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ANALYSIS OF A BULK ARRIVAL *N-POLICY* QUEUE WITH TWO-SERVICE GENRE, BREAKDOWN, DELAYED REPAIR UNDER BERNOULLI VACATION AND REPEATED SERVICE POLICY

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Abstract. This article deals with an unreliable bulk arrival single server queue rendering two-heterogeneous optional repeated service (THORS) with delayed repair, under Bernoulli Vacation Schedule (BVS) and N-policy. For this model, the joint distribution of the server's state and queue length are derived under both elapsed and remaining times. Further, probability generating function (PGF) of the queue size distribution along with the mean system size of the model are determined for any arbitrary time point and service completion epoch, besides various pivotal system characteristics. A suitable linear cost structure of the underlying model is developed, and with the help of a difference operator, a locally optimal N-policy at a lower cost is obtained. Finally, numerical experiments have been carried out in support of the theory.

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1. Introduction

Service interruption is a very demanding research topic in queueing systems. It is encountered in many day-to-day congestion scenarios noticed in the ticket counters, hospitals, banks, production systems, communication networks etc. The interruption in service due to server breakdown and server absence or vacation are nearly inescapable and inclusion of which makes a queueing model more resilient.

Most of the classical queueing models are devoted to the study of a reliable server. But in real life, existence of a perfect reliable server is practically impossible, and servers are often susceptible to unforeseen failures. White and Christie [47] were the first to study a queueing model with an unreliable server subjecting it to instantaneous repair. To cite a few papers on an unreliable server with immediate repairability, the authors refer to the works of Li *et al.* [25], Madan [29], Wang [46], Krishnamoorthy *et al.* [20], Abbas and Aissani [1]. But instantaneous repair of a broken server may be delayed due to technical difficulty, unavailability of the technician, equipment or several other reasons. The notion of delay in repair usually termed as the delayed repair, was introduced by Madan [28] for an $M/M/1$ queue with delay and repair time following general and exponential distribution, respectively. Kumar and Arumuganathan [21] studied an unreliable $M^X/G/1$ retrial

Keywords. Two genres of service, Re-service, elapsed time, remaining time, double transform, BVS, N-policy.

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queue with optional service where two different kinds of repair, *i.e.* repair time of server during patient and impatient customers, are considered. Peng *et al.* [35] analyzed an $M/G/1$ queue with pre-emptive resume priority and collisions subject to server's breakdown and delayed repair under linear retrial policy. Choudhury and Kalita [7] investigated an $M/G/1$ delayed repair queue where both the delay and repair times are assumed to follow the general distribution. Saggou *et al.* [36] examined an unreliable $M/G/1$ retrial queue with delay in repair and two kinds of customers (transit and recurrent). Singh *et al.* [38] discussed an $M^X/G/1$ queue with delayed repair where the server undergoes different compulsory phases of repair.

Queueing models with server vacation is a common instance of record perceived in many real-life queueing phenomena, incorporation of which adds more realisticity and pliability in the system. An extensive survey on vacation queue can be reported in the works of Doshi [12], Takagi [43], Tian and Zhang [45]. Different queueing models undergo different server vacations however, the vacation policy studied in this research work follows the Bernoulli schedule. Pioneering works on Bernoulli schedule can be seen in the study of Keilson and Servi [18]. Madan *et al.* [31] have studied a single server queue under BVS, where he analyzed the stationary queue size distribution besides various performance measures. Maraghi *et al.* [33] investigated a single server queue with service interruptions under BVS assuming general vacation and exponential repair time distribution and have derived the PGF of the system size. Khalaf *et al.* [19] generalized this model by introducing general repair time. Choudhury and Deka [5] developed an unreliable $M/G/1$ queue with two phases of service and BVS, where they determined the stability queue length distribution at arbitrary and departure epoch. Further, Choudhury and Deka [6] generalized their earlier model [5] by addendum of multiple vacation policy to it. Li *et al.* [27] discussed an $M/G/1$ retrial queue where the retrial times follows general distribution and the server undergoes a working vacation following the Bernoulli schedule.

Control of a queue and a vacation period is one of the most significant areas of research. To cap the queue and vacation length, various control operating policies can be used. This study opts for stationary N-policy which recieved most attention because it is analytically more easier to deal with than the other policies. Lee and Srinivasan [23] introduced the concept of N-Policy for a batch arrival queue. Later, Lee *et al.* [24] developed a suitable linear cost function to determine an optimal threshold using the system size distribution. Choudhury and Paul [8] analyzed a batch arrival queue under N-policy with a second optional service. They discussed the PGF of the queue size distribution at random and departure epoch and provided a simple procedure to derive the optimal policy. Choudhury *et al.* [9] further generalized its previous model by incorporating service interruptions and delayed repair. Ke *et al.* [17] explored an $M^X/G/1$ queue with N-policy where the server is allowed to take at most J vacations until the number of customers accumulates to N after returning from a vacation. Tadj and Yoon [41] examined an unreliable $M/G/1/N$-*policy* queue where they applied binomial schedule with k vacations instead of BVS and developed a cost structure consisting of two decision variables. Kalita and Choudhury [15] investigated an unreliable N-policy queue and analyzed the Laplace Stieltjes Transform (LST) of the system's reliability function with the mean time to first failure of the server. Lately, Lan and Tang [22] discussed a $GEO/G/1$ queue with N-policy and Bernoulli feedback under modified multiple vacations.

In this article, an $M^X/\binom{G_1}{G_2}/1$ re-service queue with service interruption and N-policy is considered. According to N-policy, the server remains idle till the queue length accumulates to N (a threshold value) and resumes service as soon as it becomes equal to or exceeds the threshold value N (≥ 1). Once the queue length is N, the service begins, and the customers are given a choice between two heterogeneous services where they have an option for re-service. Under the service mechanism THROS, the server provides two different genres of services to its customers with probabilities, say q_1/q_2 associated with each service genre, having an additional advantage of repeating the same service once with probability z_1/z_2 in case of dissatisfaction. After each busy period, the server either undergoes a vacation of random length with some probability, say p_1, or starts a new service with its complementary probability, say p_2. While in service, the server is subject to abrupt failures, which then can be fixed. The efficiency and outcome of the queueing systems are extremely affected by service interruptions.

In recent years several models have evolved with THROS. The notion of THROS was first analyzed by Madan *et al.* [30] for bulk arrival queue, where they evaluated the stationary PGF of the queue size and average waiting time in the queue/system. Tadj and Ke [40] examined a single server queue delivering two phases of service where the first phase of service offered a choice of either service to its customers and obtained an optimal control policy for it. Baruah *et al.* [3] analyzed a single server vacation queue with THROS, including the concept of balking. Kalita and Choudhury [16] examined a single server queue under THROS with randomized vacation policy and obtained important performance characteristics. Investigations so far, have considered only THORS owing to mathematical convenience, though a larger number of such services is definitely more desired. By generalizing the results of THORS findings with more than two service genres can be easily developed.

In queueing paradigm, several researchers have discussed individual or a few realistic queueing phenomenons like two genre of service, re-service, unreliable server, delayed repair, Bernoulli schedule, N-policy. But to the best of author's knowledge, no queueing literature is found, which analyzes all these features together. Owing to application of such models in real-life systems where these queueing concepts are common, there is a need for such research work that combines all these features together. So, to fulfil this research gap, this study proposes an $M^X/G/1$ queue under the realistic phenomena of (i) two genre of service, (ii) repeated service, (iii) server failure, (iv) delay in repair, (v) Bernoulli vacation and (vi) N-policy. Also, few unanswered questions elemented below have been attempted to address:

– Real life application of such model.
– Queue size PGF of an idle period.
– Joint PGF of server's state and queue length under elapsed time.
– Marginal PGF/PDF of server's state and queue length.
– Double transform under elapsed and remaining time.
– Steady-state queue size distribution at random/service completion epoch.
– Stationary system state probabilities for various states of the server.
– Steady-state availability and failure frequency of the server.
– Optimal cost policy for the underlying model.

When all the adversities are taken together, it involves functional equations. So, to tackle this difficulty and obtain the exact solution for the above-mentioned results, the well known supplementary variable technique under some suitable transformations are applied. These transformations make the calculations of the remaining time distribution results possible without even setting the Kolmogorov Backward recurrence equations. The marginal PGF and PDFs follows trivially from the joint distributions without involving much complexities of integration. However, very few works are available in the literature with this type of transformation. Takagi [42, 44] first used these kind of transformations for a time-dependent $M/G/1$ vacation queue.

The proposed model may find a potential application in the Discontinuous Reception (DRX) mechanism with flexible Transmission Time Interval (TTI) of the fifth-generation (5G) network that allows a User Equipment (UE) to enter a sleep period, thereby saving power. Here in this framework, the Short Transmission Time Interval (STTI) or the Long Transmission Time Interval (LTTI) are modelled as THROS, moving to DRX cycle or continuing transmission is modelled as BVS, and network congestion is modelled as server breakdown. As UEs are susceptible to abrupt failures and network failure affects their performance immensely, it motivated the authors to study such a system from the consideration of queueing and reliability. Hence, this research work investigates an unreliable queueing model with THORS under BVS and N-policy. The application of the underlying model is explained in details in Section 2.

The essential features of this article are: Section 2 gives a real-world justification of the proposed model. Section 3 describes the underlying stochastic model. Section 4 puts forward the Kolmogorov equations governing the model under consideration. Section 5 finds the joint distribution of the server's state and queue length under elapsed and remaining times. Section 6 obtains the system size distribution at service termination epoch, and the performance measures of the model are discussed in Section 7. Section 8 develops an optimal operating policy of the model, and Section 9 illustrates a numerical example supporting the theory. Finally, a concluding remark is summarized in Section 10.

2. Real world implementation of the model proposed

In mobile communications and wireless networks, power saving is an important issue of the UE, and there is a huge literature on it [11, 13]. The Long Term Evolution/Long Term Evolution-Advanced (LTE/LTE-A) technology uses the DRX mechanism to reduce the energy consumption of UEs by allowing them to turn off their components whenever there is no arrival of data. The DRX mechanism is such that if there are no packets in the system seeking service, the system moves to an inactivity timer state (ITS), whereas the arrival of a packet moves the system to an active state where the service is rendered to the packets. ITS is a state in which the UE waits before starting DRX. Once all the packets in the buffer are served, the system again moves to ITS and waits for new arrival for some random amount of time. If there is no packet indication in the buffer before termination of the inactivity timer (IT), it makes the system move on DRX cycle in sequence with some probability "p_1" (say) for a short period of time up to a fixed number, and after that, a long DRX cycle begins and so on until a packet arrives. If a packet arrives it gets served with the complementary probability "$p_2 = (1 - p_1)$". The structure of a DRX mechanism is explained in Figure 1.

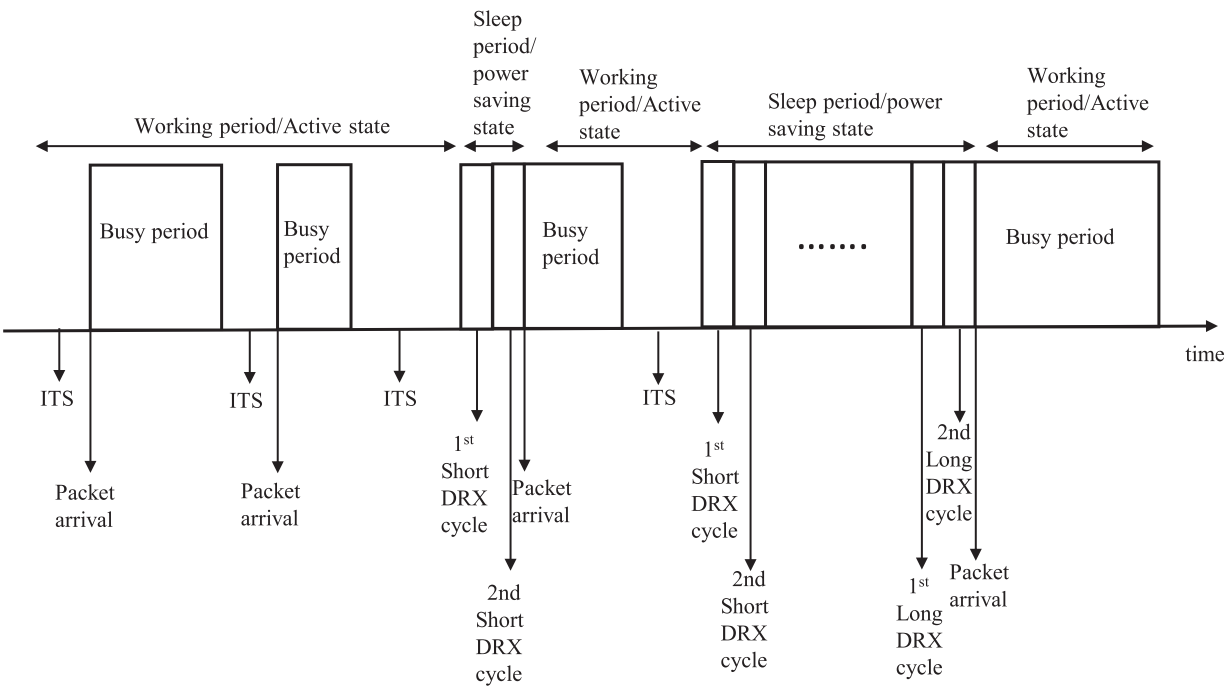

Figure 1. Structure of the DRX mechanism.

LTE/LTE-A uses fixed TTI of length 1 ms for transmission, so to deal with the multifarious data traffic increasing in various UE, it is unsuitable in its current state for 5G communications. The 5G network comprises of two flexible TTIs, namely the STTI and LTTI, to handle the diversified data traffic with various requirements of the UEs. Generally, for processing high volume data at a minimal rate, STTI is used, and for cellular network services, LTTI is used. A comprehensive survey on 5G wireless networks can be observed in the works of Agiwal *et al.* [2]. Maheshwari *et al.* [32] suggested a semi-Markov model to explore the DRX mechanism with flexible TTI for 5G communications and analyzed the power-saving factor, and proposed an algorithm for the TTI selection for different service requirements.

Employing *N-policy* in the DRX system implies that every packet that arrives in the buffer are not served immediately. The system becomes operative only when there are N or more packets in the buffer in the power-saving state of the UE. As soon as the number of packets are indicated to be N in the buffer, the service starts immediately; otherwise the short DRX cycles keep on occurring successively, followed by long DRX cycles. Gautam *et al.* [14] suggested an $M^X/G/1$ vacation queueing model with *N-policy* in the DRX mechanism of the LTE-A networks.

In this study, the packets that arrive in the buffer are assumed to follow the compound poisson process and are served only when the number of packets add up to N in the buffer. As soon as the number of packets in the buffer builds up to N, the packets can choose any of the TTI with probability q_1/q_2 (say) and repeat its transmission once if needed with probability z_1/z_2 (say) indicating THROS. After transmission and re-transmission, the system moves to a DRX cycle with probability p_1, or continue transmitting with probability p_2, thereby signifying a BVS. While transmission of a packet is in progress, it is very likely that there might be transmission interruption owing to network failure, congestion etc., implying breakdown of the service station. The failure of an UE due to network congestion can sometimes be gained by restarting the UE or sometimes by waiting till the network provider fixes it by themselves. However, fixing of a failed network may not always be immediate due to unavailability of the technician, scarcity of equipment etc., signifying a delay in repair. Thus, clubbing *N-policy* and DRX mechanism can lead to more power saving of 5G networks.

3. Description of the mathematical model

This research study considers a queueing system where the customers arrive in the system in batches of different sizes conforming to a compound Poisson process with a rate of arrival $\lambda > 0$. Let Ξ be the number of individual primary units in a batch and $\Xi_1, \Xi_2, \ldots$ be the successively arriving batch sizes which are independently and identically distributed (i.i.d) random variables (r.v). The probability mass function (PMF) and PGF of Ξ are given by $w_l = Pr[\Xi = l]; l = 1, 2, \cdots$ and $P_\Xi(s) = \sum_{l=1}^{\infty} w_l s^l$ ($\mid s \mid \leq 1$) respectively with finite factorial moment $\mu_\Xi^{(l)} = E[\Xi(\Xi-1)\cdots(\Xi-l+1)]$. Thus, if l is the number of customers in a batch, then the rate of arrival of l units in a batch is $w_l\lambda$.

The single server here provides two different service genre on a first come first serve basis. The server turns off its services when the system empties and re-establishes the service immediately upon the system size exceeding or being equal to $N(\geq 1, threshold)$. Before the start of a busy period each customer either selects the first genre of service (FGS) with probability "q_1" or the second genre of service (SGS) with probability "q_2" ($q_1 + q_2 = 1$). The service time provided in the FGS and SGS are i.i.d r.v denoted by A_1 and A_2 respectively follows the general law of probability (g.l.p) with cumulative distribution function (c.d.f) $F_{A_i}(x)$, LST $F^*_{A_i}(\eta) = \int_0^\infty e^{-\eta x}\,\mathrm{d}F_{A_i}(x)$ and l^{th} finite moment $\mu_{A_i}^{(l)}$ $(l = 1, 2, \cdots)$ $\forall\ i = 1, 2$ (i takes the value 1 for the FGS and 2 for the SGS).

The model considers that as soon as a chosen service of any genre is completed by the server, the customer may further opt to repeat the same genre of service but only once with probability z_i or leave the system with its complementary probability $(1 - z_i)$ for $i = 1, 2$. The re-service time of the server is an i.i.d r.v denoted by Q_i which follows the g.l.p with c.d.f $F_{Q_i}(x)$, LST $F^*_{Q_i}(\eta) = \int_0^\infty e^{-\eta x}\mathrm{d}F_{Q_i}(x)$ and l^{th} finite moment $\mu_{Q_i}^{(l)}$ $(l = 1, 2, \cdots)$ $\forall\ i = 1, 2$.

The server after completion of a service along with its repeated service of any genre may enter into a vacation period of random duration B with probability p_1 or continue staying in the system and serve the next unit, if any, with probability p_2 such that $p_1 + p_2 = 1$. The duration of vacation period B is an i.i.d r.v following the g.l.p with c.d.f $F_B(x)$, LST $F^*_B(\eta) = \int_0^\infty e^{-\eta x}\mathrm{d}F_B(x)$ and l^{th} finite moment $\mu_B^{(l)}$ $(l = 1, 2, \cdots)$.

As the server considered here is unreliable; therefore, the server may breakdown at any instant while providing service of any genre. The inter-arrival of breakdown time is assumed to follow an exponential distribution with breakdown rates α_1 for FGS/FGRS and α_2 for SGS/SGRS. Occurrence of a breakdown makes the server unavailable for an unspecified period of time until it is fixed (repaired). When the server breaks down the one in service waits for the server to get repaired and after that completes his remaining service. Consequently, the service time is cumulative in nature. While fixing the server many of the times it is not always immediate, rather subject to some delays owing to many reasons. The delay time is an i.i.d r.v L_i following the g.l.p with c.d.f $F_{L_i}(y)$, LST $F^*_{L_i}(\eta) = \int_0^\infty e^{-\eta y}\,\mathrm{d}F_{L_i}(y)$ and l^{th} finite moment $\mu^{(l)}_{L_i}$ $(l = 1, 2, \cdots)$. Similarly, the repair time is also an i.i.d r.v T_i following g.l.p with c.d.f $F_{T_i}(y)$, LST $F^*_{T_i}(\eta) = \int_0^\infty e^{-\eta y}\,\mathrm{d}F_{T_i}(y)$ and l^{th} finite moment $\mu^{(l)}_{T_i}$ $(l = 1, 2, \cdots)$ for $i = 1, 2$.

Further, the input process, service time, re-service time, vacation time, server's lifetime, delay time and repair time are all assumed to be mutually independent of each other.

In consideration of the above discussion a sample path of the model discussed above is depicted in Figure 2.

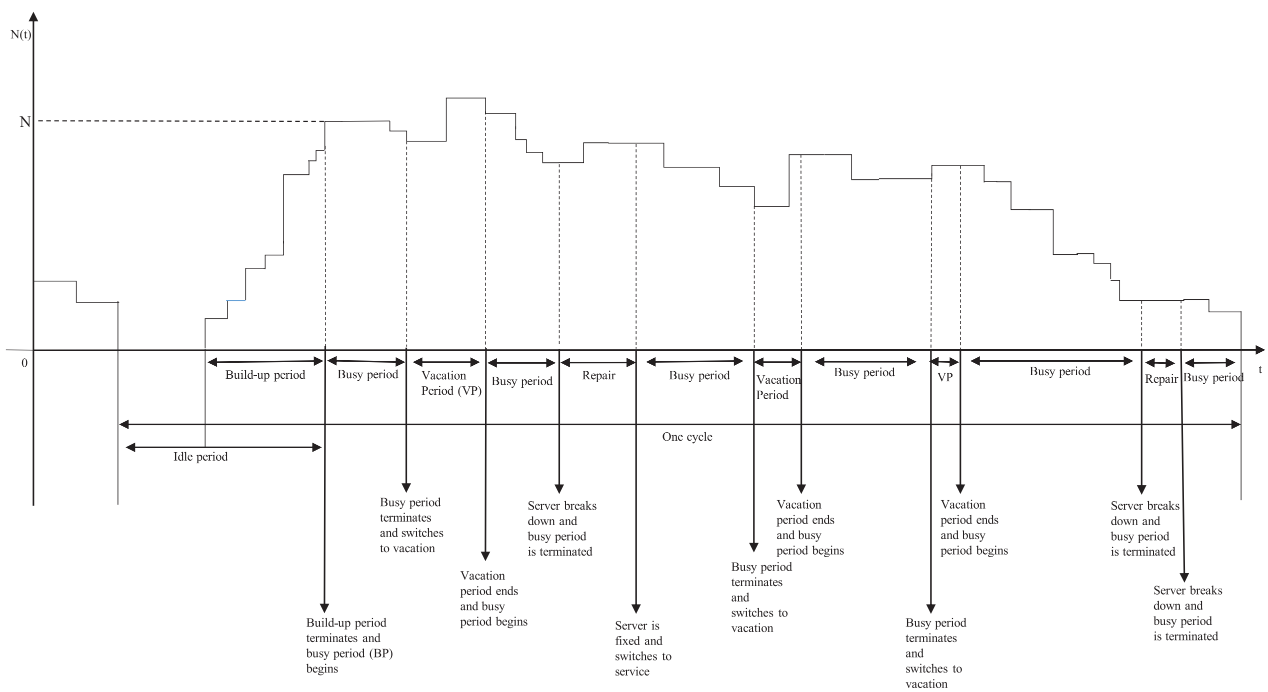

FIGURE 2. Sample path of the proposed model.

The state transition diagram for the same is represented in Figure 3 where a 2-tuple (τ, ϑ) denotes the state of the system. The variable τ denotes the number of units in the system at time t and ϑ denotes the state of the system at time t; where $\tau \in \{0, 1, \ldots\}$ and $\vartheta \in \{0, 1, 2, 3, 4, 5, 6\}$. 0 – idle period, 1 – vacation period, 2 – busy period with FGS, 3 – busy period with FGRS, 4 – busy period with SGS, 5 – busy period with SGRS, 6 – breakdown period for $\vartheta \in \{0, 1, 2, 3, 4, 5, 6\}$.

4. Governing equations

This section frames the equations governing the system states taking into account elapsed service time $A^0_i(t)$, elapsed re-service time $Q^0_i(t)$, elapsed delay time $L^0_i(t)$, elapsed repair time $T^0_i(t)$ $(i = 1, 2)$ and elapsed vacation time $B^0(t)$ at time t as supplementary variables and $A^+_i(t)$, $Q^+_i(t)$, $L^+_i(t)$, $T^+_i(t)$ $(i = 1, 2)$ and $B^+(t)$ be the corresponding remaining service, re-service, delay, repair and vacation times respectively at time t.

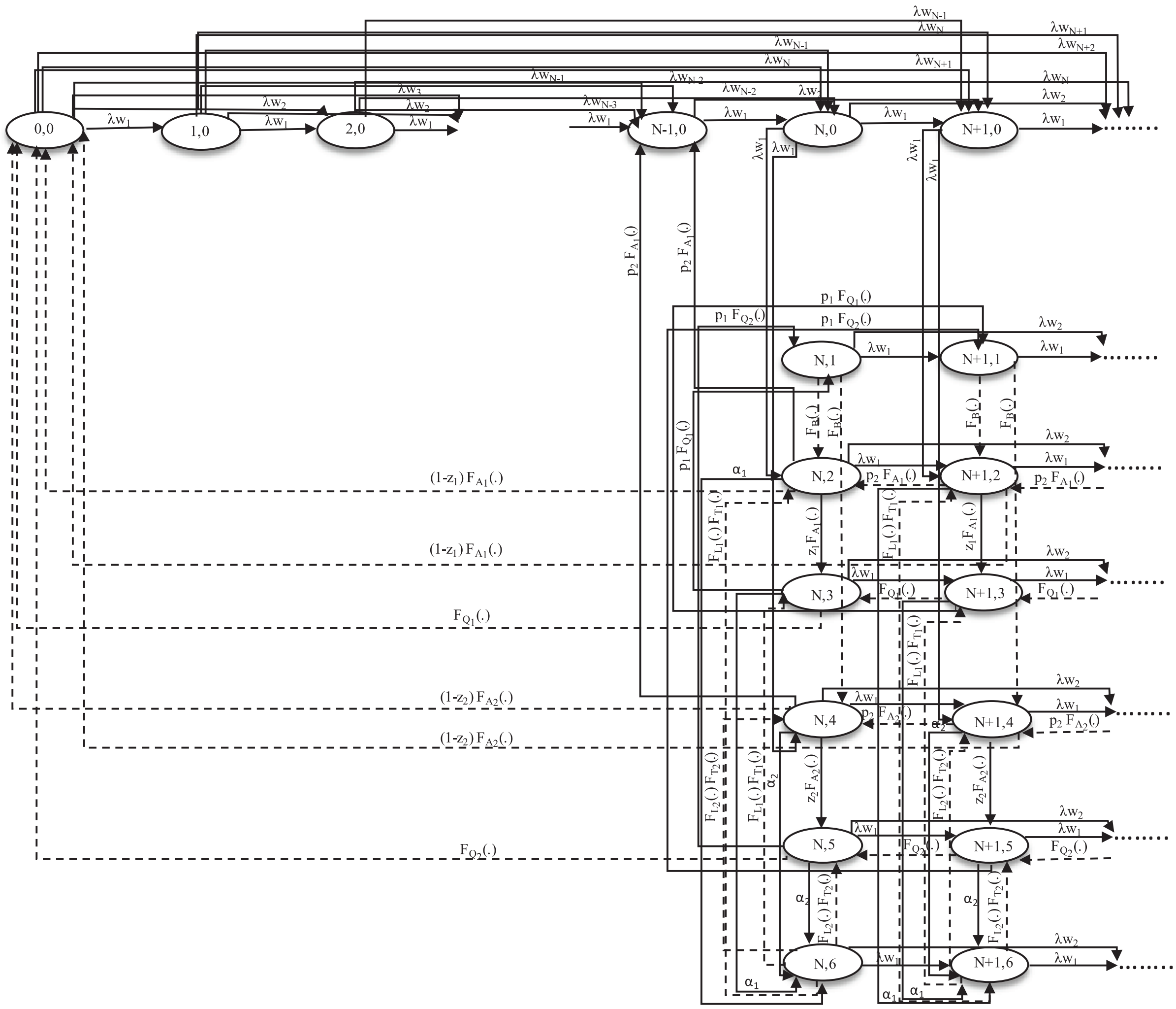

FIGURE 3. State Transition Diagram of the proposed model.

Let the random variable $\varsigma(t)$ define the various status of the server at time t. Then for $i = 1, 2$,

$$\varsigma(t) = \begin{cases} 0, & \text{idle} \\ 1, & \text{rendering the } i\text{th genre of service} \\ 2, & \text{rendering the } i\text{th genre of re-service} \\ 3, & \text{waiting for repair during } i\text{th genre of service} \\ 4, & \text{waiting for repair during } i\text{th genre of re-service} \\ 5, & \text{under repair during } i\text{th genre of service} \\ 6, & \text{under repair during } i\text{th genre of re-service} \\ 7, & \text{vacation.} \end{cases}$$

Let $N(t)$ be the number of customers present in the system at time t and $\{N(t), \varsigma(t); t \geq 0\}$ be a bivariate Markov Process, where $\{\varsigma(t), t \geq 0\}$ is an elemental process and $\{N(t), t \geq 0\}$ is a perceivable process having $\{0,1,\cdots\}$ as the state space.

The limiting probabilities for the steady-state analysis, given the elapsed and remaining times as x, y and r, v respectively, can be defined as follows:
For $n = 0, 1, \cdots N-1$,

$$I_n = \lim_{t\to\infty} Pr\{N(t) = n, \varsigma(t) = 0\}.$$

For $n = 1, 2, \cdots$ and $i = 1, 2$,

$$\begin{aligned}
f_{A_{i,n}}(x)\,\mathrm{d}x &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 1; x < A_i^0(t) \le x + \mathrm{d}x\big\}; && x > 0\\
f_{A_{i,n}^+}(r)\,\mathrm{d}r &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 1; r < A_i^+(t) \le r + \mathrm{d}r\big\}; && r > 0\\
f_{Q_{i,n}}(x)\,\mathrm{d}x &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 2; x < Q_i^0(t) \le x + \mathrm{d}x\big\}; && x > 0\\
f_{Q_{i,n}^+}(r)\,\mathrm{d}r &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 2; r < Q_i^+(t) \le r + \mathrm{d}r\big\}; && r > 0\\
f_{L_{i,n}}^{A_i}(x,y)\,\mathrm{d}y &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 3; y < L_i^0(t) \le y + \mathrm{d}y | A_i^0(t) = x\big\}; && x > 0; y > 0\\
f_{L_{i,n}^+}^{A_i}(r,v)\,\mathrm{d}v &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 3; v < L_i^+(t) \le v + \mathrm{d}v | A_i^+(t) = r\big\}; && r > 0; v > 0\\
f_{L_{i,n}}^{Q_i}(x,y)\,\mathrm{d}y &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 4; y < L_i^0(t) \le y + \mathrm{d}y | Q_i^0(t) = x\big\}; && x > 0; y > 0\\
f_{L_{i,n}^+}^{Q_i}(r,v)\,\mathrm{d}v &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 4; v < L_i^+(t) \le v + \mathrm{d}v | Q_i^+(t) = r\big\}; && r > 0; v > 0\\
f_{T_{i,n}}^{A_i}(x,y)\,\mathrm{d}y &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 5; y < T_i^0(t) \le y + \mathrm{d}y | A_i^0(t) = x\big\}; && x > 0; y > 0\\
f_{T_{i,n}^+}^{A_i}(r,v)\,\mathrm{d}v &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 5; v < T_i^+(t) \le v + \mathrm{d}v | A_i^+(t) = r\big\}; && r > 0; v > 0\\
f_{T_{i,n}}^{Q_i}(x,y)\,\mathrm{d}y &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 6; y < T_i^0(t) \le y + \mathrm{d}y | A_i^0(t) = x\big\}; && x > 0; y > 0\\
f_{T_{i,n}^+}^{Q_i}(r,v)\,\mathrm{d}v &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 6; v < T_i^+(t) \le v + \mathrm{d}v | A_i^+(t) = r\big\}; && r > 0; v > 0.
\end{aligned}$$

For $n = 0, 1, \cdots$,

$$\begin{aligned}
f_{B_n}(x)\,\mathrm{d}x &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 7; x < B^0(t) \le x + \mathrm{d}x\big\}; \quad x > 0\\
f_{B_n^+}(r)\,\mathrm{d}r &= \lim_{t\to\infty} Pr\big\{N(t) = n, \varsigma(t) = 7; r < B^+(t) \le r + \mathrm{d}r\big\}; \quad r > 0.
\end{aligned}$$

Further, in the steady-state it is assumed that $F_B(0) = 0$, $F_B(\infty) = 1$, $F_{A_i}(0) = 0$, $F_{A_i}(\infty) = 1$, $F_{Q_i}(0) = 0$, $F_{Q_i}(\infty) = 1$, $F_{L_i}(0) = 0$, $F_{L_i}(\infty) = 1$ and $F_{T_i}(0) = 0$, $F_{T_i}(\infty) = 1 (i = 1, 2)$. Also, $F_{A_i}(x)$, $F_{Q_i}(x)$ and $F_B(x)$ being continuous at $x = 0$, with $F_{L_i}(y)$ and $F_{T_i}(y)$ being continuous at $y = 0$, such that

$$\kappa_i(x)\,\mathrm{d}x = \frac{\mathrm{d}F_{A_i}(x)}{1 - F_{A_i}(x)}, \quad \omega_i(x)\,\mathrm{d}x = \frac{\mathrm{d}F_{Q_i}(x)}{1 - F_{Q_i}(x)} \quad \varphi(x)\,\mathrm{d}x = \frac{\mathrm{d}F_B(x)}{1 - F_B(x)}, \quad \chi_i(y)\,\mathrm{d}y = \frac{\mathrm{d}F_{L_i}(y)}{1 - F_{L_i}(y)}, \quad \varepsilon_i(y)\,\mathrm{d}y = \frac{\mathrm{d}F_{T_i}(y)}{1 - F_{T_i}(y)}$$

are the first order differential (hazard rate) function of A_i, Q_i, B, L_i and T_i respectively for $i = 1, 2$ [24].

4.1. Steady-state equation

Following the arguments of Takagi [43], some suitable transformations used to analyze the limiting behaviour of this model under the stability condition are stated below:

$$\begin{aligned}
\bar{\bar{f}}_{A_{i,n}}(x) &= \frac{f_{A_{i,n}}(x)}{1 - F_{A_i}(x)}; && n = 1, 2, \cdots\\
\bar{\bar{f}}_{Q_{i,n}}(x) &= \frac{f_{Q_{i,n}}(x)}{1 - F_{Q_i}(x)}; && n = 1, 2, \cdots
\end{aligned}$$

$$\bar{\bar{f}}_{B_n}(x) = \frac{f_{B_n}(x)}{1-F_B(x)}; \qquad n = 0,1,\cdots$$
$$\bar{\bar{f}}^{A_i}_{L_{i,n}}(x,y) = \frac{f^{A_i}_{L_{i,n}}(x,y)}{(1-F_{A_i}(x))(1-F_{L_i}(y))}; \quad n = 1,2,\cdots$$
$$\bar{\bar{f}}^{Q_i}_{L_{i,n}}(x,y) = \frac{f^{Q_i}_{L_{i,n}}(x,y)}{(1-F_{Q_i}(x))(1-F_{L_i}(y))}; \quad n = 1,2,\cdots$$
$$\bar{\bar{f}}^{A_i}_{T_{i,n}}(x,y) = \frac{f^{A_i}_{T_{i,n}}(x,y)}{(1-F_{A_i}(x))(1-F_{L_i}(y))}; \quad n = 1,2,\cdots$$
$$\bar{\bar{f}}^{Q_i}_{T_{i,n}}(x,y) = \frac{f^{Q_i}_{T_{i,n}}(x,y)}{(1-F_{Q_i}(x))(1-F_{L_i}(y))}; \quad n = 1,2,\cdots$$

After applying the transformations stated above the modified Kolmogorov forward equations (Cox [10]) can be set as,

$$\frac{\mathrm{d}}{\mathrm{d}x}\bar{\bar{f}}_{A_{i,n}}(x) + [\lambda+\alpha_i]\bar{\bar{f}}_{A_{i,n}}(x) = \lambda\sum_{l=1}^{n} w_l \bar{\bar{f}}_{A_{i,n-l}}(x) + \int_0^\infty \bar{\bar{f}}^{A_i}_{T_{i,n}}(x,y)\mathrm{d}F_{T_i}(y), \quad n = 1,2,\cdots \tag{4.1}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\bar{\bar{f}}_{Q_{i,n}}(x) + [\lambda+\alpha_i]\bar{\bar{f}}_{Q_{i,n}}(x) = \lambda\sum_{l=1}^{n} w_l \bar{\bar{f}}_{Q_{i,n-l}}(x) + \int_0^\infty \bar{\bar{f}}^{Q_i}_{T_{i,n}}(x,y)\mathrm{d}F_{T_i}(y), \quad n = 1,2,\cdots \tag{4.2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\bar{\bar{f}}_{B_n}(x) + \lambda\bar{\bar{f}}_{B_n}(x) = \lambda(1-\delta_{0,n})\sum_{l=1}^{n} w_l \bar{\bar{f}}_{B_{n-l}}(x), \quad n = 0,1,\cdots \tag{4.3}$$
$$\frac{\mathrm{d}}{\mathrm{d}y}\bar{\bar{f}}^{A_i}_{L_{i,n}}(x,y) + \lambda\bar{\bar{f}}^{A_i}_{L_{i,n}}(x,y) = \lambda\sum_{l=1}^{n} w_l \bar{\bar{f}}^{A_i}_{L_{i,n-l}}(x,y), \quad n = 1,2,\cdots \tag{4.4}$$
$$\frac{\mathrm{d}}{\mathrm{d}y}\bar{\bar{f}}^{Q_i}_{L_{i,n}}(x,y) + \lambda\bar{\bar{f}}^{Q_i}_{L_{i,n}}(x,y) = \lambda\sum_{l=1}^{n} w_l \bar{\bar{f}}^{Q_i}_{L_{i,n-l}}(x,y), \quad n = 1,2,\cdots \tag{4.5}$$
$$\frac{\mathrm{d}}{\mathrm{d}y}\bar{\bar{f}}^{A_i}_{T_{i,n}}(x,y) + \lambda\bar{\bar{f}}^{A_i}_{T_{i,n}}(x,y) = \lambda\sum_{l=1}^{n} w_l \bar{\bar{f}}^{A_i}_{T_{i,n-l}}(x,y), \quad n = 1,2,\cdots \tag{4.6}$$
$$\frac{\mathrm{d}}{\mathrm{d}y}\bar{\bar{f}}^{Q_i}_{T_{i,n}}(x,y) + \lambda\bar{\bar{f}}^{Q_i}_{T_{i,n}}(x,y) = \lambda\sum_{l=1}^{n} w_l \bar{\bar{f}}^{Q_i}_{T_{i,n-l}}(x,y), \quad n = 1,2,\cdots \tag{4.7}$$

$$\begin{aligned}\lambda I_n = \delta_{0,n}\Bigg[p_2\sum_{i=1}^{2}\Big\{(1-z_i)\int_0^\infty \bar{\bar{f}}_{A_{i,n+1}}(x)\,\mathrm{d}F_{A_i}(x) + \int_0^\infty \bar{\bar{f}}_{Q_{i,n+1}}(x)\,\mathrm{d}F_{Q_i}(x)\Big\} + \int_0^\infty \bar{\bar{f}}_{B_n}(x)\,\mathrm{d}F_B(x)\Bigg] \\ + \lambda(1-\delta_{0,n})\sum_{l=1}^{n} w_l I_{n-l}, \quad n = 0,1,\cdots,N-1\end{aligned} \tag{4.8}$$

where $\delta_{i,j}$ is the Kronecker's delta and $\bar{\bar{f}}_{A_{i,0}}(x) = 0$, $\bar{\bar{f}}_{Q_{i,0}}(x) = 0$, $\bar{\bar{f}}_{L_{i,0}}(y) = 0$, $\bar{\bar{f}}_{T_{i,0}}(y) = 0$, $\bar{\bar{f}}^{A_i}_{L_{i,0}}(x,y) = 0$, $\bar{\bar{f}}^{Q_i}_{L_{i,0}}(x,y) = 0$, $\bar{\bar{f}}^{A_i}_{T_{i,0}}(x,y) = 0$, $\bar{\bar{f}}^{Q_i}_{T_{i,0}}(x,y) = 0$ $(i = 1,2)$, $\bar{\bar{f}}_{B_0}(x) = 0$.

The equations established above are solved against some boundary condition set below,

at $x = 0$:

$$\bar{\bar{f}}_{A_{i,n}}(0) = p_2\left[q_i \sum_{i=1}^{2}\left\{(1-z_i)\int_0^\infty \bar{\bar{f}}_{A_{n+1,i}}(x)\,\mathrm{d}F_{A_i}(x) + \int_0^\infty \bar{\bar{f}}_{Q_{n+1,i}}(x)\,\mathrm{d}F_{Q_i}(x)\right\}\right] + q_i\int_0^\infty \bar{\bar{f}}_{B_n}(x)\,\mathrm{d}F_B(x); \quad n = 1,2,\cdots,N-1;\ (i=1,2) \tag{4.9}$$

$$\bar{\bar{f}}_{A_{i,n}}(0) = p_2\left[q_i \sum_{i=1}^{2}\left\{(1-z_i)\int_0^\infty \bar{\bar{f}}_{A_{n+1,i}}(x)\,\mathrm{d}F_{A_i}(x) + \int_0^\infty \bar{\bar{f}}_{Q_{n+1,i}}(x)\,\mathrm{d}F_{Q_i}(x)\right\}\right] + q_i\int_0^\infty \bar{\bar{f}}_{B_n}(x)\,\mathrm{d}F_B(x) + q_i\lambda\sum_{l=0}^{N-1} w_{n-l} I_l; \quad n = N, N+1,\cdots\ (i=1,2) \tag{4.10}$$

$$\bar{\bar{f}}_{Q_{n,i}}(0) = z_i\int_0^\infty \bar{\bar{f}}_{A_{n,i}}(x)\,\mathrm{d}F_{A_i}(x); \quad n = 1,2,\cdots \tag{4.11}$$

$$\bar{\bar{f}}_{B_n}(0) = p_1\left[\sum_{i=1}^{2}\left\{(1-z_i)\int_0^\infty \bar{\bar{f}}_{A_{n+1,i}}(x)\,\mathrm{d}F_{A_i}(x) + \int_0^\infty \bar{\bar{f}}_{Q_{n+1,i}}(x)\,\mathrm{d}F_{Q_i}(x)\right\}\right]; \quad n = 0,1,\cdots \tag{4.12}$$

at $y = 0$ and fixed x:

$$\bar{\bar{f}}^{A_i}_{L_{i,n}}(x,0) = \alpha_i \bar{\bar{f}}_{A_{i,n}}(x); \qquad (i=1,2) \tag{4.13}$$

$$\bar{\bar{f}}^{Q_i}_{L_{i,n}}(x,0) = \alpha_i \bar{\bar{f}}_{Q_{i,n}}(x); \qquad (i=1,2) \tag{4.14}$$

$$\bar{\bar{f}}^{A_i}_{T_{i,n}}(x;0) = \int_0^\infty \bar{\bar{f}}^{A_i}_{L_i}(x,y)\,\mathrm{d}F_{L_i}(y); \quad (i=1,2) \tag{4.15}$$

$$\bar{\bar{f}}^{Q_i}_{T_{i,n}}(x;0) = \int_0^\infty \bar{\bar{f}}^{Q_i}_{L_i}(x,y)\,\mathrm{d}F_{L_i}(y); \quad (i=1,2) \tag{4.16}$$

with the normalizing condition

$$\sum_{n=0}^{N-1} I_n + \sum_{n=1}^{\infty}\left[\sum_{i=1}^{2}\left\{\int_0^\infty \bar{\bar{f}}_{A_{i,n}}(x)[1-F_{A_i}(x)]\,\mathrm{d}x + \int_0^\infty \bar{\bar{f}}_{Q_{i,n}}(x)[1-F_{Q_i}(x)]\,\mathrm{d}x + \int_0^\infty\int_0^\infty \bar{\bar{f}}^{A_i}_{L_{i,n}}(x,y) \times[1-F_{A_i}(x)][1-F_{L_i}(y)]\,\mathrm{d}x\,\mathrm{d}y + \int_0^\infty\int_0^\infty \bar{\bar{f}}^{Q_i}_{L_{i,n}}(x,y)[1-F_{Q_i}(x)][1-F_{L_i}(y)]\,\mathrm{d}x\,\mathrm{d}y\right\}\right] + \sum_{n=1}^{\infty}\left[\sum_{i=1}^{2}\left\{\int_0^\infty\int_0^\infty \bar{\bar{f}}^{A_i}_{T_{i,n}}(x,y)[1-F_{A_i}(x)][1-F_{T_i}(y)]\,\mathrm{d}x\,\mathrm{d}y + \int_0^\infty\int_0^\infty \bar{\bar{f}}^{Q_i}_{T_{i,n}}(x,y)[1-F_{Q_i}(x)] \times[1-F_{T_i}(y)]\,\mathrm{d}x\,\mathrm{d}y\right\}\right] + \sum_{n=0}^{\infty}\int_0^\infty f_{B_n}(x)[1-F_B(x)]\mathrm{d}x = 1. \tag{4.17}$$

4.2. Solution of the model

The PGFs to solve the system of equations from (4.1) to (4.16) for $\mid s \mid < 1$ are defined below:

$$\bar{\bar{P}}_{A_i}(x;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}_{A_{n,i}}(x); \qquad \bar{\bar{P}}_{A_i}(0;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}_{A_{n,i}}(0)$$

$$\bar{\bar{P}}_{Q_i}(x;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}_{Q_{n,i}}(x); \qquad \bar{\bar{P}}_{Q_i}(0;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}_{Q_{n,i}}(0)$$

$$\bar{\bar{P}}_{B}(x;s) = \sum_{n=0}^{\infty} s^n \bar{\bar{f}}_{B_n}(x); \qquad \bar{\bar{P}}_{B}(0;s) = \sum_{n=0}^{\infty} s^n \bar{\bar{f}}_{B_n}(0)$$

$$\bar{\bar{P}}^{A_i}_{L_i}(x,y;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}^{A_i}_{L_{i,n}}(x,y); \qquad \bar{\bar{P}}^{A_i}_{L_i}(x,0;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}^{A_i}_{L_{i,n}}(x;0)$$

$$\bar{\bar{P}}^{Q_i}_{L_i}(x,y;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}^{Q_i}_{L_{i,n}}(x,y); \qquad \bar{\bar{P}}^{Q_i}_{L_i}(x,0;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}^{Q_i}_{L_{i,n}}(x;0)$$

$$\bar{\bar{P}}^{A_i}_{T_i}(x,y;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}^{A_i}_{T_{i,n}}(x,y); \qquad \bar{\bar{P}}^{A_i}_{T_i}(x,0;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}^{A_i}_{T_{i,n}}(x;0)$$

$$\bar{\bar{P}}^{Q_i}_{T_i}(x,y;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}^{Q_i}_{T_{i,n}}(x,y); \qquad \bar{\bar{P}}^{Q_i}_{T_i}(x,0;s) = \sum_{n=1}^{\infty} s^n \bar{\bar{f}}^{Q_i}_{T_{i,n}}(x;0)$$

$$I_N(s) = \sum_{n=0}^{N-1} s^n I_n.$$

Solving (4.1)–(4.7) as usual, a set of differential equation is attained as given below,

$$\bar{\bar{P}}_{A_i}(x;s) = \bar{\bar{P}}_{A_i}(0;s)\exp\{-K_i(s)x\}; \quad x > 0 \tag{4.18}$$

$$\bar{\bar{P}}_{Q_i}(x;s) = \bar{\bar{P}}_{Q_i}(0;s)\exp\{-K_i(s)x\}; \quad x > 0 \tag{4.19}$$

$$\bar{\bar{P}}_{B}(x;s) = \bar{\bar{P}}_{B}(0;s)\exp\{-\zeta(s)x\}; \quad x > 0 \tag{4.20}$$

$$\bar{\bar{P}}^{A_i}_{L_i}(x,y;s) = \bar{\bar{P}}^{A_i}_{L_i}(x,0;s)\exp\{-\zeta(s)y\}; \quad x > 0;\ y > 0 \tag{4.21}$$

$$\bar{\bar{P}}^{Q_i}_{L_i}(x,y;s) = \bar{\bar{P}}^{Q_i}_{L_i}(x,0;s)\exp\{-\zeta(s)y\}; \quad x > 0;\ y > 0 \tag{4.22}$$

$$\bar{\bar{P}}^{A_i}_{T_i}(x,y;s) = \bar{\bar{P}}^{A_i}_{T_i}(x,0;s)\exp\{-\zeta(s)y\}; \quad x > 0;\ y > 0 \tag{4.23}$$

$$\bar{\bar{P}}^{Q_i}_{T_i}(x,y;s) = \bar{\bar{P}}^{Q_i}_{T_i}(x,0;s)\exp\{-\zeta(s)y\}; \quad x > 0;\ y > 0 \tag{4.24}$$

where $K_i(s) = \zeta(s) + \alpha_i\{1 - F^*_{L_i}(\zeta(s))F^*_{T_i}(\zeta(s))\}$,$(i = 1, 2)$ and $\zeta(s) = \lambda(1 - P_{\Xi}(s))$. Equations (4.13) and (4.14) simplifies to the following,

$$\bar{\bar{P}}^{A_i}_{L_i}(x,0;s) = \alpha_i \bar{\bar{P}}_{A_i}(x;s); \quad (i = 1, 2) \tag{4.25}$$

$$\bar{\bar{P}}^{Q_i}_{L_i}(x,0;s) = \alpha_i \bar{\bar{P}}_{Q_i}(x;s); \quad (i = 1, 2). \tag{4.26}$$

Solving (4.15) and (4.16) and using (4.21) and (4.22) respectively gives the following,

$$\bar{\bar{P}}^{A_i}_{T_i}(x,0;s) = \bar{\bar{P}}^{A_i}_{L_i}(x,0;s)F^*_{L_i}(\zeta(s)); \quad (i = 1, 2) \tag{4.27}$$

$$\bar{\bar{P}}^{Q_i}_{T_i}(x,0;s) = \bar{\bar{P}}^{Q_i}_{L_i}(x,0;s)F^*_{L_i}(\zeta(s)); \quad (i = 1, 2). \tag{4.28}$$

Finally, using (4.25) and (4.26) in (4.27) and (4.28) gives,

$$\bar{\bar{P}}^{A_i}_{T_i}(x,0;s) = \alpha_i \bar{\bar{P}}_{A_i}(x;s)F^*_{L_i}(\zeta(s)); \quad (i = 1, 2) \tag{4.29}$$

$$\bar{\bar{P}}^{Q_i}_{T_i}(x,0;s) = \alpha_i \bar{\bar{P}}_{Q_i}(x;s)F^*_{L_i}(\zeta(s)); \quad (i = 1, 2). \tag{4.30}$$

Multiplying (4.11) by s^n and taking summation over $n = 1, 2, \cdots$, the following equation is derived,

$$\bar{\bar{P}}_{Q_i}(0; s) = z_i \bar{\bar{P}}_{A_i}(0; s) F^*_{A_i}(K_i(s)); \quad (i = 1, 2). \tag{4.31}$$

Similarly, equation (4.12) implies,

$$\begin{aligned} s\bar{\bar{P}}_B(0; s) = p_1 \Big[&(1 - z_1)\bar{\bar{P}}_{A_1}(0; s) F^*_{A_1}(K_1(s)) + \bar{\bar{P}}_{Q_1}(0; s) F^*_{Q_1}(K_1(s)) \Big] \\ &+ p_1 \Big[(1 - z_2)\bar{\bar{P}}_{A_2}(0; s) F^*_{A_2}(K_2(s)) + \bar{\bar{P}}_{Q_2}(0; s) F^*_{Q_2}(K_2(s)) \Big]. \end{aligned} \tag{4.32}$$

Multiplying (4.9) and (4.10) by appropriate powers of s and then summing over n, utilizing (4.8), (4.31), (4.32) with the result $\sum_{n=N}^{\infty} s^n \sum_{l=0}^{N-1} w_{n-l} I_l = \lambda I_0 - I_N(s)\zeta(s)$ the following is attained:

$$\begin{aligned} &\big[s - \{p_2 + p_1 F^*_B(\zeta(s))\}\big\{(1 - z_i) + z_i F^*_{Q_i}(K_i(s))\big\} q_i F^*_{A_i}(K_i(s))\big] \bar{\bar{P}}_{A_i}(0; s) + q_i s \zeta(s) I_N(s) \\ &\quad = \big[\{p_2 + p_1 F^*_B(\zeta(s))\}\big\{(1 - z_{i'}) + z_{i'} F^*_{Q_{i'}}(K_{i'}(s))\big\} q_i F^*_{A_{i'}}(K_{i'}(s))\big] \bar{\bar{P}}_{A_{i'}}(0; s) \end{aligned} \tag{4.33}$$

where $i = 1, 2$ and $i' = 2, 1$.

Solving (4.33) yields the following,

$$\bar{\bar{P}}_{A_i}(0; s) = \frac{s q_i \zeta(s) I_N(s)}{\Big[\{p_2 + p_1 F^*_B(\zeta(s))\}\Big\{\{(1 - z_1) + z_1 F^*_{Q_1}(K_1(s))\} q_1 F^*_{A_1}(K_1(s)) + \{(1 - z_2) + z_2 F^*_{Q_2}(K_2(s))\} q_2 F^*_{A_2}(K_2(s))\Big\} - s\Big]} (i = 1, 2). \tag{4.34}$$

Putting (4.34) in (4.31) gives,

$$\bar{\bar{P}}_{Q_i}(0; s) = \frac{s q_i z_i \zeta(s) I_N(s) F^*_{A_i}(K_i(s))}{\Big[\{p_2 + p_1 F^*_B(\zeta(s))\}\Big\{\{(1 - z_1) + z_1 F^*_{Q_1}(K_1(s))\} q_1 F^*_{A_1}(K_1(s)) + \{(1 - z_2) + z_2 F^*_{Q_2}(K_2(s))\} q_2 F^*_{A_2}(K_2(s))\Big\} - s\Big]} (i = 1, 2). \tag{4.35}$$

Substituting (4.34) and (4.35) in (4.32), results in the following,

$$\bar{\bar{P}}_B(0; s) = \frac{p_1 \zeta(s) I_N(s) \Big[\Big\{(1 - z_1) + z_1 F^*_{Q_1}(K_1(s))\Big\} q_1 F^*_{A_1}(K_1(s)) + \Big\{(1 - z_2) + z_2 F^*_{Q_2}(K_2(s))\Big\} q_2 F^*_{A_2}(K_2(s))\Big]}{\Big[\{p_2 + p_1 F^*_B(\zeta(s))\}\Big\{\{(1 - z_1) + z_1 F^*_{Q_1}(K_1(s))\} q_1 F^*_{A_1}(K_1(s)) + \Big\{(1 - z_2) + z_2 F^*_{Q_2}(K_2(s))\Big\} q_2 F^*_{A_2}(K_2(s))\Big\} - s\Big]}. \tag{4.36}$$

Utilizing (4.18), (4.25) and (4.34), provides the following,

$$\bar{\bar{P}}^{A_i}_{L_i}(x, 0; s) = \frac{s \alpha_i q_i \zeta(s) I_N(s) \exp\{-K_i(s)x\}}{\Big[\{p_2 + p_1 F^*_B(\zeta(s))\}\Big\{\{(1 - z_1) + z_1 F^*_{Q_1}(K_1(s))\} q_1 F^*_{A_1}(K_1(s)) + \{(1 - z_2) + z_2 F^*_{Q_2}(K_2(s))\} q_2 F^*_{A_2}(K_2(s))\Big\} - s\Big]} (i = 1, 2). \tag{4.37}$$

Similarly, equations (4.19), (4.26) and (4.35) gives,

$$\bar{\bar{P}}^{Q_i}_{L_i}(x, 0; s) = \frac{s \alpha_i q_i z_i \zeta(s) I_N(s) \exp\{-K_i(s)x\} F^*_{A_i}(K_i(s))}{\Big[\{p_2 + p_1 F^*_B(\zeta(s))\}\Big\{\{(1 - z_1) + z_1 F^*_{Q_1}(K_1(s))\} q_1 F^*_{A_1}(K_1(s)) + \{(1 - z_2) + z_2 F^*_{Q_2}(K_2(s))\} q_2 F^*_{A_2}(K_2(s))\Big\} - s\Big]} (i = 1, 2). \tag{4.38}$$

Equations (4.27) and (4.37) gives,

$$\bar{\bar{P}}_{T_i}^{A_i}(x,0;s) = \frac{s\alpha_i q_i \zeta(s) I_N(s) \exp\{-K_i(s)x\} F_{L_i}^*(\zeta(s))}{\begin{array}{r}\Big[\{p_2 + p_1 F_B^*(\zeta(s))\}\Big\{\{(1-z_1) + z_1 F_{Q_1}^*(K_1(s))\}q_1 F_{A_1}^*(K_1(s)) \\ + \{(1-z_2) + z_2 F_{Q_2}^*(K_2(s))\}q_2 F_{A_2}^*(K_2(s))\Big\} - s\Big]\end{array}} (i=1,2). \tag{4.39}$$

Equations (4.28) and (4.38) gives,

$$\bar{\bar{P}}_{T_i}^{Q_i}(x,0;s) = \frac{s\alpha_i q_i z_i \zeta(s) I_N(s) \exp\{-K_i(s)x\} F_{A_i}^*(K_i(s)) F_{L_i}^*(\zeta(s))}{\begin{array}{r}\Big[\{p_2 + p_1 F_B^*(\zeta(s))\}\Big\{\{(1-z_1) + z_1 F_{Q_1}^*(K_1(s))\}q_1 F_{A_1}^*(K_1(s)) \\ + \{(1-z_2) + z_2 F_{Q_2}^*(K_2(s))\}q_2 F_{A_2}^*(K_2(s))\Big\} - s\Big]\end{array}} (i=1,2). \tag{4.40}$$

Letting $s \to 1$ in (4.34) gives,

$$\bar{\bar{P}}_{A_i}(0;1) = \frac{\lambda q_i \mu_\Xi^{(1)} \left(\sum_{n=0}^{N-1} I_n\right)}{(1-\rho_u)} \tag{4.41}$$

where $\rho_u = q_1(\gamma_{A_1} + z_1\gamma_{Q_1})\Big(1 + \alpha_1\Big(\mu_{L_1}^{(1)} + \mu_{T_1}^{(1)}\Big)\Big) + q_2(\gamma_{A_2} + z_2\gamma_{Q_2})\Big(1 + \alpha_2\Big(\mu_{L_2}^{(1)} + \mu_{T_2}^{(1)}\Big)\Big) + p_1\gamma_v$ is the system's server utilization factor with $\gamma_{A_i} = \lambda\mu_\Xi^{(1)}\mu_{A_i}^{(1)}$, $\gamma_{Q_i} = \lambda\mu_\Xi^{(1)}\mu_{Q_i}^{(1)}$ $(i=1,2)$ and $\gamma_v = \lambda\mu_\Xi^{(1)}\mu_B^{(1)}$.

Subsequently, the following outcomes are derived as,

$$\bar{\bar{P}}_{A_i}(x;1) = \frac{\lambda q_i \mu_\Xi^{(1)} \left(\sum_{n=0}^{N-1} I_n\right)}{(1-\rho_u)}; \qquad (i=1,2) \tag{4.42}$$

$$\bar{\bar{P}}_{Q_i}(x;1) = \frac{\lambda q_i z_i \mu_\Xi^{(1)} \left(\sum_{n=0}^{N-1} I_n\right)}{(1-\rho_u)}; \qquad (i=1,2) \tag{4.43}$$

$$\bar{\bar{P}}_{B}(x;1) = \frac{\lambda p_1 \mu_\Xi^{(1)} \left(\sum_{n=0}^{N-1} I_n\right)}{(1-\rho_u)} \tag{4.44}$$

$$\bar{\bar{P}}_{L_i}^{A_i}(x,y;1) = \bar{\bar{P}}_{T_i}^{A_i}(x,y;1) = \frac{\lambda \alpha_i q_i \mu_\Xi^{(1)} \left(\sum_{n=0}^{N-1} I_n\right)}{(1-\rho_u)}; \qquad (i=1,2) \tag{4.45}$$

$$\bar{\bar{P}}_{L_i}^{Q_i}(x,y;1) = \bar{\bar{P}}_{T_i}^{Q_i}(x,y;1) = \frac{\lambda \alpha_i q_i z_i \mu_\Xi^{(1)} \left(\sum_{n=0}^{N-1} I_n\right)}{(1-\rho_u)}; \quad (i=1,2). \tag{4.46}$$

Equations (4.42)–(4.46) represents the server's stationary availability in the system's idle state.

5. Joint distribution of the state of the server and queue length under elapsed and remaining time

This section calculates joint and marginal PGFs of the state of the server and queue length in the form of *Theorems* stated below:

Theorem 5.1. *Under the stationary condition $\rho_u < 1$, the PGF of the queue size distribution during the idle period is given by,*

$$I_N(s) = \frac{(1-\rho_u)\left[\sum_{n=0}^{N-1} s^n \pi_n\right]}{\sum_{n=0}^{N-1} \pi_n}. \tag{5.1}$$

Proof. The queue length PGF during an idle period can be written as,

$$I_N(s)=\sum_{n=0}^{N-1} I_n s^n \tag{5.2}$$

where $I_n=\Gamma_0\pi_n$ $(n=0,1,\cdots,N-1)$ (*)

π_n being the probability that a batch of "n" units arrive in the system in an idle period and Γ_0 being the normalizing constant.

The value of the normalizing constant Γ_0 is obtained on simplification of the normalizing condition (4.17) as,

$$\Gamma_o=\frac{(1-\rho_u)}{\sum_{n=0}^{N-1}\pi_n}\cdot \tag{5.3}$$

Substituting (*) and (5.3) in (5.2) gives the required PGF. □

Note that (5.3) implies $\rho_u<1$, the stability condition for the continuation of a solution at the equilibrium of the underlying system.

Theorem 5.2. *Under the stability condition $\rho_u<1$, the joint PGF of server's state and queue size under elapsed time are given by,*

$$\bar{\bar{P}}_{A_i}(x;s)=\frac{q_i\zeta(s)(1-\rho_u)\exp\{-K_i(s)x\}\left(\sum_{n=0}^{N-1}s^{n+1}\pi_n\right)}{\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F_B^*(\zeta(s))\}\Big\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\Big\}-s\Big]}(i=1,2) \tag{5.4}$$

$$\bar{\bar{P}}_{Q_i}(x;s)=\frac{q_iz_i\zeta(s)(1-\rho_u)F_{A_i}^*(K_i(s))\exp\{-K_i(s)x\}\left(\sum_{n=0}^{N-1}s^{n+1}\pi_n\right)}{\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F_B^*(\zeta(s))\}\Big\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\Big\}-s\Big]}(i=1,2) \tag{5.5}$$

$$\bar{\bar{P}}_{B}(x;s)=\frac{p_1\zeta(s)(1-\rho_u)\exp\{-\zeta(s)x\}\left(\sum_{n=0}^{N-1}s^{n}\pi_n\right)\Big[\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s))+\{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\Big]}{\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F_B^*(\zeta(s))\}\Big\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\Big\}-s\Big]} \tag{5.6}$$

$$\bar{\bar{P}}_{L_i}^{A_i}(x,y;s)=\frac{\alpha_iq_i\zeta(s)(1-\rho_u)\exp\{-K_i(s)x\}\exp\{-\zeta(s)y\}\left(\sum_{n=0}^{N-1}s^{n+1}\pi_n\right)}{\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F_B^*(\zeta(s))\}\Big\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\Big\}-s\Big]}(i=1,2) \tag{5.7}$$

$$\bar{\bar{P}}_{L_i}^{Q_i}(x,y;s)=\frac{\alpha_iq_iz_i\zeta(s)(1-\rho_u)F_{A_i}^*(K_i(s))\exp\{-K_i(s)x\}\exp\{-\zeta(s)y\}\left(\sum_{n=0}^{N-1}s^{n+1}\pi_n\right)}{\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F_B^*(\zeta(s))\}\Big\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\Big\}-s\Big]}(i=1,2) \tag{5.8}$$

$$\bar{\bar{P}}_{T_i}^{A_i}(x,y;s)=\frac{\alpha_iq_i\zeta(s)(1-\rho_u)F_{L_i}^*(\zeta(s))\exp\{-K_i(s)x\}\exp\{-\zeta(s)y\}\left(\sum_{n=0}^{N-1}s^{n+1}\pi_n\right)}{\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F_B^*(\zeta(s))\}\Big\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\Big\}-s\Big]}(i=1,2) \tag{5.9}$$

$$\bar{\bar{P}}_{T_i}^{Q_i}(x,y;s) = \frac{\alpha_i q_i z_i \zeta(s)(1-\rho_u) F_{A_i}^*(K_i(s)) F_{L_i}^*(\zeta(s)) \exp\{-K_i(s)x\} \exp\{-\zeta(s)y\} \left(\sum_{n=0}^{N-1} s^{n+1}\pi_n\right)}{\left(\sum_{n=0}^{N-1}\pi_n\right)\left[\{p_2+p_1F_B^*(\zeta(s))\}\left\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\right\} - s\right]} (i=1,2). \quad (5.10)$$

Proof. Applying (4.34)–(4.40) in (4.18)–(4.24) gives the required PGFs defined above. □

Theorem 5.3. *Under the stationary environment $\rho_u < 1$, the double transform under elapsed time are given by,*

$$F_{A_i}^*(\eta;s) = \frac{q_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1} s^{n+1}\pi_n\right)\{1-F_{A_i}^*(\eta+K_i(s))\}}{(\eta+K_i(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\left[\{p_2+p_1F_B^*(\zeta(s))\}\left\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\right\} - s\right]} (i=1,2) \quad (5.11)$$

$$F_{Q_i}^*(\eta;s) = \frac{q_i z_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1} s^{n+1}\pi_n\right)F_{A_i}^*(K_i(s))\{1-F_{Q_i}^*(\eta+K_i(s))\}}{(\eta+K_i(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\left[\{p_2+p_1F_B^*(\zeta(s))\}\left\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\right\} - s\right]} (i=1,2) \quad (5.12)$$

$$F_B^*(\eta;s) = \frac{p_1\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1} s^{n}\pi_n\right)\{1-F_B^*(\eta+\zeta(s))\}\left[\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\right]}{(\eta+\zeta(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\left[\{p_2+p_1F_B^*(\zeta(s))\}\left\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\right\} - s\right]} \quad (5.13)$$

$$F_{L_i}^{A_i^*}(\eta,\theta;s) = \frac{\alpha_i q_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1} s^{n+1}\pi_n\right)\{1-F_{A_i}^*(\eta+K_i(s))\}\{1-F_{L_i}^*(\theta+\zeta(s))\}}{(\eta+K_i(s))(\theta+\zeta(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\left[\{p_2+p_1F_B^*(\zeta(s))\}\left\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\} q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\right\} - s\right]} (i=1,2) \quad (5.14)$$

$$F_{L_i}^{Q_i^*}(\eta,\theta;s) = \frac{\alpha_i q_i z_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1} s^{n+1}\pi_n\right)F_{A_i}^*(K_i(s))\{1-F_{Q_i}^*(\eta+K_i(s))\}\{1-F_{L_i}^*(\theta+\zeta(s))\}}{(\eta+K_i(s))(\theta+\zeta(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\left[\{p_2+p_1F_B^*(\zeta(s))\}\left\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\} q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\right\} - s\right]} (i=1,2) \quad (5.15)$$

$$F_{T_i}^{A_i^*}(\eta,\theta;s) = \frac{\alpha_i q_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1} s^{n+1}\pi_n\right)F_{L_i}^*(\zeta(s))\{1-F_{A_i}^*(\eta+K_i(s))\}\{1-F_{T_i}^*(\theta+\zeta(s))\}}{(\eta+K_i(s))(\theta+\zeta(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\left[\{p_2+p_1F_B^*(\zeta(s))\}\left\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\} q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\right\} - s\right]} (i=1,2) \quad (5.16)$$

$$F_{T_i}^{Q_i^*}(\eta,\theta;s) = \frac{\alpha_i q_i z_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1} s^{n+1}\pi_n\right)F_{A_i}^*(K_i(s))F_{L_i}^*(\zeta(s))\{1-F_{Q_i}^*(\eta+K_i(s))\}\{1-F_{T_i}^*(\theta+\zeta(s))\}}{(\eta+K_i(s))(\theta+\zeta(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\left[\{p_2+p_1F_B^*(\zeta(s))\}\left\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\} q_1F_{A_1}^*(K_1(s)) + \{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\right\} - s\right]}$$

$$(i=1,2). \quad (5.17)$$

Proof. The double transform in (5.11) is obtained as,

$$\begin{aligned} F^*_{A_i}(\eta;s) &= \sum_{n=1}^{\infty} s^n \int_0^{\infty} e^{-\eta x} f_{A_{i,n}}(x)\mathrm{d}x \\ &= \sum_{n=1}^{\infty} s^n \int_0^{\infty} e^{-\eta x} \bar{\bar{f}}_{A_{i,n}}(x)[1-F_{A_i}(x)]\mathrm{d}x \\ &= \int_0^{\infty} e^{-\eta x} \bar{\bar{P}}_{A_{i,n}}(x;s)[1-F_{A_i}(x)]\mathrm{d}x. \end{aligned}$$

Putting (5.4) in the above equation produces (5.11) on calculation.

On a similar note, equations (5.12) and (5.13) can be achieved by using (5.5) and (5.6), respectively.

Next, equation (5.14) is derived as,

$$\begin{aligned} F^{A_i^*}_{L_i}(\eta,\theta;s) &= \sum_{n=1}^{\infty} s^n \int_0^{\infty}\int_0^{\infty} e^{-\eta x}e^{-\theta y} f^{A_i}_{L_{i,n}}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{n=1}^{\infty} s^n \int_0^{\infty}\int_0^{\infty} e^{-\eta x}e^{-\theta y} \bar{\bar{f}}^{A_i}_{L_{i,n}}(x,y)[1-F_{A_i}(x)][1-F_{L_i}(y)]\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{\infty}\int_0^{\infty} e^{-\eta x}e^{-\theta y} \bar{\bar{P}}^{A_i}_{L_{i,n}}(x,y;s)[1-F_{A_i}(x)][1-F_{L_i}(y)]\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Putting (5.7) in the above expression simplifies it to (5.14).

Similarly, equation (5.15) is obtained by using (5.8).

Again, equation (5.16) is obtained as,

$$\begin{aligned} F^{A_i^*}_{T_i}(\eta,\theta;s) &= \sum_{n=1}^{\infty} s^n \int_0^{\infty}\int_0^{\infty} e^{-\eta x}e^{-\theta y} f^{A_i}_{T_{i,n}}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{n=1}^{\infty} s^n \int_0^{\infty}\int_0^{\infty} e^{-\eta x}e^{-\theta y} \bar{\bar{f}}^{A_i}_{T_{i,n}}(x,y)[1-F_{A_i}(x)][1-F_{T_i}(y)]\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{\infty}\int_0^{\infty} e^{-\eta x}e^{-\theta y} \bar{\bar{P}}^{A_i}_{T_{i,n}}(x,y;s)[1-F_{A_i}(x)][1-F_{T_i}(y)]\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Substituting (5.9) in the above expression produces (5.16) on simplification.

Likewise, equation (5.17) is derived by using (5.10). □

Theorem 5.4. *Under the stationary condition $\rho_u < 1$, the double transform under remaining time are given as,*

$$F^*_{A_i^+}(\eta;s) = \frac{q_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1} s^{n+1}\pi_n\right)\{F^*_{A_i}(K_i(s)) - F^*_{A_i}(\eta)\}}{(\eta - K_i(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s)) + \{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\} - s\Big]} (i=1,2) \tag{5.18}$$

$$F^*_{Q_i^+}(\eta;s) = \frac{q_iz_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1} s^{n+1}\pi_n\right)F^*_{A_i}(K_i(s))\{F^*_{Q_i}(K_i(s)) - F^*_{Q_i}(\eta)\}}{(\eta - K_i(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s)) + \{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\} - s\Big]} (i=1,2) \tag{5.19}$$

$$F^*_{B^+}(\eta;s)=\frac{\begin{array}{l}p_1\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1}s^n\pi_n\right)\{F^*_B(\zeta(s))-F^*_B(\eta)\}\\ \left[\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s))+\{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\right]\end{array}}{\begin{array}{r}(\eta-\zeta(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s))\\ +\{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]\end{array}} \tag{5.20}$$

$$F^{A_i^*}_{L_i^+}(\eta,\theta;s)=\frac{\alpha_iq_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1}s^{n+1}\pi_n\right)\{F^*_{A_i}(K_i(s))-F^*_{A_i}(\eta)\}\{F^*_{L_i}(\zeta(s))-F^*_{L_i}(\theta)\}}{\begin{array}{r}(\eta-K_i(s))(\theta-\zeta(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}\\ q_1F^*_{A_1}(K_1(s))+\{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]\end{array}}(i=1,2) \tag{5.21}$$

$$F^{Q_i^*}_{L_i^+}(\eta,\theta;s)=\frac{\alpha_iq_iz_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1}s^{n+1}\pi_n\right)F^*_{A_i}(K_i(s))\{F^*_{Q_i}(K_i(s))-F^*_{Q_i}(\eta)\}\{F^*_{L_i}(\zeta(s))-F^*_{L_i}(\theta)\}}{\begin{array}{r}(\eta-K_i(s))(\theta-\zeta(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}\\ q_1F^*_{A_1}(K_1(s))+\{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]\end{array}}(i=1,2) \tag{5.22}$$

$$F^{A_i^*}_{T_i^+}(\eta,\theta;s)=\frac{\alpha_iq_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1}s^{n+1}\pi_n\right)F^*_{L_i}(\zeta(s))\{F^*_{A_i}(K_i(s))-F^*_{A_i}(\eta)\}\{F^*_{T_i}(\zeta(s))-F^*_{T_i}(\theta)\}}{\begin{array}{r}(\eta-K_i(s))(\theta-\zeta(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}\\ q_1F^*_{A_1}(K_1(s))+\{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]\end{array}}(i=1,2) \tag{5.23}$$

$$F^{Q_i^*}_{T_i^+}(\eta,\theta;s)=\frac{\alpha_iq_iz_i\zeta(s)(1-\rho_u)\left(\sum_{n=0}^{N-1}s^{n+1}\pi_n\right)F^*_{A_i}(K_i(s))F^*_{L_i}(\zeta(s))\{F^*_{Q_i}(K_i(s))-F^*_{Q_i}(\eta)\}\{F^*_{T_i}(\zeta(s))-F^*_{T_i}(\theta)\}}{\begin{array}{r}(\eta-K_i(s))(\theta-\zeta(s))\left(\sum_{n=0}^{N-1}\pi_n\right)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}\\ q_1F^*_{A_1}(K_1(s))+\{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]\end{array}}$$
$$(i=1,2). \tag{5.24}$$

Proof. The double transform defined in (5.18) is derived as follows,

$$F^*_{A_i^+}(\eta;s)=\sum_{n=1}^{\infty}s^n\int_0^\infty e^{-\eta r}f_{A^+_{i,n}}(r)\,\mathrm{d}r.$$

Since service time random variable A_i has previously passed the time x, therefore the remaining service time distribution is given by,

$$Pr[r<A_i^+(t)\le r+\mathrm{d}r|A_i>x]=\frac{F_{A_i}(x+r)\,\mathrm{d}r}{\left[1-F_{A_i}(x)\right]}\cdot$$

Thus,

$$\begin{aligned}F^*_{A_i^+}(\eta;s)&=\sum_{n=1}^{\infty}s^n\int_0^\infty f_{A_{i,n}}(x)\,\mathrm{d}x\int_0^\infty e^{-\eta r}\frac{F_{A_i}(x+r)}{\left[1-F_{A_i}(x)\right]}\,\mathrm{d}r\\ &=\sum_{n=1}^{\infty}s^n\int_0^\infty \bar{\bar{f}}_{A_{i,n}}(x)\left[1-F_{A_i}(x)\right]\mathrm{d}x\int_0^\infty e^{-\eta r}\frac{F_{A_i}(x+r)}{\left[1-F_{A_i}(x)\right]}\,\mathrm{d}r\\ &=\int_0^\infty \bar{\bar{P}}_{A_i}(x;s)\,\mathrm{d}x\int_0^\infty e^{-\eta r}F_{A_i}(x+r)\,\mathrm{d}r.\end{aligned}$$

Putting the value of $\bar{\bar{P}}_{A_i}(x;s)$ from (5.4) in the above equation gives (5.18) on calculation.

Similarly, equations (5.19) and (5.20) are obtained utilizing (5.5) and (5.6) respectively.

Also,

$$F_{L_i^+}^{A_i^*}(\eta,\theta;s)=\sum_{n=1}^{\infty}s^n\int_0^{\infty}\int_0^{\infty}e^{-\eta r}e^{-\theta v}f_{L_{i,n}^+}^{A_i}(r,v)\,\mathrm{d}r\,\mathrm{d}v.$$

The remaining delay time distribution provided the service and delay time random variables A_i and L_i has already outstripped the time x and y respectively is given by,

$$Pr\big[(r<A_i^+(t)\le r+\,\mathrm{d}r|A_i>x)\cap\big(v<L_i^+(t)\le v+\mathrm{d}v|L_i>y\big)\big]=\frac{F_{A_i}(x+r)F_{L_i}(y+v)\,\mathrm{d}r\,\mathrm{d}v}{\big[1-F_{A_i}(x)\big]\big[1-F_{L_i}(y)\big]}\cdot$$

Therefore,

$$\begin{aligned}F_{L_i^+}^{A_i^*}(\eta,\theta;s)&=\sum_{n=1}^{\infty}s^n\int_0^{\infty}\int_0^{\infty}f_{L_{i,n}}^{A_i}(x,y)\,\mathrm{d}x\,\mathrm{d}y\int_0^{\infty}\int_0^{\infty}e^{-\eta r}e^{-\theta v}\frac{F_{A_i}(x+r)F_{L_i}(y+v)}{\big[1-F_{A_i}(x)\big]\big[1-F_{L_i}(y)\big]}\,\mathrm{d}r\,\mathrm{d}v\\&=\sum_{n=1}^{\infty}s^n\int_0^{\infty}\int_0^{\infty}\bar{\bar{f}}_{L_{i,n}}^{A_i}(x,y)\big[1-F_{A_i}(x)\big]\big[1-F_{L_i}(y)\big]\,\mathrm{d}x\,\mathrm{d}y\\&\quad\times\int_0^{\infty}\int_0^{\infty}e^{-\eta r}e^{-\theta v}\frac{F_{A_i}(x+r)F_{L_i}(y+v)}{\big[1-F_{A_i}(x)\big]\big[1-F_{L_i}(y)\big]}\,\mathrm{d}r\,\mathrm{d}v\\&=\int_0^{\infty}\int_0^{\infty}\bar{\bar{P}}_{L_i}^{A_i}(x,y;s)\,\mathrm{d}x\,\mathrm{d}y\int_0^{\infty}\int_0^{\infty}e^{-\eta r}e^{-\theta v}F_{A_i}(x+r)F_{L_i}(y+v)\,\mathrm{d}r\,\mathrm{d}v.\end{aligned}$$

Utilizing (5.7) in the above equation yields (5.21) on simplification.

On a similar note, equation (5.22) is derived using (5.8).

Again,

$$F_{T_i^+}^{A_i^*}(\eta,\theta;s)=\sum_{n=1}^{\infty}s^n\int_0^{\infty}\int_0^{\infty}e^{-\eta r}e^{-\theta v}f_{T_{i,n}^+}^{A_i}(r,v)\,\mathrm{d}r\,\mathrm{d}v.$$

The remaining repair time distribution provided the service and repair time random variables A_i and T_i has previously passed the time x and y respectively is given by,

$$Pr\big[\big(r<A_i^+(t)\le r+\,\mathrm{d}r|A_i>x\big)\cap\big(v<T_i^+(t)\le v+\mathrm{d}v|T_i>y\big)\big]=\frac{F_{A_i}(x+r)F_{T_i}(y+v)\,\mathrm{d}r\,\mathrm{d}v}{\big[1-F_{A_i}(x)\big]\big[1-F_{T_i}(y)\big]}\cdot$$

Therefore,

$$\begin{aligned}F_{T_i^+}^{A_i^*}(\eta,\theta;s)&=\sum_{n=1}^{\infty}s^n\int_0^{\infty}\int_0^{\infty}f_{T_{i,n}}^{A_i}(x,y)\,\mathrm{d}x\,\mathrm{d}y\int_0^{\infty}\int_0^{\infty}e^{-\eta r}e^{-\theta v}\frac{F_{A_i}(x+r)F_{T_i}(y+v)}{\big[1-F_{A_i}(x)\big]\big[1-F_{T_i}(y)\big]}\,\mathrm{d}r\,\mathrm{d}v\\&=\sum_{n=1}^{\infty}s^n\int_0^{\infty}\int_0^{\infty}\bar{\bar{f}}_{T_{i,n}}^{A_i}(x,y)\big[1-F_{A_i}(x)\big]\big[1-F_{T_i}(y)\big]\,\mathrm{d}x\,\mathrm{d}y\\&\quad\times\int_0^{\infty}\int_0^{\infty}e^{-\eta r}e^{-\theta v}\frac{F_{A_i}(x+r)F_{T_i}(y+v)}{\big[1-F_{A_i}(x)\big]\big[1-F_{T_i}(y)\big]}\,\mathrm{d}r\,\mathrm{d}v\\&=\int_0^{\infty}\int_0^{\infty}\bar{\bar{P}}_{T_i}^{A_i}(x,y;s)\,\mathrm{d}x\,\mathrm{d}y\int_0^{\infty}\int_0^{\infty}e^{-\eta r}e^{-\theta v}F_{A_i}(x+r)F_{T_i}(y+v)\,\mathrm{d}r\,\mathrm{d}v.\end{aligned}$$

Using (5.9) in the above equation gives (5.23) on calculation.

Likewise (5.24) is derived using (5.10). □

Theorem 5.5. *Under the steady-state condition $\rho_u < 1$, the marginal PGFs of the server's state and queue length are given by,*

$$\bar{\bar{P}}_{A_i}(s) = \frac{q_i\zeta(s)(1-\rho_u)\Big(\sum_{n=0}^{N-1} s^{n+1}\pi_n\Big)\{1-F^*_{A_i}(K_i(s))\}}{K_i(s)\Big(\sum_{n=0}^{N-1}\pi_n\Big)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s)) + \{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]} (i=1,2) \quad (5.25)$$

$$\bar{\bar{P}}_{Q_i}(s) = \frac{q_iz_i\zeta(s)(1-\rho_u)\Big(\sum_{n=0}^{N-1} s^{n+1}\pi_n\Big)F^*_{A_i}(K_i(s))\{1-F^*_{Q_i}(K_i(s))\}}{K_i(s)\Big(\sum_{n=0}^{N-1}\pi_n\Big)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s)) + \{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]} (i=1,2) \quad (5.26)$$

$$\bar{\bar{P}}_{B}(s) = \frac{p_1(1-\rho_u)\Big(\sum_{n=0}^{N-1} s^{n}\pi_n\Big)\{1-F^*_B(\zeta(s))\}\Big[\Big\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\Big\}q_1F^*_{A_1}(K_1(s)) + \Big\{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\Big\}q_2F^*_{A_2}(K_2(s))\Big]}{\Big(\sum_{n=0}^{N-1}\pi_n\Big)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s)) + \Big\{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\Big\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]} \quad (5.27)$$

$$\bar{\bar{P}}^{A_i}_{L_i}(s) = \frac{\alpha_iq_i(1-\rho_u)\Big(\sum_{n=0}^{N-1} s^{n+1}\pi_n\Big)\{1-F^*_{A_i}(K_i(s))\}\{1-F^*_{L_i}(\zeta(s))\}}{K_i(s)\Big(\sum_{n=0}^{N-1}\pi_n\Big)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s)) + \{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]} (i=1,2) \quad (5.28)$$

$$\bar{\bar{P}}^{Q_i}_{L_i}(s) = \frac{\alpha_iq_iz_i(1-\rho_u)\Big(\sum_{n=0}^{N-1} s^{n+1}\pi_n\Big)F^*_{A_i}(K_i(s))\{1-F^*_{Q_i}(K_i(s))\}\{1-F^*_{L_i}(\zeta(s))\}}{K_i(s)\Big(\sum_{n=0}^{N-1}\pi_n\Big)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s)) + \{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]} (i=1,2) \quad (5.29)$$

$$\bar{\bar{P}}^{A_i}_{T_i}(s) = \frac{\alpha_iq_i(1-\rho_u)\Big(\sum_{n=0}^{N-1} s^{n+1}\pi_n\Big)F^*_{L_i}(\zeta(s))\{1-F^*_{A_i}(K_i(s))\}\{1-F^*_{T_i}(\zeta(s))\}}{K_i(s)\Big(\sum_{n=0}^{N-1}\pi_n\Big)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s)) + \{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]} (i=1,2) \quad (5.30)$$

$$\bar{\bar{P}}^{Q_i}_{T_i}(s) = \frac{\alpha_iq_iz_i(1-\rho_u)\Big(\sum_{n=0}^{N-1} s^{n+1}\pi_n\Big)F^*_{A_i}(K_i(s))F^*_{L_i}(\zeta(s))\{1-F^*_{Q_i}(K_i(s))\}\{1-F^*_{T_i}(\zeta(s))\}}{K_i(s)\Big(\sum_{n=0}^{N-1}\pi_n\Big)\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s)) + \{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]} (i=1,2). \quad (5.31)$$

Proof. The marginal PGFs defined above can be procured by considering the double transform under the elapsed time or remaining time, *i.e.* by setting $\eta \to 0$ and $\theta \to 0$ either in (5.11)–(5.17) or (5.18)–(5.24). □

Theorem 5.6. *Under the stability environment $\rho_u < 1$, the LST of the marginal Probability Density Function (PDF)s of the server's state, and queue length are given by,*

$$F^*_{A_i}(\eta;1) = \frac{\lambda\mu^{(1)}_{\Xi}q_i\{1-F^*_{A_i}(\eta)\}}{\eta} \qquad (i=1,2) \quad (5.32)$$

$$F^*_{Q_i}(\eta;1) = \frac{\lambda\mu^{(1)}_{\Xi} q_i z_i\{1 - F^*_{Q_i}(\eta)\}}{\eta} \qquad (i = 1,2) \tag{5.33}$$

$$F^*_B(\eta;1) = \frac{\lambda\mu^{(1)}_{\Xi} p_1\{1 - F^*_B(\eta)\}}{\eta} \tag{5.34}$$

$$F^{A^*_i}_{L_i}(\eta,\theta;1) = \frac{\lambda\mu^{(1)}_{\Xi}\alpha_i q_i\{1 - F^*_{A_i}(\eta)\}\{1 - F^*_{L_i}(\theta)\}}{\eta\theta} \qquad (i = 1,2) \tag{5.35}$$

$$F^{Q^*_i}_{L_i}(\eta,\theta;1) = \frac{\lambda\mu^{(1)}_{\Xi}\alpha_i q_i z_i\{1 - F^*_{Q_i}(\eta)\}\{1 - F^*_{L_i}(\theta)\}}{\eta\theta} \qquad (i = 1,2) \tag{5.36}$$

$$F^{A^*_i}_{T_i}(\eta,\theta;1) = \frac{\lambda\mu^{(1)}_{\Xi}\alpha_i q_i\{1 - F^*_{A_i}(\eta)\}\{1 - F^*_{T_i}(\theta)\}}{\eta\theta} \qquad (i = 1,2) \tag{5.37}$$

$$F^{Q^*_i}_{T_i}(\eta,\theta;1) = \frac{\lambda\mu^{(1)}_{\Xi}\alpha_i q_i z_i\{1 - F^*_{Q_i}(\eta)\}\{1 - F^*_{T_i}(\theta)\}}{\eta\theta} \qquad (i = 1,2). \tag{5.38}$$

Proof. The LST of the joint PDFs defined above follows by considering the double transform under the elapsed time or remaining time, *i.e.* either by letting $s \to 1$ in (5.11)–(5.17) or (5.18)–(5.24). □

6. Distribution of server's state and queue length at random and service completion epoch

This section finds the PGF of the queue length distribution at random and departure epoch along with the mean system size of the underlying system.

Theorem 6.1. *Under the stationary condition $\rho_u < 1$, the PGF of server's state and queue length*

(i) *at random epoch $P_N(s)$ is given by,*

$$P_N(s) = \left[\frac{\sum_{n=0}^{N-1} s^n\pi_n}{\sum_{n=0}^{N-1}\pi_n}\right] \frac{\begin{aligned}&(1-\rho_u)(1-s)[p_2 + p_1F^*_B(\zeta(s))]\\ &\left[\{(1-z_1) + z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s)) + \{(1-z_2) + z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\right]\end{aligned}}{\begin{aligned}&\Big[\{p_2 + p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1) + z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s))\\ &\quad + \{(1-z_2) + z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\} - s\Big]\end{aligned}} \tag{6.1}$$

(ii) *at service completion epoch $P_\Psi(s)$ is given by,*

$$P_\Psi(s) = \left[\frac{\sum_{n=0}^{N-1} s^n\pi_n}{\sum_{n=0}^{N-1}\pi_n}\right] \frac{\begin{aligned}&(1-\rho_u)(1-P_\Xi(s))[p_2 + p_1F^*_B(\zeta(s))]\\ &\left[\{(1-z_1) + z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s)) + \{(1-z_2) + z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\right]\end{aligned}}{\begin{aligned}&\mu^{(1)}_{\Xi}\Big[\{p_2 + p_1F^*_B(\zeta(s))\}\Big\{\{(1-z_1) + z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s))\\ &\quad + \{(1-z_2) + z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\} - s\Big]\end{aligned}}. \tag{6.2}$$

Proof. (i) Since π_n is defined as the probability of arrival of a batch of "n" units in the system during an idle period, it satisfies the recursive relation elemented below:

$\pi_n = \sum_{l=1}^{n} w_l\pi_{n-l}$ $(n = 1,2,\cdots,N-1)$ and $\pi_0 = 1$.

Let, ϕ_n $(n = 0,1,\cdots,N-1)$ be the probability that a batch of n units are already there in the system during an idle period. This results in conditioning the number of arrivals in $(I_n = \Gamma_0\pi_n)$ and as such the following is achieved,

$$\phi_n = \frac{I_n}{\sum_{n=0}^{N-1} I_n} = \frac{\pi_n}{\sum_{n=0}^{N-1}\pi_n};\ n = 0,1,\cdots,(N-1) \tag{6.3}$$

where $\sum_{n=0}^{N-1}\pi_n$ represents the mean number of batch arrivals in an idle period.
Thus,

$$\phi_{N}(s)=\sum_{n=0}^{N-1}s^n\phi_n=\frac{\sum_{n=0}^{N-1}s^n\pi_n}{\sum_{n=0}^{N-1}\pi_n}\,[\text{using (6.3)}]. \tag{6.4}$$

Equating (6.4) in (5.1) gives

$$I_{N}(s)=\phi_{N}(s)(1-\rho_u). \tag{6.5}$$

The PGF at random epoch follows directly by equating (6.5) and (5.25)–(5.31) in the formula (**) (given below).

$$P_N(s)=I_N(s)+s\bar{\bar{P}}_B(s)+\sum_{i=1}^{2}\Big[\bar{\bar{P}}_{A_i}(s)+\bar{\bar{P}}_{Q_i}(s)+\bar{\bar{P}}^{A_i}_{L_i}(s)+\bar{\bar{P}}^{Q_i}_{L_i}(s)+\bar{\bar{P}}^{A_i}_{T_i}(s)+\bar{\bar{P}}^{Q_i}_{T_i}(s)\Big]. \tag{**}$$

(ii) Following the argument of PASTA [48], an unit just after a departure witnesses "m" units in the queue iff there were "$m+1$" units either in FGS(FGRS)/SGS(SGRS) or vacation just before departing.
If Ψ_m is the probability of "m" units being present in the queue at service completion epoch, then,

$$\begin{aligned}\Psi_m=\epsilon\Bigg[&p_2\sum_{i=1}^{2}\Bigg\{(1-z_i)\left(\int_0^\infty \bar{\bar{f}}_{A_{m+1,i}}(x)\,\mathrm{d}F_{A_i}(x)\right)+\left(\int_0^\infty \bar{\bar{f}}_{Q_{m+1,i}}(x)\,\mathrm{d}F_{Q_i}(x)\right)\Bigg\}\\&+\int_0^\infty \bar{\bar{f}}_{B_m}(x)\,\mathrm{d}F_B(x)\Bigg]\ m=0,1,\cdots\end{aligned} \tag{6.6}$$

where ϵ is a normalizing constant.

Multiplying (6.6) by s^m and taking summation overall values of $(m=0,1,\cdots)$, and after that using (4.18), (4.19) and (4.20) results in the following:

$$P_\Psi(s)=\frac{\begin{aligned}&\epsilon\zeta(s)\Big[\textstyle\sum_{n=0}^{N-1}s^n\pi_n\Big][p_2+p_1F_B^*(\zeta(s))]\\&\Big[\Big\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\Big\}q_1F^*_{A_1}(K_1(s))+\Big\{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\Big\}q_2F^*_{A_2}(K_2(s))\Big]\end{aligned}}{\begin{aligned}\Big[\textstyle\sum_{n=0}^{N-1}\pi_n\Big]\Big[&\{p_2+p_1F_B^*(\zeta(s))\}\Big\{\{(1-z_1)+z_1F^*_{Q_1}(K_1(s))\}q_1F^*_{A_1}(K_1(s))\\&+\{(1-z_2)+z_2F^*_{Q_2}(K_2(s))\}q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]\end{aligned}}\cdot \tag{6.7}$$

Letting $s\to1$ in (6.7) and using the L'Hospital's rule gives the value of the normalizing constant as,

$$\epsilon=\frac{(1-\rho_u)}{\lambda\mu_\Xi^{(1)}}\cdot$$

Thus, replacing ϵ by $\frac{(1-\rho_u)}{\lambda\mu_\Xi^{(1)}}$ in (6.7) gives the departure epoch PGF defined in (6.2). □

Corollary 6.2. *If Ψ_0 stands for the probability of no unit waiting in the system at departure epoch. Then setting $s=0$ in (6.2),*

$$\Psi_0=P_\Psi(0)=\frac{(1-\rho_u)\pi_0}{\mu_\Xi^{(1)}\Big[\sum_{n=0}^{N-1}\pi_n\Big]}\cdot$$

Utilizing (5.3) in the above equation establishes a relation expressed below,

$$\Gamma_o=\mu_\Xi^{(1)}\Psi_0$$

which implies that a random observer is more likely to find the system empty than a departing customer leaving the system.

Remark 6.3. (i) $P_N(s)$ is the decomposed PGF of two independent random variables:

- PGF of the steady-state queue size distribution at random epoch of an $M^X/\binom{G_1}{G_2}/1$ queue with re-service approach subject to service interruption and delayed repair.
- Queue length PGF owing to N-policy.

(ii) $P_\Psi(s)$ is the convolution of two independent r.v as given below,

$$P_\Psi(s) = P_C(s) \times P_N(s) \tag{6.8}$$

where, $P_C(s) = \frac{1-P_\Xi(s)}{\mu_\Xi^{(1)}(1-s)}$, is the PGF of the number of units placed before a random test unit in a batch, in which the test unit arrives. This is the backward recurrence time of the discrete-time renewal process where renewal points are generated by the arrival size r.v owing to the randomness of the arriving batch size. and $P_N(s)$ is the random epoch PGF.

Equation (6.8) entails the decomposition property, which holds for different vacation models, also holds for the model under consideration.

Theorem 6.4. *Let Ω be the system size either at a departure epoch or at the termination epoch of an idle period. Then under the stability condition $\rho_u < 1$, its PGF $P_\Omega(s)$ is given by,*

$$P_\Omega(s) = \frac{\begin{aligned}(1-\rho_u)\Big(\textstyle\sum_{n=0}^{N-1} s^n\pi_n\Big)\Big[\{p_2 + p_1 F_B^*(\zeta(s))\}\Big\{\big\{(1-z_1) + z_1 F_{Q_1}^*(K_1(s))\big\}q_1 F_{A_1}^*(K_1(s))\\ + \big\{(1-z_2) + z_2 F_{Q_2}^*(K_2(s))\big\}q_2 F_{A_2}^*(K_2(s))\Big\} - sP_\Xi(s)\Big]\end{aligned}}{\begin{aligned}\Big(1+\mu_\Xi^{(1)}-\rho_u\Big)\Big(\textstyle\sum_{n=0}^{N-1}\pi_n\Big)\Big[\{p_2 + p_1 F_B^*(\zeta(s))\}\Big\{\big\{(1-z_1) + z_1 F_{Q_1}^*(K_1(s))\big\}q_1 F_{A_1}^*(K_1(s))\\ + \big\{(1-z_2) + z_2 F_{Q_2}^*(K_2(s))\big\}q_2 F_{A_2}^*(K_2(s))\Big\} - s\Big]\end{aligned}}. \tag{6.9}$$

Proof. The result follows straight from the decomposition property of an $M^X/\binom{G_1}{G_2}/1(UR)/G(BS)/Vs/N-Policy$. Thus,

$$P_\Omega(s) = \phi_N(s) P_{\Omega_0}(s) \tag{6.10}$$

where $P_{\Omega_0}(s)$ is the stationary PGF of the queue size distribution at departure epoch of a customer or at termination epoch of an idle period for an $M^X/\binom{G_1}{G_2}/1(UR)/G(BS)/Vs$. It is easily obtainable by following the well established result of Gross and Harris [37] stated below,

$$P_{\Omega_0}(s) = \frac{\tau[P_\xi(s) - sP_\Xi(s)]}{[P_\xi(s) - s]} \tag{6.11}$$

where τ is the normalizing constant to be determined and $P_\xi(s)$ is the PGF of a batch of customers arrived during the actual service time A. Therefore,

$$\begin{aligned}P_\xi(s) = \{p_2 + p_1 F_B^*(\zeta(s))\}\Big\{\big\{(1-z_1) + z_1 F_{Q_1}^*(K_1(s))\big\}q_1 F_{A_1}^*(K_1(s))\\ + \big\{(1-z_2) + z_2 F_{Q_2}^*(K_2(s))\big\}q_2 F_{A_2}^*(K_2(s))\Big\}.\end{aligned} \tag{6.12}$$

Utilizing (6.12) in (6.11) yields,

$$P_{\Omega_0}(s) = \frac{\begin{aligned}\tau\Big[\{p_2 + p_1 F_B^*(\zeta(s))\}\Big\{\big\{(1-z_1) + z_1 F_{Q_1}^*(K_1(s))\big\}q_1 F_{A_1}^*(K_1(s))\\ + \big\{(1-z_2) + z_2 F_{Q_2}^*(K_2(s))\big\}q_2 F_{A_2}^*(K_2(s))\Big\} - sP_\Xi(s)\Big]\end{aligned}}{\begin{aligned}\Big[\{p_2 + p_1 F_B^*(\zeta(s))\}\Big\{\big\{(1-z_1) + z_1 F_{Q_1}^*(K_1(s))\big\}q_1 F_{A_1}^*(K_1(s))\\ + \big\{(1-z_2) + z_2 F_{Q_2}^*(K_2(s))\big\}q_2 F_{A_2}^*(K_2(s))\Big\} - s\Big]\end{aligned}}. \tag{6.13}$$

Since, $P_{\Omega_0}(1) = 1$, therefore,

$$\tau = \frac{(1-\rho_u)}{(1+\mu_{\Xi}^{(1)}-\rho_u)}\cdot \tag{6.14}$$

Hence, the PGF (6.9) follows by substituting (6.13), (6.14) and (6.4) in (6.10). □

Theorem 6.5. *Under the steady-state condition $\rho_u < 1$, the mean number of customers present in the system is given by,*

(i) *at random epoch*

$$\begin{aligned}\mu_N = \rho_u &+ \frac{\left(\lambda\mu_{\Xi}^{(1)}\right)^2\left[q_1\left(\mu_{A_1}^{(2)}+z_1\mu_{Q_1}^{(2)}\right)\left\{1+\alpha_1\left(\mu_{L_1}^{(1)}+\mu_{T_1}^{(1)}\right)\right\}^2+q_2\left(\mu_{A_2}^{(2)}+z_2\mu_{Q_2}^{(2)}\right)\left\{1+\alpha_2\left(\mu_{L_2}^{(1)}+\mu_{T_2}^{(1)}\right)\right\}^2+p_1\mu_B^{(2)}\right]}{2(1-\rho_u)}\\
&+\frac{\left(\lambda\mu_{\Xi}^{(1)}\right)^2\left[\alpha_1 q_1\left(\mu_{A_1}^{(1)}+z_1\mu_{Q_1}^{[1]}\right)\left(\mu_{L_1}^{(2)}+\mu_{T_1}^{(2)}+2\mu_{L_1}^{(1)}\mu_{T_1}^{(1)}\right)+\alpha_2 q_2\left(\mu_{A_2}^{(1)}+z_2\mu_{Q_2}^{(1)}\right)\left(\mu_{L_2}^{(2)}+\mu_{T_2}^{(2)}+2\mu_{L_2}^{(1)}\mu_{T_2}^{(1)}\right)\right]}{2(1-\rho_u)}\\
&+\frac{\left(\lambda\mu_{\Xi}^{(1)}\right)^2\left[p_1\mu_B^{(1)}\left\{q_1 z_1\mu_{A_1}^{(1)}\mu_{Q_1}^{(1)}\left\{1+\alpha_1\left(\mu_{L_1}^{(1)}+\mu_{T_1}^{(1)}\right)\right\}^2+q_2 z_2\mu_{A_2}^{(1)}\mu_{Q_2}^{(1)}\left\{1+\alpha_2\left(\mu_{L_2}^{(1)}+\mu_{T_2}^{(1)}\right)\right\}^2\right\}\right]}{(1-\rho_u)}\\
&+\frac{\rho_u\mu_{\Xi}^{(2)}}{2\mu_{\Xi}^{(1)}(1-\rho_u)}+\frac{\sum_{n=0}^{N-1}n\pi_n}{\sum_{n=0}^{N-1}\pi_n}\end{aligned} \tag{6.15}$$

(ii) *at departure epoch*

$$\mu_D = \mu_N + \frac{\mu_{\Xi}^{(2)}}{2\mu_{\Xi}^{(1)}}\cdot \tag{6.16}$$

Proof. Equations (6.15) and (6.16) are attained by differentiating (6.1) and (6.2), *i.e.* the random and departure epoch PGF respectively with respect to s and then setting $s \to 1$.

Special cases:

(1) $M^X/\binom{G_1}{G_2}/1(UR)/Re-service/G(BS)/V_s$ queue *i.e.*, if there is no threshold in the system. In that case $N = 1$ and consequently (6.1) reduces to the following,

$$P_N(s) = \frac{(1-\rho_u)(1-s)[p_2+p_1F_B^*(\zeta(s))]\left[\left\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\right\}q_1F_{A_1}^*(K_1(s))+\left\{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\right\}q_2F_{A_2}^*(K_2(s))\right]}{\left[\{p_2+p_1F_B^*(\zeta(s))\}\left\{\{(1-z_1)+z_1F_{Q_1}^*(K_1(s))\}q_1F_{A_1}^*(K_1(s))+\{(1-z_2)+z_2F_{Q_2}^*(K_2(s))\}q_2F_{A_2}^*(K_2(s))\right\}-s\right]}$$

where the server utilization factor is $\rho_u = q_1(\gamma_{A_1}+z_1\gamma_{Q_1})\left(1+\alpha_1\left(\mu_{L_1}^{(1)}+\mu_{T_1}^{(1)}\right)\right) + q_2(\gamma_{A_2}+z_2\gamma_{Q_2})\left(1+\alpha_2\left(\mu_{L_2}^{(1)}+\mu_{T_2}^{(1)}\right)\right) + p_1\gamma_v$. The result thus obtained coincides with the result of Begum and Choudhury [4] (without re-service).

(2) $M^X/\begin{pmatrix}G_1\\G_2\end{pmatrix}/1/Re-service$ queue when there is no operating policy, vacation and breakdown *i.e.*, $N=1$, $p_1=0$ and $\alpha_i=0 (i=1,2)$ then (6.1) reduces to,

$$P_N(s)=\frac{(1-\rho_u)(1-s)\Big[\Big\{(1-z_1)+z_1F^*_{Q_1}(\zeta(s))\Big\}q_1F^*_{A_1}(\zeta(s))+\Big\{(1-z_2)+z_2F^*_{Q_2}(\zeta(s))\Big\}q_2F^*_{A_2}(\zeta(s))\Big]}{\Big[\Big\{\big\{(1-z_1)+z_1F^*_{Q_1}(\zeta(s))\big\}q_1F^*_{A_1}(\zeta(s))+\big\{(1-z_2)+z_2F^*_{Q_2}(\zeta(s))\big\}q_2F^*_{A_2}(\zeta(s))\big\}-s\Big]}$$

with the server utilization factor as $\rho_u=q_1(\gamma_{A_1}+z_1\gamma_{Q_1})+q_2(\gamma_{A_2}+z_2\gamma_{Q_2})$ which is persistent with the existing literature obtained by Madan *et al.* [30].

(3) $M^X/\begin{pmatrix}G_1\\G_2\end{pmatrix}/1(UR)/G(BS)/V_s$ if the re-service is not allowed and there is no threshold in the system, *i.e.* if $N=1$ and $z_i=0$ $(i=1,2)$ then (6.1) can be written as,

$$P_N(s)=\frac{(1-\rho_u)(1-s)[p_2+p_1F^*_B(\zeta(s))]\big[q_1F^*_{A_1}(K_1(s))+q_2F^*_{A_2}(K_2(s))\big]}{\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{q_1F^*_{A_1}(K_1(s))+q_2F^*_{A_2}(K_2(s))\Big\}-s\Big]}$$

where $\rho_u=q_1\gamma_{A_1}\Big(1+\alpha_1\Big(\mu^{(1)}_{L_1}+\mu^{(1)}_{T_1}\Big)\Big)+q_2\gamma_{A_2}\Big(1+\alpha_2\Big(\mu^{(1)}_{L_2}+\mu^{(1)}_{T_2}\Big)\Big)+p_1\gamma_v$, and the result tallies with the findings of Begum and Choudhury [4].

(4) $M^X/\begin{pmatrix}G_1\\G_2\end{pmatrix}/1/G(BS)/V_s$ queue with no operating policy, re-service queue and service interruption *i.e.* $N=1$, $z_i=0$ and $\alpha_i=0$ $(i=1,2)$, then (6.1) is equal to the following,

$$P_N(s)=\frac{(1-\rho_u)(1-s)[p_2+p_1F^*_B(\zeta(s))]\big[q_1F^*_{A_1}(\zeta(s))+q_2F^*_{A_2}(\zeta(s))\big]}{\Big[\{p_2+p_1F^*_B(\zeta(s))\}\Big\{q_1F^*_{A_1}(\zeta(s))+q_2F^*_{A_2}(\zeta(s))\Big\}-s\Big]}$$

where $\rho_u=q_1\gamma_{A_1}+q_2\gamma_{A_2}+p_1\gamma_v$ and the results matches up to the findings of Madan *et al.* [31].

(5) $M^X/\begin{pmatrix}G_1\\G_2\end{pmatrix}/1$ queue if the threshold policy, option for repeated service, service interruption and vacation are omitted *i.e.*, $N=1$, $z_i=0$, $\alpha_i=0$ $(i=1,2)$ and $p_1=0$ then the expression (6.1) becomes,

$$P_N(s)=\frac{(1-\rho_u)(1-s)\big[q_1F^*_{A_1}(\zeta(s))+q_2F^*_{A_2}(\zeta(s))\big]}{\big[\{q_1F^*_{A_1}(\zeta(s))+q_2F^*_{A_2}(\zeta(s))\}-s\big]}$$

with $\rho_u=q_1\gamma_1+q_2\gamma_2$ which represents the stationary PGF at random epoch for classical M/G/1 queue with two service types [34].

(6) $M^X/G/1$ when the number of services is reduced to one from two *i.e.*, $q_1=0$ and $q_2=1$ or $q_1=1$ and $q_2=0$ and there is no operating policy ($N=1$), no repeated service ($z_i=0$), no breakdown ($\alpha_i=0$), no vacation ($p_1=0$) then (6.1) results in the following,

$$P_N(s)=\frac{(1-\rho_u)(1-s)\big[F^*_{A_1}(\zeta(s))\big]}{\big[F^*_{A_1}(\zeta(s))-s\big]}\quad\text{or}\quad\frac{(1-\rho_u)(1-s)\big[F^*_{A_2}(\zeta(s))\big]}{\big[F^*_{A_2}(\zeta(s))-s\big]}$$

where the server utilization is $\rho_u=\lambda\mu^{(1)}_{A_1}$ or $\lambda\mu^{(1)}_{A_2}$ and it is the classical Pollaczek Khinchin formula for an $M^X/G/1$ queue [34].

Thus, the results achieved in this study can be witnessed as a classical generalization of the Pollaczek-Khinchin formula for an unreliable $M^X/G/1$ queue providing two-heterogeneous services with optional re-service under Bernoulli vacation system and N-policy. □

7. Performance measures

This subsection derives the system state probabilities along with the availability and failure frequency (FF) of the server under the steady-state regime.

Theorem 7.1. *Under the steady-state condition $\rho_u < 1$, the system state probabilities are defined as,*

(i) *Probability that the server is busy with the* ith *genre of service:*

$$P_{A_i} = q_i \gamma_{A_i}; i = 1, 2.$$

(ii) *Probability that the server is busy with the* ith *genre of re-service:*

$$P_{Q_i} = q_i z_i \gamma_{Q_i}; i = 1, 2.$$

(iii) *Probability that the server is on vacation:*

$$P_B = p_1 \gamma_v.$$

(iv) *Probability that the server is waiting for repair during the FGS/SGS:*

$$P_{L_i}^{A_i} = \alpha_i q_i \gamma_{A_i} \mu_{L_i}^{(1)}\,;\ i = 1, 2.$$

(v) *Probability that the server is waiting for repair during the FGRS/SGRS:*

$$P_{L_i}^{Q_i} = \alpha_i q_i z_i \gamma_{Q_i} \mu_{L_i}^{(1)}\,;\ i = 1, 2.$$

(vi) *Probability that the server is under repair during the FGS/SGS:*

$$P_{T_i}^{A_i} = \alpha_i q_i \gamma_{A_i} \mu_{T_i}^{(1)}\,;\ i = 1, 2.$$

(vii) *Probability that the server is under repair during the FGRS/SGRS:*

$$P_{T_i}^{Q_i} = \alpha_i q_i z_i \gamma_{Q_i} \mu_{T_i}^{(1)}\,;\ i = 1, 2.$$

(viii) *Probability that the system is idle:*

$$P_I = 1 - q_1(\gamma_{A_1} + z_1\gamma_{Q_1})\left\{1 + \alpha_1\left(\mu_{L_1}^{(1)} + \mu_{T_1}^{(1)}\right)\right\} - q_2(\gamma_{A_2} + z_2\gamma_{Q_2})\left\{1 + \alpha_2\left(\mu_{L_2}^{(1)} + \mu_{T_2}^{(1)}\right)\right\} - p_1\gamma_v.$$

Proof. (i)–(vii) are attained by letting $\eta \to 0$ and $\theta \to 0$ in (5.32)–(5.38). And (viii) is procured by algebraic calculation of the expression $P_I = 1 - \sum_{i=1}^{2}\left\{P_{A_i} + P_{Q_i} + P_{L_i}^{A_i} + P_{L_i}^{Q_i} + P_{T_i}^{A_i} + P_{T_i}^{Q_i}\right\} - P_B$. □

Theorem 7.2. *Under the stationary condition $\rho_u < 1$, the steady-state availability of the server is given by,*

$$S_a = 1 - \left[q_1\alpha_1(\gamma_{A_1} + z_1\gamma_{Q_1})\left(\mu_{L_1}^{(1)} + \mu_{T_1}^{(1)}\right) + q_2\alpha_2(\gamma_{A_2} + z_2\gamma_{Q_2})\left(\mu_{L_2}^{(1)} + \mu_{T_2}^{(1)}\right) + p_1\gamma_v\right]. \tag{7.1}$$

Proof. Equation (7.1) is obtained by applying (5.25) and (5.26) in (***) (given below):

$$S_a = \sum_{n=0}^{N-1} I_n + \lim_{s\to 1}\sum_{i=2}\left[\bar{\bar{P}}_{A_i}(s) + \bar{\bar{P}}_{Q_i}(s)\right]. \tag{***}$$

□

Theorem 7.3. *Under the steady-state environment $\rho_u < 1$, the stationary failure frequency of the server denoted by S_f is given by,*

$$S_f = q_1\alpha_1(\gamma_{A_1} + z_1\gamma_{Q_1}) + q_2\alpha_2(\gamma_{A_2} + z_2\gamma_{Q_2}). \tag{7.2}$$

Proof. Following the argument of Li *et al.* [26], FF of a server is given by:

$$S_f = \sum_{i=1}^{2}\alpha_i\left[\int_0^\infty \bar{\bar{P}}_{A_i}(x;1)[1-F_{A_i}(x)]\mathrm{d}x + \int_0^\infty \bar{\bar{P}}_{Q_i}(x;1)[1-F_{Q_i}(x)]\mathrm{d}x\right]. \tag{****}$$

Since,

$$\int_0^\infty [1-F_i(x)]\mathrm{d}x = \int_0^\infty x\,\mathrm{d}F_i(x) = \mu_{F_i} \quad \text{(notations implying their usual meaning)} \tag{*****}$$

a standard renewal theory result. □

Therefore, utilizing (4.42), (4.43), and (*****), in the above expression (****), FF of the server is accomplished.

8. Optimal cost structure

This section formulates an analogous long term average cost function per unit time for the system under study (refer, [23,24,39]), which can be extensively used by the system engineers for evaluating the optimal value of N that minimizes the average cost of operation per unit time. The different expenses incurred for operating the system are as follows,

C_h: holding cost for each customer that arrives in the system/ unit time.
C_o: operating cost for keeping the server on and in operation/ unit time.
C_s: start-up cost/ busy cycle.

Let the average cost per unit time, AC(N), be defined as,

$$\mathrm{AC}(N) = C_o\rho_u + C_h\mu_{_N} + \frac{C_s}{\mu_{bc}}; \tag{8.1}$$

where ρ_u, $\mu_{_N}$ are already defined in Sections 4, 6, respectively, and $\mu_{_{bc}}$ is the mean length of a busy cycle of the model under consideration. To derive $\mu_{_{bc}}$ the following relation is considered,

$$\mu_{_{bc}} = \mu_{_{ip}} + \mu_{_{bp}}, \tag{8.2}$$

where $\mu_{_{ip}}$ and $\mu_{_{bp}}$ are the mean length of an idle and busy period of the system, respectively. The mean length of an idle period is obtained as $\mu_{_{ip}} = \frac{\sum_{n=0}^{N-1}\pi_n}{\lambda}$, dividing the average number of batches during an idle period by the rate of arrival. The average number of arrivals during an idle period is $\lambda\mu_{_{ip}}\mu_{\Xi}^{(1)}$. Hence, the average length of a busy period equals to $\mu_{_{bp}} = \frac{\rho_u\mu_{ip}}{(1-\rho_u)}$. Finally, using (8.2), $\mu_{_{bc}}$ is obtained as,

$$\mu_{_{bc}} = \frac{\sum_{n=0}^{N-1}\pi_n}{\lambda(1-\rho_u)}\cdot$$

Thus, AC(N) can be rewritten as,

$$\begin{aligned}
&\mathrm{AC}(N)\\
&= (C_o + C_h)\Big[p_1\gamma_v + q_1(\gamma_{A_1} + z_1\gamma_{Q_1})\Big(1+\alpha_1\Big(\mu_{L_1}^{(1)} + \mu_{T_1}^{(1)}\Big)\Big) + q_2(\gamma_{A_2} + z_2\gamma_{Q_2})\Big(1+\alpha_2\Big(\mu_{L_2}^{(1)} + \mu_{T_2}^{(1)}\Big)\Big)\Big]\\
&\quad + \frac{C_h\sum_{n=0}^{N-1}n\pi_n + C_s\lambda\Big[1 - p_1\gamma_v - q_1(\gamma_{A_1} + z_1\gamma_{Q_1})\Big(1+\alpha_1\Big(\mu_{L_1}^{(1)} + \mu_{T_1}^{(1)}\Big)\Big) - q_2(\gamma_{A_2} + z_2\gamma_{Q_2})\Big(1+\alpha_2\Big(\mu_{L_2}^{(1)} + \mu_{T_2}^{(1)}\Big)\Big)\Big]}{\sum_{n=0}^{N-1}\pi_n}
\end{aligned}$$

$$+ C_h \left[\frac{\left(\lambda\mu_{\Xi}^{(1)}\right)^2 \left[q_1\left(\mu_{A_1}^{(2)} + z_1\mu_{Q_1}^{(2)}\right)\left\{1+\alpha_1\left(\mu_{L_1}^{(1)}+\mu_{T_1}^{(1)}\right)\right\}^2 + q_2\left(\mu_{A_2}^{(2)} + z_2\mu_{Q_2}^{(2)}\right)\left\{1+\alpha_2\left(\mu_{L_2}^{(1)}+\mu_{T_2}^{(1)}\right)\right\}^2 + p_1\mu_B^{(2)}\right]}{2(1-\rho_u)} \right.$$

$$+ \frac{\left(\lambda\mu_{\Xi}^{(1)}\right)^2 \left[\alpha_1 q_1\left(\mu_{A_1}^{(1)} + z_1\mu_{Q_1}^{(1)}\right)(\mu_{L_1}^{(2)} + \mu_{T_1}^{(2)} + 2\mu_{L_1}^{(1)}\mu_{T_1}^{(1)}) + \alpha_2 q_2\left(\mu_{A_2}^{(1)} + z_2\mu_{Q_2}^{(1)}\right)\left(\mu_{L_2}^{(2)} + \mu_{T_2}^{(2)} + 2\mu_{L_2}^{(1)}\mu_{T_2}^{(1)}\right)\right]}{2(1-\rho_u)}$$

$$+ \frac{\left(\lambda\mu_{\Xi}^{(1)}\right)^2 \left[p_1\mu_B^{(1)}\left\{q_1 z_1\mu_{A_1}^{(1)}\mu_{Q_1}^{(1)}\{1+\alpha_1\left(\mu_{L_1}^{(1)}+\mu_{T_1}^{(1)}\right)\}^2 + q_2 z_2\mu_{A_2}^{(1)}\mu_{Q_2}^{(1)}\left\{1+\alpha_2\left(\mu_{L_2}^{(1)}+\mu_{T_2}^{(1)}\right)\right\}^2\right\}\right]}{(1-\rho_u)}$$

$$\left. + \frac{\rho_u\mu_{\Xi}^{(2)}}{2\mu_{\Xi}^{(1)}(1-\rho_u)} \right]. \tag{8.3}$$

For notational convenience, let $R(j) = \sum_{n=0}^{j} \pi_n$, $S(j) = \sum_{n=0}^{j} n\pi_n$ and the optimal value of N be denoted by N^*. Then to determine N^* the cost function AC(N) needs to be shown as a convex function, but the non-linearity and complexity of AC(N) make it a tough job. However, an alternative procedure is presented here in the form of a *Theorem* stated below that makes the calculation of N^* possible.

Theorem 8.1. *Under the long term average cost structure, the optimal threshold for an* $M^X/\binom{G_1}{G_2}/1(UR)/Re-service/G(BS)/V_s/N-Policy$ *queue is given by,*

$$N^* = \min\left[j \geq 1 \mid \sum_{n=0}^{j-1}(j-n)\pi_n > \frac{\lambda(1-\rho_u)C_s}{C_h}\right]. \tag{8.4}$$

Proof. Let,

$$\begin{aligned} \mathrm{AC}(j+1) - \mathrm{AC}(j) &= C_h\left[\frac{S(j)}{R(j)} - \frac{S(j-1)}{R(j-1)}\right] + \lambda(1-\rho_u)C_s\left[\frac{1}{R(j)} - \frac{1}{R(j-1)}\right] \\ &= \frac{\pi_j}{R(j)R(j-1)}[C_h\{jR(j) - S(j)\} - \lambda C_s(1-\rho_u)]. \end{aligned}$$

Since $C_h\{jR(j) - S(j)\} > 0$ and $\frac{\pi_j}{R(j)R(j-1)} > 0$, the function $h(j) = C_h\{jR(j) - S(j)\} - \lambda C_s(1-\rho_u)$ rules whether AC(N) increases or decreases.

Let "k" be the first "j" such that $h(j) > 0$. Then

$$\begin{aligned} h(k+1) &= C_h[(k+1)R(k+1) - S(k+1)] - \lambda C_s(1-\rho_u) \\ &= C_h[(k+1)R(k) - S(k)] - \lambda C_s(1-\rho_u) \\ &= h(k) + C_h R(k) \\ \implies h(k+1) &> h(k). \end{aligned} \tag{8.5}$$

Thus, it is observed that AC(n) > AC(k) for some $n > k$. Hence,

$$\begin{aligned} N^* &= \text{first } j \text{ such that } h(j) > 0 \\ &= \min\left[j \geq 1 \mid \sum_{n=0}^{j-1}(j-n)\pi_n > \frac{\lambda(1-\rho_u)C_s}{C_h}\right]. \end{aligned} \tag{8.6}$$

□

Remark 8.2. It is to be noted here that if $\frac{C_h}{C_s} > \frac{\lambda(1-\rho_u)}{\sum_{n=0}^{j-1}(j-n)\pi_n}$, the optimal threshold value of N^* is always equal to 1, which implies that it is not beneficial to have a control policy if the holding cost/unit time is greater than the start-up cost/ unit time.

9. Numerical experiment

This section performs a quantitative analysis of the system's survivability attributes instead of exact model parameterizations. As the exact values of the system parameters for the underlying model are not known at this time point so assumed parametric values are taken into consideration.

For illustrative purpose, the service, re-service, vacation, delay and repair time r.v are all assumed to be exponentially distributed with parameters σ_i τ_i, ϑ, d_i and r_i $(i = 1, 2)$ respectively. The corresponding PDFs are $f_{A_i}(x) = \sigma_i e^{-\sigma_i x}(x > 0)$, $f_{Q_i}(x) = \tau_i e^{-\tau_i x}(x > 0)$, $f_B(x) = \vartheta e^{-\vartheta x}(x > 0)$, $f_{L_i}(y) = d_i e^{-d_i y}(y > 0)$ and $f_{T_i}(y) = r_i e^{-r_i y}(y > 0)$ respectively; LST $F^*_{A_i}(\eta) = \frac{\sigma_i}{\sigma_i+\eta}$, $F^*_{Q_i}(\eta) = \frac{\tau_i}{\tau_i+\eta}$, $F^*_B(\eta) = \frac{\vartheta}{\vartheta+\eta}$, $F^*_{L_i}(\eta) = \frac{d_i}{d_i+\eta}$ and $F^*_{T_i}(\eta) = \frac{r_i}{r_i+\eta}$ respectively; mean $\mu^{(1)}_{A_i} = \frac{1}{\sigma_i}$, $\mu^{(1)}_{Q_i} = \frac{1}{\tau_i}$, $\mu^{(1)}_B = \frac{1}{\vartheta}$, $\mu^{(1)}_{L_i} = \frac{1}{d_i}$ and $\mu^{(1)}_{T_i} = \frac{1}{r_i}$ respectively; second moment $\mu^{(2)}_{A_i} = \frac{2}{\sigma_i^2}$, $\mu^{(2)}_{Q_i} = \frac{2}{\tau_i^2}$, $\mu^{(2)}_B = \frac{2}{\vartheta^2}$, $\mu^{(2)}_{L_i} = \frac{2}{d_i^2}$ and $\mu^{(2)}_{T_i} = \frac{2}{r_i^2}$ respectively.

The arrival batch size is presumed to follow geometric distribution with parameter w $(0 < w < 1)$. The corresponding PMF is $P(\Xi = l) = w(1-w)^{(l-1)}$ $(l = 1, 2, \cdots; 0 < w < 1)$, mean $\mu^{(1)}_\Xi = \frac{1}{w}$ and second moment $\mu^{(2)}_\Xi = \frac{2-w}{w^2}$ respectively.

For the sake of computational convenience, the assumed non-monetary and monetary values of the system parameters are summarized in Tables 1 and 2, respectively.

Table 1. Parametric non-monetary values of the model.

Processes	Parameters	Parametric values
Arrival	λ	0.3
	w	0.2
FGS/FGRS	q_1	0.5
	z_1	0.1
	σ_1	4
	τ_1	3
	α_1	0.022
	d_1	30
	r_1	20
SGS/SGRS	q_2	0.5
	z_2	0.15
	σ_2	5
	τ_2	4
	α_2	0.025
	d_2	35
	r_2	25
Vacation	p_1	0.4
	ϑ	11

Table 2. Parametric monetary values of the model.

Parameters		Parametric values (in Rs.)
Costs	C_h	240
	C_o	500
	C_s	2000
		5000
		9000

9.1. Influence of the reliability factors α_1 and α_2

The effect of breakdown rates $\alpha_i (i = 1, 2)$ on the important reliability measure of the system *viz*, system state availability and failure frequency of the server are presented in Table 3 with the help of the given data in Table 1.

TABLE 3. Impact of breakdown rates on server availability and failure frequency.

α_1	α_2	S_a	S_f
0	0	0.95	1
0	0.025	0.94569	0.00445
0.02	0.025	0.94534	0.00870
0.04	0.025	0.94498	0.01294
0.06	0.025	0.94463	0.01719
0.08	0.025	0.94428	0.02143
0.022	0	0.945612	0.00467
0.022	0.02	0.945367	0.00823
0.022	0.04	0.945121	0.01179
0.022	0.06	0.944875	0.01536
0.022	0.08	0.944629	0.01892
0.08	0.08	0.943607	0.03123

Table 3 clearly shows that a higher value of $\alpha_i (i = 1, 2)$, *i.e.* breakdown rate results in lower server availability, *i.e.* S_a, and higher failure frequency, *i.e.* S_f. The server's stationary system availability is found as 95% with failure frequency less than 1% for the model under study.

9.2. Optimal policy

This subsection describes how the decision regarding the optimal threshold of N to minimize the average cost is made with the help of the cost structure defined in (8.3) based on the data given in Tables 1 and 2.

To determine the optimal value of N, the recursive relationship $\pi_n = \sum_{\iota=1}^{n} w_\iota \pi_{n-\iota}$ and $\pi_0 = 1$ is ulitized to calculate $\sum_{n=0}^{j-1} (j - n) \pi_n$ and is shown in Table 4.

TABLE 4. Different values of $\sum_{n=0}^{j-1} (j - n) \pi_n$.

n	π_n	$\sum_{n=0}^{j-1} (j - n) \pi_n$
1	0.8	1
2	0.8	2.8
3	0.8	5.4
4	0.8	8.8
5	0.8	13
6	0.8	18
7	0.8	23.8
8	0.8	30.4
9	0.8	37.8
10	0.8	46

Further, keeping $C_h = 240$ as fixed and varying the probability of p_1 from 0 to 1 such that the utilization factor $\rho_u < 1$ always satisfies, Table 5 represents some numerical results of the cost ratio $\frac{\lambda(1-\rho_u)C_s}{C_h}$ for three different values of C_s considered in Table 2.

TABLE 5. Different values of $\frac{\lambda(1-\rho_u)C_s}{C_h}$.

p_1	ρ_u	$\frac{\lambda(1-\rho_u)C_s}{C_h}$		
		$C_s = 2000$	$C_s = 5000$	$C_s = 9000$
0	0.39107	1.46143	3.65358	6.57645
0.1	0.40457	1.42903	3.57258	6.430651
0.2	0.41807	1.39663	3.49158	6.28485
0.3	0.43157	1.36423	3.41058	6.13905
0.4	0.44507	1.33183	3.32958	5.99325
0.5	0.45857	1.29943	3.24858	5.84745
0.6	0.47207	1.26703	3.16758	5.70165
0.7	0.48557	1.23463	3.08658	5.55585
0.8	0.49907	1.20223	3.00558	5.41005
0.9	0.51257	1.16983	2.92458	5.26425
1	0.52607	1.13743	2.84358	5.11845

TABLE 6. Optimal values of N^* for different C_s.

C_s	Optimal threshold of N
2000	$N^* = 2$; $0 \leq p_1 \leq 1$
5000	$N^* = 3$; $0 \leq p_1 \leq 1$
9000	$N^* = 4$; $0 \leq p_1 \leq 0.8$
	$N^* = 3$; $0.9 \leq p_1 \leq 1$

Finally, based on the result achieved in Theorem 8.1 to determine the optimal value of N, the values of $\sum_{n=0}^{j-1}(j-n)\,\pi_n$ and $\frac{\lambda(1-\rho_u)C_s}{C_h}$ are compared from Tables 4 and 5 respectively, and the optimal N^* is procured and presented in Table 6.

The effect of the system parameter p_1 on the mean queue length μ_N of the underlying model is computed for $N = 2, 3$ and 4 obtained in Table 6 and is presented in Table 7, followed by a graphical representation for the same in Figure 4.

TABLE 7. Mean queue size for optimal thresholds of N and different values of p_1.

p_1	μ_N		
	$N = 2$	$N = 3$	$N = 4$
0	5.96368	6.19583	6.48749
0.1	6.28727	6.51941	6.81108
0.2	6.62524	6.85738	7.14905
0.3	6.97862	7.21077	7.50243
0.4	7.34855	7.58069	7.87236
0.5	7.73624	7.96839	8.26005
0.6	8.14308	8.37522	8.66689
0.7	8.57055	8.8027	9.09436
0.8	9.02034	9.25249	9.54415
0.9	9.4943	9.72645	10.0181
1	9.99449	10.2266	10.5183

From both Table 7 and Figure 4, it is clear that as p_1 increases, the mean queue size increases for different values of N and the same law follows for any higher value of N.

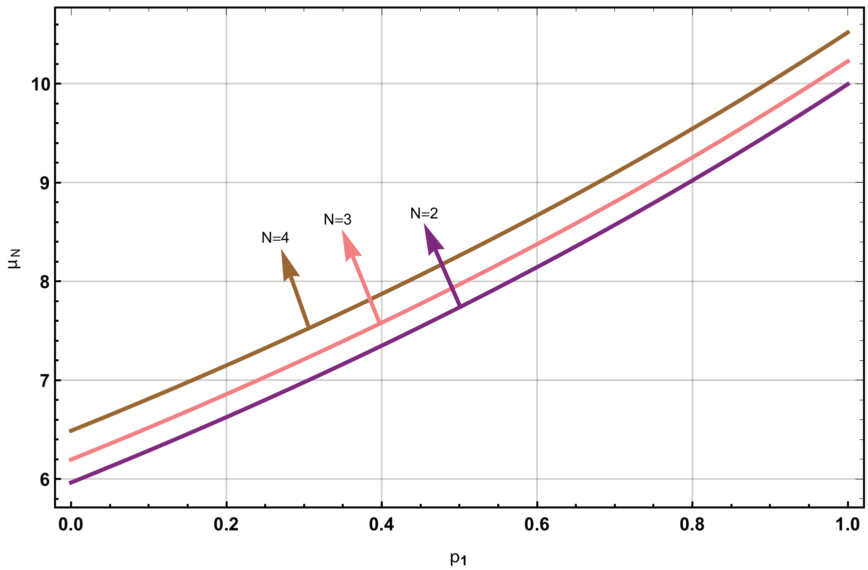

FIGURE 4. Mean system size *vs.* threshold level of N.

Based on the parameter setting of Table 1 and 2, the average cost per unit of time for different values of C_s and N is presented graphically in Figure 5.

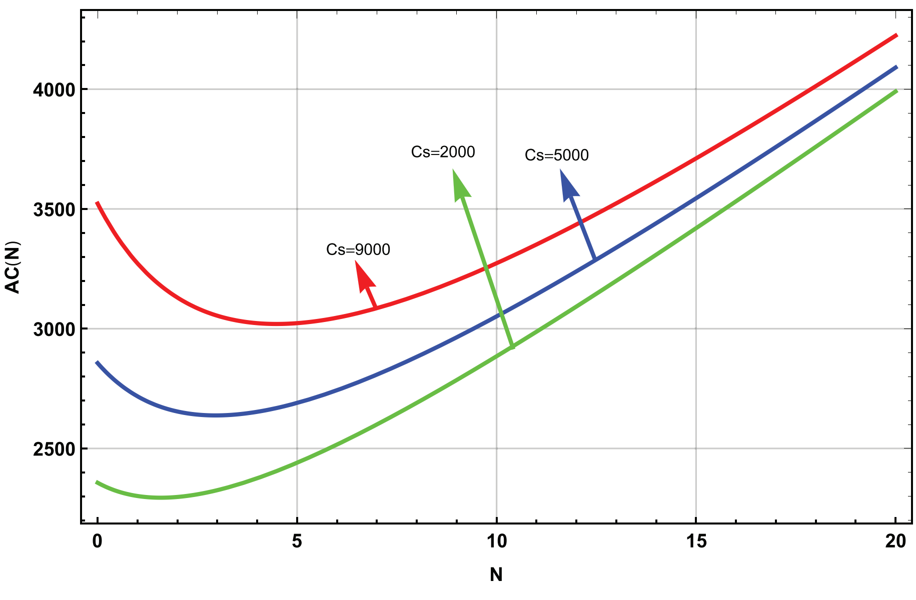

FIGURE 5. Average cost *vs.* different Cs and threshold level of N.

Figure 5 is a convex plot depicting the impact of threshold levels on the average cost per unit time for all the three different start-up costs considered in this study. The minimum average cost per unit is achieved as

AC(N^*) = 2297.5, AC(N^*) = 2637.95 and AC(N^*) = 3023.02 against $N^* = 2$ for $C_s = 2000$, $N^* = 3$ for $C_s = 5000$ and $N^* = 4$ for $C_s = 9000$ respectively. This upholds the conclusion of Theorem 8.1 and Table 6.

10. Conclusion

In this manuscript, a non-Markovian model under N-policy is developed to study the joint distribution of server's state and queue size in both elapsed and remaining times, assuming general distribution of the service, re-service, vacation, repair and delay times for a bulk arrival queueing model. The underlying queueing system takes into consideration the supplementary variable technique under some suitable transformations to deliver the stationary queue size distribution at arbitrary and service completion epoch and mean system size besides various pivotal performance measures. An optimal operating policy under a linear cost structure has been put forward in the form of a theorem. Finally, with the help of some numerical experiments, the applicability of this theorem is shown, and the optimal thresholds of the model under consideration are obtained for three different start-up costs. These will furnish information to the system designers and system engineers about the efficiency of the model and help them use the proposed model while designing different digital systems, production systems, and inventory systems.

It would be interesting to explore a similar model with more than two heterogeneous services, set-up time, multi servers, customers impatience, modified vacation policy etc., making it a more resilient queueing model with more general results.

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