}p_1, \\ \dfrac{\partial^2 H(p_1,p_2)}{\partial p_1^2} = \sigma_1^2\big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big), \\ \dfrac{\partial^2 H(p_1,p_2)}{\partial p_2^2} = \sigma_2^2\big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big), \\ \dfrac{\partial^2 H(p_1,p_2)}{\partial p_1\partial p_2} = \big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)\lambda\mu_{1Z_1}\mu_{1Z_2}. \end{cases}$$

The Hessian matrix is

$$H=\begin{pmatrix} \frac{\partial^2 H(p_1,p_2)}{\partial p_1^2} & \frac{\partial^2 H(p_1,p_2)}{\partial p_1\partial p_2} \\ \frac{\partial^2 H(p_1,p_2)}{\partial p_1\partial p_2} & \frac{\partial^2 H(p_1,p_2)}{\partial p_2^2} \end{pmatrix} = \begin{pmatrix} \sigma_1^2\big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big) & \big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)\lambda\mu_{1Z_1}\mu_{1Z_2} \\ \big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)\lambda\mu_{1Z_1}\mu_{1Z_2} & \sigma_2^2\big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big) \end{pmatrix}.$$

Because of

$$\begin{aligned} |H| &= \Big[\sigma_1^2\sigma_2^2-\big(\lambda\mu_{1Z_1}\mu_{1Z_2}\big)^2\Big]\times\big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)^2 \\ &= \sigma_1^2\sigma_2^2(1-\rho^2)\big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)^2\geq 0, \end{aligned}$$

with

$$\rho := \frac{\lambda\mu_{1Z_1}\mu_{1Z_2}}{\sqrt{(\lambda_1+\lambda)\mu_{2Z_1}(\lambda_2+\lambda)\mu_{2Z_2}}}\in(0,1), \tag{3.18}$$

the maximizer (p_1^*,p_2^*) of $H(p_1,p_2)$ is the solution of

$$\begin{cases} \sigma_1^2\big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)p_1+\big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)\lambda\mu_{1Z_1}\mu_{1Z_2}p_2 \\ = -2\Lambda V_x\big(\sigma_1^2+\lambda\mu_{1Z_1}\mu_{1Z_2}\big), \\ \big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)\lambda\mu_{1Z_1}\mu_{1Z_2}p_1+\sigma_2^2\big(-2\Lambda V_x+V_{xx}-\gamma g_x^2\big)p_2 \\ = -2\Lambda V_x\big(\sigma_2^2+\lambda\mu_{1Z_1}\mu_{1Z_2}\big). \end{cases}$$

Solving the above equation, we have

$$p_1^*=p_2^*=\frac{\widetilde{B}}{\widetilde{A}}=\frac{-2\Lambda V_x}{-2\Lambda V_x+V_{xx}-\gamma g_x^2}=\frac{2\Lambda V_x}{2\Lambda V_x-V_{xx}+\gamma g_x^2}. \tag{3.19}$$

Since the reserve process has the linear structure, and in accordance with the forms of the boundary conditions, we conjecture that

$$\begin{aligned} V(t,x) &= A(t)x+B(t), \\ g(t,x) &= a(t)x+b(t). \end{aligned}$$

Then, we have

$$V_t = A_t x + B_t,\ V_x = A(t),\ V_{xx} = 0, \tag{3.20}$$

and

$$g_t = a_t x + b_t,\ g_x = a(t),\ g_{xx} = 0. \tag{3.21}$$

Inserting them into (3.17) and (3.19), we have

$$u^* = \frac{r_1 - r_0}{\sigma^2}\frac{A(t)}{\gamma a^2(t)}, \tag{3.22}$$

and

$$p_1^* = p_2^* = \frac{2\Lambda A(t)}{2\Lambda A(t) + \gamma a^2(t)}. \tag{3.23}$$

Inserting (3.20)–(3.23) into (3.16) and (3.18), we have

$$f(u^*) = \frac{1}{2}\big(V_{xx} - G_{yy}g_x^2\big)\sigma^2(u^*)^2 + V_x(r_1 - r_0)u^* = \frac{(r_1 - r_0)^2}{2\gamma\sigma^2} \times \frac{A^2(t)}{a^2(t)}, \tag{3.24}$$

and

$$H(p_1^*, p_2^*) = -\frac{\Lambda\gamma A(t)a^2(t)}{2\Lambda A(t) + \gamma a^2(t)}\big(\sigma_1^2 + \sigma_2^2 + 2\lambda\mu_{1Z_1}\mu_{1Z_2}\big). \tag{3.25}$$

Then (3.15) becomes

$$A_t x + B_t + A(t)[r_0 x + c_0 - a_1 - a_2] + f(u^*) + H(p_1^*, p_2^*) = 0.$$

That is,

$$A_t x + B_t + A(t)[r_0 x + c_0 - a_1 - a_2] + \frac{(r_1 - r_0)^2}{2\gamma\sigma^2}\frac{A^2(t)}{a^2(t)} - \frac{\Lambda\gamma A(t)a^2(t)}{2\Lambda A(t) + \gamma a^2(t)}\big(\sigma_1^2 + \sigma_2^2 + 2\lambda\mu_{1Z_1}\mu_{1Z_2}\big) = 0.$$

By separating variables, we obtain the following ordinary differential equations

$$\begin{cases} A_t + r_0 A = 0, \\ A(T) = 1, \end{cases}$$

$$\begin{cases} B_t + A(t)[c_0 - a_1 - a_2] + \dfrac{(r_1 - r_0)^2}{2\gamma\sigma^2}\dfrac{A^2(t)}{a^2(t)} \\ -\dfrac{\Lambda\gamma A(t)a^2(t)}{2\Lambda A(t) + \gamma a^2(t)}\big(\sigma_1^2 + \sigma_2^2 + 2\lambda\mu_{1Z_1}\mu_{1Z_2}\big) = 0, \\ B(T) = 0. \end{cases}$$

Moreover, the second equation in (2.7) becomes

$$\begin{aligned} g_t(t,x) &+ \Big\{ r_0 x + (r_1 - r_0)u^* + c_0 - a_1 - a_2 \\ &- \Lambda\big[(1 - p_1^*)^2\sigma_1^2 + (1 - p_2^*)^2\sigma_2^2 + 2(1 - p_1^*)(1 - p_2^*)\lambda\mu_{1Z_1}\mu_{1Z_2}\big]\Big\} g_x(t,x) \\ &+ \frac{1}{2}\big[\sigma_1^2(p_1^*)^2 + \sigma_2^2(p_2^*)^2 + 2p_1^* p_2^*\lambda\mu_{1Z_1}\mu_{1Z_2} + \sigma^2(u^*)^2\big] g_{xx}(t,x) = 0. \end{aligned} \tag{3.26}$$

Inserting (3.21)–(3.23) into (3.26), we have

$$
\begin{aligned}
&a_t x + b_t + a(t)(r_0 x + c_0 - a_1 - a_2) + a(t)(r_1 - r_0)\frac{r_1 - r_0}{\sigma^2}\frac{A}{\gamma a^2} \\
&\quad - \Lambda a(\sigma_1^2 + \sigma_2^2 + 2\lambda\mu_{1Z_1}\mu_{1Z_2})\left(\frac{\gamma a^2}{2\Lambda A + \gamma a^2}\right)^2 = 0.
\end{aligned} \tag{3.27}
$$

By separating variables again, we obtain

$$
\begin{cases}
a_t + r_0 a = 0, \\
a(T) = 1,
\end{cases}
$$

$$
\begin{cases}
b_t + a(t)[c_0 - a_1 - a_2] + \dfrac{(r_1 - r_0)^2}{\sigma^2}\dfrac{A}{\gamma a} \\
\quad - \Lambda a(\sigma_1^2 + \sigma_2^2 + 2\lambda\mu_{1Z_1}\mu_{1Z_2})\left(\dfrac{\gamma a^2}{2\Lambda A + \gamma a^2}\right)^2 = 0, \\
b(T) = 0.
\end{cases}
$$

Then we have (3.12)–(3.14). This completes the proof. □

Remark 3.4. The optimal proportions we obtained in Theorem 3.3 satisfy $p_1^*(t) \in [0,1]$ and $p_2^*(t) \in [0,1]$. Then the risk sharing functions satisfy $g_1^*(t, Z_1) \in [0, Z_1]$ and $g_2^*(t, Z_2) \in [0, Z_2]$. Moreover, the corresponding equilibrium value function satisfies the assumptions in Theorem 3.2, *i.e.*, $V_x \geq 0$, $V_{xx} \leq 0$.

Remark 3.5. Under the variance premium principle, the proportional reinsurance is optimal. And the optimal proportions for the first class and the second class of claims are the same, which just depend on the reinsurer's safety loading Λ of the reinsurance business, the interest rate r_0 of the risk-free asset, and the risk aversion coefficient γ of the insurer. The equilibrium investment strategy depends on the interest rate r_0 of the risk-free asset, the appreciation rate r_1, the volatility coefficient σ of the risky asset, and the risk aversion coefficient γ of the insurer. Moreover, owing to the fact that the risk aversion coefficient γ of the insurer is a constant, the equilibrium reinsurance-investment strategy is independent of the state variable x.

4. Equilibrium strategy under expected value premium principle

In this section, we assume that the reinsurance premium is calculated according to the expected value principle. That is,

$$
\delta(q_1, q_2) = (1 + \eta_1)(\lambda + \lambda_1)(\mu_{1Z_1} - \mathbb{E}g_1(t, Z_1)) + (1 + \eta_2)(\lambda + \lambda_2)(\mu_{1Z_2} - \mathbb{E}g_2(t, Z_2)), \tag{4.1}
$$

where η_1 and η_2 are the reinsurer's safety loadings of the insurance businesses.

The usual infinitesimal generator $\mathcal{A}$ becomes

$$
\begin{aligned}
\mathcal{A}^\pi \phi(t, x) = \phi_t &+ \Big[r_0 x + (r_1 - r_0)u + c_0 - (1 + \eta_1)(\lambda + \lambda_1)\mu_{1Z_1} \\
&- (1 + \eta_2)(\lambda + \lambda_2)\mu_{1Z_2} + \eta_1(\lambda + \lambda_1)\mathbb{E}g_1(t, Z_1) + \eta_2(\lambda + \lambda_2)\mathbb{E}g_2(t, Z_2)\Big]\phi_x \\
&+ \frac{1}{2}\Big\{(\lambda + \lambda_1)\mathbb{E}\big[g_1(t, Z_1)^2\big] + (\lambda + \lambda_2)\mathbb{E}\big[g_2(t, Z_2)^2\big] + 2\lambda\mathbb{E}g_1(t, Z_1)\mathbb{E}g_2(t, Z_2) + \sigma^2 u^2\Big\}\phi_{xx}.
\end{aligned} \tag{4.2}
$$

By the similar calculations as those in Section 3, we obtain the following result about the extended HJB system of equations under the expected value premium principle.

Proposition 4.1. *The first and second equations of the extended HJB system of equations (2.7) can be simplified as follows:*

$$
\begin{aligned}
V_t + \sup_{\pi\in\Pi}\bigg\{ &\big[r_0 x + (r_1 - r_0)u + c_0 - (1+\eta_1)(\lambda+\lambda_1)\mu_{1Z_1} \\
&- (1+\eta_2)(\lambda+\lambda_2)\mu_{1Z_2} + \eta_1(\lambda+\lambda_1)\mathbb{E}g_1(t,Z_1) + \eta_2(\lambda+\lambda_2)\mathbb{E}g_2(t,Z_2)\big]V_x \\
&+ \frac{1}{2}\big(V_{xx} - G_{yy}g_x^2\big)\big[(\lambda+\lambda_1)\mathbb{E}(g_1(t,Z_1)^2) + (\lambda+\lambda_2)\mathbb{E}(g_2(t,Z_2)^2) \\
&+ 2\lambda\mathbb{E}g_1(t,Z_1)\mathbb{E}g_2(t,Z_2)\big] + \frac{1}{2}\big(V_{xx} - G_{yy}g_x^2\big)\sigma^2 u^2\bigg\} = 0,
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
g_t(t,x) + \bigg[&r_0 x + (r_1 - r_0)u^* + c_0 - (1+\eta_1)(\lambda+\lambda_1)\mu_{1Z_1} \\
&- (1+\eta_2)(\lambda+\lambda_2)\mu_{1Z_2} + \eta_1(\lambda+\lambda_1)\mathbb{E}g_1^*(t,Z_1) + \eta_2(\lambda+\lambda_2)\mathbb{E}g_2^*(t,Z_2)\bigg]g_x(t,x) \\
&+ \frac{1}{2}\bigg\{(\lambda+\lambda_1)\mathbb{E}[g_1^*(t,Z_1)^2] + (\lambda+\lambda_2)\mathbb{E}\big[g_2^*(t,Z_2)^2\big] + 2\lambda\mathbb{E}g_1^*(t,Z_1)\mathbb{E}g_2^*(t,Z_2) \\
&+ \sigma^2(u^*)^2\bigg\}g_{xx}(t,x) = 0.
\end{aligned}
\tag{4.4}
$$

Proof. The proof of this proposition is similar to that of Proposition 3.1, so we omit it here. □

Next we will find the equilibrium reinsurance-investment strategy and the corresponding value function under the expected value premium principle.

Theorem 4.2. *Assume that the value function satisfies $V_x \geq 0$, $V_{xx} \leq 0$. Under the expected value premium principle, the optimal reinsurance-investment strategy $\pi^* = (g_1^*, g_2^*, u^*)$ for the mean-variance criterion is in the form of*

$$
g_1^*(t,Z_1) = \min\bigg\{Z_1, \max\bigg\{0, \frac{V_x}{G_{yy}g_x^2 - V_{xx}} \times \frac{(\lambda+\lambda_2)[(\lambda+\lambda_1)\eta_1 - \lambda\eta_2]}{(\lambda+\lambda_1)(\lambda+\lambda_2)-\lambda^2}\bigg\}\bigg\},
\tag{4.5}
$$

$$
g_2^*(t,Z_2) = \min\bigg\{Z_2, \max\bigg\{0, \frac{V_x}{G_{yy}g_x^2 - V_{xx}} \times \frac{(\lambda+\lambda_1)[(\lambda+\lambda_2)\eta_2 - \lambda\eta_1]}{(\lambda+\lambda_1)(\lambda+\lambda_2)-\lambda^2}\bigg\}\bigg\},
\tag{4.6}
$$

$$
u^* = \frac{r_1 - r_0}{\sigma^2}\frac{V_x}{\gamma g_x^2 - V_{xx}}.
\tag{4.7}
$$

In other words, the optimal risk sharing function is excess-of-loss reinsurance.

Proof. From (4.3), we have

$$
u^* = \frac{r_1 - r_0}{\sigma^2}\frac{V_x}{\gamma g_x^2 - V_{xx}}.
\tag{4.8}
$$

Set

$$\begin{aligned}\hat{H}(g_1,g_2):=&\frac{1}{2}\big(V_{xx}-G_{yy}g_x^2\big)(\lambda+\lambda_1)g_1(t,Z_1)^2\\&+\frac{1}{2}\big(V_{xx}-G_{yy}g_x^2\big)(\lambda+\lambda_2)g_2(t,Z_2)^2+\big(V_{xx}-G_{yy}g_x^2\big)\lambda g_1(t,Z_1)g_2(t,Z_2)\\&+V_x\eta_1(\lambda+\lambda_1)g_1(t,Z_1)+V_x\eta_2(\lambda+\lambda_2)g_2(t,Z_2).\end{aligned}$$

Then (4.3) becomes

$$\begin{aligned}V_t+\sup_{\pi\in\Pi}\Big\{&\big[r_0x+(r_1-r_0)u+c_0-(1+\eta_1)(\lambda+\lambda_1)\mu_{1Z_1}\\&-(1+\eta_2)(\lambda+\lambda_2)\mu_{1Z_2}\big]V_x+\mathbb{E}\hat{H}(g_1,g_2)+\frac{1}{2}\big(V_{xx}-G_{yy}g_x^2\big)\sigma^2u^2\Big\}=0.\end{aligned}$$

Differentiating the function $\hat{H}(g_1,g_2)$ with respect to g_1 and g_2 respectively, we obtain

$$\begin{cases}\dfrac{\partial\hat{H}(g_1,g_2)}{\partial g_1}=\big(V_{xx}-G_{yy}g_x^2\big)(\lambda+\lambda_1)g_1+\big(V_{xx}-G_{yy}g_x^2\big)\lambda g_2+V_x\eta_1(\lambda+\lambda_1),\\ \dfrac{\partial\hat{H}(g_1,g_2)}{\partial g_2}=\big(V_{xx}-G_{yy}g_x^2\big)(\lambda+\lambda_2)g_2+\big(V_{xx}-G_{yy}g_x^2\big)\lambda g_1+V_x\eta_2(\lambda+\lambda_2),\\ \dfrac{\partial^2\hat{H}(g_1,g_2)}{\partial g_1^2}=\big(V_{xx}-G_{yy}g_x^2\big)(\lambda+\lambda_1),\\ \dfrac{\partial^2\hat{H}(g_1,g_2)}{\partial g_2^2}=\big(V_{xx}-G_{yy}g_x^2\big)(\lambda+\lambda_2),\\ \dfrac{\partial^2\hat{H}(g_1,g_2)}{\partial g_1\partial g_2}=\big(V_{xx}-G_{yy}g_x^2\big)\lambda.\end{cases}$$

The Hessian matrix is

$$\hat{H}=\begin{pmatrix}\frac{\partial^2\hat{H}(g_1,g_2)}{\partial g_1^2}&\frac{\partial^2\hat{H}(g_1,g_2)}{\partial g_1\partial g_2}\\ \frac{\partial^2\hat{H}(g_1,g_2)}{\partial g_1\partial g_2}&\frac{\partial^2\hat{H}(g_1,g_2)}{\partial g_2^2}\end{pmatrix}=\begin{pmatrix}\big(V_{xx}-G_{yy}g_x^2\big)(\lambda+\lambda_1)&\big(V_{xx}-G_{yy}g_x^2\big)\lambda\\ \big(V_{xx}-G_{yy}g_x^2\big)\lambda&\big(V_{xx}-G_{yy}g_x^2\big)(\lambda+\lambda_2)\end{pmatrix}.$$

Because of $V_{xx}\le 0$ and

$$\Big|\hat{H}\Big|=\big(V_{xx}-G_{yy}g_x^2\big)^2\big[(\lambda+\lambda_1)(\lambda+\lambda_2)-\lambda^2\big]\ge 0,$$

it is easy to see that the maximizer candidate of $\hat{H}(g_1,g_2)$ is the solution of the equations

$$\begin{cases}\big(V_{xx}-G_{yy}g_x^2\big)(\lambda+\lambda_1)g_1+\big(V_{xx}-G_{yy}g_x^2\big)\lambda g_2+V_x\eta_1(\lambda+\lambda_1)=0,\\ \big(V_{xx}-G_{yy}g_x^2\big)(\lambda+\lambda_2)g_2+\big(V_{xx}-G_{yy}g_x^2\big)\lambda g_1+V_x\eta_2(\lambda+\lambda_2)=0.\end{cases}\tag{4.9}$$

The solution of equation (4.9) is

$$\begin{cases}\hat{g}_1^*=\dfrac{V_x}{G_{yy}g_x^2-V_{xx}}\times\dfrac{(\lambda+\lambda_2)[(\lambda+\lambda_1)\eta_1-\lambda\eta_2]}{(\lambda+\lambda_1)(\lambda+\lambda_2)-\lambda^2},\\ \hat{g}_2^*=\dfrac{V_x}{G_{yy}g_x^2-V_{xx}}\times\dfrac{(\lambda+\lambda_1)[(\lambda+\lambda_2)\eta_2-\lambda\eta_1]}{(\lambda+\lambda_1)(\lambda+\lambda_2)-\lambda^2}.\end{cases}\tag{4.10}$$

Because of the constraints of $0\le g_1(t,X)\le X$, $0\le g_2(t,Y)\le Y$, we get (4.5)–(4.7). □

Theorem 4.3. *The equilibrium reinsurance-investment strategy* $\pi^* = (g_1^*, g_2^*, u^*)$ *for problem (2.5) with (2.4) under expected value premium principle (4.1) is*

$$g_1^*(t, Z_1) = \min\left\{Z_1, \max\left\{0, \frac{(\lambda+\lambda_2)[(\lambda+\lambda_1)\eta_1 - \lambda\eta_2]}{(\lambda+\lambda_1)(\lambda+\lambda_2)-\lambda^2} \times \frac{1}{\gamma e^{r_0(T-t)}}\right\}\right\}, \tag{4.11}$$

$$g_2^*(t, Z_2) = \min\left\{Z_2, \max\left\{0, \frac{(\lambda+\lambda_1)[(\lambda+\lambda_2)\eta_2 - \lambda\eta_1]}{(\lambda+\lambda_1)(\lambda+\lambda_2)-\lambda^2} \times \frac{1}{\gamma e^{r_0(T-t)}}\right\}\right\}, \tag{4.12}$$

$$u^* = \frac{r_1 - r_0}{\sigma^2}\frac{1}{\gamma e^{r_0(T-t)}}. \tag{4.13}$$

The corresponding equilibrium value function of the extended HJB system of equations (2.7) is

$$\begin{aligned} V(t,x) &= A(t)x + B(t), \\ g(t,x) &= a(t)x + b(t), \end{aligned}$$

with

$$A(t) = a(t) = e^{r_0(T-t)}, \tag{4.14}$$

$$\begin{aligned} B(t) = \int_t^T \Big\{ & e^{r_0(T-\tau)}\big[c_0 - (1+\eta_1)(\lambda+\lambda_1)\mu_{1Z_1} - (1+\eta_2)(\lambda+\lambda_2)\mu_{1Z_2} \\ & + \eta_1(\lambda+\lambda_1)\mathbb{E}g_1^*(t,Z_1) + \eta_2(\lambda+\lambda_2)\mathbb{E}g_2^*(t,Z_2)\big] + \frac{1}{2}\frac{(r_1-r_0)^2}{\gamma\sigma^2} \\ & - \frac{1}{2}\gamma e^{2r_0(T-\tau)}\Big[(\lambda+\lambda_1)\mathbb{E}[g_1^*(t,Z_1)^2] + (\lambda+\lambda_2)\mathbb{E}[g_2^*(t,Z_2)^2] \\ & + 2\lambda\mathbb{E}g_1^*(t,Z_1)\mathbb{E}g_2^*(t,Z_2)\Big]\Big\}\mathrm{d}\tau, \end{aligned} \tag{4.15}$$

$$\begin{aligned} b(t) = \int_t^T \Big\{ & e^{r_0(T-\tau)}\big[c_0 - (1+\eta_1)(\lambda+\lambda_1)\mu_{1Z_1} - (1+\eta_2)(\lambda+\lambda_2)\mu_{1Z_2}\big] \\ & + \frac{(r_1-r_0)^2}{\gamma\sigma^2} + e^{r_0(T-\tau)}\Big[\eta_1(\lambda+\lambda_1)\mathbb{E}g_1^*(t,Z_1) + \eta_2(\lambda+\lambda_2)\mathbb{E}g_2^*(t,Z_2)\Big]\Big\}\mathrm{d}\tau. \end{aligned} \tag{4.16}$$

Proof. Since the wealth process has the linear structure, and in accordance with the forms of the boundary conditions, we conjecture that

$$\begin{aligned} V(t,x) &= A(t)x + B(t), \\ g(t,x) &= a(t)x + b(t). \end{aligned}$$

Then, we have

$$V_t = A_t x + B_t,\ V_x = A(t),\ V_{xx} = 0, \tag{4.17}$$

and

$$g_t = a_t x + b_t,\ g_x = a(t),\ g_{xx} = 0. \tag{4.18}$$

Inserting (4.17) and (4.5)–(4.7) into (4.3), we have

$$A_t x + B_t + \Big[r_0 x + (r_1 - r_0)\frac{r_1 - r_0}{\sigma^2}\frac{A(t)}{\gamma a^2(t)} + c_0 - (1+\eta_1)(\lambda+\lambda_1)\mu_{1Z_1}$$

$$
\begin{aligned}
&- (1+\eta_2)(\lambda+\lambda_2)\mu_{1Z_2} + \eta_1(\lambda+\lambda_1)\mathbb{E}g_1^*(t,Z_1) + \eta_2(\lambda+\lambda_2)\mathbb{E}g_2^*(t,Z_2)\Big]A(t) \\
&- \frac{1}{2}\gamma a^2(t)\bigg[\frac{(r_1-r_0)^2}{\sigma^2}\left(\frac{A(t)}{\gamma a^2(t)}\right)^2 + (\lambda+\lambda_1)\mathbb{E}\big[g_1^*(t,Z_1)^2\big] \\
&+ (\lambda+\lambda_2)\mathbb{E}\big[g_2^*(t,Z_2)^2\big] + 2\lambda\mathbb{E}g_1^*(t,Z_1)\mathbb{E}g_2^*(t,Z_2)\bigg] = 0. \qquad (4.19)
\end{aligned}
$$

By separating variables, we obtain the following ordinary differential equations

$$
\begin{cases}
A_t + r_0 A = 0, \\
A(T) = 1,
\end{cases}
$$

$$
\begin{cases}
B_t + A(t)[c_0 - (1+\eta_1)(\lambda+\lambda_1)\mu_{1Z_1} - (1+\eta_2)(\lambda+\lambda_2)\mu_{1Z_2} \\
\quad + \eta_1(\lambda+\lambda_1)\mathbb{E}g_1^*(t,Z_1) + \eta_2(\lambda+\lambda_2)\mathbb{E}g_2^*(t,Z_2)] + \dfrac{1}{2}\dfrac{(r_1-r_0)^2}{\sigma^2}\dfrac{A^2(t)}{\gamma a^2(t)} \\
\quad - \dfrac{1}{2}\gamma a^2(t)\bigg[(\lambda+\lambda_1)\mathbb{E}[g_1^*(t,Z_1)^2] + (\lambda+\lambda_2)\mathbb{E}[g_2^*(t,Z_2)^2] \\
\quad + 2\lambda\mathbb{E}g_1^*(t,Z_1)\mathbb{E}g_2^*(t,Z_2)\bigg] = 0, \\
B(T) = 0.
\end{cases}
$$

Inserting (4.17), (4.18) and (4.5)–(4.7) into (4.4), we have

$$
\begin{aligned}
a_t x + b_t + a(t)\bigg[& r_0 x + (r_1-r_0)\frac{r_1-r_0}{\sigma^2}\frac{A(t)}{\gamma a^2(t)} + c_0 - (1+\eta_1)(\lambda+\lambda_1)\mu_{1Z_1} \\
&- (1+\eta_2)(\lambda+\lambda_2)\mu_{1Z_2} + \eta_1(\lambda+\lambda_1)\mathbb{E}g_1^*(t,Z_1) + \eta_2(\lambda+\lambda_2)\mathbb{E}g_2^*(t,Z_2)\bigg] = 0.
\end{aligned}
$$

By separating variables again, we obtain

$$
\begin{cases}
a_t + r_0 a = 0, \\
a(T) = 1,
\end{cases}
$$

$$
\begin{cases}
b_t + a(t)[c_0 - (1+\eta_1)(\lambda+\lambda_1)\mu_{1Z_1} - (1+\eta_2)(\lambda+\lambda_2)\mu_{1Z_2}] \\
\quad + \dfrac{A}{\gamma a}\dfrac{(r_1-r_0)^2}{\sigma^2} + a \times \bigg[\eta_1(\lambda+\lambda_1)\mathbb{E}g_1^*(t,Z_1) + \eta_2(\lambda+\lambda_2)\mathbb{E}g_2^*(t,Z_2)\bigg] = 0, \\
b(T) = 0.
\end{cases}
$$

Solving these equations, we obtain (4.14)–(4.16). Inserting the result into (4.5)–(4.7), we obtain (4.11)–(4.13). This completes the proof. □

In the following, we give explicit expression of the equilibrium reinsurance strategy. Note that $\frac{\bar{\rho} a_2 b_1}{a_1 b_2} < 1 < \frac{a_2 b_1}{\bar{\rho} a_1 b_2}$, we need to discuss the following three cases:

Case 1. $\eta_1 \le \frac{\lambda}{\lambda+\lambda_1}\eta_2$, which leads to $\hat{g_1} \le 0$ and $\hat{g_2} > 0$;
Case 2. $\frac{\lambda}{\lambda+\lambda_1}\eta_2 < \eta_1 < \frac{\lambda+\lambda_2}{\lambda}\eta_2$, which leads to $\hat{g_1} > 0$ and $\hat{g_2} > 0$;
Case 3. $\eta_1 \ge \frac{\lambda+\lambda_2}{\lambda}\eta_2$, which leads to $\hat{g_1} > 0$ and $\hat{g_2} \le 0$.

Proposition 4.4. *The equilibrium reinsurance strategies $g_1^*,\ g_2^*$ under the expected value premium principle are*

$$g_1^*(t,Z_1)=\begin{cases}0, & \textbf{Case 1},\\ \frac{(\lambda+\lambda_2)[(\lambda+\lambda_1)\eta_1-\lambda\eta_2]}{[(\lambda+\lambda_1)(\lambda+\lambda_2)-\lambda^2]\gamma e^{r_0(T-t)}}\wedge Z_1, & \textbf{Case 2},\\ \frac{(\lambda+\lambda_2)[(\lambda+\lambda_1)\eta_1-\lambda\eta_2]}{[(\lambda+\lambda_1)(\lambda+\lambda_2)-\lambda^2]\gamma e^{r_0(T-t)}}\wedge Z_1, & \textbf{Case 3},\end{cases}$$

$$g_2^*(t,Z_2)=\begin{cases}\frac{(\lambda+\lambda_1)[(\lambda+\lambda_2)\eta_2-\lambda\eta_1]}{[(\lambda+\lambda_1)(\lambda+\lambda_2)-\lambda^2]\gamma e^{r_0(T-t)}}\wedge Z_2, & \textbf{Case 1},\\ \frac{(\lambda+\lambda_1)[(\lambda+\lambda_2)\eta_2-\lambda\eta_1]}{[(\lambda+\lambda_1)(\lambda+\lambda_2)-\lambda^2]\gamma e^{r_0(T-t)}}\wedge Z_2, & \textbf{Case 2},\\ 0, & \textbf{Case 3}.\end{cases}$$

Remark 4.5. Under the expected value premium principle, the excess-of-loss reinsurance is optimal. The equilibrium excess-of-loss retention limits depend on the intensity parameters $\lambda_1,\ \lambda_2,\lambda$ of the Poisson processes, the reinsurer's safety loadings $\eta_1,\ \eta_2$, the risk aversion coefficient γ of the insurer, and the interest rate r_0 of the risk-free asset.

Remark 4.6. The equilibrium investment strategies under the two different reinsurance premium principles are the same, depending on the interest rate r_0 of the risk-free asset, the appreciation rate r_1, the volatility coefficient σ of the risky asset, and the risk aversion coefficient γ of the insurer.

The equilibrium investment strategy is independent of the parameters λ_1, λ_2, λ, μ_{1Z_1}, μ_{1Z_2}, μ_{2Z_1}, μ_{2Z_2} in the reserve process, and the safety loadings η_1 and η_2 of the reinsurance business. This shows that the equilibrium investment strategy is unaffected by the price of reinsurance and the price of the primary insurance.

5. Numerical illustrations

In this section, we present some numerical examples to illustrate our results obtained in Sections 3 and 4. Without loss of generality, we assume that all the parameters of the financial market and the insurer are constants.

In Sections 3 and 4, we have derived the explicit expressions of the equilibrium reinsurance-investment strategies under both variance premium principle and expected value premium principle given in Theorems 3.3 and 4.3, respectively. In the following, we provide the numerical illustrations of the equilibrium reinsurance-investment strategies under the two premium principles according to Theorems 3.3 and 4.3. We use MATLAB to obtain the following numerical results.

5.1. The equilibrium strategy in Section 3

The conclusion in Theorem 3.3 shows that the optimal proportions are increasing with respect to the safety loading Λ of the reinsurance company, and decreasing with respect to the interest rate r_0 of the risk-free asset and the risk aversion coefficient γ of the insurer. Moreover, the equilibrium investment strategy is increasing with respect to the appreciation rate r_1 of the risky asset, and decreasing with respect to the interest rate r_0 of the risk-free asset, the volatility coefficient σ of the risky asset and the risk aversion coefficient γ of the insurer.

Set $T=4$, $r_0=0.02$, $r_1=0.05$, $\sigma=0.2$, $\Lambda=2$, $\gamma=0.5$. The equilibrium reinsurance-investment strategy is shown in Figure 1.

5.2. The equilibrium strategy in Section 4

Theorem 4.3 give the expression of the candidates of equilibrium reinsurance strategy $(\hat{g}_1^*(t),\ \hat{g}_2^*(t))$. $(\hat{g}_1^*(t),\ \hat{g}_2^*(t))$ is decreasing with respect to the interest rate r_0 of the risk-free asset and the risk aversion coefficient γ of the insurer. Next we focus on how $(\hat{g}_1^*(t),\ \hat{g}_2^*(t))$ changes with λ, η_1 and η_2. Set $T=4$, $r_0=0.02$, $\gamma=0.5$, $\lambda_1=2$, $\lambda_2=3$.

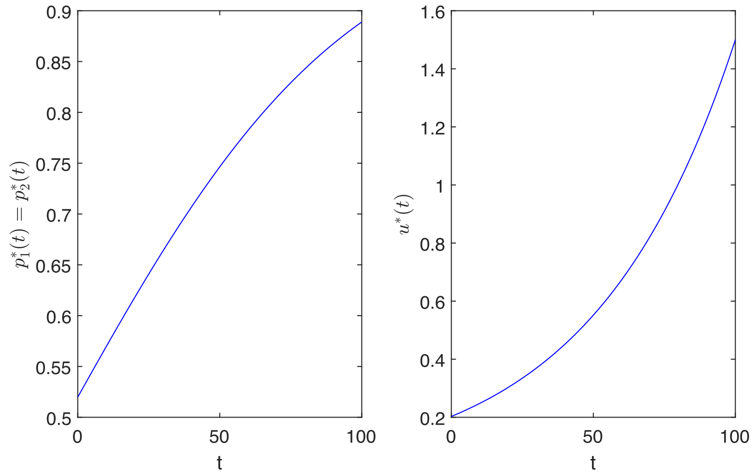

FIGURE 1. $p_1^*(t),\ p_2^*(t)$ and $u^*(t)$ in Section 3.

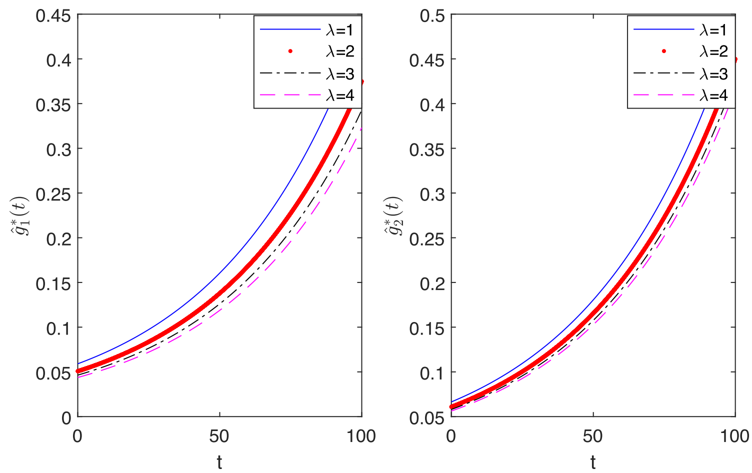

FIGURE 2. $\hat{g}_1^*(t),\ \hat{g}_2^*(t)$ with $\eta_1=\eta_2=0.3$ under different λ in Section 4.

First, we show $(\hat{g}_1^*(t),\ \hat{g}_2^*(t))$ under expected value premium principle with $\eta_1 = \eta_2 = 0.3$ for different common Poisson intensity $\lambda\ (=1,\ 2,\ 3,\ 4)$ in Figure 2. It can be seen from Figure 2 that both $\hat{g}_1^*(t)$ and $\hat{g}_2^*(t)$ are decreasing as λ increases. This is due to that a large value of λ means a riskier insurance market, so the insurer will reserve less risk share of the claims.

Second, we show $(\hat{g}_1^*(t),\ \hat{g}_2^*(t))$ under expected value premium principle with $\lambda = 2$ and $\eta_2 = 0.3$ for different $\eta_1\ (=0.1,\ 0.5,\ 0.8)$ in Figure 3. It can be seen from Figure 3 that $\hat{g}_1^*(t)$ is increasing as η_1 increases. In other words, the retained claim increases as the reinsurance safety loading increases. It is reasonable because as the reinsurance policy becomes more expensive, the insurer retains more insurance risk. On the other hand, $\hat{g}_2^*(t)$ is decreasing as η_1 increases.

Third, we show $(\hat{g}_1^*(t),\ \hat{g}_2^*(t))$ under expected value premium principle with $\lambda = 2$ and $\eta_1 = 0.3$ for different $\eta_2\ (=0.1,\ 0.4,\ 0.7)$ in Figure 4. It can be seen from Figure 4 that $\hat{g}_1^*(t)$ is decreasing as η_2 increases and $\hat{g}_2^*(t)$ is increasing as η_2 increases.

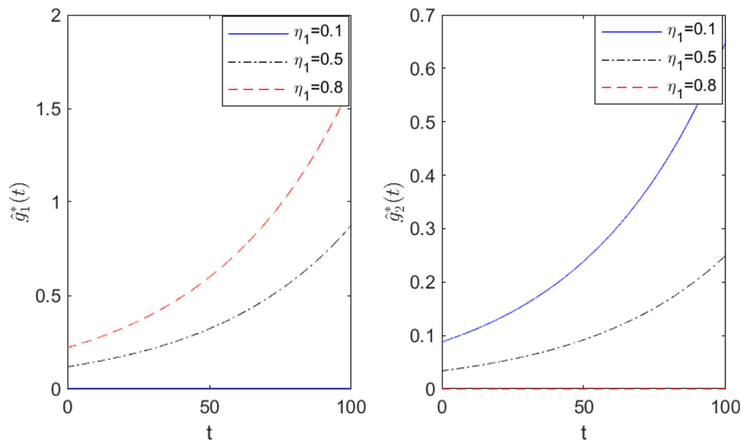

FIGURE 3. $\hat{g}_1^*(t),\ \hat{g}_2^*(t)$ with $\lambda = 2$, $\eta_2 = 0.3$ under different η_1 in Section 4.

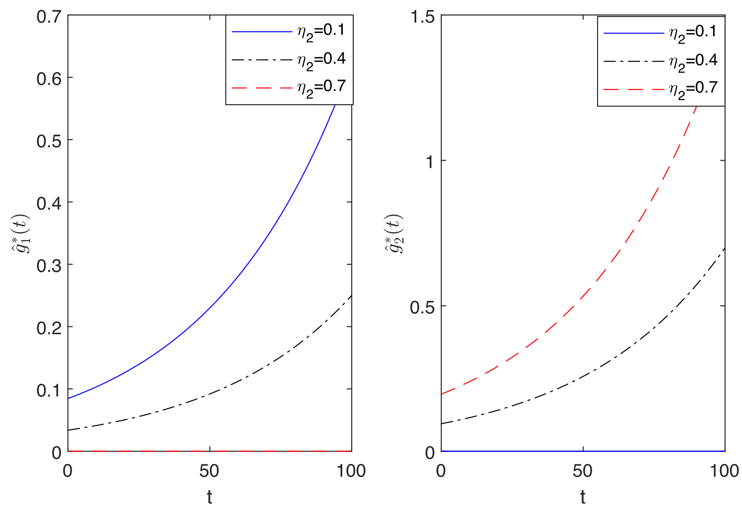

FIGURE 4. $\hat{g}_1^*(t),\ \hat{g}_2^*(t)$ with $\lambda = 2$, $\eta_1 = 0.3$ under different η_2 in Section 4.

6. Concluding remarks

This paper studies the equilibrium reinsurance-investment strategies for a mean-variance insurer with common shock dependence under two kinds of premium principles: the variance premium principle and the expected value premium principle. Using the technique of stochastic control theory and the corresponding extended HJB system of equations, within a game theoretic framework, we derive the closed-form expressions of the equilibrium reinsurance-investment strategies and the equilibrium value functions. We find that: under the variance premium principle, the optimal reinsurance contract is a proportional reinsurance; on the other hand, under the expected value premium principle, the optimal reinsurance contract is not a proportional reinsurance but an excess-of-loss reinsurance.

It would be interesting to extend our analysis to some other situations. In recent years, socially responsible investing has become a popular subject with both private and institutional investors. We can propose a modification allowing to incorporate not only the risk-free and risky assets but also a social responsibility measure into the investment decision making process, and reconsider the mean-variance reinsurance-investment problem. Of course, these problems are more complicated. To solve such problems, we need to adopt much more sophisticated techniques. We will explore these problems in the future research.

Acknowledgements. The work is supported by National Natural Science Foundation of China (Grant Nos. 11871220, 11801179, 11901201, 11871219, 11971172, 12071147, 71931004), the "Chenguang Program" supported by Shanghai Education Development Foundation and Shanghai Municipal Education Commission (No. 18CG26), and 111 Project (B14019).

References

[1] L. Bai and H. Zhang, Dynamic mean-variance problem with constrained risk control for the insurers. *Math. Methods. Oper. Res.* **68** (2008) 181–205.

[2] L. Bai, J. Cai and M. Zhou, Optimal reinsurance policies for an insurer with a bivariate reserve risk process in a dynamic setting. *Insur. Math. Econ.* **53** (2013) 664–670.

[3] N. Bäuerle, Benchmark and mean-variance problems for insurers. *Math. Methods. Oper. Res.* **62** (2005) 159–165.

[4] J. Bi and J. Cai, Optimal investment-reinsurance strategies with state dependent risk aversion and VaR constraints in correlated markets. *Insur. Math. Econ.* **85** (2019) 1–14.

[5] J. Bi, Z. Liang and F. Xu, Optimal mean-variance investment and reinsurance problems for the risk model with common shock dependence. *Insur. Math. Econ.* **70** (2016) 245–258.

[6] T. Björk and A. Murgoci, A general theory of Markovian time inconsistent stochastic control problems. Working Paper (2010).

[7] T. Björk and A. Murgoci, A theory of Markovian time-inconsistent stochastic control in discrete time. *Finan. Stoch.* **18** (2014) 545–592.

[8] T. Björk, A. Murgoci and X. Zhou, Mean-variance portfolio optimization with state-dependent risk aversion. *Math. Finan.* **24** (2014) 1–24.

[9] T. Björk, M. Khapko and A. Murgoci, On time-inconsistent stochastic control in continuous time. *Finan. Stoch.* **21** (2017) 331–360.

[10] S. Browne, Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. *Math. Oper. Res.* **20** (1995) 937–957.

[11] J. Grandell, Aspects of Risk Theory. Springer-Verlag, New York (1991).

[12] A. Gu, F.G. Viens and B. Yi, Optimal reinsurance and investment strategies for insurers with mispricing and model ambiguity. *Insur. Math. Econ.* **72** (2017) 235–249.

[13] C. Hipp and M. Taksar, Optimal non-proportional reinsurance control. *Insur. Math. Econ.* **47** (2010) 246–254.

[14] M. Kaluszka, Optimal reinsurance under mean-variance premium principles. *Insur. Math. Econ.* **28** (2001) 61–67.

[15] Y. Li and Z. Li, Optimal time-consistent investment and reinsurance strategies for mean-variance insurers with state dependent risk aversion. *Insur. Math. Econ.* **53** (2013) 86–97.

[16] D. Li, D. Li and V.R. Young, Optimality of excess-loss reinsurance under a mean-variance criterion. *Insur. Math. Econ.* **75** (2017) 82–89.

[17] Z. Liang and K.C. Yuen, Optimal dynamic reinsurance with dependent risks: variance premium principle. *Scand. Actuar. J.* **2016** (2016) 18–36.

[18] Z. Liang, K.C. Yuen and C. Zhang, Optimal reinsurance and investment in a jump-diffusion financial market with common shock dependence. *J. Appl. Math. Comput.* **56** (2018) 637–664.

[19] J. Liu and Z. Chen, Time consistent multi-period robust risk measures and portfolio selection models with regime-switching. *Eur. J. Oper. Res.* **268** (2018) 373–385.

[20] B. Liu, H. Meng and M. Zhou, Optimal investment and reinsurance policies for an insurer with ambiguity aversion. *N. Am. J. Econ. Finan.* **55** (2021) 101303.

[21] X. Peng, F. Chen and W. Wang, Robust optimal investment and reinsurance for an insurer with inside information. *Insur. Math. Econ.* **96** (2021) 15–30.

[22] D. Promislow and V.R. Young, Minimizing the probability of ruin when claims follow Brownian motion with drift. *N. Am. Actuar. J.* **9** (2005) 110–128.

[23] H. Schmidli, On minimizing the ruin probability by investment and reinsurance. *Ann. Appl. Probab.* **12** (2002) 890–907.

[24] J. Wei and T. Wang, Time-consistent mean-variance asset-liability management with random coefficients. *Insur. Math. Econ.* **77** (2017) 84–96.

[25] H. Wu and Y. Zeng, Equilibrium investment strategy for defined-contribution pension schemes with generalized mean-variance criterion and mortality risk. *Insur. Math. Econ.* **64** (2015) 396–408.

[26] L. Xu, L. Zhang and D. Yao, Optimal investment and reinsurance for an insurer under Markov-modulated financial market. *Insur. Math. Econ.* **74** (2017) 7–19.

[27] P. Yang, Z. Chen and X. Cui, Equilibrium reinsurance strategies for n insurers under a unified competition and cooperation framework. *Scand. Actuar. J.* **2021** (2021) 969–997.

[28] K.C. Yuen, J. Guo and X. Wu, On a correlated aggregate claim model with Poisson and Erlang risk process. *Insur. Math. Econ.* **31** (2002) 205–214.
[29] K.C. Yuen, J. Guo and X. Wu, On the first time of ruin in the bivariate compound Poisson model. *Insur. Math. Econ.* **38** (2006) 298–308.
[30] K.C. Yuen, Z. Liang and M. Zhou, Optimal proportional reinsurance with common shock dependence. *Insur. Math. Econ.* **64** (2015) 1–13.
[31] Y. Zeng and Z. Li, Optimal time-consistent investment and reinsurance policies for mean-variance insurers. *Insur. Math. Econ.* **49** (2011) 145–154.
[32] X. Zeng and S. Luo, Stochastic pareto-optimal reinsurance policies. *Insur. Math. Econ.* **53** (2013) 671–677.
[33] X. Zhang, H. Meng and Y. Zeng, Optimal investment and reinsurance strategies for insurers with generalized mean-variance premium principle and no-short selling. *Insur. Math. Econ.* **67** (2016) 125–132.
[34] C. Zhang and Z. Liang, Portfolio optimization for jump-diffusion risky assets with common shock dependence and state dependent risk aversion. *Optim. Contr. Appl. Met.* **38** (2017) 229–246.