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A NOTE ON THE PAPER "NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS USING CONVEXIFACTORS FOR MATHEMATICAL PROGRAMS WITH EQUILIBRIUM CONSTRAINTS"

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Abstract. In this work, some counterexamples are given to refute some results in the paper by Kohli [*RAIRO:OR* **53** (2019) 1617–1632]. We correct the fault in some of his results.

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1. Introduction

Mathematical programs with equilibrium constraints have been investigated by many authors. In the paper [8], the author investigated the following mathematical programs with equilibrium constraints

$$\text{(MPEC)}: \begin{cases} \text{Minimize} \quad f(x) \\ \text{s.t.} \begin{cases} g(x) \leq 0,\ h(x) = 0, \\ G(x) \geq 0,\ H(x) \geq 0,\ G(x)^\top H(x) = 0, \end{cases} \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^p$, $G : \mathbb{R}^n \to \mathbb{R}^l$ and $H : \mathbb{R}^n \to \mathbb{R}^l$. Under a nonsmooth constraint qualification ($\partial^* - GCQ$) given in terms of convexifactors, the author established first order necessary optimality condition for (MPEC). The main theorem, where the author gave necessary optimality conditions, is Theorem 4.4 of [8].

In this article, we show that necessary optimality conditions given by Kohli [8] are not correct. In support of our remarks, some counterexamples are given (see Example 3.1 and Rem. 3.3) and some reasoning mistakes in the proof of the main result ([8], Thm. 4.4) are highlighted (see Rems. 3.2, 3.3 and 4.2). Finally, we present corrected versions of his results. Theorem 4.5 is actually a corrected version of Theorem 4.4 in [8].

The rest of the paper is organized in this way: Section 2 contains basic definitions and preliminary material. Counterexamples and comments are given in Section 3. Section 4 addresses our main results (corrected optimality conditions). A conclusion is given in Section 5.

Keywords. Convexifactor, constraint qualifications, mathematical programs with equilibrium constraints, optimality conditions.

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2. Preliminaries

Throughout this section, let $\mathbb{R}^n$ be the usual n-dimensional Euclidean space. Given a nonempty subset S of $\mathbb{R}^n$, the closure, convex hull, and convex cone (including the origin) generated by S are denoted respectively by $cl\ S$, $conv\ S$ and $pos\ S$. The negative polar cone of S is defined by

$$S^- := \{v \in \mathbb{R}^n \mid \langle x, v\rangle \leq 0,\ \forall x \in S\}.$$

The contingent cone $T(S, x)$ to S at $x \in cl\ S$ is defined by

$$T(S, x) = \{v \in \mathbb{R}^n : \exists t_n\ \downarrow 0 \text{ and } \exists v_n \to v \text{ such that } x + t_n v_n \in S,\ \forall n \in \mathbb{N}\}.$$

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function and let $x \in \mathbb{R}^n$ where $f(x)$ is finite. The expressions

$$f_d^-(x, v) = \liminf_{t \searrow 0}\ [f(x + tv) - f(x)]/t \text{ and } f_d^+(x, v) = \limsup_{t \searrow 0}\ [f(x + tv) - f(x)]/t$$

signify, respectively, the lower and upper Dini directional derivatives of f at x in the direction v.

Definition 2.1 ([2])**.** The function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have an upper convexifactor $\partial^u f(x)$ at x if $\partial^u f(x) \subseteq \mathbb{R}^n$ is closed and, for each $v \in \mathbb{R}^n$,

$$f_d^-(x, v) \leq \sup_{x^* \in \partial^u f(x)} \langle x^*, v\rangle.$$

The function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have an upper semiregular convexifactor $\partial^{us} f(x)$ at x if $\partial^{us} f(x)$ is an upper convexifactor at x and, for each $v \in \mathbb{R}^n$,

$$f_d^+(x, v) \leq \sup_{x^* \in \partial^{us} f(x)} \langle x^*, v\rangle.$$

3. Counterexamples and comments

The following example shows that Theorem 4.4 of [8] is not correct.

Example 3.1. Consider the optimization problem (MPEC) where

$$f(x_1, x_2, x_3) := x_1 + x_2 - 2x_3,\ g(x_1, x_2, x_3) := x_3,$$
$$h(x_1, x_2, x_3) := 0,\ G_1(x_1, x_2, x_3) := x_1,\ G_2(x_1, x_2, x_3) := x_2,\ H_1(x_1, x_2, x_3) := x_2 \text{ and } H_2(x_1, x_2, x_3) := x_1.$$

On the one hand, the origin is the unique minimizer of (MPEC). On the other hand, it can be seen that $\partial^{us} f(\overline{x}) := \{(1, 1, -2)\}$ is a bounded upper semiregular convexifactor of f at $\overline{x} := (0, 0, 0)$. Moreover,

$$\partial^u g(\overline{x}) := \{(0, 0, 1)\},\ \partial^u h(\overline{x}) := \{(0, 0, 0)\},$$
$$\partial^u G_1(\overline{x}) := \{(1, 0, 0)\},\ \partial^u(-G_1)(\overline{x}) := \{(-1, 0, 0)\},\ \partial^u G_2(\overline{x}) := \{(0, 1, 0)\},\ \partial^u(-G_2)(\overline{x}) := \{(0, -1, 0)\},$$
$$\partial^u H_1(\overline{x}) := \{(0, 1, 0)\},\ \partial^u(-H_1)(\overline{x}) := \{(0, -1, 0)\},\ \partial^u H_2(\overline{x}) := \{(1, 0, 0)\} \text{ and } \partial^u(-H_2)(\overline{x}) := \{(-1, 0, 0)\}$$

are upper convexifactors of g, h, G_1, $-G_1$, G_2, $-G_2$, H_1, $-H_1$, H_2 and $-H_2$ at $\overline{x}$ respectively. Remark that $B = \{1, 2\}$.

- The feasible set K of (MPEC) is $K = (\{0\} \times \mathbb{R}^+ \times \mathbb{R}^-) \cup (\mathbb{R}^+ \times \{0\} \times \mathbb{R}^-)$. Consequently,

$$T(K, \overline{x}) = \big(\{0\} \times \mathbb{R}^+ \times \mathbb{R}^-\big) \cup \big(\mathbb{R}^+ \times \{0\} \times \mathbb{R}^-\big) \text{ and } cl\,conv\ T(K, \overline{x}) = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^-.$$

- $\partial^* - GCQ(B_1, B_2)$ holds for all $(B_1, B_2) \in P(B)$ at $\overline{x}$.

- If $B_1 = \{1\}$ and $B_2 = \{2\}$, then $\partial^* - GCQ(B_1, B_2)$ holds at $\overline{x}$. Indeed,

$$\begin{pmatrix} (conv\ \partial^u g(\overline{x}))\bigcup(conv\ \partial^u h(\overline{x}))\bigcup(conv\ \partial^u G_2(\overline{x}) \cup conv\ \partial^u(-G_2)(\overline{x})) \\ \bigcup(conv\ \partial^u H_1(\overline{x}) \cup conv\ \partial^u(-H_1)(\overline{x}))\bigcup conv\ \partial^u(-G_1)(\overline{x})\bigcup(conv\ \partial^u(-H_2)(\overline{x})) \end{pmatrix}^- = \mathbb{R}^+ \times \{0\} \times \mathbb{R}^-.$$

- If $B_1 = \{2\}$ and $B_2 = \{1\}$, then $\partial^* - GCQ(B_1, B_2)$ holds at $\overline{x}$. Indeed,

$$\begin{pmatrix} (conv\ \partial^u g(\overline{x}))\bigcup(conv\ \partial^u h(\overline{x}))\bigcup(conv\ \partial^u G_1(\overline{x}) \cup conv\ \partial^u(-G_1)(\overline{x})) \\ \bigcup(conv\ \partial^u H_2(\overline{x}) \cup conv\ \partial^u(-H_2)(\overline{x}))\bigcup conv\ \partial^u(-G_2)(\overline{x})\bigcup(conv\ \partial^u(-H_1)(\overline{x})) \end{pmatrix}^- = \{0\} \times \mathbb{R}^+ \times \mathbb{R}^-.$$

- If $B_1 = \emptyset$ and $B_2 = \{1,2\}$, then $\partial^* - GCQ(B_1, B_2)$ holds at $\overline{x}$. Indeed,

$$\begin{pmatrix} (conv\ \partial^u g(\overline{x}))\bigcup(conv\ \partial^u h(\overline{x}))\bigcup(conv\ \partial^u G_1(\overline{x}) \cup conv\ \partial^u(-G_1)(\overline{x})) \\ \bigcup(conv\ \partial^u G_2(\overline{x}) \cup conv\ \partial^u(-G_2)(\overline{x}))\bigcup conv\ \partial^u(-H_1)(\overline{x})\bigcup(conv\ \partial^u(-H_2)(\overline{x})) \end{pmatrix}^- = \{0\} \times \{0\} \times \mathbb{R}^-.$$

- If $B_1 = \{1,2\}$ and $B_2 = \emptyset$, then $\partial^* - GCQ(B_1, B_2)$ holds at $\overline{x}$. Indeed,

$$\begin{pmatrix} (conv\ \partial^u g(\overline{x}))\bigcup(conv\ \partial^u h(\overline{x}))\bigcup(conv\ \partial^u H_1(\overline{x}) \cup conv\ \partial^u(-H_1)(\overline{x})) \\ \bigcup(conv\ \partial^u H_2(\overline{x}) \cup conv\ \partial^u(-H_2)(\overline{x}))\bigcup conv\ \partial^u(-G_1)(\overline{x})\bigcup(conv\ \partial^u(-G_2)(\overline{x})) \end{pmatrix}^- = \{0\} \times \{0\} \times \mathbb{R}^-.$$

– $\partial^* - GCQ$ holds at $\overline{x}$. Indeed,

$$\begin{pmatrix} (conv\ \partial^u g(\overline{x}))\bigcup(conv\ \partial^u h(\overline{x})) \\ \bigcup(conv\ \partial^u(-G_1)(\overline{x}) \cup conv\ \partial^u(-G_2)(\overline{x})) \\ \bigcup(conv\ \partial^u(-H_1)(\overline{x}) \cup conv\ \partial^u(-H_2)(\overline{x})) \end{pmatrix}^- = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^- \subseteq cl\, conv\ T(K, \overline{x}).$$

– Observe that all hypotheses of Theorem 4.4 in [8] are satisfied, but $\overline{x}$ is not a ∂^*-strong stationary point as defined by Kohli ([8], Def. 4.1). Indeed, if there exists a vector $0 \neq \left(\lambda^g, \lambda^h, \lambda^G, \lambda^H, \mu^G, \mu^H\right) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ such that

$$\lambda^g,\ \lambda^h,\ \lambda_1^G,\ \lambda_2^G,\ \lambda_1^H,\ \lambda_2^H,\ \mu_1^G,\ \mu_2^G,\ \mu_1^H,\ \mu_2^H \geq 0, \tag{3.1}$$

$$\lambda^g + \lambda^h + \lambda_1^G + \lambda_2^G + \lambda_1^H + \lambda_2^H + \mu_1^G + \mu_2^G + \mu_1^H + \mu_2^H = 1 \tag{3.2}$$

and

$$0 \in cl \begin{bmatrix} conv\ \partial^{us} f(\overline{x}) + \lambda^g\ conv\ \partial^u g(\overline{x}) + \lambda^h\ conv\ \partial^u h(\overline{x}) + \lambda_1^G\ conv\ \partial^u(-G_1)(\overline{x}) \\ +\lambda_2^G\ conv\ \partial^u(-G_2)(\overline{x}) + \lambda_1^H\ conv\ \partial^u(-H_1)(\overline{x}) + \lambda_2^H\ conv\ \partial^u(-H_2)(\overline{x}) \\ +\mu_1^G\ conv\ \partial^u G_1(\overline{x}) + \mu_1^H\ conv\ \partial^u H_1(\overline{x}) + \mu_2^G\ conv\ \partial^u G_2(\overline{x}) + \mu_2^H\ conv\ \partial^u H_2(\overline{x}) \end{bmatrix}$$

we get

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in cl \left\{ \begin{pmatrix} 1 - \lambda_1^G - \lambda_2^H + \mu_1^G + \mu_2^H \\ 1 - \lambda_2^G - \lambda_1^H + \mu_1^H + \mu_2^G \\ -2 + \lambda^g \end{pmatrix} \right\}.$$

Then,

$$\begin{cases} 0 = 1 - \lambda_1^G - \lambda_2^H + \mu_1^G + \mu_2^H, \\ 0 = 1 - \lambda_2^G - \lambda_1^H + \mu_1^H + \mu_2^G, \\ 0 = -2 + \lambda^g. \end{cases}$$

We have $\lambda^g = 2$ while $\lambda^g \leq 1$ due to (3.1) and (3.2). A contradiction.

Remark 3.2. Contrary to what is stated on page 1625 (line -1), it is impossible to deduce

$$\lim_{k\to\infty}\left[\sum_{i\in I(\overline{x})}\lambda_{ik}^{g}+\sum_{i=1}^{p}\lambda_{ik}^{h}+\sum_{i\in A\cup B_1\cup B_2}\lambda_{ik}^{G}+\sum_{i\in D\cup B_1\cup B_2}\lambda_{ik}^{H}+\sum_{i\in A\cup B_2}\mu_{ik}^{G}+\sum_{i\in D\cup B_1}\mu_{ik}^{H}\right]:=1. \tag{3.3}$$

The author did not pay attention to the cone that precedes the convex hull in the previous formula (see line-6 on page 1625). This error has seriously impacted the remaining of the proof of Theorem 4.4 from [8]. Since (3.3) is an essential part of the definition of the ∂^*-strong stationarity property, Theorem 4.4 of [8] is also not correct. Notice that the boundedness of the sequence of the multipliers is neither acquired nor insured.

Remark 3.3. The main result ([8], Thm. 4.4), is based on Lemma 2.3 of [8]. However, this latter ([8], Lem. 2.3) is clearly incorrect, as setting

$$A:=\left\{(x,y)\in\mathbb{R}^2: x<0,\ y<0\right\}\cup\{(0,0)\} \text{ and } B:=\{(1,0)\}$$

yields a simple counterexample. Unfortunately, this error impacted ([8], Thm. 4.4) and forced the author to add useless and cumbersome closures and convex hulls.

The following result is a corrected version of Lemma 2.3 from [8]. Being standard, the proof has been omitted.

Lemma 3.4. *Let $\mathcal{B}$ a nonempty, convex and compact set and $\mathcal{A}$ be a convex cone. If*

$$\sup_{v\in\mathcal{B}}\langle v,d\rangle\geq 0,\quad \text{for all } d\in\mathcal{A}^-$$

then $0\in\mathcal{B}+cl\mathcal{A}$.

4. Optimality conditions

In the following definition, we recall the generalized alternatively stationarity concept given by Ardali *et al.* ([1], Def. 4.3).

Definition 4.1 ([1])**.** A feasible point $\overline{x}$ of MPEC is said to be a generalized alternatively stationary point if there exists a vector $\left(\lambda^g,\lambda^h,\mu^h,\lambda^G,\lambda^H,\mu^G,\mu^H\right)\in\mathbb{R}^m\times\mathbb{R}^{2p}\times\mathbb{R}^{2l}\times\mathbb{R}^{2l}$ such that

$$0\in\left[\begin{array}{l} conv\,\partial^{us}f(\overline{x})+\sum\limits_{i=1}^{m}\lambda_i^g\,conv\,\partial^u g_i(\overline{x})+\sum\limits_{i\in I'}\mu_i^h\,conv\,\partial^u h_i(\overline{x})+\sum\limits_{i\in I'}\lambda_i^h\,conv\,\partial^u(-h_i)(\overline{x})+\sum\limits_{i=1}^{l}\lambda_i^G\\ \times conv\,\partial^u(-G_i)(\overline{x})+\sum\limits_{i=1}^{l}\lambda_i^H\,conv\,\partial^u(-H_i)(\overline{x})+\sum\limits_{i=1}^{l}\mu_i^G\,conv\,\partial^u G_i(\overline{x})+\sum\limits_{i=1}^{l}\mu_i^H\,conv\,\partial^u H_i(\overline{x})\end{array}\right] \tag{4.1}$$

with

$$\lambda_i^g\ g_i(\overline{x})=0,\ \forall i\in I \tag{4.2}$$

and

$$\begin{cases}\mu_i^G=0 \text{ or } \mu_i^H=0,\ \forall i\in B,\\ \lambda_i^G=0,\ \mu_i^G=0,\ \forall i\in D,\\ \lambda_i^H=0,\ \mu_i^H=0,\ \forall i\in A,\\ \lambda_i^G,\ \lambda_i^H,\ \mu_i^G,\ \mu_i^H\geq 0,\ \forall i\in\{1,\ldots,l\},\\ \lambda_i^g\geq 0,\ \forall i\in I=\{1,\ldots,m\},\ \text{and } \lambda_i^h\geq 0,\ \mu_i^h\geq 0,\ \forall i\in I'=\{1,\ldots,p\}.\end{cases} \tag{4.3}$$

Here,

$$\begin{aligned}A&:=\{i\in\{1,\ldots,l\}:G_i(\overline{x})=0,\ H_i(\overline{x})>0\},\\ B&:=\{i\in\{1,\ldots,l\}:G_i(\overline{x})=0,\ H_i(\overline{x})=0\},\\ D&:=\{i\in\{1,\ldots,l\}:G_i(\overline{x})>0,\ H_i(\overline{x})=0\}.\end{aligned}$$

Remark 4.2. Contrary to Definition 10 of [8],

$$\sum_{i=1}^{m}\lambda_i^g + \sum_{i=1}^{p}\lambda_i^h + \sum_{i=1}^{l}\lambda_i^G + \sum_{i=1}^{l}\lambda_i^H + \sum_{i=1}^{l}\mu_i^G + \sum_{i=1}^{l}\mu_i^H = 1$$

is not an integral part of Definition 4.1. It is this equality that distorted Kohli's result. Notice that Remark 4.2 of [8] is not correct since condition for the sum of multipliers does not exist in [4,6,10].

Remark 4.3. Notice that if all the functions are differentiable and the upper convexifactor is replaced by the upper regular convexifactor in the above stationary notion, then this notion reduces to the A-stationary condition given by Flegel and Kanzow [6] and by Flegel [3].

We shall need the following nonsmooth constraint qualification.

Definition 4.4. Let $\overline{x} \in K$ and (B_1, B_2) be a partition of $B \neq \emptyset$. Suppose that g_i, $i \in I$, h_i, $-h_i$, $i \in J$, $-G_i$, G_i, $i \in A \cup B$, $-H_i$, H_i, $i \in D \cup B$, admit upper convexifactors $\partial^u g_i(\overline{x})$, $i \in I$, $\partial^u h_i(\overline{x})$, $\partial^u(-h_i)(\overline{x})$, $i \in J$, $\partial^u(-G_i)(\overline{x})$, $\partial^u G_i(\overline{x})$, $i \in A \cup B$, $\partial^u(-H_i)(\overline{x})$, $\partial^u H_i(\overline{x})$, $i \in D \cup B$, respectively at $\overline{x}$. We say that $\partial^* - ACQ(B_1, B_2)$ holds at $\overline{x}$ if

$$\mathcal{A}^- \subseteq \ cl\, conv(T(K, \overline{x})),$$

where K is the feasible set of (MPEC) and

$$\begin{aligned}\mathcal{A} := &\left(\bigcup_{i\in I(\overline{x})} conv\ \partial^u g_i(\overline{x})\right) \cup \left(\bigcup_{i\in I'} conv\ \partial^u h_i(\overline{x})\right) \cup \left(\bigcup_{i\in I'} conv\ \partial^u(-h_i)(\overline{x})\right)\\ &\cup \left(\bigcup_{i\in A\cup B_2} (conv\ \partial^u G_i(\overline{x}) \cup conv\ \partial^u(-G_i)(\overline{x}))\right) \cup \left(\bigcup_{i\in D\cup B_1} (conv\ \partial^u H_i(\overline{x}) \cup conv\ \partial^u(-H_i)(\overline{x}))\right)\\ &\cup \left(\bigcup_{i\in B_1} conv\ \partial^u(-G_i)(\overline{x})\right) \cup \left(\bigcup_{i\in B_2} conv\ \partial^u(-H_i)(\overline{x})\right).\end{aligned}$$

The following result is the corrected version of Theorem 4.4 from [8].

Theorem 4.5. *Let $\overline{x}$ be a local optimal solution of* MPEC. *Assume that f is locally Lipschitz and admits a bounded upper semiregular convexifactor $\partial^{us} f(\overline{x})$ at $\overline{x}$. Let g_i, $i \in I$, $-h_i$, h_i, $i \in I'$, $-G_i$, G_i, $i \in A \cup B$, $-H_i$, H_i, $i \in D \cup B$, admit upper convexifactors $\partial^u g_i(\overline{x})$, $i \in I$, $\partial^u(-h_i)(\overline{x})$, $\partial^u h_i(\overline{x})$, $i \in I'$, $\partial^u(-G_i)(\overline{x})$, $\partial^u G_i(\overline{x})$, $i \in A \cup B$, $\partial^u(-H_i)(\overline{x})$, $\partial^u H_i(\overline{x})$, $i \in D \cup B$, respectively at $\overline{x}$. Suppose that pos $\mathcal{A}$ is closed and that there exists a partition (B_1, B_2) of B such that $\partial^* - ACQ(B_1, B_2)$ holds at $\overline{x}$. Then, $\overline{x}$ is a generalized alternatively stationary point.*

Proof. The beginning of the proof of Theorem 4.4 from [8] remains correct. However, from line 6 on page 1624 until the end of the proof, the argument should be corrected as the following.

$$\sup_{\eta\in conv\partial^{us} f(\overline{x})} \langle \eta, v\rangle \geq 0, \quad \text{for all } v \in \mathcal{A}^-.$$

– Since $\mathcal{A} \subseteq pos\ \mathcal{A}$, we get

$$\sup_{\eta\in conv\partial^{us} f(\overline{x})} \langle \eta, v\rangle \geq 0, \quad \text{for all } v \in (pos\ \mathcal{A})^-.$$

– Since $\partial^{us} f(\overline{x})$ is also a closed set, $conv\ \partial^{us} f(\overline{x})$ is a compact set (see [7], Thm. 1.4.3). By Lemma 3.4, we get

$$0 \in conv\ \partial^{us} f(\overline{x}) + cl(pos\ \mathcal{A}).$$

- Since $pos\ \mathcal{A}$ is closed, we obtain

$$0 \in conv\ \partial^{us} f(\overline{x}) + pos\ \mathcal{A}.$$

Then, there exist scalars $\lambda_i^g \geq 0,\ i \in I(\overline{x}),\ \mu_i^h \geq 0,\ \lambda_i^h \geq 0,\ i \in I',\ \mu_i^G \geq 0,\ i \in A \cup B_2,\ \lambda_i^G \geq 0,\ i \in A \cup B,\ \mu_i^H \geq 0,\ i \in D \cup B_1$, and $\lambda_i^H \geq 0,\ i \in D \cup B$, such that

$$0 \in \begin{bmatrix} conv\ \partial^{us} f(\overline{x}) + \sum\limits_{i \in I(\overline{x})} \lambda_i^g\ conv\ \partial^u g_i(\overline{x}) + \sum\limits_{i \in I'} \mu_i^h\ conv\ \partial^u h_i(\overline{x}) + \sum\limits_{i \in I'} \lambda_i^h\ conv\ \partial^u (-h_i)(\overline{x}) \\ + \sum\limits_{i \in A \cup B_2} \mu_i^G\ conv\ \partial^u G_i(\overline{x}) + \sum\limits_{i \in A \cup B} \lambda_i^G\ conv\ \partial^u (-G_i)(\overline{x}) \\ + \sum\limits_{i \in D \cup B_1} \mu_i^H\ conv\ \partial^u H_i(\overline{x}) + \sum\limits_{i \in D \cup B} \lambda_i^H\ conv\ \partial^u (-H_i)(\overline{x}) \end{bmatrix}.$$

- Setting

$$\begin{cases} \mu_i^G = 0, & \forall i \in D \cup B_1 \\ \mu_i^H = 0, & \forall i \in A \cup B_2 \\ \lambda_i^G = 0, & \forall i \in D \\ \lambda_i^H = 0, & \forall i \in A \end{cases}$$

we obtain (4.1), (4.2) and (4.3). The proof is then finished.

□

5. Conclusions

In the paper [8], the author investigated a mathematical programs with equilibrium constraints. The main result, Theorem 4.4 of [8], and the lemma ([8], Lem. 2.3) on which the author is based are false. In this work, counterexamples are given to refute Theorem 4.4 of [8] and Lemma 2.3 of [8]. Furthermore, we correct the flaws.

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