OPTIMALITY CONDITIONS FOR NONSMOOTH MULTIOBJECTIVE BILEVEL OPTIMIZATION USING TANGENTIAL SUBDIFFERENTIALS

Mohsine Jennane[1], El Mostafa Kalmoun[2,*] and Lhoussain El Fadil[1]

Abstract. In combining the value function approach and tangential subdifferentials, we establish necessary optimality conditions of a nonsmooth multiobjective bilevel programming problem under a suitable constraint qualification. The upper level objectives and constraint functions are neither assumed to be necessarily locally Lipschitz nor convex.

1. Introduction

Bilevel programming is considered to be one of the important areas in optimization and operations research due to its many applications in various fields (economics, logistics, transportation, engineering and computer science, etc). The problem is stated at two levels of hierarchy in which the feasible set of decisions at the upper level is implicitly related with the solution set at the lower level.

This paper treats the case of bilevel programs in which the upper level decision requires minimizing a vector function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, while the lower level involves a single objective $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. This program has the following form

$$
\begin{aligned}
\mathbb{R}^p_+ - \min_{x,y} \quad & F(x,y) = (F_1, \ldots, F_p)(x,y) \\
\text{subject to} \quad & G_j(x,y) \le 0, && \text{for all } j \in \mathbb{J} := \{1, \ldots, q\}, \\
& y \in S(x),
\end{aligned}
\tag{1.1}
$$

with the upper level constraints being written in terms of the leader's constraint function $G_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$, $j \in \mathbb{J}$, as well as the follower's region of rational reactions $S(x)$, which represents the solution set to

$$
\begin{aligned}
\min_{y} \quad & f(x,y) \\
\text{subject to} \quad & g_i(x,y) \le 0, && \text{for all } i \in \mathbb{I} := \{1, \ldots, l\},
\end{aligned}
\tag{1.2}
$$

[1] FSDM, Department of Mathematics, Sidi Mohamed Ben Abdellah University, Fez, Morocco.
[2] School of Science and Engineering, Al Akhawayn University in Ifrane, PO Box 104, Ifrane 53000, Morocco.
*Corresponding author: E.Kalmoun@aui.ma; kalmoun@gmail.com

with $g_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$, $i \in \mathbb{I}$, denote the lower level constraints.

The same problem has been investigated by many authors. In employing both the value function as well as the KKT conditions of (1.2), necessary optimality conditions were established in [17] under smooth settings. In [4,9], necessary optimality conditions were derived for Pareto optimal solutions using the Hiriart-Urruty scalarization [7], under the use of Clarke's generalized Jacobian with the nonsmooth Mangasarian–Fromovitz constraint qualification in the first paper [4], and convexificators with a generalized Abadie's constraint qualification in the latter paper [9].

In this work, we take up the general concept of local weakly efficiency with respect to Problem (1.1) which is multiobjective. Letting

$$\overline{S} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in S(x),\, G_j(x,y) \leq 0,\ \text{ for all } j \in \mathbb{J}\}$$

be the leader's feasible region, a pair $(\widehat{x}, \widehat{y}) \in \overline{S}$ is called a local weakly efficient solution of (1.1) and (1.2) if there is an open set $U_0 \times V_0$ containing $(\widehat{x}, \widehat{y})$ such that

$$F(x,y) - F(\widehat{x}, \widehat{y}) \notin -\mathrm{int}\,\mathbb{R}^n_+ \quad \forall (x,y) \in \overline{S} \cap (U_0 \times V_0). \tag{1.3}$$

Our aim is to apply the value function approach as well as the notion of tangential subdifferentials as mathematical tools in order to establish the optimality conditions that must hold at any local weakly efficient solution of the problem (1.1) and (1.2). The latter tool includes many types of subdifferentials like Gâteaux derivatives or convex subdifferentials and coincides with those of Clarke and Michel-Penot, in the case of locally Lipschitz functions that are Clarke regular and MP regular, respectively. It coincides also with upper regular convexificator in the case of tangentially convex functions. Moreover, we employ a specific constraint qualification which is weaker than the Zangwill constraint qualification [19] and other known constraint qualifications like Cottle, Mangasarian–Fromovitz, Kuhn–Tucker, etc.

After this introduction, our paper has the following structure. In Section 2, we give the notations and definitions needed in the sequel. In Section 3, we reformulate the bilevel program (1.1) and (1.2) and we present some constraint qualifications in terms of tangential subdifferentials with a statement of the relationship between them. In Section 4, necessary conditions are derived under a suitable constraint qualification. An example is given at the end to clarify the main result.

2. Notations and preliminaries

We follow the standard notation employed in nonlinear optimization. First, letting $\emptyset \neq \mathcal{C} \subseteq \mathbb{R}^n$, by the sets $\mathrm{int}\,\mathcal{C}$, $\mathrm{co}\,\mathcal{C}$, $\mathcal{C}^\circ$, $\mathcal{C}^s$ we mean the interior, convex hull, negative and strictly negative polar cone of $\mathcal{C}$, respectively.

The convex cone generated by $\mathcal{C}$ contains exactly all conic combinations of the elements of $\mathcal{C}$, it can be expressed as follows:

$$\mathrm{cone}(\mathcal{C}) := \left\{ \xi \in \mathbb{R}^n : \xi = \sum_{i=1}^r \alpha_i \xi_i,\ \alpha_i \geq 0,\ \xi_i \in \mathcal{C},\ i = 1, 2, \dots, r,\ r \geq 1 \right\}.$$

Recall that for any two sets $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbb{R}^n$ one has

$$\mathrm{cone}(\mathcal{C}_1 \cup \mathcal{C}_2) = \mathrm{cone}(\mathcal{C}_1) + \mathrm{cone}(\mathcal{C}_2). \tag{2.1}$$

We recall also three notions of tangent cones: feasible directions, weak feasible directions and contingent, which are given with respect to $\mathcal{C}$ and an element $z \in \mathrm{cl}\,\mathcal{C}$ as follows

$$D(\mathcal{C}, z) := \{\xi \in \mathbb{R}^n : \exists \delta > 0,\ \forall \alpha \in (0, \delta),\ z + \alpha \xi \in \mathcal{C}\},$$
$$WD(\mathcal{C}, z) := \{\xi \in \mathbb{R}^n : \exists t_n \downarrow 0,\ \forall n \in \mathbb{N},\ z + t_n \xi \in \mathcal{C}\},$$
$$T(\mathcal{C}, z) := \{\xi \in \mathbb{R}^n : \exists t_k \downarrow 0,\ \exists \xi_k \to \xi,\ z + t_k \xi_k \in \mathcal{C}\}.$$

It is a direct matter to check that $D(\mathcal{C}, z) \subseteq WD(\mathcal{C}, z) \subseteq T(\mathcal{C}, z)$. Note that the cone $D(\mathcal{C}, z)$ is in general not convex nor closed, whereas $T(\mathcal{C}, z)$ is closed but not necessarily convex, and when $\mathcal{C}$ is convex, $D(\mathcal{C}, z) = WD(\mathcal{C}, z)$

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$. The function φ is said to be tangentially convex at $z \in \mathbb{R}^n$ [15] if its directional derivative (also known as Dini derivative) at z,

$$\varphi'(z, \xi) := \lim_{t \downarrow 0} \frac{\varphi(z + t\xi) - \varphi(z)}{t},$$

is finite for any direction $\xi \in \mathbb{R}^n$ and convex in this argument. Observe that we have sublinearity of the directional derivative of any tangentially convex function since it is positive homogeneous. Realize that if φ is Hadamard directionally differentiable, then its Hadamard directional derivative reduces to it directional derivative. For the converse, φ is Hadamard directionally differentiable at z in ξ if φ is locally Lipschitz at z and directionally differentiable. On the other hand, the tangential subdifferential of $\varphi : \mathbb{R}^n \to \mathbb{R}$ at $z \in \mathbb{R}^n$ is given by [10, 15]

$$\partial_T \varphi(z) := \left\{ y^* \in \mathbb{R}^n : \langle y^*, \xi \rangle \leq \varphi'(z, \xi) \; \forall \xi \in \mathbb{R}^n \right\}. \tag{2.2}$$

For a tangentially convex function, this subdifferential is nonempty, compact and convex (see [11]). Furthermore, tangentially convex functions constitute a large class that contains convex functions on open domains where the tangential subdifferential falls into the classical Fréchet subdifferential, Gâteaux differentiable functions on open domains with a tangential subdifferential reduced to the gradient. This class also includes locally Lipschitz functions that are either Clarke regular [1] or Michel-Penot regular [13], and their tangential subdifferential is equal to that of Clarke in the first case and Michel-Penot in the second.

Notice that if φ is tangentially convex at z, then it follows from the sublinearity of $\varphi'(z, .)$ that (2.2) is equivalent to

$$\varphi'(z, \xi) = \max_{z^* \in \partial_T \varphi(z)} \langle z^*, \xi \rangle, \forall \; \xi \in \mathbb{R}^n,$$

which implies that

$$\inf_{z^* \in \partial_T \varphi(z)} \langle z^*, \xi \rangle \leq \varphi_d^+(z, \xi) = \varphi'(z, \xi) = \varphi_d^-(z, \xi) \leq \sup_{z^* \in \partial_T \varphi(z)} \langle z^*, \xi \rangle, \forall \; \xi \in \mathbb{R}^n,$$

where $\varphi_d^+(z, \xi)$ and $\varphi_d^-(z, \xi)$ denote respectively the upper and lower Dini directional derivatives of φ at z in the direction ξ. Then, $\partial_T \varphi(z)$ is a convexificator of φ at z.

Note also that the definition of tangential subdifferential coincides with that of upper regular convexificator in the case of tangentially convex functions since $\varphi'(z, \xi) = \varphi_d^+(z, \xi) = \sup_{z^* \in \partial_T \varphi(z)} \langle z^*, \xi \rangle, \forall \; \xi \in \mathbb{R}^n$.

3. Reformulation of the bilevel program and constraint qualifications

In order to convert the bilevel program (1.1) to an equivalent single-level program, we are going to employ the value function approach [14, 18]. Let us first denote the value function of the program (1.2) by

$$\phi(x) := \inf_y \left\{ f(x, y) : g_i(x, y) \leq 0 \; \text{ for all } i \in \mathbb{I} \right\}, \tag{3.1}$$

where by convention $\inf \emptyset = +\infty$. The leader program (1.1) can be restated as follows

$$\begin{aligned}
\mathbb{R}_+^p - \min_{x, y} \quad & F(x, y) = (F_1, \dots, F_p)(x, y) \\
\text{subject to} \quad & G_j(x, y) \leq 0, && \text{for all } j \in \mathbb{J}, \\
& g_i(x, y) \leq 0, && \text{for all } i \in \mathbb{I},
\end{aligned}$$

$$f(x, y) - \phi(x) \leq 0. \tag{3.2}$$

The feasible set of (3.2) is

$$\Xi := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid G_j(x, y) \leq 0, j \in \mathbb{J}, g_i(x, y) \leq 0, i \in \mathbb{I}, \ f(x, y) - \phi(x) \leq 0\}.$$

Given a point $(\widehat{x}, \widehat{y}) \in \Xi$, we consider the index sets

$$\mathbb{I}(\widehat{x}, \widehat{y}) := \{i \in \mathbb{I} \ : \ g_i(\widehat{x}, \widehat{y}) = 0\} \quad \text{and} \quad \mathbb{J}(\widehat{x}, \widehat{y}) := \{j \in \mathbb{J} \ : \ G_j(\widehat{x}, \widehat{y}) = 0\}.$$

Suppose that F_k, $k \in \mathbb{K} := \{1, \ldots, p\}$ and G_j, $j \in \mathbb{J}(\widehat{x}, \widehat{y})$ are tangentially convex at $(\widehat{x}, \widehat{y})$, and that f and g_i, $i \in \mathbb{I}(\widehat{x}, \widehat{y})$ are convex. We can easily show the convexity of ϕ.

In the following definition, we give three constraint qualifications:

Definition 3.1. Let

$$\Gamma(\widehat{x}, \widehat{y}) := \bigcup_{i \in \mathbb{I}(\widehat{x}, \widehat{y})} \partial_T g_i(\widehat{x}, \widehat{y}) \ \cup \bigcup_{j \in \mathbb{J}(\widehat{x}, \widehat{y})} \partial_T G_j(\widehat{x}, \widehat{y}) \ \cup (\partial_T f(\widehat{x}, \widehat{y}) - \partial_T \phi(\widehat{x}) \times \{0\}).$$

We say that:

– Zangwill constraint qualification $(\partial_T\text{-ZCQ})$ is verified at $(\widehat{x}, \widehat{y})$ if

$$(\Gamma(\widehat{x}, \widehat{y}))^\circ \ \subseteq \ D(\Xi, (\widehat{x}, \widehat{y})).$$

– Weak Zangwill constraint qualification $(\partial_T\text{-WZCQ})$ is verified at $(\widehat{x}, \widehat{y})$ if

$$(\Gamma(\widehat{x}, \widehat{y}))^\circ \ \subseteq \ WD(\Xi, (\widehat{x}, \widehat{y})).$$

Remark 3.2. Since $D(\Xi, (\widehat{x}, \widehat{y})) \subseteq WD(\Xi, (\widehat{x}, \widehat{y}))$, then $\partial_T\text{-ZCQ} \Rightarrow \partial_T\text{-WZCQ}$.

4. NECESSARY OPTIMALITY CONDITIONS

Before we proceed to establish our new necessary optimality conditions for local weakly efficient solution of (1.1) by means of the tangential subdifferential, we recall this lemma from [6].

Lemma 4.1 ([6]). *Consider two arbitrary index sets V and W (possibly infinite), and two maps $\varphi : V \to \mathbb{R}^n$ and $\psi : W \to \mathbb{R}^n$.*

If the set $\mathrm{co}\{\varphi(v), v \in V\} + \mathrm{cone}\{\psi(w), w \in W\}$ is closed, then the following two statements are equivalent:

(i) $\begin{cases} \langle \varphi(v), d \rangle < 0, \ v \in V, \ V \neq \emptyset, \\ \langle \psi(w), d \rangle \leq 0, \ w \in W, \end{cases}$ *has no solution $d \in \mathbb{R}^n$.*

(ii) $0 \in \mathrm{co}\{\varphi(v), v \in V\} + \mathrm{cone}\{\psi(w), w \in W\}$.

Theorem 4.2. *Assume the point $(\widehat{x}, \widehat{y})$, a local weakly efficient solution of (1.1), is such that*

(1) *F_k, $k \in \mathbb{K}$, and G_j, $j \in \mathbb{J}(\widehat{x}, \widehat{y})$ are tangentially convex at $(\widehat{x}, \widehat{y})$, and that f and g_i, $i \in \mathbb{I}(\widehat{x}, \widehat{y})$ are convex.*
(2) *The ∂_T-WZCQ holds at $(\widehat{x}, \widehat{y})$.*
(3) *The set*

$$D := \mathrm{cone} \left(\bigcup_{i \in \mathbb{I}(\widehat{x}, \widehat{y})} \partial_T g_i(\widehat{x}, \widehat{y}) \cup \bigcup_{j \in \mathbb{J}(\widehat{x}, \widehat{y})} \partial_T G_j(\widehat{x}, \widehat{y}) \cup (\partial_T f(\widehat{x}, \widehat{y}) - \partial_T \phi(\widehat{x}) \times \{0\}) \right)$$

is closed.

Then there exists $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m_+$ *with* $\sum_{k=1}^{p} \alpha_k = 1$, *such that*

$$0 \in \sum_{k=1}^{p} \alpha_i \ \partial_T F_k(\widehat{x}, \widehat{y}) + \sum_{i \in \mathbb{I}(\widehat{x}, \widehat{y})} \mathrm{cone} \ \partial_T g_i(\widehat{x}, \widehat{y}) + \sum_{j \in \mathbb{J}(\widehat{x}, \widehat{y})} \mathrm{cone} \ \ \partial_T G_j(\widehat{x}, \widehat{y}) \tag{4.1}$$

$$+ \mathrm{cone}(\partial_T f(\widehat{x}, \widehat{y}) - \partial_T \phi(\widehat{x}) \times \{0\}). \tag{4.2}$$

Proof. First, by definition of local weak efficiency, there is an open set $U \times V$ of $(\widehat{x}, \widehat{y})$ such that there is no $(x, y) \in \Xi \cap (U \times V)$ with

$$F_k(x, y) < F_k(\widehat{x}, \widehat{y}); \quad \forall k \in \mathbb{K}.$$

We claim that

$$\left(\bigcup_{k \in \mathbb{K}} \partial_T F_k(\widehat{x}, \widehat{y}) \right)^s \cap WD(\Xi, (\widehat{x}, \widehat{y})) = \emptyset. \tag{4.3}$$

Indeed, contrary to our claim, suppose there exists

$$z^* = (z_1^*, z_2^*) \in \left(\bigcup_{k \in \mathbb{K}} \partial_T F_k(\widehat{x}, \widehat{y}) \right)^s \cap WD(\Xi, (\widehat{x}, \widehat{y})),$$

then we get $z^* \in \left(\bigcup_{k \in \mathbb{K}} \partial_T F_k(\widehat{x}, \widehat{y}) \right)^s$ and $z^* \in WD(\Xi, (\widehat{x}, \widehat{y}))$.

Note first that, from $z^* \in \left(\bigcup_{k \in \mathbb{K}} \partial_T F_k(\widehat{x}, \widehat{y}) \right)^s$, it follows that

$$\langle w^*, z^* \rangle < 0, \quad \forall w^* \in \partial_T F_k(\widehat{x}, \widehat{y}), \ \forall k \in \mathbb{K}. \tag{4.4}$$

For each $k \in \mathbb{K}$, define $\varphi_k : \partial_T F_k(\widehat{x}, \widehat{y}) \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ as

$$\varphi_k(w^*) = \langle w^*, z^* \rangle, \ \forall w^* \in \partial_T F_k(\widehat{x}, \widehat{y}).$$

The continuity of φ_k on $\partial_T F_k(\widehat{x}, \widehat{y})$, which is compact, implies the existence of $\overline{w}^* \in \partial_T F_k(\widehat{x}, \widehat{y})$ with $\varphi_k(\overline{w}^*) = \max_{w^* \in \partial_T F_k(\widehat{x}, \widehat{y})} \langle w^*, z^* \rangle$. Hence, according to (4.4), we obtain for every $k \in \mathbb{K}$

$$F_k'(\widehat{x}, \widehat{y}, z^*) = \max_{w^* \in \partial_T F_k(\widehat{x}, \widehat{y})} \langle w^*, z^* \rangle = \langle \overline{w}^*, z^* \rangle < 0. \tag{4.5}$$

Now, since $z^* = (z_1^*, z_2^*) \in WD(\Xi, (\widehat{x}, \widehat{y}))$, there is $t_s \downarrow 0$ satisfying $(\widehat{x}, \widehat{y}) + t_s(z_1^*, z_2^*) \in \Xi$ for all $s \in \mathbb{N}$. Taking into account that the point $(\widehat{x}, \widehat{y})$ is a local weakly efficient solution of (1.1), we obtain that there is $(\widehat{x}, \widehat{y}) + t_s(z_1^*, z_2^*) \in B((\widehat{x}, \widehat{y}), \delta)$, for some $\delta > 0$ and for s large enough, such that there is $\overline{k} \in \mathbb{K}$ satisfying $F_{\overline{k}}((\widehat{x}, \widehat{y}) + t_s(z_1^*, z_2^*)) \geq F_{\overline{k}}(\widehat{x}, \widehat{y})$. In combining this with the fact that

$$F_{\overline{k}}'((\widehat{x}, \widehat{y}), (z_1^*, z_2^*)) = \lim_{s \to \infty} \frac{F_{\overline{k}}((\widehat{x}, \widehat{y}) + t_s(z_1^*, z_2^*)) - F_{\overline{k}}(\widehat{x}, \widehat{y})}{t_s},$$

we obtain $F_{\overline{k}}'((\widehat{x}, \widehat{y}), (z_1^*, z_2^*)) \geq 0$ for $\overline{k} \in \mathbb{K}$, which contradicts (4.5), and consequently, (4.3) is fulfilled. Thus,

$$\mathrm{int} \left(\bigcup_{k \in \mathbb{K}} \partial_T F_k(\widehat{x}, \widehat{y}) \right)^s \cap \mathrm{cl}(WD(\Xi, (\widehat{x}, \widehat{y}))) = \left(\bigcup_{k \in \mathbb{K}} \partial_T F_k(\widehat{x}, \widehat{y}) \right)^s \cap \mathrm{cl}(WD(\Xi, (\widehat{x}, \widehat{y}))) = \emptyset.$$

On the basis of ∂_T-WZCQ, we have

$$\left(\bigcup_{k \in \mathbb{K}} \partial_T F_k(\widehat{x}, \widehat{y}) \right)^s \cap \left(\bigcup_{i \in \mathbb{I}(\widehat{x}, \widehat{y})} \partial_T g_i(\widehat{x}, \widehat{y}) \right)^{\circ} \cap \left(\bigcup_{j \in \mathbb{J}(\widehat{x}, \widehat{y})} \partial_T G_j(\widehat{x}, \widehat{y}) \right)^{\circ} \cap (\partial_T f(\widehat{x}, \widehat{y}) - \partial_T \phi(\widehat{x}) \times \{0\})^{\circ} = \emptyset. \tag{4.6}$$

Then, from (4.6), we see that the system

$$
\begin{cases}
\langle \zeta_k, z^* \rangle < 0, & \forall \, k \in \mathbb{K}, \ \forall \zeta_k \in \partial_T F_k(\widehat{x}, \widehat{y}), \\
\langle \vartheta_j, z^* \rangle \leq 0, & \forall \, j \in \mathbb{J}, \ \forall \vartheta_j \in \partial_T G_j(\widehat{x}, \widehat{y}), \\
\langle \theta_i, z^* \rangle \leq 0, & \forall \, i \in \mathbb{I}, \ \forall \zeta_i \in \partial_T g_i(\widehat{x}, \widehat{y}), \\
\langle \eta, z^* \rangle \leq 0, & \forall \eta_k \in \partial_T f(\widehat{x}, \widehat{y}) - \partial_T \phi(\widehat{x}) \times \{0\},
\end{cases}
$$

has no solution $z^* \in \mathbb{R}^n \times \mathbb{R}^m$. On the other hand, since $\partial_T F_k(\widehat{x}, \widehat{y})$ is compact for each $k \in \mathbb{K}$, the set $\bigcup_{k=1}^p (\partial_T F_k(\widehat{x}, \widehat{y}))$ is also compact, and hence $\bigcup_{k=1}^p (\partial_T F_k(\widehat{x}, \widehat{y})) + D$ is closed because so is D. Thus, by virtue of Lemma 4.1, we are led to

$$
(0,0) \in \mathrm{co}\left(\bigcup_{k=1}^p \partial_T F_k(\widehat{x}, \widehat{y}) \right) + \mathrm{cone}\left(\bigcup_{i \in \mathbb{I}(\widehat{x}, \widehat{y})} \partial_T g_i(\widehat{x}, \widehat{y}) \cup \bigcup_{j \in \mathbb{J}(\widehat{x}, \widehat{y})} \partial_T G_j(\widehat{x}, \widehat{y}) \cup (\partial_T f(\widehat{x}, \widehat{y}) - \partial_T \phi(\widehat{x}) \times \{0\}) \right).
$$

Taking into account that tangential subdifferentials are convex, there exist nonnegative scalars α_k, $k \in \mathbb{K}$ with $\sum_{k=1}^p \alpha_k = 1$ such that

$$
(0,0) \in \sum_{k=1}^p \alpha_k \, \partial_T F_k(\widehat{x}, \widehat{y}) + \mathrm{cone}\left(\bigcup_{i \in \mathbb{I}(\widehat{x}, \widehat{y})} \partial_T g_i(\widehat{x}, \widehat{y}) \cup \bigcup_{j \in \mathbb{J}(\widehat{x}, \widehat{y})} \partial_T G_j(\widehat{x}, \widehat{y}) \cup (\partial_T f(\widehat{x}, \widehat{y}) - \partial_T \phi(\widehat{x}) \times \{0\}) \right).
$$

Consequently by (2.1) that yields

$$
(0,0) \in \sum_{k=1}^p \alpha_i \, \partial_T F_k(\widehat{x}, \widehat{y}) + \sum_{i \in \mathbb{I}(\widehat{x}, \widehat{y})} \mathrm{cone}\, \partial_T g_i(\widehat{x}, \widehat{y}) + \sum_{j \in \mathbb{J}(\widehat{x}, \widehat{y})} \mathrm{cone}\, \partial_T G_j(\widehat{x}, \widehat{y}) + \mathrm{cone}(\partial_T f(\widehat{x}, \widehat{y}) - \partial_T \phi(\widehat{x}) \times \{0\}).
$$

$\square$

Remark 4.3. From Remark 3.2, our main result remains true if we replace ∂_T-WZCQ with ∂_T-ZCQ.

Remark 4.4. In (3.2), without the two last inequalities, we obtain the same problem studied in many papers as [11, 12, 16] with different constraint qualifications. In the future, it is possible to consider finding new results related to bilevel problems based on these studies.

Next, following [3, 8] we present an upper estimate for the subdifferential of the value function ϕ (which coincides with $\partial_T \phi(x)$ since ϕ is convex) in terms of the initial data of (1.2).

Proposition 4.5. *Suppose that*

(i) *f is a convex and continuous function.*
(ii) *The argminimum map S is inner semicontinuous at $(\widehat{x}, \widehat{y})$; that is, for any sequence $(x_k) \in \mathrm{dom}\, S$ that converges to $\widehat{x}$ there exists a sequence $(y_k) \in S(x_k)$ that converges to $\widehat{y}$.*
(iii) *$(\widehat{x}, \widehat{y})$ is lower-level regular; that is, the following implication holds*

$$
\left[\sum_{i \in \mathbb{I}(\widehat{x}, \widehat{y})} \beta_i \widetilde{y}_i = 0, \beta_i \geq 0 \right] \Rightarrow [\beta_i = 0 \text{ for each } i \in \mathbb{I}(\widehat{x}, \widehat{y})]
$$

when $(\widetilde{x}_i, \widetilde{y}_i) \in \mathrm{co}(\partial_T g_i(\widehat{x}, \widehat{y}))$ with some $\widetilde{x}_i \in \mathbb{R}^n$ as $i \in \mathbb{I}(\widehat{x}, \widehat{y})$.

Then, an efficient upper estimate of the subdifferential of the value function can be determined as follows:

$$
\partial \phi(\widehat{x}) \subseteq \bigcup_{(\theta_1, \theta_2, \ldots, \theta_q) \in \Theta(\widehat{x}, \widehat{y})} \left(\partial_x f(\widehat{x}, \widehat{y}) + \sum_{i \in \mathbb{I}} \theta_i \partial_x g_i(\widehat{x}, \widehat{y}) \right)
$$

where

$$\Theta(\widehat{x},\widehat{y}) := \left\{ (\theta_1, \theta_2, \ldots, \theta_q) \in \mathbb{R}^q \mid 0 \in \partial_y f(\widehat{x},\widehat{y}) + \sum_{i \in \mathbb{I}} \theta_i \partial_y g_i(\widehat{x},\widehat{y}), \theta_i \geq 0, \ \theta_i g_i(\widehat{x},\widehat{y}) = 0, i \in \mathbb{I} \right\}.$$

Now, we present an example illustrating the main result.

Example 4.6. Consider $n = m = 1$, $p = q = l = 2$ and let the leader's objectives and constraints be given by

$$F_1(x,y) = \begin{cases} \frac{x^3}{y} - x, & y \neq 0, \\ -x, & y = 0, \end{cases} \qquad F_2(x,y) = \max\{x,y\}$$

$$G_1(x,y) = -x, \qquad G_2(x,y) = x^3 - 4x.$$

The follower's objective and constraints are as follows

$$f(x,y) = |x| + |y|, \qquad g_1(x,y) = y - 1 \qquad g_2(x,y) = -y.$$

Then, after straightforward calculations, one has $\Xi = [0,2] \times \{0\}$. The convexity of f and g_1 can be easily checked as well as the tangential convexity of F_1, G_1 and G_2 at the local weakly efficient point $(\widehat{x},\widehat{y}) := (0,0)$.

In this example, the follower's region of rational reactions S and the function ϕ are

$$S(x) = \{0\}, \qquad \phi(x) = |x|.$$

Let us show that $\mathbb{I}(\widehat{x},\widehat{y}) = \{2\}$ and $\mathbb{J}(\widehat{x},\widehat{y}) = \{1,2\}$. Moreover, the tangential subdifferentials of F_1, F_2, f, ϕ, G_1, G_2 and g_2, are given by

$$\partial_T F_1(\widehat{x},\widehat{y}) = \{(-1,0)\}, \quad \partial_T F_2(\widehat{x},\widehat{y}) = \mathrm{co}\{(0,1); (1,0)\},$$
$$\partial_T f(\widehat{x},\widehat{y}) = [-1,1] \times [-1,1], \quad \partial_T \phi(\widehat{x}) = [-1,1],$$
$$\partial_T G_1(\widehat{x},\widehat{y}) = \{(-1,0)\}, \quad \partial_T G_2(\widehat{x},\widehat{y}) = \{(-4,0)\}, \quad \partial_T g_2(\widehat{x},\widehat{y}) = \{(0,-1)\}.$$

The weak feasible directions cone w.r.t. Ξ and $(\widehat{x},\widehat{y})$ is given as follows $\Xi = WD(\Xi, (\widehat{x},\widehat{y})) = \mathbb{R}^+ \times \{0\}$. A simple calculation can lead to

$$(\Gamma(\widehat{x},\widehat{y}))^\circ \subseteq WD(\Xi, (\widehat{x},\widehat{y})),$$

which mean our constraint qualification ∂_T-WZCQ holds at $(\widehat{x},\widehat{y})$.

We conclude that all assumptions of Theorem 4.2 are fulfilled. Taking $\alpha_1 = \alpha_2 = \frac{1}{2}$, we can easily verify the optimality condition given in this theorem:

$$(0,0) \in \alpha_1\, \partial_T F_1(\widehat{x},\widehat{y}) + \alpha_2\, \partial_T F_2(\widehat{x},\widehat{y}) + \mathrm{cone}\, \partial_T G_1(\widehat{x},\widehat{y}) + \mathrm{cone}\, \partial_T G_2(\widehat{x},\widehat{y})$$
$$+ \mathrm{cone}\, \partial_T g_2(\widehat{x},\widehat{y}) + \mathrm{cone}\left(\partial_T f(\widehat{x},\widehat{y}) - \partial_T \phi(\widehat{x}) \times \{0\} \right).$$

In the above example, we can obtain a similar result if we have only $F = F_1$. In this case, it's worth noting that Theorem 5.1 of [8] and Theorem 1 of [5] cannot be employed because F_1 is not locally Lipschitz at the local weakly efficient $(\widehat{x},\widehat{y}) = (0,0)$. It follows that our main result is more general since the leader's objective can be multiobjective and not necessarily locally Lipschitz. On the other hand, since the functions F_1 and G_2 are not convex, we cannot apply the results that require the convexity of the functions in the upper level such as Theorem 4.1 of [2].

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