RAIRO-Oper. Res. 55 (2021) 3715–3742
https://doi.org/10.1051/ro/2021174

RAIRO Operations Research
www.rairo-ro.org

DESIGNING A SINGLE-VENDOR AND MULTIPLE-BUYERS' INTEGRATED PRODUCTION INVENTORY MODEL FOR INTERVAL TYPE-2 FUZZY DEMAND AND FUZZY RULE BASED DETERIORATION

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Abstract. In this paper, a single-vendor and multiple-buyers' integrated production inventory model is investigated where demand of the item at the buyers' location is considered as interval type-2 fuzzy number (IT2FN). Deterioration rate of the item is assumed to change in accordance with the weather conditions of a particular region. It relies upon the values of certain attributes that have a direct influence on the extent of deterioration. These parameter values are easily forecasted and thereby can be utilized to determine the item depletion rate, which is executed here using Mamdani fuzzy inference scheme. Besides, a nearest interval approximation formula for the defuzzification of IT2FN is developed and applied in the proposed integrated production inventory model. The model optimizes the total number of shipments to be made to the buyers within a complete cycle so as to minimize the overall integrated cost incurred. A detailed illustration of the theoretical results is further demonstrated with the help of numerical example, followed by sensitivity analysis which provides insights into better decision making.

Mathematics Subject Classification. 90B06.

Received March 26, 2021. Accepted November 21, 2021.

1. Introduction

Inventory modelling and supply chain management constitute the key issues in logistics system planning and are among the most developed fields of operations management. An interesting and widely studied aspect of inventory theory includes the mathematical modelling of deteriorating items. Different patterns of deterioration rates have been widely suggested by researchers till date, which include constant, time-varying, probabilistic, fuzzy, non-instantaneous, etc. Some notable contributions in this regard are discussed in Section 2. It is observed in practical situations that most of the items have a deterioration rate which does not remain fixed in all circumstances; instead, items get depleted at different rates depending upon how the weather conditions of a particular region are or how good are the storage facilities for the items. So, it would be more practical to

Keywords. Interval type-2 fuzzy number, nearest interval approximation, vendor–buyer coordination, production inventory, Fuzzy rule based deterioration.

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assume that the same item deteriorates at different rates when stocked in places having variations in their weather conditions. In order to handle similar scenario, fuzzy rule base technique is implemented in this study. Such a situation of fuzzy rule based deterioration could not be found in the literature till date.

Demand is known to be one of the major parameters in inventory modelling which depends on various uncertain and unreliable activities of market as well as past records [13]. Thus, there can arise situations where consideration of constant demand, time-varying demand or even fuzzy demand with crisp membership grade will not be suitable at all. For instance, suppose the demand of a certain item is to be estimated by a group of experts. After analyzing the previous demand patterns and predicting forthcoming scenarios, each expert individually suggests a fuzzy demand with a certain grade of membership. This is because, considering all the experts' opinion, the membership grade of a particular demand value also turns fuzzy, so that the demand is finally estimated as a type-2 fuzzy set (T2FS). The same is examined in the proposed model. Rout *et al.* [45] investigated the possibility of occurrence of type-2 fuzziness in inventory parameters. Specifically, the authors dealt with discrete type-2 fuzzy deterioration rate in their proposed inventory model. In this paper, we intend to develop an integrated production inventory model for a single vendor and multiple buyers focusing on two new ideas, namely, interval type-2 fuzzy demand and weather (of a specific location) dependent deterioration rate. This varying deterioration rate at different locations is handled using fuzzy rule base approach in order to derive its specific value. Another novel aspect of the paper lies in the development of a methodology for the defuzzification of IT2FN and thereby incorporating it in the proposed inventory model for defuzzification of interval type-2 fuzzy demand rate. A brief overview of related existing studies in the literature is carried out in the next section.

The remainder of the paper is structured as follows: A comprehensive review on the literature of item deterioration, integrated supply chain network and implementation of Fuzzy set theory (FST) in inventory models is presented in Section 2. The proposed methodology of nearest interval approximation of IT2FN is discussed in Section 3. Section 4 provides a situation description for the model with a detailed overview of the notations and assumptions followed throughout the paper. The adopted policy is mathematically formulated in Section 5. In Section 6, the model is developed in the fuzzy environment with a comprehensive discussion on the procedure of determining the demand and deterioration rates. Section 7 exemplifies the developed theory with the help of numerical experiments, followed by sensitivity analysis of some key parameters which is presented in Section 8. Finally, some concluding remarks are drawn in Section 9, thereby identifying certain areas for future research.

2. Literature review

Deterioration in inventory modelling was first examined by Ghare [19] in the form of an exponentially decaying inventory. After that, many authors extrapolated Ghare and Schrader's work presenting both Economic Order Quantity (EOQ) and Economic Production Quantity (EPQ) models for more complex scenarios. A constant rate of deterioration has been considered in the research works of Widyadana *et al.* [62], Chan *et al.* [3], Pando *et al.* [43] and Rout *et al.* [48]. Skouri *et al.* [56] presented ramp-type demand rate with time-dependent rate of deterioration. The idea of non-instantaneous deterioration was adopted by Wu *et al.* [63], Ouyang *et al.* [41] and Sharma *et al.* [54]. Lee and Dye [30] presented an inventory model with stock-dependent demand and controllable deterioration rate. Mohanty *et al.* [37] discussed a two-warehouse inventory model for non-instantaneously deteriorating items with partial backlogging over a stochastic planning horizon. Tai *et al.* [57] developed an inventory system with deterioration rate depending upon the maximum lifetime of items. A joint pricing, replenishment and preservation technology investment problem was studied by Li *et al.* [32] for non-instantaneous deteriorating items. Sarkar *et al.* [50] investigated a profit maximization model considering selling-price and credit-period dependent demand and time-varying deterioration rate for the concerned products.

Inventory modelling incorporating vendor–buyer integrated approach has gained remarkable attention in the recent decades. Yang and Wee [65] noticed that collaborative approach of both the vendor and the buyers can further minimize the overall integrated cost in comparison to the independent approach by either of the two.

Other relevant works in this direction include those of Rau *et al.* [44], Yao and Chiou [66], Lo *et al.* [34], Yan *et al.* [64], Taleizadeh *et al.* [58], Jia *et al.* [25], Mohanty *et al.* [38] and Sarkar *et al.* [51]. A single-manufacturer and single-buyer production model was developed by Kumar *et al.* [29] under fuzzy random demand of customers. Recently, Chen [5] developed an EPQ model for deteriorating items comprising of a single manufacturer and multiple retailers. Recently, a sustainable single-vendor single-buyer production model was investigated by Rout *et al.* [46] incorporating emission regulation strategies. Pal *et al.* [42] studied an imperfect production inventory model consisting of a manufacturer and a retailer for deteriorating items, where the deterioration occurs at different rates in the manufacturer's and the retailer's level considering a fixed lifetime of the product. Some recent notable contributions in this regard include the works of Dey *et al.* [15] and Khanna *et al.* [27] which efficiently deal with the integrated approach of vendor and buyers.

In recent years, incorporation of fuzzy sets and its variants, such as intuitionistic fuzzy set, fuzzy random variable, random fuzzy variable, etc., has been widely carried out in inventory modelling problems. A detailed literature survey focusing on "fuzzy inventory modelling" was carried out by Shekarian *et al.* [55]. Inventory models considering fuzzy parameters have been extensively studied by a large number of researchers till date. Fuzzy rate of deterioration was established in the works of De and Goswami [11], De *et al.* [12], among others. Dutta *et al.* [17, 18], Chang *et al.* [4], Dey and Chakraborty [13, 14], Kumar and Goswami [28], Kumar *et al.* [29] and Chakraborty and Bhuiya [1] are some milestones in the literature addressing inventory models in fuzzy random environment. Among the most recent studies, Rout *et al.* [46] demonstrated scenario-dependent demand pattern based on historical records which is achieved using Mamdani fuzzy inference scheme. However, it is observed that there is hardly any work done considering type-2 fuzzy demand rate which can also be the scenario in certain situations, as discussed in this paper.

FST has always been beneficial in modelling and transforming imprecise information effectively. However, sometimes it is required to approximate a given fuzzy set by a crisp quantity. Recently, Rout *et al.* [45] proposed a production inventory model for items with type-2 fuzzy deterioration rate. A complete review of the available research works related to T2FS defuzzification techniques can be obtained in Torshizi *et al.* [60]. Some notable contributions in this aspect are presented in Section 3. In the literature, numerous studies are carried out with constant, ramp type, random and fuzzy demand rates but type-2 fuzzy demand has not been implemented in inventory problems as such. So, interval type-2 fuzzy demand rate is incorporated in the proposed model that tends to fill this research gap in literature.

The purpose of this study is twofold: first one is to develop an integrated production inventory model of a single vendor and multiple buyers by considering demand rate as IT2FN. In this process, a novel method of defuzzification of IT2FN is proposed which approximates it directly to a crisp interval without any intermediate type reduction phase. The second objective is to model a real life situation of weather-dependent deterioration in a supply chain network. Deterioration rates of certain items like volatile liquids, iron products, etc., usually depend upon weather conditions of the location, the preserving facilities where these items are stored and several other parameters. These uncertain components perturb the deterioration situation. Hence, fuzzy rule base technique is employed to forecast the rate of deterioration. To the best of our knowledge, such an inventory model with the aforementioned assumptions could not be found in the literature.

3. Proposed methodology: nearest interval approximation of IT2FN

In this section, we will discuss a novel defuzzification approach of IT2FN just after briefly reviewing the existing methods of others. For the defuzzification of fuzzy numbers, Grzegorzewski [23] derived an interval approximation operator with respect to a distance measure between fuzzy numbers. For type-2 fuzzy numbers (T2FNs), Karnik and Mendel [26] introduced the centroid method of defuzzification through the intermediate phase of type reduction. The traditional defuzzification methods, which include Karnik and Mendel [26] and Nie and Tan [40] algorithms and the sampling method of defuzzification by Greenfield *et al.* [22], involve quite a high computational complexity as far as the centroid calculation is concerned. Coupland and John [10] suggested a fast geometric method in order to defuzzify T2FSs. The collapsing method of defuzzification of discretized interval type-2 fuzzy sets (IT2FSs) was developed by Greenfield *et al.* [21]. Signed distance method of type-1 fuzzy set (T1FS) is extended for IT2FN by Chen *et al.* [7]. Torshizi and Zarandi [59] developed a direct defuzzification method for general T2FSs, based on collapsing procedure and α-plane decomposition. Runkler *et al.* [49] suggested some mathematical properties of type reduction, and proposed two methods of type reduction of IT2FN, namely, consistent linear type reduction (CLTR) and consistent quadratic type reduction (CQTR). Greenfield and Chiclana [20] proposed type reduction of continuous IT2FN by introducing the concepts of truncation and truncation grade. Moreno *et al.* [39] developed a defuzzification methodology for IT2FN, based on descriptive statistics and granular computing theory. With the purpose of reducing the computational complexity involved in the process, Nie and Tan [40] developed a type-reduction operator having a simple closed-form representation, computing the average of the upper and lower bounds of the footprint of uncertainty. In 2017, Li *et al.* [31] proved that the Nie-Tan operator is actually an accurate method for defuzzifying IT2FSs. However, using a defuzzification operator which replaces a T2FS by a single crisp number might generally result in the loss of certain important information. Therefore, a crisp set approximation of a fuzzy set is often advisable [23]. In this approach, we substitute a given IT2FN by a crisp interval, which is in some sense close to the former one.

Type-reduction is considered to be a defuzzification bottleneck, the reason being the computational complexity involved in the process. However, the proposed methodology reduces an IT2FN directly into a crisp interval, instead of a single value, through alpha-cut computations. Unlike the iterative algorithms present in the literature, it develops closed-form formulae for computing the end-points of the interval so that it does not require any centroid calculation for an extraordinarily large number of T1FSs (embedded sets), and that also without discretization of the continuous domains. The proposed methodology does not involve any intermediate type reduction phase and is therefore comparatively much less laborious for handling continuous T2FS. The method thus put forward is illustrated in the present section with the validity of the same. Throughout this paper, tilde "$\sim$" and double tilde "$\approx$" over an alphabet represent a T1FS and T2FS respectively [6].

Our aim to find the nearest interval approximation requires the distance between the fuzzy number and the corresponding interval to be minimum, which is achieved through the computations that follow henceforth. Let $\tilde{\tilde{A}} = \left(\tilde{A}^L, \tilde{A}^U\right)$ be an IT2FN where $\tilde{A}^L$ and $\tilde{A}^U$ represent the Lower Membership Function (LMF) and Upper Membership Function (UMF) with heights $h\left(\tilde{A}^L\right)$ and $h\left(\tilde{A}^U\right)$ respectively [35]. The α-cut of $\tilde{\tilde{A}}$, where $\alpha \in [0,1]$ is given by $A_\alpha = \left(A_\alpha^L, A_\alpha^U\right) = \left(\left[{}^lA_\alpha^L, {}^rA_\alpha^L\right], \left[{}^lA_\alpha^U, {}^rA_\alpha^U\right]\right)$. Following the idea of Grzegorzewski [23], we define the distance metric d between $\tilde{\tilde{A}}$ and a closed interval $C_d\left(\tilde{\tilde{A}}\right) = [C_L, C_R]$ as

$$d\left(\tilde{\tilde{A}}, C_d\left(\tilde{\tilde{A}}\right)\right) = \left[\int_0^{h\left(\tilde{A}^L\right)} \left(C_L - {}^lA_\alpha^L\right)^2 \mathrm{d}\alpha + \int_0^{h\left(\tilde{A}^L\right)} \left(C_L - {}^lA_\alpha^U\right)^2 \mathrm{d}\alpha + \int_0^{h\left(\tilde{A}^L\right)} \left(C_R - {}^rA_\alpha^L\right)^2 \mathrm{d}\alpha \right.$$
$$\left. + \int_0^{h\left(\tilde{A}^L\right)} \left(C_R - {}^rA_\alpha^U\right)^2 \mathrm{d}\alpha + \int_{h\left(\tilde{A}^L\right)}^{h\left(\tilde{A}^U\right)} \left(C_L - {}^lA_\alpha^U\right)^2 \mathrm{d}\alpha + \int_{h\left(\tilde{A}^L\right)}^{h\left(\tilde{A}^U\right)} \left(C_R - {}^rA_\alpha^U\right)^2 \mathrm{d}\alpha \right]^{1/2}. \quad (3.1)$$

Given $\tilde{\tilde{A}}$, the objective is to find its nearest closed interval $C_d\left(\tilde{\tilde{A}}\right)$ with respect to the metric d. It requires to minimize $d\left(\tilde{\tilde{A}}, C_d\left(\tilde{\tilde{A}}\right)\right)$ for which it would be sufficient to minimize $D(C_L, C_R) = d^2\left(\tilde{\tilde{A}}, C_d\left(\tilde{\tilde{A}}\right)\right)$ given by:

$$D(C_L, C_R) = \int_0^{h\left(\tilde{A}^L\right)} \left(C_L - {}^lA_\alpha^L\right)^2 \mathrm{d}\alpha + \int_0^{h\left(\tilde{A}^L\right)} \left(C_L - {}^lA_\alpha^U\right)^2 \mathrm{d}\alpha + \int_0^{h\left(\tilde{A}^L\right)} \left(C_R - {}^rA_\alpha^L\right)^2 \mathrm{d}\alpha$$
$$+ \int_0^{h\left(\tilde{A}^L\right)} \left(C_R - {}^rA_\alpha^U\right)^2 \mathrm{d}\alpha + \int_{h\left(\tilde{A}^L\right)}^{h\left(\tilde{A}^U\right)} \left(C_L - {}^lA_\alpha^U\right)^2 \mathrm{d}\alpha + \int_{h\left(\tilde{A}^L\right)}^{h\left(\tilde{A}^U\right)} \left(C_R - {}^rA_\alpha^U\right)^2 \mathrm{d}\alpha.$$

The first order partial derivatives of $D(C_L, C_R)$ obtained by the application of Leibniz integral rule are as follows:

$$\frac{\partial D(C_L, C_R)}{\partial C_L} = 2\int_0^{h(\tilde{A}^L)} \left(C_L - {}^l A^L_\alpha\right) \mathrm{d}\alpha + 2\int_0^{h(\tilde{A}^L)} \left(C_L - {}^l A^U_\alpha\right) \mathrm{d}\alpha + 2\int_{h(\tilde{A}^L)}^{h(\tilde{A}^U)} \left(C_L - {}^l A^U_\alpha\right) \mathrm{d}\alpha.$$

$$\frac{\partial D(C_L, C_R)}{\partial C_R} = 2\int_0^{h(\tilde{A}^L)} \left(C_R - {}^r A^L_\alpha\right) \mathrm{d}\alpha + 2\int_0^{h(\tilde{A}^L)} \left(C_R - {}^r A^U_\alpha\right) \mathrm{d}\alpha + 2\int_{h(\tilde{A}^L)}^{h(\tilde{A}^U)} \left(C_R - {}^r A^U_\alpha\right) \mathrm{d}\alpha.$$

The necessary conditions for the minimum to exist are given by $\frac{\partial D(C_L, C_R)}{\partial C_L} = 0$ and $\frac{\partial D(C_L, C_R)}{\partial C_R} = 0$ which imply

$$C_L = \frac{1}{h\left(\tilde{A}^L\right) + h\left(\tilde{A}^U\right)} \left[\int_0^{h(\tilde{A}^L)} {}^l A^L_\alpha \,\mathrm{d}\alpha + \int_0^{h(\tilde{A}^U)} {}^l A^U_\alpha \,\mathrm{d}\alpha\right] \text{ and} \tag{3.2}$$

$$C_R = \frac{1}{h\left(\tilde{A}^L\right) + h\left(\tilde{A}^U\right)} \left[\int_0^{h(\tilde{A}^L)} {}^r A^L_\alpha \,\mathrm{d}\alpha + \int_0^{h(\tilde{A}^U)} {}^r A^U_\alpha \,\mathrm{d}\alpha\right]. \tag{3.3}$$

Moreover,

$$\det\begin{bmatrix} \frac{\partial^2 D(C_L, C_R)}{\partial C_L^2} & \frac{\partial^2 D(C_L, C_R)}{\partial C_L \partial C_R} \\ \frac{\partial^2 D(C_L, C_R)}{\partial C_L \partial C_R} & \frac{\partial^2 D(C_L, C_R)}{\partial C_R^2} \end{bmatrix} = \det\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = 36 > 0$$

and

$$\frac{\partial^2 D(C_L, C_R)}{\partial C_L^2} = 6 > 0.$$

Therefore C_L and C_R as expressed in (3.2) and (3.3) actually minimize $D(C_L, C_R)$. In other words, they minimize $d\left(\tilde{\tilde{A}}, C_d\left(\tilde{\tilde{A}}\right)\right)$. So, the interval with minimum distance from $\tilde{\tilde{A}}$ is obtained as

$$C_d\left(\tilde{\tilde{A}}\right) = \frac{1}{h\left(\tilde{A}^L\right) + h\left(\tilde{A}^U\right)} \left[\int_0^{h(\tilde{A}^L)} {}^l A^L_\alpha \,\mathrm{d}\alpha + \int_0^{h(\tilde{A}^U)} {}^l A^U_\alpha \,\mathrm{d}\alpha, \int_0^{h(\tilde{A}^L)} {}^r A^L_\alpha \,\mathrm{d}\alpha + \int_0^{h(\tilde{A}^U)} {}^r A^U_\alpha \,\mathrm{d}\alpha\right]. \tag{3.4}$$

Now, it remains to prove that $C_d\left(\tilde{\tilde{A}}\right)$ is indeed the nearest interval approximation of $\tilde{\tilde{A}}$. Grzegorzewski [23] suggested certain criteria required to be fulfilled by an operator C to be an interval approximation of a fuzzy number $\tilde{A}$. These are summarized as follows:

(C1) $C\left(\tilde{A}\right) \subseteq \text{support}\left(\tilde{A}\right)$,

(C2) $\text{core}\left(\tilde{A}\right) \subseteq C\left(\tilde{A}\right)$,

(C3) C is a continuous interval approximation operator.

Based on the notion of fuzzy set as introduced by Dubois and Prade [16], the UMF of an IT2FN $\tilde{\tilde{A}}$ is represented by four numbers $a^U_1, a^U_2, a^U_3, a^U_4 \in \mathbb{R}$ and two functions $L_{A^U}, R_{A^U} : \mathbb{R} \to [0, 1]$, where $\mathbb{R}$ denotes the real line, L_{A^U} is non-decreasing and R_{A^U} is non-increasing, such that a membership function $\mu_{\tilde{A}^U}$ can be defined in the following manner:

$$\mu_{\tilde{A}^U}(x) = \begin{cases} 0, & \text{if } x < a_1^U \\ L_{A^U}(x), & \text{if } a_1^U \leq x < a_2^U \\ h\left(\tilde{A}^U\right), & \text{if } a_2^U \leq x \leq a_3^U \\ R_{A^U}(x), & \text{if } a_3^U < x \leq a_4^U \\ 0, & \text{if } a_4^U < x. \end{cases}$$

Functions L_{A^U} and R_{A^U} are called the left and right sides of the fuzzy number $\tilde{A}^U$ respectively. Similar arguments also hold for the LMF $\tilde{A}^L$.

Theorem 3.1. *Consider an IT2FN $\tilde{\tilde{A}}$ with continuous and strictly monotonic sides L_{A^U}, R_{A^U} and L_{A^L}, R_{A^L} for the UMF and LMF respectively and $[C_L, C_R]$ be its nearest interval approximation. Then, $[C_L, C_R] \subseteq \left[a_1^U, a_4^U\right]$.*

Proof. Using the well-known formulae of integration by substitution, derivative of the inverse function and integration by parts as suggested by Grzegorzewski [23], we have,

$$\begin{aligned} C_L &= \frac{1}{h\left(\tilde{A}^L\right) + h\left(\tilde{A}^U\right)} \left[\int_0^{h\left(\tilde{A}^L\right)} {}^{l}A_\alpha^L \,\mathrm{d}\alpha + \int_0^{h\left(\tilde{A}^U\right)} {}^{l}A_\alpha^U \,\mathrm{d}\alpha\right] \\ &= \frac{1}{h\left(\tilde{A}^L\right) + h\left(\tilde{A}^U\right)} \left[a_2^L - \int_{a_1^L}^{a_2^L} L_{A^L}(x)\,\mathrm{d}x + a_2^U - \int_{a_1^U}^{a_2^U} L_{A^U}(x)\,\mathrm{d}x\right] \\ &\geq \frac{a_1^L + a_1^U}{h\left(\tilde{A}^L\right) + h\left(\tilde{A}^U\right)}. \end{aligned}$$

The inequality follows from the fact that $L_{A^L}(x) \leq 1$ for all x and that L_{A^L} is continuous so that $\int_{a_1^L}^{a_2^L} L_{A^L}(x)\,\mathrm{d}x \leq \int_{a_1^L}^{a_2^L} \mathrm{d}x$ holds. Moreover, we know that the height of UMF or LMF must not exceed 1. Hence, $h\left(\tilde{A}^L\right) \leq 1, h\left(\tilde{A}^U\right) \leq 1$ and $a_1^L \geq a_1^U$ is trivially true. Therefore,

$$C_L \geq a_1^U. \tag{3.5}$$

With similar arguments, it can be shown that

$$C_R \leq a_4^U. \tag{3.6}$$

Combining the results of (3.5) and (3.6), we can conclude that

$$[C_L, C_R] \subseteq \left[a_1^U, a_4^U\right].$$

This completes the proof. □

Theorem 3.2. *The operator $C_d : \mathbb{IF}_2(\mathbb{R}) \to \mathbb{P}(\mathbb{R})$ defined by (3.4) is a continuous interval approximation operator where $\mathbb{IF}_2(\mathbb{R})$ denotes the space of all IT2FNs and $\mathbb{P}(\mathbb{R})$ denotes the family of all closed intervals on the real line.*

Proof. It is required to prove that if two IT2FNs $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are close, then their interval approximations are also close which means the following condition must be satisfied:

for every $\epsilon > 0, \exists\delta > 0$ such that $d\left(\tilde{\tilde{A}}, \tilde{\tilde{B}}\right) < \delta \implies d\left(C_d\left(\tilde{\tilde{A}}\right), C_d\left(\tilde{\tilde{B}}\right)\right) < \epsilon$
(it is assumed that $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ have same corresponding heights for LMF and UMF).

Given $d\left(\tilde{\tilde{A}}, \tilde{\tilde{B}}\right) < \delta \implies d^2\left(\tilde{\tilde{A}}, \tilde{\tilde{B}}\right) < \delta^2$. So, we have,

$$\begin{aligned}
&\left[\int_0^{h(\tilde{A}^L)} \left({}^lA_\alpha^L - {}^lB_\alpha^L\right) \mathrm{d}\alpha\right]^2 + \left[\int_0^{h(\tilde{A}^L)} \left({}^lA_\alpha^U - {}^lB_\alpha^U\right) \mathrm{d}\alpha\right]^2 \\
&\quad + \left[\int_0^{h(\tilde{A}^L)} \left({}^rA_\alpha^L - {}^rB_\alpha^L\right) \mathrm{d}\alpha\right]^2 + \left[\int_0^{h(\tilde{A}^L)} \left({}^rA_\alpha^U - {}^rB_\alpha^U\right) \mathrm{d}\alpha\right]^2 \\
&\quad + \left[\int_{h(\tilde{A}^L)}^{h(\tilde{A}^U)} \left({}^lA_\alpha^U - {}^lB_\alpha^U\right) \mathrm{d}\alpha\right]^2 + \left[\int_{h(\tilde{A}^L)}^{h(\tilde{A}^U)} \left({}^rA_\alpha^U - {}^rB_\alpha^U\right) \mathrm{d}\alpha\right]^2 \\
&\quad \leq \int_0^{h(\tilde{A}^L)} \left({}^lA_\alpha^L - {}^lB_\alpha^L\right)^2 \mathrm{d}\alpha + \int_0^{h(\tilde{A}^L)} \left({}^lA_\alpha^U - {}^lB_\alpha^U\right)^2 \mathrm{d}\alpha + \int_0^{h(\tilde{A}^L)} \left({}^rA_\alpha^L - {}^rB_\alpha^L\right)^2 \mathrm{d}\alpha \\
&\quad\quad + \int_0^{h(\tilde{A}^L)} \left({}^rA_\alpha^U - {}^rB_\alpha^U\right)^2 \mathrm{d}\alpha + \int_{h(\tilde{A}^L)}^{h(\tilde{A}^U)} \left({}^lA_\alpha^U - {}^lB_\alpha^U\right)^2 \mathrm{d}\alpha + \int_{h(\tilde{A}^L)}^{h(\tilde{A}^U)} \left({}^rA_\alpha^U - {}^rB_\alpha^U\right)^2 \mathrm{d}\alpha \\
&\quad < \delta^2. \qquad (3.7)
\end{aligned}$$

Inequality (3.7) indicates that the sum of certain square terms is less than δ^2 which implies that each square term must be less than δ^2 *i.e.*,

$$\left[\int_0^{h(\tilde{A}^L)} \left({}^lA_\alpha^L - {}^lB_\alpha^L\right) \mathrm{d}\alpha\right]^2 < \delta^2 \implies -\delta < \int_0^{h(\tilde{A}^L)} \left({}^lA_\alpha^L - {}^lB_\alpha^L\right) \mathrm{d}\alpha < \delta \qquad (3.8)$$

and the same is true for the remaining five terms also. Using (3.4) and the results obtained above, the following can be established:

$$d^2(C_d(A), C_d(B)) = \int_0^1 [C_L(A) - C_L(B)]^2 \,\mathrm{d}\alpha + \int_0^1 [C_R(A) - C_R(B)]^2 \,\mathrm{d}\alpha$$

which means,

$$\begin{aligned}
d^2(C_d(A), C_d(B)) &= [C_L(A) - C_L(B)]^2 + [C_R(A) - C_R(B)]^2 \\
&\quad \text{[since the integrands are independent of } \alpha\text{]} \\
&< \frac{1}{\left[h\left(\tilde{A}^L\right) + h\left(\tilde{A}^U\right)\right]^2}\left[\delta^2 + 2\left(\delta^2 + \delta^2 + \delta^2 + \delta^2 + \delta^2 + \delta^2\right)\right] \\
&= \frac{13\delta^2}{\left[h\left(\tilde{A}^L\right) + h\left(\tilde{A}^U\right)\right]^2} \\
&= \epsilon^2 \quad \text{(say)}. \qquad (3.9)
\end{aligned}$$

Therefore, if $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are close enough, then it is proved that their nearest interval approximations obtained by operator C_d are also close *i.e.*, C_d is a continuous interval approximation operator. This completes the proof. □

Thus, it is evident from the results of Theorems 3.1 and 3.2 that the deduced formula (3.4) for $[C_L, C_R]$ is indeed the nearest interval approximation of $\tilde{\tilde{A}}$.

TABLE 1. Computational results.

Proposed operator		Karnik–Mendel	
C_L	C_R	c_l	c_r
3.7467	6.2533	3.5955	6.4045

Notes. c_l denotes the minimum of the centroids of embedded sets, c_r denotes the maximum of the centroids of embedded sets.

3.1. Comparison with the Karnik–Mendel algorithm [36]

Symmetric Gaussian Membership Functions with uncertain deviation:

$$\tilde{A}^L = \exp\left[-\frac{1}{2}\{4(x-5)\}^2\right], \qquad 0 \le x \le 10$$
$$\tilde{A}^U = \exp\left[-\frac{1}{2}\left\{\frac{4}{7}(x-5)\right\}^2\right], \quad 0 \le x \le 10.$$

Here, $h\left(\tilde{A}^L\right) = h\left(\tilde{A}^U\right) = 1$. For $\alpha \in [0,1]$, the α-cuts of $\tilde{A}^L$ and $\tilde{A}^U$ are $\left[5 - \frac{1}{2\sqrt{2}}\sqrt{\ln\left(\frac{1}{\alpha}\right)}, 5 + \frac{1}{2\sqrt{2}}\sqrt{\ln\left(\frac{1}{\alpha}\right)}\right]$ and $\left[5 - \frac{7}{2\sqrt{2}}\sqrt{\ln\left(\frac{1}{\alpha}\right)}, 5 + \frac{7}{2\sqrt{2}}\sqrt{\ln\left(\frac{1}{\alpha}\right)}\right]$ respectively so that the nearest interval approximation of $\tilde{\tilde{A}} = \left(\tilde{A}^L, \tilde{A}^U\right)$ is computed as

$$C_d = [C_L, C_R] = [3.7467, 6.2533].$$

The results are in good agreement with the well-known Karnik–Mendel approach as shown in Table 1. The mean value of the interval $[C_L, C_R]$ *i.e.*, $\frac{C_L + C_R}{2}$ accurately matches the center of the centroid *i.e.*, $\frac{c_l + c_r}{2}$.

4. MODEL FORMULATION

This section illustrates the proposed model, thereby presenting the notations, assumptions and problem description as follows:

4.1. Notations

Listed below are the terminologies followed throughout the paper. Some additional notations, wherever required, will be listed accordingly in the paper.

4.2. Situation description and assumptions

This paper investigates a supply chain model for a single vendor and multiple buyers, trading over an infinite planning horizon. Depending upon experts' prediction on the customers' demand pattern, buyers place their respective orders to the vendor. The latter procures raw materials from a supplier (who is not a part of this integrated supply chain) and manufactures the finished product which is then delivered to the buyers in multiple shipments. The item under consideration deteriorates both at the vendor and the buyers' warehouses with different rates, depending upon the temperature, humidity, etc., of the concerned location.

Following are some assumptions taken for the development of the model:

(1) Integrated production inventory model for a single vendor and multiple buyers is developed.
(2) Single type of item is taken into consideration.

Parameters	
Notation	Description
N	Number of buyers
D_i	Demand rate of the ith buyer (units per unit time), $i = 1, 2, \ldots, N$
P	Production rate (units per unit time) $\left(P > \sum_{i=1}^{N} D_i\right)$
β	Proportion of good quality items manufactured $(0 < \beta \leq 1)$
θ_v	Item deterioration rate per unit time at the vendor location $(0 < \theta_v < 1)$
θ_i	Item deterioration rate per unit time at the ith buyer location $(0 < \theta_i < 1)$, $i = 1, 2, \ldots, N$
c_v	Unit production cost for the vendor ($/unit)
c_b	Unit purchase price for the buyers ($/unit)
K_v	Production setup cost for the vendor ($/setup)
K_b	Ordering cost for the buyers ($/order)
h_v	Holding cost for the vendor ($/unit/unit time)
h_{bi}	Holding cost for the ith buyer ($/unit/unit time), $i = 1, 2, \ldots, N$
c_s	Scrapping cost ($/unit)
Decision variables	
Notation	Description
n_i	Number of deliveries to the ith buyer per cycle, a positive integer, $i = 1, 2, \ldots, N$
T	Cycle length
T_1	Production run time in cycle T
Other terminologies	
Notation	Description
$I_v(t)$	Inventory level for the vendor at time t
$I_{bi}(t)$	Inventory level for the ith buyer at time t, $i = 1, 2, \ldots, N$
I_{mv}	Maximum inventory level of the vendor
I_{mi}	Maximum inventory level of the ith buyer, $i = 1, 2, \ldots, N$
TC_1	Total integrated inventory cost per cycle ($)
TC	Total integrated inventory cost per unit time ($)

(3) Shortages are not allowed.
(4) Production rate is constant and the number of perfectly produced items is greater than the sum of the demands of all the buyers.
(5) Machine turns faulty after multiple uses, so certain imperfections in the produced items are considered which are instantly scrapped assuming that they are non-reworkable.
(6) Item deterioration rate varies from region to region depending upon the weather conditions.
(7) Deteriorated inventory is non-recoverable *i.e.*, there is no replacement or repair of deteriorated items [47].
(8) Experts provide their opinion regarding the demand rates at the buyers' locations. Based on the same, resulting demand patterns are visualized as IT2FNs.

5. Mathematical modelling

The proposed model is schematically illustrated in Figures 1 and 2 which respectively demonstrate the instantaneous inventory behaviours at the vendor and the buyers' locations over a complete cycle [65]. The

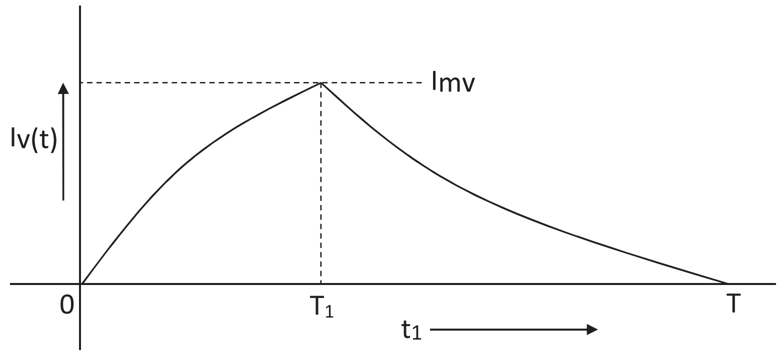


FIGURE 1. Inventory level for the vendor during a complete cycle.

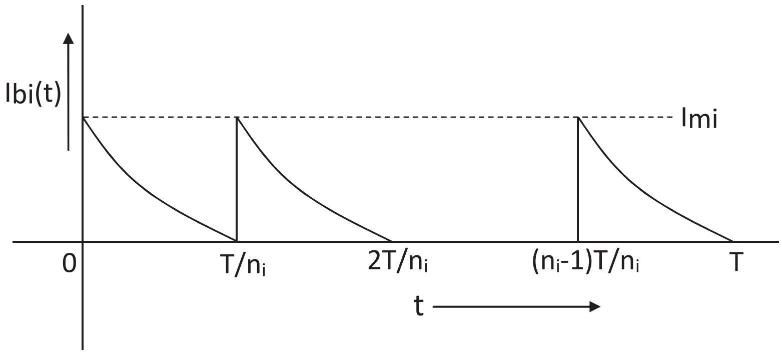


FIGURE 2. Inventory level for the ith buyer during a complete cycle.

elapsed time and the instantaneous level of inventory are respectively denoted by the horizontal and vertical axes.

The instantaneous states of the level of inventory are described in the differential equations that follow:

$$\frac{\mathrm{d}I_{v1}(t_1)}{\mathrm{d}t_1} = \beta P - \sum_{i=1}^{N} D_i - \theta_v I_{v1}(t_1), \quad 0 \le t_1 \le T_1, \quad I_{v1}(0) = 0. \tag{5.1}$$

$$\frac{\mathrm{d}I_{v2}(t_1)}{\mathrm{d}t_1} = -\sum_{i=1}^{N} D_i - \theta_v I_{v2}(t_1), \quad T_1 \le t_1 \le T, \quad I_{v2}(T) = 0. \tag{5.2}$$

$$\frac{\mathrm{d}I_{bi}(t)}{\mathrm{d}t} = -D_i - \theta_i I_{bi}(t), \quad 0 \le t \le T/n_i, \quad I_{bi}(T/n_i) = 0 \quad (i = 1, 2, \ldots, N). \tag{5.3}$$

Solutions to the corresponding differential equations are obtained as

$$I_{v1}(t_1) = \frac{\beta P - \sum_{i=1}^{N} D_i}{\theta_v}\left(1 - e^{-\theta_v t_1}\right), \quad 0 \le t_1 \le T_1. \tag{5.4}$$

$$I_{v2}(t_1) = \frac{\sum_{i=1}^{N} D_i}{\theta_v}\left[e^{\theta_v (T - t_1)} - 1\right], \quad T_1 \le t_1 \le T. \tag{5.5}$$

$$I_{bi}(t) = \frac{D_i}{\theta_i}\left[e^{\theta_i (T/n_i - t)} - 1\right], \quad 0 \le t \le T/n_i \quad (i = 1, 2, \ldots, N). \tag{5.6}$$

Deterioration rates being very small quantities, their second and higher powers can be neglected for ease of computation. Using the boundary conditions $I_{v2}(T_1) = I_{mv}$ and $I_{bi}(0) = I_{mi}$ in (5.5) and (5.6) respectively and applying the Taylor's series expansion as used by Widyadana and Wee [61], the following relations appear to hold:

$$I_{mv} = \sum_{i=1}^{N} D_i(T - T_1)\left[1 + \frac{\theta_v}{2}(T - T_1)\right] \text{ and} \tag{5.7}$$

$$I_{mi} = \frac{D_i T}{n_i}\left[1 + \frac{\theta_i T}{2n_i}\right], \quad (i = 1, 2, \ldots, N). \tag{5.8}$$

Following similar arguments, the continuity of inventory at T_1 *i.e.*, $I_{v1}(T_1) = I_{v2}(T_1)$ establishes the following relation:

$$T \approx \frac{T_1}{\sum_{i=1}^{N} D_i}\left[\frac{\theta_v}{2}T_1\sum_{i=1}^{N} D_i + \beta P\left(1 - \frac{\theta_v}{2}T_1\right)\right] \quad \text{(assuming } \theta_v, \theta_i \ll 1). \tag{5.9}$$

Our objective is to construct the overall integrated cost incurred by both the vendor and the buyers. Accordingly, the cost components related to various operations are separately listed below:

Costs incurred by the vendor per cycle:

$$\begin{aligned}
\text{Production cost} &= c_v P T_1.\\
\text{Setup cost} &= K_v.\\
\text{Scrapping cost} &= c_s(1-\beta)PT_1.\\
\text{Holding cost} &= h_v\left[\int_0^{T_l} I_{v1}(t_1)\mathrm{d}t_1 + \int_{T_1}^{T_2} I_{v2}(t_1)\mathrm{d}t_1 - \sum_{i=1}^{N} n_i \int_0^{T/n_i} I_{bi}(t)\mathrm{d}t\right]\\
&= h_v\left[\frac{\beta P - \sum_{i=1}^{N} D_i}{\theta_v}\int_0^{T_l}(1 - e^{-\theta_v t_1})\mathrm{d}t_1 + \frac{\sum_{i=1}^{N} D_i}{\theta_v}\int_{T_1}^{T_2}\left[e^{\theta_v(T-t_1)} - 1\right]\mathrm{d}t_1\right.\\
&\quad \left. - \sum_{i=1}^{N} n_i \frac{D_i}{\theta_i}\int_0^{T/n_i}\left[e^{\theta_i(T/n_i - t)} - 1\right]\mathrm{d}t\right].\\
\text{Deterioration cost} &= c_v\left[\beta P T_1 - \sum_{i=1}^{N} n_i I_{mi}\right].
\end{aligned}$$

Costs incurred by all the buyers per cycle:

$$\begin{aligned}
\text{Purchase price} &= c_b \sum_{i=1}^{N} n_i I_{mi}.\\
\text{Ordering cost (includes transportation cost)} &= K_b \sum_{i=1}^{N} n_i.\\
\text{Holding cost} &= \sum_{i=1}^{N} h_{bi} n_i \int_0^{T/n_i} I_{bi}(t)\mathrm{d}t
\end{aligned}$$

$$= \sum_{i=1}^{N} h_{bi} n_i \frac{D_i}{\theta_i} \int_0^{T/n_i} \left[e^{\theta_i (T/n_i - t)} - 1 \right] \mathrm{d}t.$$

$$\text{Deterioration cost} = c_b \sum_{i=1}^{N} n_i \left(I_{mi} - D_i \frac{T}{n_i} \right) \cdot$$

Summing up all the components and simplifying the same, the overall integrated inventory cost per cycle can be obtained as a function of T_1 (production run time) and n_i (number of deliveries to the ith buyer) as given below:

$$\begin{aligned} \mathrm{TC}_1(n_i, T_1) = {} & c_v(1+\beta)PT_1 + (c_b - c_v)T \sum_{i=1}^{N} D_i \left(1 + \frac{\theta_i T}{2n_i}\right) + c_b \frac{T^2}{2} \sum_{i=1}^{N} D_i \frac{\theta_i}{n_i} \\ & + K_v + K_b \sum_{i=1}^{N} n_i + c_s(1-\beta)PT_1 + \frac{T^2}{2} \sum_{i=1}^{N} (h_{bi} - h_v) \frac{D_i}{n_i} \left(1 + \frac{\theta_i T}{3n_i}\right) \\ & + h_v \frac{\beta P T_1^2}{2} \left(1 - \frac{\theta T_1}{3}\right) + h_v \frac{T^2}{2} \left\{1 + \frac{\theta}{3}(T - T_1)\right\} \sum_{i=1}^{N} D_i + h_v \frac{\theta}{6} \\ & \times T_1^2 T \sum_{i=1}^{N} D_i - h_v T T_1 \left(1 + \frac{\theta}{3}(T - T_1)\right) \sum_{i=1}^{N} D_i. \end{aligned} \tag{5.10}$$

6. Model in fuzzy environment

In this section, the mathematical model derived in Section 5 is extended to fuzzy environment by considering buyers' demand patterns as IT2FNs and through the application of fuzzy rule base technique to forecast the deterioration rate. As discussed in the introduction section, deterioration rate depends upon several attributes such as temperature, humidity and amount of rainfall of a region, so that the item depletion rates at the vendor's and buyers' locations can be determined with the application of suitable fuzzy rule base scheme.

Besides, when the buyers plan to place their orders, they may not know exactly the upcoming demand of customers. They would depend on the past experiences or data sets available regarding buyers' ordering behaviour. Such uncertain and vague information encourages an expert to suggest fuzzy demand with certain grade of membership. The fuzzy opinions may vary from expert to expert so that the membership function itself turns fuzzy. In such a situation, the resultant demand pattern is modelled as an IT2FN represented by $\tilde{\tilde{D}}_i$ (say). Similar scenario is taken into consideration in this paper. Therefore, the demand and deterioration rates are the fuzzy parameters in the model. The procedures to compute their values from the available data are elaborately discussed in the following two subsections.

6.1. Determination of deterioration rates

Records regarding certain attributes, namely, temperature, pressure, humidity, precipitation, etc., of a concerned location are readily available, which are known to have a direct influence on the deterioration rate of the item produced. Depending upon the weather conditions and preserving facilities at different locations as well as the nature of the item concerned, knowledge can be gathered as to how the deterioration rate of the item will get affected (as can be seen in the work of Liang and Zhou [33] where the same item deteriorates at a lower rate at the rented warehouse compared to the own warehouse due to better preserving facilities at the former). These help to formulate a set of fuzzy if-then rules that can be utilized to infer the corresponding deterioration rates by the application of a suitable fuzzy inference scheme.

Consider a manufacturing company which transports the finished products to N buyers located at different places having variations in their weather conditions. The production schedule is to be made by the vendor for a certain period (say the coming month) when the exact deterioration rates of the product at the buyers' locations are not known to him/her. However, depending on the weather conditions of the respective region, the same can be determined with the application of Mamdani fuzzy inference scheme [9]. Taking into account the extent of deterioration of the items based on, say, m parameters, a set of p fuzzy if-then rules can be described as given below:

$\mathfrak{R}_j$: if x_1 is $\widetilde{A}_{1j}$ and x_2 is $\widetilde{A}_{2j} \ldots x_m$ is $\widetilde{A}_{mj}$ then θ_v is $\widetilde{C}_{0j}, \theta_1$ is $\widetilde{C}_{1j}, \ldots, \theta_N$ is $\widetilde{C}_{Nj}$.
Input: x_1 is y_1 and x_2 is y_2 ... x_m is y_m
Output: θ_k is θ_{kM}

where $\widetilde{A}_{qj}$ and $\widetilde{C}_{kj}$ represent the term sets containing linguistic values for the linguistic variables x_q and θ_k respectively for $j = 1, 2, \ldots, p$, $q = 1, 2, \ldots, m$ and $k = 0, 1, \ldots, N$. The notations clearly indicate that $\widetilde{A}_{qj}$ and $\widetilde{C}_{kj}$ take the form of T1FSs. The crisp output θ_{kM} is calculated from the crisp input vector $(y_1, y_2, \ldots, y_m)$ with the application of Mamdani fuzzy inference scheme. In this context, Takagi-Sugeno approach is not helpful since the consequent of each fuzzy if-then rule is also fuzzy in nature. Again, Tsukamoto's inference scheme also fails in this regard because the consequent fuzzy membership function is not necessarily strictly monotonic [2]. Accordingly, Mamdani approach is found most suitable for the present scenario and is therefore selected over others. The corresponding procedure of obtaining θ_{kM} from $y = (y_1, y_2, \ldots, y_m)$ is briefly outlined below:

(1) Initially, a set of fuzzy if-then rules $\{\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_p\}$ is determined.
(2) Fuzzification: using input membership functions, the crisp inputs y_q are fuzzified, $q = 1, 2, \ldots, m$.
(3) Fuzzy operations: a rule strength is established by combining the fuzzified inputs according to the rules. The formula

$$l_j = t\Big(\mu_{\widetilde{A}_{j1}}(y_1), \mu_{\widetilde{A}_{j2}}(y_2), \ldots, \mu_{\widetilde{A}_{jm}}(y_m)\Big), \tag{6.1}$$

determines the degree to which the input matches the jth rule $\mathfrak{R}_j$, $j = 1, 2, \ldots, p$. Here t represents product or minimum operator.
(4) Implication: rule strength is combined with output membership function to obtain the consequence. The output membership function gets truncated at height l_j.
(5) Aggregation: an output distribution is obtained by the combination of all the consequences for all the applicable rules using maximum operator.
(6) Defuzzification: centroid of area formula is finally applied to defuzzify the output distribution in order to obtain the crisp output θ_{kM}.

6.2. Determination of demand rates

As mentioned earlier, the demand of buyer i is in the form of IT2FN $\widetilde{\widetilde{D}}_i, i = 1, 2, \ldots, N$ according to the experts' opinion. Therefore, for a given demand rate $\widetilde{\widetilde{D}}_i$, the nearest approximated interval denoted by $[D_{iL}, D_{iR}]$ $(i = 1, 2, \ldots, N)$ can be computed from the formula (3.4) as derived in Section 3, where D_{iL} and D_{iR} respectively denote the lower and upper limits of the nearest interval. Accordingly, the cycle length T also turns into an interval because of its dependence upon demand. Therefore, using interval arithmetic, the cycle length T given by (5.9) and the cost function TC_1 given by (5.10) can be rewritten as

$$T = [T_L, T_R] = \left[\frac{\beta P T_1}{\sum_{i=1}^{N} D_{iR}}\left\{1 - \frac{\theta_{0M}}{2}T_1 + \frac{\theta_{0M}}{2\beta P}T_1\sum_{i=1}^{N} D_{iL}\right\}, \frac{\beta P T_1}{\sum_{i=1}^{N} D_{iL}}\left\{1 - \frac{\theta_{0M}}{2}T_1 + \frac{\theta_{0M}}{2\beta P}T_1\sum_{i=1}^{N} D_{iR}\right\}\right]. \tag{6.2}$$

$\mathrm{TC}_1 = [\mathrm{TC}_{1L}, \mathrm{TC}_{1R}]$.

Here, TC_{1L} and TC_{1R} represent the costs calculated corresponding to the demand $[D_{iL}, D_{iR}]$, the expressions for which are provided below:

$$\begin{aligned}\mathrm{TC}_{1L}(n_i, T_1) = {} & c_v(1+\beta)PT_1 + (c_b - c_v)T_L \sum_{i=1}^{N} D_{iL}\left(1+\frac{\theta_{iM}T_L}{2n_i}\right) + c_b\frac{T_L^2}{2}\sum_{i=1}^{N} D_{iL} \\ & \times \frac{\theta_{iM}}{n_i} + K_v + K_b\sum_{i=1}^{N} n_i + c_s(1-\beta)PT_1 + \frac{T_L^2}{2}\sum_{i=1}^{N}(h_{bi}-h_v)\frac{D_{iL}}{n_i} \\ & \times \left(1+\frac{\theta_{iM}T_L}{3n_i}\right) + h_v\frac{T_L^2}{2}\left\{1+\frac{\theta_{0M}}{3}(T_L - T_1)\right\}\sum_{i=1}^{N} D_{iL} + h_v\frac{\beta PT_1^2}{2} \\ & \times \left(1-\frac{\theta_{0M}T_1}{3}\right) - h_vT_RT_1\left\{1+\frac{\theta_{0M}}{3}(T_R - T_1)\right\}\sum_{i=1}^{N} D_{iR} + h_v\frac{\theta_{0M}}{6}\times T_1^2T_L\sum_{i=1}^{N} D_{iL}. \end{aligned} \quad (6.3)$$

$$\begin{aligned}\mathrm{TC}_{1R}(n_i, T_1) = {} & c_v(1+\beta)PT_1 + (c_b - c_v)T_R \sum_{i=1}^{N} D_{iR}\left(1+\frac{\theta_{iM}T_R}{2n_i}\right) + c_b\frac{T_R^2}{2}\sum_{i=1}^{N} D_{iR} \\ & \times \frac{\theta_{iM}}{n_i} + K_v + K_b\sum_{i=1}^{N} n_i + c_s(1-\beta)PT_1 + \frac{T_R^2}{2}\sum_{i=1}^{N}(h_{bi}-h_v)\frac{D_{iR}}{n_i} \\ & \times \left(1+\frac{\theta_{iM}T_R}{3n_i}\right) + h_v\frac{T_R^2}{2}\left\{1+\frac{\theta_{0M}}{3}(T_R - T_1)\right\}\sum_{i=1}^{N} D_{iR} + h_v\frac{\beta PT_1^2}{2} \\ & \times \left(1-\frac{\theta_{0M}T_1}{3}\right) - h_vT_LT_1\left\{1+\frac{\theta_{0M}}{3}(T_L - T_1)\right\}\sum_{i=1}^{N} D_{iL} + h_v\frac{\theta_{0M}}{6}\times T_1^2T_R\sum_{i=1}^{N} D_{iR}. \end{aligned} \quad (6.4)$$

Therefore, overall integrated cost per unit time can be determined using basic interval arithmetic operations [53] as follows:

$$\mathrm{TC} = [\mathrm{TC}_L, \mathrm{TC}_R] = \frac{[\mathrm{TC}_{1L}, \mathrm{TC}_{1R}]}{[T_L, T_R]} = \left[\frac{\mathrm{TC}_{1L}}{T_R}, \frac{\mathrm{TC}_{1R}}{T_L}\right]. \quad (6.5)$$

Our objective is to minimize it and determine the optimal policy to be followed corresponding to the minimum cost.

6.3. Solution procedure

For the minimization problem, certain assumptions are taken into account which are summarized below [52]:

(1) Low cost is better than high cost.
(2) More certainty is better than less certainty.
(3) If less cost is associated with more uncertainty, a Decision Maker (DM) makes a trade-off between the two.
(4) To a pessimistic (optimistic) DM, assumption 2 (1) is somewhat more important than assumption 1 (2).

Now the basic problem reduces to the minimization of an interval objective function given by (6.5). Based upon the formulation of a general non-linear optimization problem with interval valued parameters [52], the model is transformed using linear weighted sum method to develop a composite goal, thereby defining the composite objective function as provided below:

$$\begin{cases} \text{Minimize} & Z = \{\lambda\mathrm{TC}_m + (1-\lambda)\mathrm{TC}_w\} \\ \text{subject to} & n_i > 0,\ \text{discrete variables}, \\ & T_1 > 0,\ \text{a continuous variable}, \\ & \lambda \in [0,1] \end{cases} \quad (6.6)$$

where, $\mathrm{TC}_m = m(\mathrm{TC}) = \frac{1}{2}[\mathrm{TC}_L + \mathrm{TC}_R]$ (mid-value of the interval objective function) and $\mathrm{TC}_w = w(\mathrm{TC}) = \frac{1}{2}[\mathrm{TC}_R - \mathrm{TC}_L]$ (half-width of the interval objective function). The factor λ defines the DMs pessimistic or optimistic bias. The DM is more inclined towards optimism for a value of λ closer to unity whereas the DMs pessimistic bias is reflected by smaller values of λ closer to zero. Therefore, a Pareto front is obtained which indicates a set of feasible solutions for the corresponding problem.

It is to be mentioned here that in absence of any uncertainty in the parameters *i.e.*, if a constant demand rate D_i is considered for the ith buyer with a fixed deterioration rate θ at all locations, then the results would have been reduced to the following:

$$\mathrm{TC}(n_i, T_1) = \frac{\mathrm{TC}_1(n_i, T_1)}{T} \tag{6.7}$$

where, T and $\mathrm{TC}_1(n_i, T_1)$ are as expressed in (5.9) and (5.10). These are in good agreement with the results described in [65] (assuming $\beta = 1$).

We will illustrate the theoretical results with the help of a numerical example in the next section.

7. Numerical illustration

In this section, we present a numerical example to demonstrate the model. The data set is hypothetically generated as per requirement. Consider the production of a volatile liquid by a manufacturing company which transports the finished products to 2 buyers located at different places having different weather conditions. The physical depletion of the liquid by evaporation can be regarded as deterioration in this case. The different parameter values are summarized below:
$N = 2$, $P = 65\,000$ gallons/month, $\beta = 0.98$, $c_v = \$8$/gallon, $c_b = \$10$/gallon, $K_v = \$2000$/production setup, $K_b = \$100$/order, $h_v = \$1.5$/gallon/month, $h_{b1} = \$2$/gallon/month, $h_{b2} = \$2.4$/gallon/month, $c_s = \$3$/gallon.

Approximate temperature and humidity at the vendor location for the coming month = 36° and 51%, approximate temperature and humidity at buyer 1 location for the coming month = 24° and 86%, approximate temperature and humidity at buyer 2 location for the coming month = 46° and 12%.

Every linguistic variable is interpreted with the help of a term set {very low, low, medium, high, very high} where each term is characterized by a triangular fuzzy number. Rules are constructed according to the fact that the rate of deterioration in the present scenario is high under high temperature and low humidity.

Tables 2–4 display each linguistic term with its corresponding scale which is represented by a triangular fuzzy number. The L-R representation of every fuzzy number is expressed in the tables.

The rate of evaporation is found to increase with rise in temperature and fall in humidity (see `https://serc.carleton.edu/196548`). Based on the aforementioned facts, Table 5 presents a complete list of fuzzy if-then rules which shows how the deterioration rate θ (θ represents any one of θ_k for $k = 0, 1, 2$) varies according to the variations in the two stated factors. The temperature and humidity values at the ith buyer location act as input vector for the fuzzy rule base system given by $\{y_{1i}, y_{2i}\}$ for $i = 1, 2, \ldots, 3$.

TABLE 2. Term set of x_1 (Temperature).

Term	Fuzzy number
Very Low (VL)	$(0; 0, 12.5)$
Low (L)	$(12.5; 12.5, 12.5)$
Medium (M)	$(25; 12.5, 12.5)$
High (H)	$(37.5; 12.5, 12.5)$
Very High (VH)	$(50; 12.5, 0)$

TABLE 3. Term set of x_2 (Humidity).

Term	Fuzzy number
Very Low (VL)	$(0; 0, 25)$
Low (L)	$(25; 25, 25)$
Medium (M)	$(50; 25, 25)$
High (H)	$(75; 25, 25)$
Very High (VH)	$(100; 25, 0)$

TABLE 4. Term set of θ (Deterioration rate).

Term	Fuzzy number
Very Low (VL)	$(0; 0, 0.025)$
Low (L)	$(0.025; 0.025, 0.025)$
Medium (M)	$(0.05; 0.025, 0.025)$
High (H)	$(0.075; 0.025, 0.025)$
Very High (VH)	$(0.1; 0.025, 0)$

TABLE 5. Fuzzy if-then rules for x_1, x_2 and θ.

if	x_1	and x_2	then θ	if	x_1	and x_2	then θ
$\Re_1$	VH	VH	M	$\Re_{14}$	M	L	M
$\Re_2$	VH	H	H	$\Re_{15}$	M	VL	H
$\Re_3$	VH	M	H	$\Re_{16}$	L	VH	VL
$\Re_4$	VH	L	VH	$\Re_{17}$	L	H	L
$\Re_5$	VH	VL	VH	$\Re_{18}$	L	M	L
$\Re_6$	H	VH	M	$\Re_{19}$	L	L	M
$\Re_7$	H	H	M	$\Re_{20}$	L	VL	M
$\Re_8$	H	M	H	$\Re_{21}$	VL	VH	VL
$\Re_9$	H	L	H	$\Re_{22}$	VL	H	VL
$\Re_{10}$	H	VL	VH	$\Re_{23}$	VL	M	L
$\Re_{11}$	M	VH	L	$\Re_{24}$	VL	L	L
$\Re_{12}$	M	H	L	$\Re_{25}$	VL	VL	M
$\Re_{13}$	M	M	M				

Given a set of inputs for the temperature and humidity of a region, Mamdani inference procedure, based upon the defined fuzzy if-then rules, can be implemented to obtain the desired rate of deterioration. Considering the vendor location, following are the rules contributing to the scheme for the input vector $\{y_{11}, y_{21}\} = \{36°, 51\%\}$:

$\Re_7$: if x_1 is high and x_2 is high then θ is medium

$\Re_8$: if x_1 is high and x_2 is medium then θ is high

$\Re_{12}$: if x_1 is medium and x_2 is high then θ is low

$\Re_{13}$: if x_1 is medium and x_2 is medium then θ is medium.

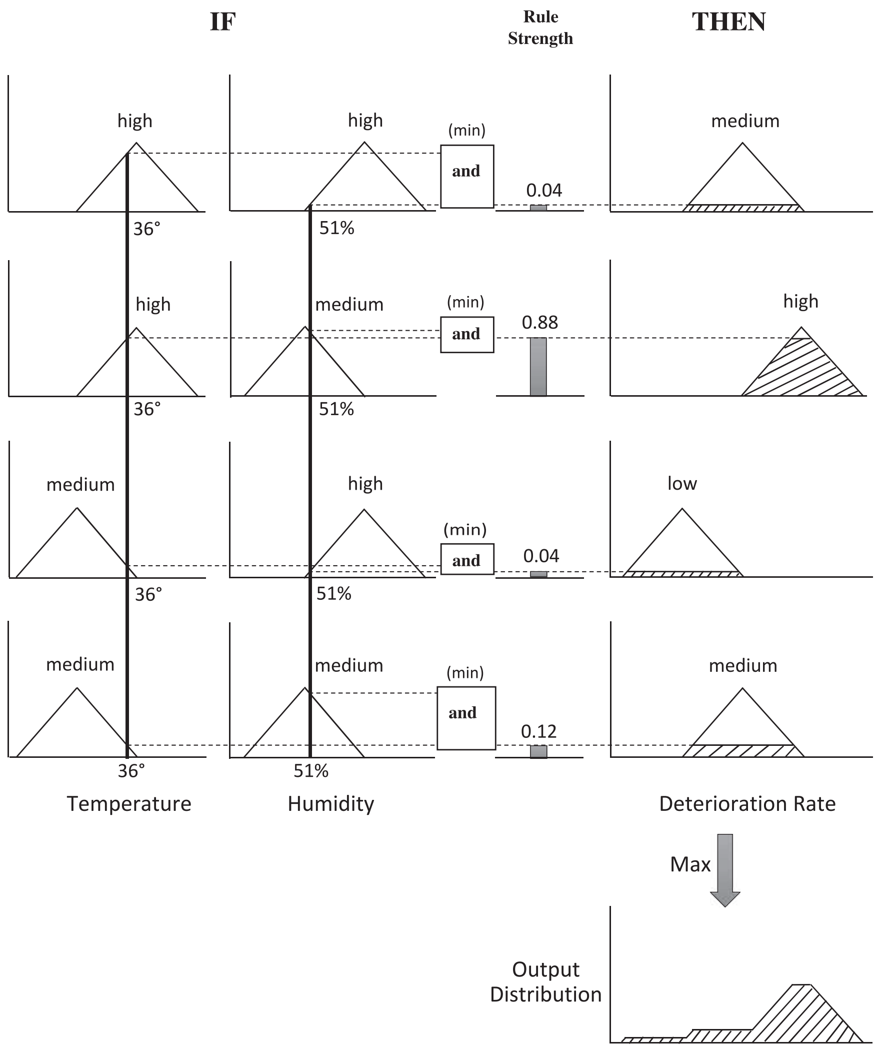


FIGURE 3. Mamdani Fuzzy Inference System for the first set of inputs (temperature = 36° and humidity = 51%).

Mamdani fuzzy inference scheme is graphically represented in Figure 3. With the application of centroid of area formula, the crisp output (rate of deterioration) is obtained as

$$\theta_{kM} = \frac{\int \theta\mu_\theta \,\mathrm{d}\theta}{\int \mu_\theta \,\mathrm{d}\theta} \qquad (k = 0, 1, 2). \tag{7.1}$$

Here, μ_θ represents the membership function of θ and the integration is taken over the entire shaded region of output distribution as shown in Figure 3. The defuzzified output corresponding to the first set of inputs is therefore computed as:

$$\theta_{0M} = \left(\int_0^{0.001} \frac{\theta}{0.025}\theta \,\mathrm{d}\theta + \int_{0.001}^{0.026} 0.04\theta \,\mathrm{d}\theta + \int_{0.026}^{0.028} \frac{\theta - 0.025}{0.025}\theta \,\mathrm{d}\theta + \int_{0.028}^{0.053} 0.12\theta \,\mathrm{d}\theta \right.$$

$$
\begin{aligned}
&+ \int_{0.053}^{0.072} \frac{\theta - 0.05}{0.025}\theta\,\mathrm{d}\theta + \int_{0.072}^{0.078} 0.88\theta\,\mathrm{d}\theta + \int_{0.078}^{0.1} \frac{0.1-\theta}{0.025}\theta\,\mathrm{d}\theta\Bigg) \Bigg/ \Bigg(\int_{0}^{0.001} \frac{\theta}{0.025}\,\mathrm{d}\theta \\
&+ \int_{0.001}^{0.026} 0.04\,\mathrm{d}\theta + \int_{0.026}^{0.028} \frac{\theta - 0.025}{0.025}\,\mathrm{d}\theta + \int_{0.028}^{0.053} 0.12\,\mathrm{d}\theta + \int_{0.053}^{0.072} \frac{\theta-0.05}{0.025}\,\mathrm{d}\theta \\
&+ \int_{0.072}^{0.078} 0.88\,\mathrm{d}\theta + \int_{0.078}^{0.1} \frac{0.1-\theta}{0.025}\,\mathrm{d}\theta\Bigg) \\
&= 0.069. \qquad (7.2)
\end{aligned}
$$

[Integration is carried out using the formula given by (7.1) for smaller sections of the shaded output distribution, which are then summed up.]

Similarly, the rules which contribute for the second set of inputs $\{y_{12}, y_{22}\} = \{24°, 86\%\}$ related to buyer 1 are listed below:

$\mathfrak{R}_{11}$: if x_1 is medium and x_2 is very high then θ is low

$\mathfrak{R}_{12}$: if x_1 is medium and x_2 is high then θ is low

$\mathfrak{R}_{16}$: if x_1 is low and x_2 is very high then θ is very low

$\mathfrak{R}_{17}$: if x_1 is low and x_2 is high then θ is low

and those for the third set of inputs $\{y_{13}, y_{23}\} = \{46°, 12\%\}$ are:

$\mathfrak{R}_{4}$: if x_1 is very high and x_2 is low then θ is very high

$\mathfrak{R}_{5}$: if x_1 is very high and x_2 is very low then θ is very high

$\mathfrak{R}_{9}$: if x_1 is high and x_2 is low then θ is high

$\mathfrak{R}_{10}$: if x_1 is high and x_2 is very low then θ is very high.

Following similar procedure as in (7.2), the corresponding crisp outputs for the second and third cases are obtained as $\theta_{1M} = 0.025$ and $\theta_{2M} = 0.079$. So, the product deteriorates at the rates of $\theta_{0M} = 0.069$, $\theta_{1M} = 0.025$ and $\theta_{2M} = 0.079$ at the vendor, buyer 1 and buyer 2 locations respectively.

After observing the previous demand records and experts' opinions, demand for buyer 1 is modelled as an IT2FN $\tilde{\tilde{D}}_1$ with UMF_1 and LMF_1 given by Type 1 Gaussian Fuzzy Numbers as presented below:

$$
\mathrm{UMF}_1 = \begin{cases} e^{-\frac{1}{2}\left(\frac{x-14\,500}{155}\right)^2}, & 14\,000 \le x \le 14\,500 \\ e^{-\frac{1}{2}\left(\frac{x-14\,500}{90}\right)^2}, & 14\,500 \le x \le 14\,800 \\ 0, & \text{otherwise.} \end{cases}
$$

$$
\mathrm{LMF}_1 = \begin{cases} 0.8e^{-\frac{1}{2}\left(\frac{x-14\,470}{80}\right)^2}, & 14\,200 \le x \le 14\,470 \\ 0.8e^{-\frac{1}{2}\left(\frac{x-14\,470}{60}\right)^2}, & 14\,470 \le x \le 14\,700 \\ 0, & \text{otherwise.} \end{cases}
$$

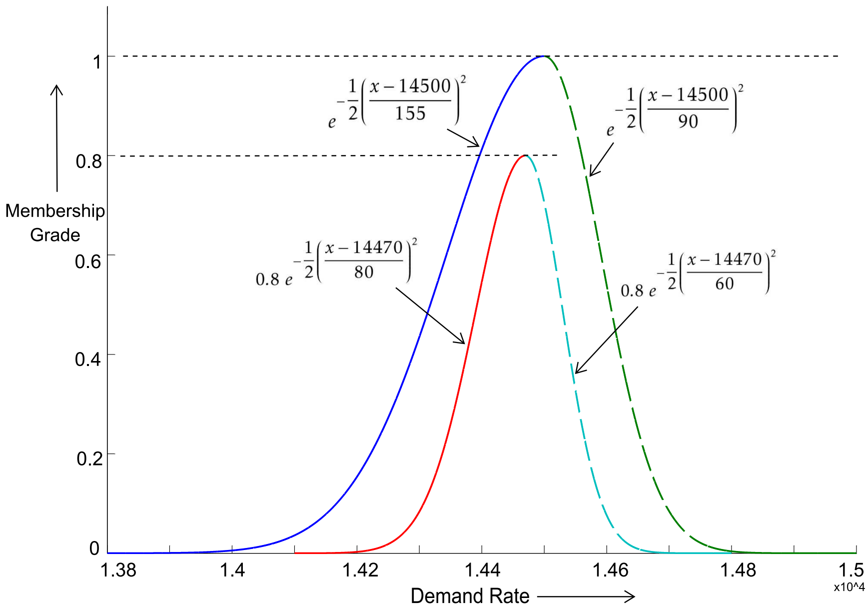


FIGURE 4. Demand pattern $\tilde{\tilde{D}}_1$ for buyer 1.

Similarly, the demand pattern for buyer 2 is modelled as an IT2FN $\tilde{\tilde{D}}_2$ with UMF_2 and LMF_2 given by

$$\text{UMF}_2 = \begin{cases} e^{-\frac{1}{2}\left(\frac{x-19\,740}{100}\right)^2}, & 19\,400 \le x \le 19\,740 \\ e^{-\frac{1}{2}\left(\frac{x-19\,740}{170}\right)^2}, & 19\,740 \le x \le 20\,300 \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{LMF}_2 = \begin{cases} 0.65e^{-\frac{1}{2}\left(\frac{x-19\,810}{90}\right)^2}, & 19\,500 \le x \le 19\,810 \\ 0.65e^{-\frac{1}{2}\left(\frac{x-19\,810}{70}\right)^2}, & 19\,810 \le x \le 20\,050 \\ 0, & \text{otherwise.} \end{cases}$$

Figures 4 and 5 pictorially represent the demand patterns for buyers 1 and 2 respectively.

For $\alpha \in [0, 1]$, the α-cut of UMF_1 is

$$\left[14\,500 - 155\sqrt{2\ \ln\ \frac{1}{\alpha}}, 14\,500 + 90\sqrt{2\ \ln\ \frac{1}{\alpha}}\right]$$

and that of LMF_1 is

$$\left[14\,470 - 80\sqrt{2\ \ln\ \frac{0.8}{\alpha}}, 14\,470 + 60\sqrt{2\ \ln\ \frac{0.8}{\alpha}}\right].$$

Therefore, $D_{1L} = 14\,334.18$ and $D_{1R} = 14\,582.75$ which suggest that the nearest interval approximation is $[14\,334.18, 14\,582.75]$.

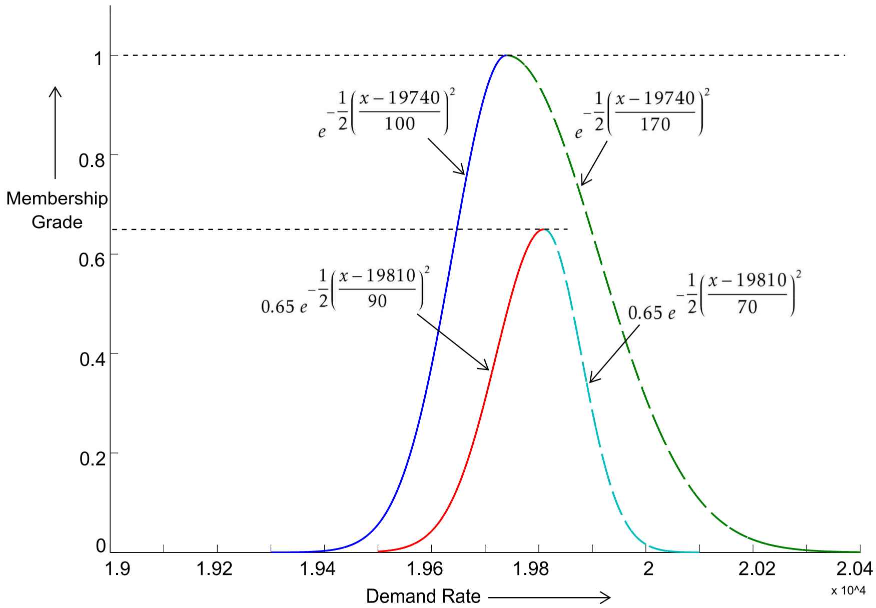


FIGURE 5. Demand pattern $\tilde{\tilde{D}}_2$ for buyer 2.

Likewise, the α-cut of UMF_2 is

$$\left[19\,740 - 100\sqrt{2\ \ln\ \frac{1}{\alpha}}, 19\,740 + 170\sqrt{2\ \ln\ \frac{1}{\alpha}}\right]$$

and that of LMF_2 is

$$\left[19\,810 - 90\sqrt{2\ \ln\ \frac{0.65}{\alpha}}, 19\,810 + 70\sqrt{2\ \ln\ \frac{0.65}{\alpha}}\right].$$

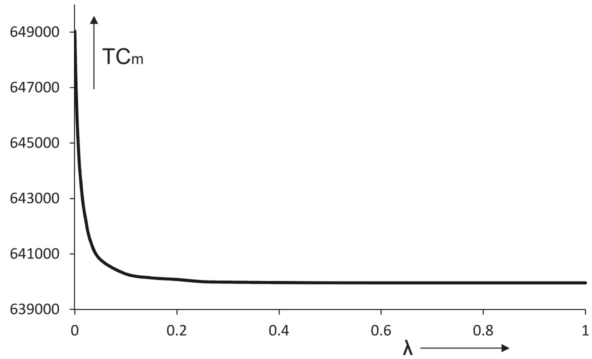
The nearest interval approximation is therefore computed for buyer 2 and is expressed as $[D_{2L}, D_{2R}] = [19\,647.18, 19\,931.27]$. Thus the interval valued demand for both the buyers are given by

$$[D_{1L}, D_{1R}] = [14\,334.18, 14\,582.75]$$
$$\text{and } [D_{2L}, D_{2R}] = [19\,647.18, 19\,931.27].$$

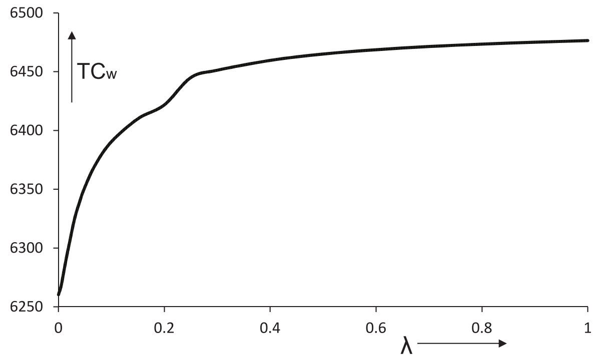
Based on the discussions made in Section 6.2, the reduced optimization problem expressed by (6.6) needs to be solved subject to 2 discrete variables n_1, n_2 and a continuous variable T_1, where the value of $\lambda \in [0,1]$ represents the DMs pessimistic and optimistic attitude. Both TC_m and TC_w are observed to undergo changes with alterations in the value of the weighting coefficient λ. Such variations are demonstrated through plots presented in Figure 6.

Within the range $[0,1]$ for λ, a set of optimal solutions is obtained in the form of a Pareto front. The Pareto optimal solutions for the formulated problem are marked with a continuous blue curve in Figure 7.

For an elaborate discussion, we select a particular solution from the Pareto front corresponding to $\lambda = 0.8$ *i.e.*, when the DM wishes to give more importance to the minimization of the mid-value of the interval objective function compared to the half-width.



(a)



(b)

FIGURE 6. Variations in TC_m and TC_w with λ.

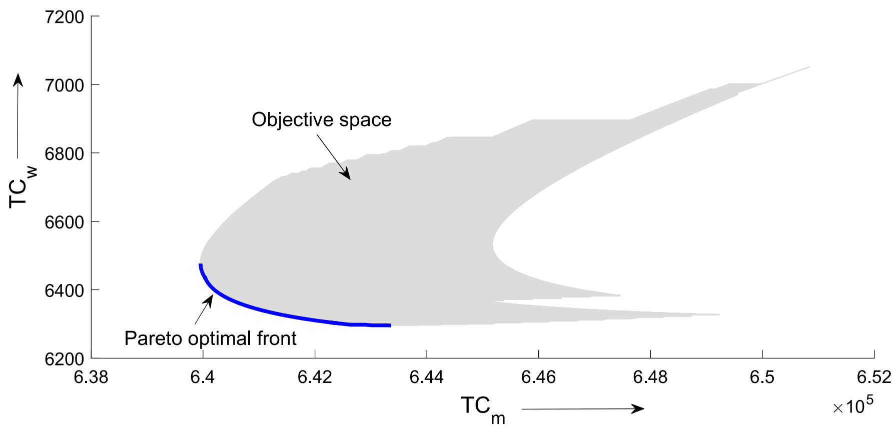


FIGURE 7. Pareto optimal front.

TABLE 6. Optimal solution for $\lambda = 0.8$.

Min Z	n_1	n_2	T_1	T	TC_m	TC_w	TC
513 259.61	3	5	0.1898	[0.349, 0.355]	639 956	6473.46	[633 483, 646 430]

The optimal values n_1^* and n_2^* for the single objective optimization problem corresponding to $\lambda = 0.8$ are derived when the following is satisfied:

$$Z(n_1^* - 1, n_2^* - 1, T_1) \geq Z(n_1^*, n_2^*, T_1) \leq Z(n_1^* + 1, n_2^* + 1, T_1)$$

where, $Z = 0.8\text{TC}_m + 0.2\text{TC}_w$.

The result presented in Table 6 shows that the solution to the proposed optimization problem corresponding to $\lambda = 0.8$ is $\text{TC} = [\text{TC}_L, \text{TC}_R] = [633\,483, 646\,430]$ with $(n_1, n_2, T_1) = (3, 5, 0.1898)$.

Convexity of the functions TC_m and TC_w is investigated by computing the leading principal minors (LPM) of the Hessian matrices H_m and H_w respectively at the point $(3, 5, 0.1898)$. All of these calculations are carried out in MATLAB R2019a and the obtained results are as follows:

First LPM of $H_m = 151.13 > 0$,
Second LPM of $H_m = 15\,611.3 > 0$,
Third LPM of $H_m = 4.89 \times 10^9 > 0$ and similarly,

First LPM of $H_w = 4.84 > 0$,
Second LPM of $H_w = 15.29 > 0$,
Third LPM of $H_w = 33\,463.7 > 0$.

This proves the positive definiteness of the Hessian matrices which indicates that the functions TC_m and TC_w, for the adopted set of numerical data, are convex at the point $(n_1, n_2, T_1) = (3, 5, 0.1898)$.

However, if the changes in the deterioration rate of the item due to changes in the weather conditions are ignored, that is if $\theta_v = \theta_1 = \theta_2 = 0.069$ with all the other parameter values being kept unchanged, the resultant total cost is computed as $[\text{TC}_L, \text{TC}_R] = [633\,851, 646\,784]$. This suggests that if the vendor assumes the item to deteriorate at the same rate in any other region as it does at his location, then the cost incurred per unit time is found to be comparatively high. In the next section, we conduct the sensitivity analysis by changing the values of input parameters of the numerical example for providing better insights into decision making.

8. Sensitivity analysis and managerial implication

In a decision-making environment, due to uncertainties related to dynamic market conditions, variations inevitably occur in some parameter values. Sensitivity analysis in this regard is of immense help to encounter the impact of such changes in the values of the concerned parameters. Same is carried out in this section by deviating the value of each parameter from −20% to +20%, and the corresponding impact on the decision variables n_1, n_2, T_1 and cost function $\text{TC}(n_1, n_2, T_1)$ is taken into account. A single parameter value is changed at a time, when all the others are kept fixed and the resultant solution is computed. The outcomes of the conducted sensitivity analysis are presented in Figure 8 and the corresponding observations are summarized accordingly.

With an increase in the production rate P, the optimal production time tends to decrease. This brings reduction in the production lot per cycle as well, so that the quantities delivered to the buyers I_{m1} and I_{m2} tend to decrease, provided the optimal number of shipments n_1 and n_2 remain unchanged. For a 20% increment in P, the reduction in optimal n_1 and n_2 explains the respective rise in I_{m1} and I_{m2}. Decrease in the cycle time increases the total integrated cost per unit time.

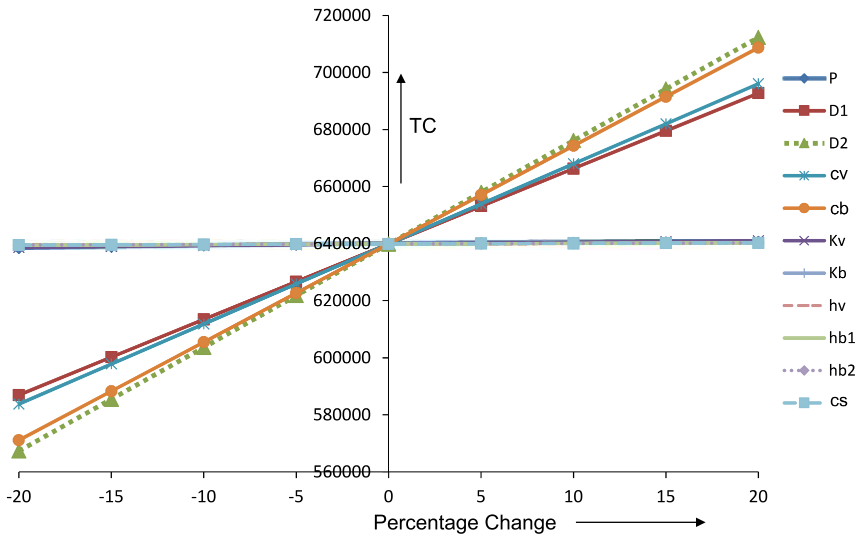


FIGURE 8. Variations in TC with changes in parameter values.

If the demand rate D_1 rises while all the other factors remain the same, the quantity delivered to the first buyer I_{m1} is found to increase accordingly in order to satisfy the increased demand. Similarly, D_2 has a direct effect on I_{m2}.

Further analysis shows that an increase in the unit production cost c_v reduces the maximum inventory of the vendor in order to mitigate the deterioration and holding cost pressure. Likewise, it is evident that an increment in the purchase price c_b results in a lower shipment size for the buyers as no discounts are offered from the vendor for procuring a larger lot.

When the production setup cost K_v is increased, it is observed that the optimal production run time T_1 increases accordingly thereby increasing the maximum inventory level I_{mv} of the vendor. Greater cycle length T reduces the setup cost per unit time. Likewise, with an increase in the ordering cost K_b for the buyers, the delivery quantities I_{m1} and I_{m2} tend to increase.

The holding costs of the item for both the vendor and the buyers are found to have an inverse effect on each of I_{mv}, I_{m1} and I_{m2} so as to counteract other cost components.

Variations in the scrapping cost c_s hardly bring any change in the production run time T_1 or any of the other factors. Only its increment tends to increase the total cost.

Some of the parameters such as β, θ_v, θ_1, θ_2 and λ can only assume values restricted within the closed interval $[0, 1]$. Therefore, instead of changing by percentage, a separate analysis is carried out by keeping their values within the permissible range as illustrated in Table 7. The percentage of increase index (PII) is defined as $\dfrac{\text{TC} - \text{TC}^*}{\text{TC}^*} \times 100\%$ where TC^* denotes the solution corresponding to $\lambda = 0.8$.

When there is an increase in the value of the proportion β, a larger fraction of the produced items is obtained as good quality. The maximum inventory level I_{mv} for the vendor tends to increase in a shorter production time. Slight changes are noticed due to variations in the values of the deterioration rates. The total cost is observed to increase with higher deterioration rates.

It is evident from the tabulated results that variation in the value of λ from 0 to 1 accordingly reflects the change from DM's pessimistic to optimistic bias. It is clearly delineated from the graphs plotted in Figure 6 that the minimum values of TC_m and TC_w are respectively attained at $\lambda = 1$ and $\lambda = 0$. Therefore, assigning a smaller value to λ close to zero indicates that the DM wants to put more importance on the minimization of the half-width compared to the mid-value, thereby reflecting his/her pessimistic bias. In a similar manner, more

TABLE 7. Sensitivity analysis of parameters β, θ_v, θ_1, θ_2, λ.

β	0.9	0.95	0.98	0.99	1.0
n_1	3	3	3	3	3
n_2	5	5	5	5	5
T_1	0.212	0.197	0.189	0.187	0.185
T	$[0.358, 0.363]$	$[0.352, 0.358]$	$[0.349, 0.355]$	$[0.348, 0.354]$	$[0.348, 0.353]$
TC	$[666\,980, 680\,511]$	$[645\,389, 658\,542]$	$[633\,483, 646\,430]$	$[629\,673, 642\,554]$	$[625\,939, 638\,756]$
PII	$[+5.288, +5.272]$	$[+1.879, +1.874]$	$[0, 0]$	$[-0.601, -0.600]$	$[-1.191, -1.187]$
θ_v	0.06	0.065	0.069	0.075	0.08
n_1	3	3	3	3	3
n_2	5	5	5	5	5
T_1	0.192	0.191	0.189	0.188	0.187
T	$[0.354, 0.360]$	$[0.351, 0.357]$	$[0.349, 0.355]$	$[0.346, 0.351]$	$[0.343, 0.349]$
TC	$[633\,257, 646\,207]$	$[633\,383, 646\,331]$	$[633\,483, 646\,430]$	$[633\,631, 646\,577]$	$[633\,754, 646\,698]$
PII	$[-0.036, -0.034]$	$[-0.016, -0.015]$	$[0, 0]$	$[+0.023, +0.023]$	$[+0.043, +0.041]$
θ_1	0.015	0.02	0.025	0.03	0.035
n_1	2	3	3	3	3
n_2	5	5	5	5	5
T_1	0.184	0.190	0.189	0.189	0.189
T	$[0.340, 0.344]$	$[0.350, 0.356]$	$[0.349, 0.355]$	$[0.348, 0.354]$	$[0.347, 0.352]$
TC	$[633\,391, 646\,319]$	$[633\,432, 646\,378]$	$[633\,483, 646\,430]$	$[633\,534, 646\,481]$	$[633\,585, 646\,532]$
PII	$[-0.015, -0.017]$	$[-0.008, -0.008]$	$[0, 0]$	$[+0.008, +0.008]$	$[+0.016, +0.016]$
θ_2	0.07	0.075	0.079	0.085	0.09
n_1	3	3	3	3	3
n_2	5	5	5	5	5
T_1	0.191	0.190	0.189	0.189	0.189
T	$[0.351, 0.356]$	$[0.350, 0.355]$	$[0.349, 0.355]$	$[0.348, 0.354]$	$[0.347, 0.353]$
TC	$[633\,407, 646\,353]$	$[633\,449, 646\,396]$	$[633\,483, 646\,430]$	$[633\,533, 646\,480]$	$[633\,575, 646\,522]$
PII	$[-0.012, -0.012]$	$[-0.005, -0.005]$	$[0, 0]$	$[+0.008, +0.008]$	$[+0.015, +0.014]$
λ	0	0.2	0.5	0.8	1
n_1	2	2	3	3	3
n_2	3	4	5	5	5
T_1	0.061	0.161	0.186	0.189	0.191
T	$[0.113, 0.115]$	$[0.296, 0.301]$	$[0.342, 0.348]$	$[0.349, 0.355]$	$[0.352, 0.357]$
TC	$[642\,773, 655\,294]$	$[633\,658, 646\,502]$	$[633\,496, 646\,426]$	$[633\,483, 646\,430]$	$[633\,479, 646\,432]$
PII	$[+1.466, +1.371]$	$[+0.028, +0.011]$	$[+0.002, -0.001]$	$[0, 0]$	$[-0.001, +0.0003]$

emphasis is put on the minimization of TC_m when the DM chooses a value of λ closer to unity. It is observed from both the plots that the minimal value of TC_m leads to quite a high value for TC_w and vice versa, so that the DM has to make a trade-off between the two depending upon his priorities.

TC is observed to be more sensitive to the parameters D_1, D_2, c_v and c_b. In other words, even a slight change in the values of these parameters tends to affect the cost of the system. TC is found to be much less sensitive to variations in the values of the rest of the parameters involved. This can also be clearly visualized from the sensitivity graph presented in Figure 8 where all plots are obtained by reducing the interval costs to their mean values.

8.1. Managerial implications

This study aims at developing a novel method of defuzzification of IT2FNs, thereby incorporating the same in our proposed supply chain production model. As already discussed in Section 3, the proposed methodology will be more convenient (both in terms of computational time and accuracy) compared to the existing techniques. Accordingly, our study is supposed to be highly beneficial for researchers dealing with IT2FNs in their studies. Besides, modeling customer demand rate in the form of an interval type-2 fuzzy number will allow authors to visualize and model such crucial inventory parameters in several novel patterns.

Moreover, keeping in mind one of the major difficulties that is being faced by the decision makers, the novel approach presented in this study manages to more accurately predict the deterioration rate for a certain item. Specifically for items facing variations in their rates of depletion due to fluctuations in weather parameters, the proposed strategy will be far beneficial to the firms in forecasting the deterioration rate compared to the existing ones.

9. Concluding remarks

The novelty of this study lies in two aspects: firstly, it takes into account a situation in which the rate of deterioration of the item is dependent upon certain attributes such as temperature and humidity of the region. Based on the forecasted values of these parameters, it is possible to determine the exact values of deterioration rates employing fuzzy rule base technique. As it is unrealistic to assume a constant rate of deterioration of a product in every environmental condition, the scenario presented in this paper serves better to encounter more practical situations. The outcomes reveal that ignoring the influence of temperature and humidity on item deterioration results in an increased cost per unit time as compared to the situation when the effect of weather is taken into account. Secondly, an interval approximation method is developed for the defuzzification of IT2FN, and the same is implemented in the proposed model. The suggested methodology is more convenient than the existing ones as it converts an IT2FN to a crisp interval instead of a single crisp quantity and that also without any intermediate type reduction phase. Application of the same is brought off by considering imprecise demand patterns in the form of IT2FNs.

Imperfections in the produced items have been considered in the model without any rework process for the same, they are scrapped assuming non-reworkable. So, remodelling it including rework setup can be considered as a future scope for researchers [61, 67]. The approach taken in this work can also be further extended to include even more complicated scenarios. Introducing probabilistic theory while considering fuzzy rule base in deterioration or assuming demand parameter as generalized type-2 fuzzy number instead of IT2FN may be included among few ideas for future research. In the current scenario, one of the major concerns of the firms is to reduce carbon emissions generated through various operations. Based on some recent findings by Chen *et al.* [8], Hovelaque and Bironneau [24] and Rout *et al.* [46], carbon emission constraints can also be imposed into the proposed model.

Acknowledgements. The authors would like to acknowledge the support provided by the Indian Institute of Technology Kharagpur yielding facilities for research. The first and third authors are grateful to Ministry of Human Resource Development for supporting their scientific studies with the Institute Research Assistantship.

Funding. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Conflict of interest/Competing interests. The authors declare that they have no conflict of interest.

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