

## AN ESTIMATION OF EFFECTS OF MEMORY AND LEARNING EXPERIENCE ON THE EOQ MODEL WITH PRICE DEPENDENT DEMAND

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**Abstract.** In this article, an economic order quantity model has been studied in view of joint impacts of the memory and learning due to experiences on the decision-making process where demand is considered as price dependant function. The senses of memory and experience-based learning are accounted by the fractional calculus and dense fuzzy lock set respectively. Here, the physical scenario is mathematically captured and presented in terms of fuzzy fractional differential equation. The  $\alpha$ -cut defuzzification technique is used for dealing with the crisp representative of the objective function. The main credit of this article is the introduction of a smart decision-making technique incorporating some advanced components like memory, self-learning and scopes for alternative decisions to be accessed simultaneously. Besides the dynamics of the EOQ model under uncertainty is described in terms of fuzzy fractional differential equation which directs toward a novel approach for dealing with the lot-sizing problem. From the comparison of the numerical results of different scenarios (as particular cases of the proposed model), it is perceived that strong memory and learning experiences with appropriate keys in the hand of the decision maker can boost up the profitability of the retailing process.

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### 1. INTRODUCTION

The maintenance of stocks is very crucial issue in a supply chain management problem. Inadequate on hand stocks may cause the discontinuity in the supply flow. On the contrary, uncontrolled gathering of product in the store may increase the maintenance cost and blockage of fund. For both of the cases, the result acts as opposite to the objectivity of the supplier/retailer. So, there is an urge to develop aptly fitted model for scheduling optimal lots of stocks in this business scenario. Inventory control management can fulfil the needs in this regard.

Among different techniques of inventory control problems, lot sizing problems have drawn huge attentions of the researchers and the decision makers for its simplicity and efficiency to capture the dynamics of the model. The classical economic order quantity model (EOQ) introduced by Haris [25] was the pioneering establishment in this filed. In the classical EOQ model, a simple problem was formulated under the assumptions of deterministic

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*Keywords.* Riemann–Liouville differentiability, Caputo differentiability, Laplace transformation, EOQ model, selling price, lock fuzzy dense, memory.

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and constant demand with no shortage. Gradually, the literature of the lot sizing modelling was improved and matured incorporating more reliable assumptions serving different purposes [24, 46]. In reality, the demand and pricing are a very crucial factor in any retailing business model. The demand rate may vary with respect to time [19, 61], stock level [21], selling price [37], various dealing policy and so many other components in different supply situations. In a developing country, generally, the customers pay attentions towards a low-priced product even sacrificing over the quality in terms of durability and reliability [34–36]. So, the demand rate of product in these circumstances can be viewed as a negatively proportional to the selling price. In general, the demand pattern is imprecise in nature. So, the optimal business planning forecasting the demand is matter of great challenge for the manager. If an organization or an individual decision-maker is enriched with past memory of dealing with the similar type of circumstance, the overall present dealing management can be more effective in the fulfilment of its objectivity. Again, the learning through the experiences of the repeated tasks can make clear perceptions of the decision maker on the demand pattern and the measure of vagueness over the demand pattern can be reduced.

On the basis of above-mentioned intuitions, the following research questionnaires motivate the authors for introducing the theory in this present article:

- (i) What will be the optimal dealing strategy for an economic order quantity (EOQ) model motivated retailing scenario with price dependent imprecise demand pattern?
- (ii) How does the vagueness regarding the demand pattern create hindrances on the decision-making policy?
- (iii) How much the experience gained through doing repeated tasks can help the decision-maker for installing a better policy to attain the goal?
- (iv) How much the memory of the system (retailing organization) can create impacts in making the optimal strategy in favour of the profit maximization objectivity?
- (v) What will be the simultaneous effects of the memory and experience-based learning on the EOQ model?
- (vi) If the decision maker has keys in his/her hand, how much the keys can regulate the retailing process towards the profit maximization objectivity?

Focusing on the mentioned research questions, the above-mentioned intuitions on the decision-making phenomena are mathematically modelled and established in this present paper through the study of an EOQ model with price sensitive demand rate under the fuzzy fractional differential equation of Caputo type and triangular dense fuzzy lock set (TDFLS) decision making setup. The next section provides the details on the theoretical background of the present article.

## 2. THEORETICAL BACKGROUND

### 2.1. Literature review

#### 2.1.1. Recent advancement on inventory modelling

As it is mentioned earlier that the selling price has very significant impacts on the demand pattern in a business scenario, following the footstep of Kim *et al.* [37], Polatoglu and Sahin [51] also considered price dependent demand in one of their studies. Mukhopadhyay *et al.* [45] include the deterioration of items beside the price dependency of the demand rate in the list of the assumptions. In another major finding on the consideration of the price dependent demand, Pal *et al.* [47] discussed the problem with no shortage. The investigation for an EOQ model with price depended demand and fully backlogged shortage was credited by Sana [58]. In this context, the time varying holding cost and discount policy in inventory model with price dependent demand had been implemented by Alfares and Ghaithan [6]. Mishra *et al.* [42] considered an inventory model with stock and price dependent demand rate, controllable rate of deterioration under preservation technology. Mashud *et al.* [40] contributed a worthy investigation on an inventory control problem with stock and price dependent demand rate, deterioration and partial backlogged shortage. An inventory control management strategy for the perishable items was developed by Feng *et al.* [20] assuming the dependency of the demand rate on price, displayed stock, freshness of items simultaneously. On the other hand, Hendalianpour [26] developed an inventory control problem

with price dependent demand in game theoretic approach using the double interval grey number for tracing the consumer behavior more accurately. The study of the distribution management of blood products (one kind of perishable items) was mathematically modeled under transshipment and uncertain demand consideration and optimized through heuristic algorithm by Liu *et al.* [38]. The potential of different contracts (like wholesale-price, revenue-share, quantity discount etc.) under the competitive structures of two manufactures and two retailers with price sensitive stochastic grey demand was examined by Hendalianpour *et al.* [27]. Also, Dey *et al.* [17] considered the discrete set up cost for an integrated inventory model with price dependent demand.

### 2.1.2. Fuzzy lot-sizing models

In the real-world decision-making system, a decision maker must have to tackle the ambiguities involved in the parameters which have great impacts on the decision. To define, describe and measure the real-world uncertainty/ambiguity, fuzzy concept is regarded as one of the finest tools. Zadeh [63] was the first to introduce the concept of fuzzy uncertainty. This novel concept was incorporated for decision making problem by Bellman and Zadeh [9]. Park [49] was the pioneer to manifest the economic order quantity (EOQ) model under fuzzy uncertainty. Consequently, some worthy findings on the fuzzy inventory models were contributed by the researchers like Vujosevic *et al.* [62] and Hojati [29]. Chen *et al.* [13] was the first to build a fuzzy economic order quantity model with back order. Following their contribution, the advancement and enrichment of the literature of the fuzzy inventory model has been done in the recent decades. A very detailed and versatile review on the inventory model under fuzzy uncertainty was contributed by Shekarian *et al.* [60] in this regard. On the other hand, fuzzy arithmetic was quite different from the crisp counterpart. So, the calculus of fuzzy variables and fuzzy valued functions were developed separately. Chang and Zadeh [12] gave an insight of fuzzy mapping. Later, Kaleva [30] introduced the concept of fuzzy differential equation. Then, lots of improvements and advancements [4, 7, 8, 11, 43] were done in this direction. While discussing the inventory problems under fuzzy uncertainty, it is expected to use the fuzzy differential equation in order to describe the uncertain dynamics of the model. Few papers [14, 22, 23, 39, 44] used fuzzy differential equation approach to interpret the inventory control problems under fuzzy uncertainty.

### 2.1.3. Learning and memory-based decision making of inventory model

Experiences and past memory are two major effective components for self-learning and advancement of human knowledge. So, memory from the past experience may help the decision maker to accurate the dealing strategy. Also, earning the experiences through executing the repeated practice of similar kind of job may make the decision maker more mature in his/her field. So, these two features can be implemented in the study of the inventory control problems to develop more optimal business planning in favor of its objectivity. In this context, one of the closest physical interpretations of the fractional calculus is assumed that it represents the memory of a system through its iterative kernels. The theory of fractional calculus [1, 18, 41, 50] gained attention of the scholar engaging on the exploration of truth in different research directions of science and technology. Nature and emotion-based dynamic cannot be described aptly by the classical integer order calculus. Thus, the theory of fractional differential equation has been implemented to estimate the effect of memory on inventory control problems [48, 52, 53]. In addition, if the function is fuzzy valued, then it called the fuzzy fractional differential equation. The fractional differential equation was first discussed in uncertain phenomena by Agarwal *et al.* [2]. Then, the theory of fuzzy Laplace transformation and fuzzy fractional differential equation in the sense of Riemann–Liouville and Caputo had been developed gradually by several novel investigations [3, 5, 28, 56, 57]. Very recently, the production phase of an EPQ model has been analysed by the fuzzy fractional differential equation under the Riemann–Liouville derivative by Rahaman *et al.* [54]. The theories of learning were implemented on the study of inventory control problems in different ways. In this context, Kazemi *et al.* [33] incorporated the sense of human learning to study an economic order quantity (EOQ) model with back order under fuzzy uncertainty. Also, Shekarian *et al.* [59] considered the fuzzy uncertainty and learning to discuss an EOQ model for the items of imperfect quality. Bousdekis *et al.* [10] developed sensor driven learning approach for the perspective analysis of time dependent parameters that leads to self-optimization through learning and feedback. An alternative

intuition for the fuzzy learning-based decision making was established through the introduction of the triangular dense fuzzy set (TDFS) by De and Beg [16]. The concept of experience-based learning was further enriched with the key and lock facility in terms of the triangular lock fuzzy dense set (TLFDS) by De [15]. Consequently, the concept of the TDFS and TLFDS were aptly utilized to explore the environments favoured sustainable decision making of inventory control problems [31, 32].

## 2.2. Motivation of the present study

### 2.2.1. Research gap in the existing literature

Based on the existing literature related to our above-mentioned research questionaries, the following research gaps have been identified:

- (i) The past memory very frequently affects the present decision-making process of an organization in present time. Optimal strategy for the present dealing situation can be adapted from previous experiences. Thus, the study on the memory concerned decision making of the inventory management problems [48, 52, 53] is very new trend and many things can be explored in this arena. For example, memory effect with uncertainty can turn the decision-making process more interesting and reliable.
- (ii) The mathematical quantification of experience-based learning doing repeated tasks in terms of dense and lock fuzzy sets [15, 16, 31, 32] provided a very effective alternative to analyse the uncertain decision-making situation under self-optimization. The combined effects of the memory and experiences-based learning may be more reliable scenario in reality. In the existing literature, we have not noticed much work (except the article [55]) where the memory and learning has been considered simultaneously to describe an EOQ model.
- (iii) Fuzzy numbers and variables are being run according to their own arithmetic rule and calculus that are different from its crisp counterpart. Thus, it is very much desirable to describe the dynamics of a fuzzy inventory model through the analysis of fuzzy differential equations. The literature of fuzzy inventory models expressed by the fuzzy differential equations [14, 22, 23, 39, 44] is a little bit inferior in the whole literature of fuzzy inventory control problems. Memory effected inventory models are seen to be described by the fractional differential equation. Thus, fuzzy fractional differential equations are expected to be used for dealing the memory concerned model under fuzzy uncertainty. In this context, we identified an article concerning the application of the differential equation under Riemann–Liouville derivative on an EPQ model [54]. However, the definition of initial conditions associated with the Riemann–Liouville fractional differential equations is little abstract in terms of the physical significance. Thus, the fuzzy fractional differential equation can be better alternative to fill the gap up.
- (iv) The fuzzy fractional differential equations have an enriched literature in the theoretical point of view. The applications of the notions of fuzzy fractional differential equations are rarely discussed (at least in the operation research domain). So, there is a gap to explore the meaning of fuzzy fractional differential equation with respect to the decision-making scenario of lot sizing problems.

### 2.2.2. Present study as a junction of different research directions

In this article, an EOQ model of price dependent demand rate is developed and analysed incorporating impacts of the past memory and learning due to experiences on the decision-making process. Intuitively, the optimal dealing strategy for price sensitive business scenario is subjected to the learning and training facility of the decision maker(s). Mathematically, the present study connects different research disciplines according to the intuitive meanings. The sense of memory is adapted in the study in terms of fractional calculus and uncertainty with associated parameters and variables is represented by the fuzzy set up. We, therefore, utilized Caputo type fuzzy fractional differential equations to trace the dynamical behaviour of the system in the present study. The decreasing natures of the vagueness regarding the demand pattern as the time forwards (through repetitions of similar kind of the decision-making phenomena) are accounted by the theory of TDFLS. Thus, the introduction of present study may be viewed as a theoretical amalgamation of distinct research directions.

### 2.2.3. Novelty of the present study

The main goal of this paper is to consider the demand to be selling price dependent in a TDPLS set up. Price is a very crucial factor in marketing-retailer scenario. Low price of product can attract the customer attention. So, the demand rate can be boosted up by lowering the selling price. Thus, the demands were regarded as the functions of selling price in the existing literature. But, in reality the demand pattern is not precise to the retailer and thus the selling price may not be crisps. In this circumstance, the decision maker has to find the optimal retailing policy anticipating the demand pattern more precisely. Suppose, the selling price is primarily traced with the association of impreciseness and is given loosely as {price is not suitable for the expected profit, price is suitable for the expected price, price is suitable for the profit above the expectation}. The decision maker(s) had to take an optimal decision to choose a selling price which can meet an expected level of the profit at least. The whole discussion of this paper is established on this novel intuition of managerial phenomena. Secondly, the present study chooses the fuzzy fractional differential equation to describe the proposed EOQ model under uncertainty and memory sensitivity. To describe the dynamics of the fuzzy parameters carrying memory, the theory of the fuzzy fractional differential equation under Caputo derivative is utilized for the first time (according to the authors' knowledge) in the discussion of an inventory control problem. Thirdly, the proposed theory is represented in this paper as a more generalized phenomenon of interpretation over the other phenomena on the different combinations of crisps, fuzzy, dense fuzzy, memory and so many components as the decision-making features. Also, the superiority of the memory motivated inventory model under TDPLS learning consideration over the models discussed as the particular cases is established here.

## 3. DIFFERENT NOTATIONS AND ASSUMPTIONS

The following notations are used to denote the parameters, decision variables and objective functions to develop and optimize the model:

$\tilde{D}$ :	Fuzzy demand (units)
$\tilde{p}$ :	Fuzzy selling price (\$/units)
$\tilde{q}(t)$ :	Fuzzy inventory level at time $t$
$\tilde{Q}$ :	Fuzzy lot size (units)
$Q_{\text{defuzz}}$ :	Defuzzified lot size (units) (decision variable)
$T$ :	Total time cycle (month) (decision variable)
$\alpha$ :	Differential memory index (decision variables)
$\beta$ :	Integral memory index (decision variables)
$h$ :	Holding cost (per unit per unit time)
$C$ :	Ordering cost (per complete cycle)
$TAP_{\alpha, \beta}$ :	Total average profit (in fuzzy)
$TAP_{\text{defuzz}}$ :	Total average profit (after defuzzification) (objective function)

The proposed EOQ model is developed based on the following assumptions:

- The demand  $(\tilde{D})$  is selling price dependent *i.e.*,  $\tilde{D} = a - b\tilde{p}$ , where  $a$  and  $b$  are two positive crisp constants and  $\tilde{p}$  is a triangular dense fuzzy lock number.
- Production is instantaneous.
- Shortages are not allowed.
- Lead time is zero.
- EOQ model is memory sensitive *i.e.*, the demand pattern is motivated by the memory of the customer with the previous experience concerned with the behaviour of the shopkeeper or the quality of the product etc.

#### 4. MATHEMATICAL MODELLING OF THE EOQ

##### 4.1. Formulation of the model

The classical EOQ model under uncertainty can be described by the fuzzy differential equation with the terminal conditions as follows:

$$\begin{cases} \tilde{q}'(t) = -\tilde{D} \\ \tilde{q}(0) = \tilde{Q} \\ \tilde{q}(T) = \tilde{0}. \end{cases} \quad (4.1)$$

The equation (4.1) represents a memory free uncertain economic order quantity model. The sense of memory can be added in this regard by reconstructing the mathematical model with the help of the following differ-integral equations:

$$\begin{cases} \tilde{q}'(t) = -\int_0^t K(t, z) \tilde{D} dz \\ \tilde{q}(0) = \tilde{Q} \\ \tilde{q}(T) = \tilde{0}. \end{cases} \quad (4.2)$$

In the equation (4.2),  $K(t, z)$  is an iterative kernel carrying the notion of memory. If the kernel  $K(t, z)$  is chosen to be  $\frac{(t-z)^{\alpha-2}}{\Gamma(\alpha-1)}$ , then the equation (4.2) can be rewritten as:

$$\tilde{q}'(t) = -{}^{RL}I_t^{\alpha-1} \tilde{D}$$

i.e.,

$${}^{RL}I_t^{1-\alpha}[\tilde{q}'(t)] = -\tilde{D}$$

i.e.,

$${}^C D_t^\alpha \tilde{q}(t) = -\tilde{D}. \quad (4.3)$$

Thus, incorporating the sense of memory in the fuzzy EOQ model given by the equation (4.1), the memory motivated uncertain scenario can be depicted by the following fuzzy fractional differential equation under Caputo differentiability along with the terminal conditions:

$$\begin{cases} {}^C D_t^\alpha \tilde{q}(t) = -\tilde{D}, \text{ } 0 \leq t \leq T \text{ and } 0 < \alpha \leq 1 \text{ be a real number} \\ \tilde{q}(0) = \tilde{Q} \\ \tilde{q}(T) = \tilde{0}. \end{cases} \quad (4.4)$$

The fuzzy demand is taken as a function of the fuzzy selling price in the assumptions for developing the proposed model. So, putting  $\tilde{D} = a - b\tilde{p}$  (where the selling price  $\tilde{p}$  is a fuzzy number), the system represented by the equation (4.4) can be rewritten as:

$$\begin{cases} {}^C D_t^\alpha \tilde{q}(t) = -(a - b\tilde{p}) \\ \tilde{q}(0) = \tilde{Q} \\ \tilde{q}(T) = \tilde{0}. \end{cases} \quad (4.5)$$

##### 4.2. Existence, uniqueness of the solution

Now, one of the major concerns about the system given by the equation (4.5) is existence and uniqueness of its solution. Here, the theory provided by the Theorem 5.1 of [54] is imitated to discuss the existence and uniqueness criteria of the solution of the system given by the equation (4.5). The following notes about the equation (4.5) can be observed:

- (a) Suppose, we consider a region  $\mathcal{R} = [0, h] \times B_\varepsilon(\tilde{Q})$ , where  $B_\varepsilon(\tilde{Q})$  is the closed ball with centre at  $\tilde{Q}$  and radius  $\varepsilon$ .
- (b) Obviously,  $-(a - b\tilde{p})$  is a fuzzy constant and hence continuous on  $\mathcal{R}$ . Let us consider that  $\psi(t, \tilde{q}(t)) = -(a - b\tilde{p})$ .
- (c) If we restrict the fuzzy demand  $\tilde{D} = (a - b\tilde{p})$  to be bounded quantity, then we can find a positive integer  $M_1$  such that  $D_H(\tilde{0}, \psi(t, \tilde{q}(t))) \leq M_1$ , for all  $(t, \tilde{q}(t)) \in \mathcal{R}$ .
- (d) We take  $\varphi(t, z) = A_1 z$ , where  $A_1$  is a positive crisp constant. Then  $\varphi$  is non-decreasing in  $z$  and  $\varphi(t, 0) = 0$ .
- (e) If we consider  $q(t)$  is bounded for the whole-time cycle then,  $0 \leq \varphi(t, q(t)) = A_1 q(t) \leq M_2$  for all  $t \in [0, h]$ ,  $0 \leq q(t) \leq \varepsilon$  and  ${}^C D_t^\alpha q(t) = -A_1 q(t)$ ,  $q(0) = 0$  has the unique solution.
- (f)  $D_H\left(\frac{\psi(t, \tilde{q}(t))}{(t_1 - t)^{1-\alpha}}, \frac{\psi(t, \tilde{q}_1(t))}{(t_2 - t)^{1-\alpha}}\right) \leq \left|(t_1 - t)^{\alpha-1} - (t_2 - t)^{\alpha-1}\right| \varphi(t, D_H(\tilde{q}(t), \tilde{q}_1(t)))$ .

Then, the Theorem 5.1 of [54] provides the guarantee of the existence and uniqueness of the solution of the fuzzy fractional differential equation (4.5). Thus, the system represented by the equation (4.5) can produce a unique solution, provided the demand to have a bounded value.

### 4.3. Analytical solution

For the analytical solution of the equation (4.5), the parametric representation approach is utilized in this subsection. Suppose, the  $r$ -cuts of the fuzzy parameters and variables  $\tilde{q}(t)$ ,  $\tilde{D}$ ,  $\tilde{Q}$  and  $\tilde{p}$  are given as:

$$\begin{aligned} \tilde{q}(t, r) &= [q_L(t, r), q_R(t, r)]; \tilde{D}(r) = [D_L(r), D_R(r)]; \tilde{Q}(r) = [Q_L(r), Q_R(r)] \\ &= [q_L(0, r), q_R(0, r)] \quad \text{and} \quad \tilde{p}(r) = [p_L(r), p_R(r)]. \end{aligned}$$

Then, the components of the parametric representation of the demand rate  $\tilde{D}(r)$  are derived as:

$$\begin{cases} D_L(r) = a - b p_R(r) \\ D_R(r) = a - b p_L(r). \end{cases} \quad (4.6)$$

Taking fuzzy Laplace transformation [3, 56, 57] of the equation (4.5):

$$\mathcal{L}\{D_a^\alpha \tilde{q}(t); s\} = -\mathcal{L}\{\tilde{D}; s\}. \quad (4.7)$$

Here, two different cases are considered based on the fuzzy fractional Caputo differentiability [5] of the inventory level function  $\tilde{q}(t)$ .

**Case I.** When  $\tilde{q}(t)$  is  ${}^C[(1) - \alpha]$  differentiable

Then, from the equation (4.7), the following equation is derived:

$$s^\alpha \mathcal{L}\{\tilde{q}(t); s\} \ominus_{gH} s^{\alpha-1} \tilde{q}(0) = -\mathcal{L}\{\tilde{D}; s\}. \quad (4.8)$$

The parametric representation of the equation (4.8) can be written as:

$$\begin{aligned} &\begin{cases} s^\alpha \ell\{q_L(t, r); s\} - s^{\alpha-1} q_L(0, r) = -\ell\{D_R(r); s\} \\ s^\alpha \ell\{q_R(t, r); s\} - s^{\alpha-1} q_R(0, r) = -\ell\{D_L(r); s\} \end{cases} \\ i.e., &\begin{cases} s^\alpha \ell\{q_L(t, r); s\} - s^{\alpha-1} Q_L(r) = -\frac{\{a - b p_L(r)\}}{s} \\ s^\alpha \ell\{q_R(t, r); s\} - s^{\alpha-1} Q_R(r) = -\frac{\{a - b p_R(r)\}}{s} \end{cases} \\ i.e., &\begin{cases} \ell\{q_L(t, r); s\} = \frac{Q_L(r)}{s} - \frac{\{a - b p_L(r)\}}{s^{\alpha+1}} \\ \ell\{q_R(t, r); s\} = \frac{Q_R(r)}{s} - \frac{\{a - b p_R(r)\}}{s^{\alpha+1}} \end{cases}. \end{aligned} \quad (4.9)$$

Taking the inverse Laplace transformation of the system represented by the equation (4.9), the components of the parametric representation of the fuzzy inventory level are given by:

$$\begin{cases} q_L(t, r) = Q_L(r) - \frac{\{a-bp_L(r)\}t^\alpha}{\Gamma(\alpha+1)} \\ q_R(t, r) = Q_R(r) - \frac{\{a-bp_R(r)\}t^\alpha}{\Gamma(\alpha+1)}. \end{cases} \quad (4.10)$$

Using the initial conditions in the equation (4.10), the components of the parametric representation of the fuzzy lot size are given by:

$$\begin{cases} Q_L(r) = \frac{\{a-bp_L(r)\}T^\alpha}{\Gamma(\alpha+1)} \\ Q_R(r) = \frac{\{a-bp_R(r)\}T^\alpha}{\Gamma(\alpha+1)}. \end{cases} \quad (4.11)$$

Then, ultimately the equation (4.10) takes the form:

$$\begin{cases} q_L(t, r) = \frac{\{a-bp_L(r)\}(T^\alpha - t^\alpha)}{\Gamma(\alpha+1)} \\ q_R(t, r) = \frac{\{a-bp_R(r)\}(T^\alpha - t^\alpha)}{\Gamma(\alpha+1)}. \end{cases} \quad (4.12)$$

Now, the parametric values of different costs and revenue are counted as follow:  
Holding Cost,  $HC_{\alpha, \beta} = [HC_L(r), HC_R(r)]$ , where

$$\begin{aligned} HC_L(r) &= \frac{h}{\Gamma(\beta)} \int_0^T (T-t)^{\beta-1} q_L(t, r) dt \\ &= \frac{h\{a-bp_L(r)\}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ T^\alpha \int_0^T (T-t)^{\beta-1} dt - \int_0^T t^\alpha (T-t)^{\beta-1} dt \right] \\ &= \frac{h\{a-bp_L(r)\}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ \frac{T^{\alpha+\beta}}{\beta} - T^{\alpha+\beta} \int_0^1 t^\alpha (1-t)^{\beta-1} dt \right] \\ &= \frac{h\{a-bp_L(r)\}T^{\alpha+\beta}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ \frac{1}{\beta} - B(\alpha+1, \beta) \right] \\ &= h\{a-bp_L(r)\}T^{\alpha+\beta} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right] \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} HC_R(r) &= \frac{h}{\Gamma(\beta)} \int_0^T (T-t)^{\beta-1} q_R(t, r) dt \\ &= \frac{h\{a-bp_R(r)\}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ T^\alpha \int_0^T (T-t)^{\beta-1} dt - \int_0^T t^\alpha (T-t)^{\beta-1} dt \right] \\ &= \frac{h\{a-bp_R(r)\}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ \frac{T^{\alpha+\beta}}{\beta} - T^{\alpha+\beta} \int_0^1 t^\alpha (1-t)^{\beta-1} dt \right] \\ &= \frac{h\{a-bp_R(r)\}T^{\alpha+\beta}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ \frac{1}{\beta} - B(\alpha+1, \beta) \right] \\ &= h\{a-bp_R(r)\}T^{\alpha+\beta} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right]. \end{aligned} \quad (4.14)$$

Total earned revenue:

$$ER_{\alpha, \beta} = [ER_L(r), ER_R(r)], \text{ where}$$

$$\begin{aligned}
\text{ER}_L(r) &= \frac{p_L(r)}{\Gamma(\beta)} \int_0^T (T-t)^{\beta-1} D_L(r) dt \\
&= \frac{p_L(r)\{a-bp_R(r)\}}{\Gamma(\beta)} \int_0^T (T-t)^{\beta-1} dt \\
&= \frac{p_L(r)\{a-bp_R(r)\}T^\beta}{\Gamma(\beta+1)}
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
\text{ER}_R(r) &= \frac{p_R(r)}{\Gamma(\beta)} \int_0^T (T-t)^{\beta-1} D_R(r) dt \\
&= \frac{p_R(r)\{a-bp_L(r)\}}{\Gamma(\beta)} \int_0^T (T-t)^{\beta-1} dt \\
&= \frac{p_R(r)\{a-bp_L(r)\}T^\beta}{\Gamma(\beta+1)}.
\end{aligned} \tag{4.16}$$

So, total average profit is given by:

$$\text{TAP}_{\alpha,\beta} = [\text{TAP}_L, \text{TAP}_R] = \frac{1}{T} \{\text{ER}_{\alpha,\beta} - (C + \text{HC}_{\alpha,\beta})\}.$$

And therefore, the components of the parametric representation of  $\text{TAP}_{\alpha,\beta}$  are the follows:

$$\begin{aligned}
\text{TAP}_L &= \frac{\text{ER}_L(r) - C - \text{HC}_R(r)}{T} \\
&= \frac{p_L(r)\{a-bp_R(r)\}T^{\beta-1}}{\Gamma(\beta+1)} - \frac{C}{T} - h\{a-bp_L(r)\}T^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right]
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
\text{TAP}_R &= \frac{\text{ER}_R(r) - C - \text{HC}_L(r)}{T} \\
&= \frac{p_R(r)\{a-bp_L(r)\}T^{\beta-1}}{\Gamma(\beta+1)} - \frac{C}{T} - h\{a-bp_R(r)\}T^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right].
\end{aligned} \tag{4.18}$$

**Case II.** When  $\tilde{q}(t)$  is  $^C[(2) - \alpha]$  differentiable.

Then, proceeding in the similar path as of case I (details are given in Appendix A), the components of the parametric representation of the total average profit ( $\text{TAP}_{\alpha,\beta}$ ) and lot size

$$\begin{cases} \text{TAP}_L = \frac{p_L(r)\{a-bp_R(r)\}T^{\beta-1}}{\Gamma(\beta+1)} - \frac{C}{T} - h\{a-bp_R(r)\}T^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right] \\ \text{TAP}_R = \frac{p_R(r)\{a-bp_L(r)\}T^{\beta-1}}{\Gamma(\beta+1)} - \frac{C}{T} - h\{a-bp_L(r)\}T^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right] \end{cases} \tag{4.19}$$

$$\text{and, } \begin{cases} Q_L(r) = \frac{\{a-bp_L(r)\}T^\alpha}{\Gamma(\alpha+1)} \\ Q_R(r) = \frac{\{a-bp_R(r)\}T^\alpha}{\Gamma(\alpha+1)}. \end{cases} \tag{4.20}$$

#### 4.4. TDPLS decision making

In this subsection, the triangular dense fuzzy lock set approach is incorporated to make a proper decision in the uncertain environment using the learning experience and memory. Here, the unit selling price is assumed to

be triangular dense fuzzy lock number [15] with two keys and the other involved parameters are assumed to be constants.

Let, the selling price, a triangular lock fuzzy dense number is given by:

$\tilde{p} = \left\langle p \left\{ 1 - \rho \left( \frac{1}{K_1} - \frac{1}{n+1} \right) \right\}, p, p \left\{ 1 + \sigma \left( \frac{1}{K_2} - \frac{1}{n+1} \right) \right\} \right\rangle$  with its membership function

$$\mu_{\tilde{p}}(x) = \begin{cases} \frac{x - p \left\{ 1 - \rho \left( \frac{1}{K_1} - \frac{1}{n+1} \right) \right\}}{p \rho \left( \frac{1}{K_1} - \frac{1}{n+1} \right)}, & \text{when } p \left\{ 1 - \rho \left( \frac{1}{K_1} - \frac{1}{n+1} \right) \right\} \leq x \leq p \\ \frac{p \left\{ 1 + \sigma \left( \frac{1}{K_2} - \frac{1}{n+1} \right) \right\} - x}{p \sigma \left( \frac{1}{K_2} - \frac{1}{n+1} \right)}, & \text{when } p \leq x \leq p \left\{ 1 + \sigma \left( \frac{1}{K_2} - \frac{1}{n+1} \right) \right\} \\ 0, & \text{otherwise.} \end{cases} \quad (4.21)$$

Therefore, in the  $r$ -cut representation the unit selling price can be given by:

$$\begin{aligned} \tilde{p} &= [p_L(r), p_R(r)], \text{ where} \\ p_L(r) &= p - (1 - r)p\rho \left( \frac{1}{K_1} - \frac{1}{n+1} \right) \end{aligned} \quad (4.22)$$

and

$$p_R(r) = p + (1 - r)p\sigma \left( \frac{1}{K_2} - \frac{1}{n+1} \right). \quad (4.23)$$

For the case of  $C[(1) - \alpha]$  differentiability of  $\tilde{q}(t)$ , the defuzzified value of the total average profit is obtained using the  $\alpha$ -cut defuzzification approach as follows (details are given in Appendix B):

$$\begin{aligned} \text{TAP}_{\text{defuzz1}} &= \frac{\sum_{n=0}^N \int_0^1 \{\text{TAP}_L + \text{TAP}_R\} dr}{2N} \\ &= \frac{T^{\beta-1}}{\Gamma(\beta+1)} \frac{\sum_{n=0}^N \int_0^1 \{a(p_L(r) + p_R(r)) - 2bp_L(r)p_R(r)\} dr}{2N} - \frac{C}{T} \\ &\quad - hT^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right] \left\{ a - b \frac{\sum_{n=0}^N \int_0^1 (p_L(r) + p_R(r)) dr}{2N} \right\} \\ &= \frac{T^{\beta-1}}{\Gamma(\beta+1)} p \left[ (a - bp) + \left( \frac{a}{4} - \frac{bp}{2} \right) \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) + \frac{bp\sigma\rho}{3K_1K_2} \right. \\ &\quad \left. - \left\{ \frac{a(\sigma - \rho)}{4} - \frac{bp(\sigma - \rho)}{2} - \frac{bp\sigma\rho}{3} \left( \frac{1}{K_1} + \frac{1}{K_2} \right) \right\} \frac{1}{N} \sum_{n=0}^N \frac{1}{n+1} + \frac{bp\sigma\rho}{3} \frac{1}{N} \sum_{n=0}^N \left( \frac{1}{n+1} \right)^2 \right. \\ &\quad \left. - \frac{C}{T} - hT^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right] \right. \\ &\quad \left. \times \left[ (a - bp) - \frac{bp}{4} \left\{ \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] \right]. \end{aligned} \quad (4.24)$$

Also, the defuzzified value of the lot size in the case of  $C[(1) - \alpha]$  differentiability of  $\tilde{q}(t)$  will be:

$$\begin{aligned} Q_{\text{defuzz1}} &= \frac{\sum_{n=0}^N \int_0^1 \{Q_L(r) + Q_R(r)\} dr}{2N} \\ &= \frac{\left\{ a - b \frac{\sum_{n=0}^N \int_0^1 \{p_L(r) + p_R(r)\} dr}{2N} \right\} T^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

$$= \left[ (a - bp) - \frac{bp}{4} \left\{ \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] \frac{T^\alpha}{\Gamma(\alpha+1)}. \quad (4.25)$$

For the case of  $C[(2) - \alpha]$  differentiability of  $\tilde{q}(t)$ , the defuzzified value of the total average profit is obtained using the  $\alpha$ -cut defuzzification approach as follows (details are given in Appendix B):

$$\begin{aligned} \text{TAP}_{\text{defuzz2}} &= \frac{\sum_{n=0}^N \int_0^1 \{\text{TAP}_L + \text{TAP}_R\} dr}{2N} \\ &= \frac{T^{\beta-1}}{\Gamma(\beta+1)} \frac{\sum_{n=0}^N \int_0^1 \{a(p_L(r) + p_R(r)) - 2bp_L(r)p_R(r)\} dr}{2N} - \frac{C}{T} \\ &\quad - hT^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right] \left\{ a - b \frac{\sum_{n=0}^N \int_0^1 (p_L(r) + p_R(r)) dr}{2N} \right\} \\ &= \text{TAP}_{\text{defuzz1}}. \end{aligned}$$

Also,

$$Q_{\text{defuzz1}} = Q_{\text{defuzz2}}.$$

Interestingly, there is no distinction between two cases regarding fuzzy fractional differentiability of  $\tilde{q}(t)$  after the defuzzifications in the above mentioned technique. So, without the loss of generality,  $Q_{\text{defuzz1}}$  and  $\text{TAP}_{\text{defuzz1}}$  are replaced by the notations  $Q_{\text{defuzz}}$  and  $\text{TAP}_{\text{defuzz}}$  respectively.

So, the optimization problem will be

$$\begin{cases} \text{Max } \text{TAP}_{\text{defuzz}} \\ Q_{\text{defuzz}} = \left[ (a - bp) - \frac{bp}{4} \left\{ \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] \frac{T^\alpha}{\Gamma(\alpha+1)} \\ \text{TAP}_{\text{defuzz}} \text{ is given by (4.21)} \\ T > 0 \\ 0 < \alpha, \beta \leq 1. \end{cases} \quad (4.26)$$

This is a non-linear single objective multiple constrained optimization problem. Instead of going through the analytical optimization technique, the numerical optimization techniques are chosen in this paper. The numerical optimization of the system given by the equation (4.26) is done by using LINGO 18.0 software as the working tool.

## 5. SPECIAL CASES FOR OPTIMIZATION OF THE PROBLEMS

In this section, some popular models are seen as particular cases of the proposed inventory model under memory and learning based study with double keys in the hand of the decision maker. The approach of deduction may be specified through the values of the memory index or the values of the involved parameters to present the degree of fuzziness. The categorization of the proposed problem is based on the memory index which corresponds to the main problem itself and its integer order version (stands for memory free situation). For both the fractional and integer order model, more categorizations are carried out on the basis of the values of keys and the variance of the parameters of dense lock fuzzy system.

### 5.1. Fractional order sub problems

#### 5.1.1. When $K_1 = K_2 = K$

Then

$$\text{TAP}_{\text{defuzz}} = \frac{T^{\beta-1}}{\Gamma(\beta+1)} p \left[ (a - bp) + \left( \frac{a}{4} - \frac{bp}{2} \right) \left( \frac{\sigma - \rho}{K} \right) + \frac{bp\sigma\rho}{3K^2} - \left\{ \frac{a(\sigma - \rho)}{4} - \frac{bp(\sigma - \rho)}{2} - \frac{2bp\sigma\rho}{3K} \right\} \right]$$

$$\begin{aligned} & \times \left[ \frac{1}{N} \sum_{n=0}^N \frac{1}{n+1} + \frac{bp\sigma\rho}{3} \frac{1}{N} \sum_{n=0}^N \left( \frac{1}{n+1} \right)^2 \right] - \frac{C}{T} - hT^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right] \\ & \times \left[ (a - bp) - \frac{bp}{4} \left\{ \left( \frac{\sigma - \rho}{K} \right) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] \end{aligned}$$

and

$$Q_{\text{defuzz}} = \left[ (a - bp) - \frac{bp}{4} \left\{ \frac{\sigma - \rho}{K} - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] \frac{T^\alpha}{\Gamma(\alpha+1)}.$$

This is the case of fractional model in the lock fuzzy dense environment with single key.

#### 5.1.2. When $K_1 = K_2 = 1$

$$\begin{aligned} \text{TAP}_{\text{defuzz}} = & \frac{T^{\beta-1}}{\Gamma(\beta+1)} p \left[ (a - bp) + \left( \frac{a}{4} - \frac{bp}{2} \right) (\sigma - \rho) + \frac{bp\sigma\rho}{3} - \left\{ \frac{a(\sigma - \rho)}{4} - \frac{bp(\sigma - \rho)}{2} - \frac{2bp\sigma\rho}{3} \right\} \right. \\ & \times \left. \frac{1}{N} \sum_{n=0}^N \frac{1}{n+1} + \frac{bp\sigma\rho}{3} \frac{1}{N} \sum_{n=0}^N \left( \frac{1}{n+1} \right)^2 \right] - \frac{C}{T} - hT^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right] \\ & \times \left[ (a - bp) - \frac{bp}{4} \left\{ (\sigma - \rho) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] \end{aligned}$$

and

$$Q_{\text{defuzz}} = \left[ (a - bp) - \frac{bp}{4} \left\{ (\sigma - \rho) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] \frac{T^\alpha}{\Gamma(\alpha+1)}.$$

This is the case of fractional model in the fuzzy dense environment.

#### 5.1.3. When $K_1 = K_2 = 1$ and $N \rightarrow \infty$

$$\begin{aligned} \text{TAP}_{\text{defuzz}} = & \frac{T^{\beta-1}}{\Gamma(\beta+1)} p \left[ (a - bp) + \left( \frac{a}{4} - \frac{bp}{2} \right) (\sigma - \rho) + \frac{bp\sigma\rho}{3} \right] \\ & - \frac{C}{T} - hT^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right] \left[ (a - bp) - \frac{bp(\sigma - \rho)}{4} \right] \end{aligned}$$

and

$$Q_{\text{defuzz}} = \left[ (a - bp) - \frac{bp(\sigma - \rho)}{4} \right] \frac{T^\alpha}{\Gamma(\alpha+1)}.$$

This is the case of fractional in the general fuzzy environment.

#### 5.1.4. When $K_1 = K_2 = 1$ , $N \rightarrow \infty$ , $\sigma = \rho = 0$

$$\text{TAP}_{\text{defuzz}} = \frac{T^{\beta-1}}{\Gamma(\beta+1)} p(a - bp) - \frac{C}{T} - hT^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right] (a - bp)$$

and

$$Q_{\text{defuzz}} = \frac{(a - bp)T^\alpha}{\Gamma(\alpha+1)}.$$

This is the case of fractional in the crisp environment.

## 5.2. Integer order sub problems

When  $\alpha = \beta = 1$ , then

$$\begin{aligned} \text{TAP}_{\text{defuzz}} &= p \left[ (a - bp) + \left( \frac{a}{4} - \frac{bp}{2} \right) \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) + \frac{bp\sigma\rho}{3K_1K_2} - \left\{ \frac{a(\sigma - \rho)}{4} - \frac{bp(\sigma - \rho)}{2} - \frac{bp\sigma\rho}{3} \left( \frac{1}{K_1} + \frac{1}{K_2} \right) \right\} \right. \\ &\quad \times \left. \frac{1}{N} \sum_{n=0}^N \frac{1}{n+1} + \frac{bp\sigma\rho}{3} \frac{1}{N} \sum_{n=0}^N \left( \frac{1}{n+1} \right)^2 \right] \\ &\quad - \frac{C}{T} - \frac{hT}{2} \left[ (a - bp) - \frac{bp}{4} \left\{ \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] \end{aligned}$$

and

$$Q_{\text{defuzz}} = \left[ (a - bp) - \frac{bp}{4} \left\{ \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] T.$$

### 5.2.1. When $K_1 = K_2 = K$

Then

$$\begin{aligned} \text{TAP}_{\text{defuzz}} &= p \left[ (a - bp) + \left( \frac{a}{4} - \frac{bp}{2} \right) \left( \frac{\sigma - \rho}{K} \right) + \frac{bp\sigma\rho}{3K^2} - \left\{ \frac{a(\sigma - \rho)}{4} - \frac{bp(\sigma - \rho)}{2} - \frac{2bp\sigma\rho}{3K} \right\} \right. \\ &\quad \times \left. \frac{1}{N} \sum_{n=0}^N \frac{1}{n+1} + \frac{bp\sigma\rho}{3} \frac{1}{N} \sum_{n=0}^N \left( \frac{1}{n+1} \right)^2 \right] \\ &\quad - \frac{C}{T} - \frac{hT}{2} \left[ (a - bp) - \frac{bp}{4} \left\{ \left( \frac{\sigma - \rho}{K} \right) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] \end{aligned}$$

and

$$Q_{\text{defuzz}} = \left[ (a - bp) - \frac{bp}{4} \left\{ \frac{\sigma - \rho}{K} - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] T.$$

This is the case of integer order model in the lock fuzzy dense environment with single key.

### 5.2.2. When $K_1 = K_2 = 1$

$$\begin{aligned} \text{TAP}_{\text{defuzz}} &= p \left[ (a - bp) + \left( \frac{a}{4} - \frac{bp}{2} \right) (\sigma - \rho) + \frac{bp\sigma\rho}{3} - \left\{ \frac{a(\sigma - \rho)}{4} - \frac{bp(\sigma - \rho)}{2} - \frac{2bp\sigma\rho}{3} \right\} \right. \\ &\quad \times \left. \frac{1}{N} \sum_{n=0}^N \frac{1}{n+1} + \frac{bp\sigma\rho}{3} \frac{1}{N} \sum_{n=0}^N \left( \frac{1}{n+1} \right)^2 \right] \\ &\quad - \frac{C}{T} - \frac{hT}{2} \left[ (a - bp) - \frac{bp}{4} \left\{ (\sigma - \rho) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] \end{aligned}$$

and

$$Q_{\text{defuzz}} = \left[ (a - bp) - \frac{bp}{4} \left\{ (\sigma - \rho) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \right] T.$$

This is the case of integer order model in the fuzzy dense environment.

5.2.3. When  $K_1 = K_2 = 1$  and  $N \rightarrow \infty$

$$\text{TAP}_{\text{defuzz}} = p \left[ (a - bp) + \left( \frac{a}{4} - \frac{bp}{2} \right) (\sigma - \rho) + \frac{bp\sigma\rho}{3} \right] - \frac{C}{T} - \frac{hT}{2} \left[ (a - bp) - \frac{bp(\sigma - \rho)}{4} \right]$$

and

$$Q_{\text{defuzz}} = \left[ (a - bp) - \frac{bp(\sigma - \rho)}{4} \right] T.$$

This is the case of integer order model in the general fuzzy environment.

5.2.4. When  $K_1 = K_2 = 1$ ,  $N \rightarrow \infty$ ,  $\sigma = \rho = 0$

$$\text{TAP}_{\text{defuzz}} = p(a - bp) - \frac{C}{T} - \frac{hT}{2}(a - bp)$$

and

$$Q_{\text{defuzz}} = (a - bp)T.$$

This is the case of integer order model in the crisp environment.

### 5.3. Graphical depiction of the sense of generalization

The senses of generalization discussed in the Sections 5.1 and 5.2 are presented graphically by Figures 1 and 2. Figure 1 represents the intuitions for the generalization of a crisp model (may be integer or fractional order) to complicated model under dense fuzzy lock system. On the other hand, Figure 2 explores the fact that every integer order model is a particular case of the corresponding fractional order model (better to say the arbitrary order model).

## 6. NUMERICAL EXPLORATION

In this section, some numerical problems are analysed to study the effect of memory and learning experience-based decision making in an inventory control problem with the selling price dependent demand rate. The following algorithm developed in the first subsection of the present section is used to measure the impact of memory and learning experiences.

### 6.1. Solution algorithm

**Step 1.** Input the value of the parameters  $C, h, p$  and constant  $a, b$ .

**Step 2.** Set  $\beta = 1$ .

**Step 3.** For  $\alpha = 1$  to 0.1, calculate  $\text{TAP}_{\text{defuzz}}^*$ ,  $Q_{\text{defuzz}}^*$ ,  $T^*$ .

**Step 4.** Check feasible values of  $Q_{\text{defuzz}}^*$  and  $T^*$ .

**Step 5.** Select  $\text{TAP}_{\text{defuzz1}}^* = \max_{\alpha} \{\text{TAP}_{\text{defuzz}}^*(\alpha)\}$ . Go to step 13.

**Step 6.** Set  $\alpha = 1$ .

**Step 7.** For  $\beta = 1$  to 0.1, calculate  $\text{TAP}_{\text{defuzz}}^*$ ,  $Q_{\text{defuzz}}^*$ ,  $T^*$ .

**Step 8.** Check feasible values of  $Q_{\text{defuzz}}^*$  and  $T^*$ .

**Step 9.** Select  $\text{TAP}_{\text{defuzz2}}^* = \max_{\beta} \{\text{TAP}_{\text{defuzz}}^*(\beta)\}$ . Go to step 13.

**Step 10.** For  $\alpha, \beta = 1$  to 0.1, calculate  $\text{TAP}_{\text{defuzz}}^*$ ,  $Q_{\text{defuzz}}^*$ ,  $T^*$ .

**Step 11.** Check feasible values of  $Q_{\text{defuzz}}^*$  and  $T^*$ .

**Step 12.** Select  $\text{TAP}_{\text{defuzz3}}^* = \max_{\alpha, \beta} \{\text{TAP}_{\text{defuzz}}^*(\alpha, \beta)\}$ . Go to step 13.

**Step 13.** Do fuzzification of the selected crisp model. Input  $\sigma$  and  $\rho$ .

**Step 14.** Calculate  $\begin{cases} \text{TAP}_{\text{defuzz4}}^* = \text{TAP}_{\text{defuzz1}}^* & \text{in general fuzzy environment} \\ \text{TAP}_{\text{defuzz5}}^* = \text{TAP}_{\text{defuzz2}}^* & \text{in general fuzzy environment} \\ \text{TAP}_{\text{defuzz6}}^* = \text{TAP}_{\text{defuzz3}}^* & \text{in general fuzzy environment.} \end{cases}$

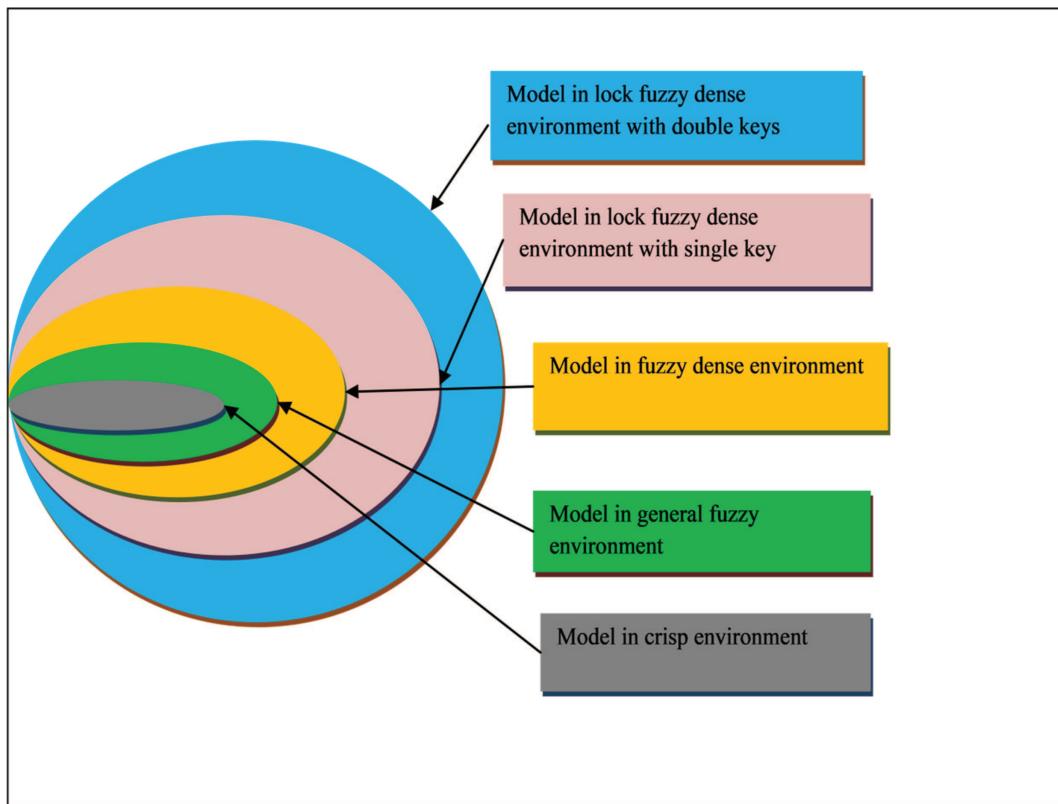


FIGURE 1. Sense of generalization through different aspects of fuzziness.

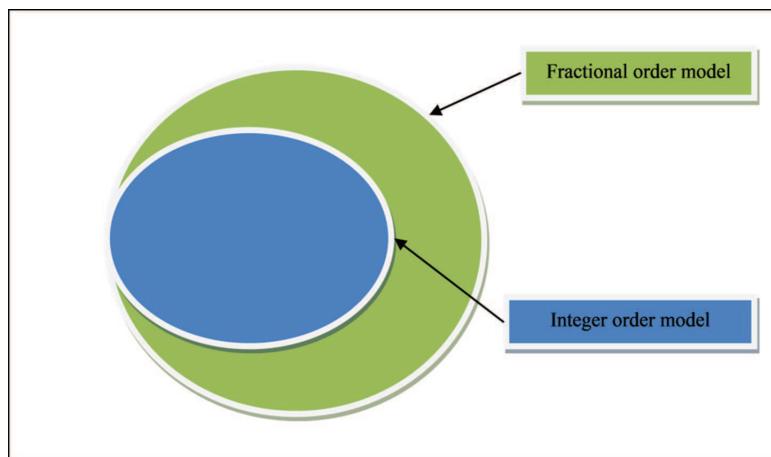


FIGURE 2. Sense of generalization through fractional calculus.

**Step 15.** Apply dense fuzzy environment on the selected crisp model. Input  $\sigma$  and  $\rho$ .

**Step 16.** For value  $N = 1$  to 4.

Calculate  $\left\{ \begin{array}{l} \text{TAP}_{\text{defuzz7}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy environment } (N = 1) \\ \text{TAP}_{\text{defuzz8}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy environment } (N = 2) \\ \text{TAP}_{\text{defuzz9}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy environment } (N = 3) \\ \text{TAP}_{\text{defuzz10}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy environment } (N = 4) \\ \text{TAP}_{\text{defuzz11}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy environment } (N = 1) \\ \text{TAP}_{\text{defuzz12}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy environment } (N = 2) \\ \text{TAP}_{\text{defuzz13}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy environment } (N = 3) \\ \text{TAP}_{\text{defuzz14}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy environment } (N = 4) \\ \text{TAP}_{\text{defuzz15}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy environment } (N = 1) \\ \text{TAP}_{\text{defuzz16}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy environment } (N = 2) \\ \text{TAP}_{\text{defuzz17}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy environment } (N = 3) \\ \text{TAP}_{\text{defuzz18}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy environment } (N = 4). \end{array} \right.$

**Step 17.** Apply dense fuzzy lock (single key) environment on the selected crisp model. Input  $\sigma$ ,  $\rho$  and  $K$ .

**Step 18.** For value  $N = 1$  to 4.

Calculate  $\left\{ \begin{array}{l} \text{TAP}_{\text{defuzz19}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy lock (single key) environment } (N = 1) \\ \text{TAP}_{\text{defuzz20}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy lock (single key) environment } (N = 2) \\ \text{TAP}_{\text{defuzz21}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy lock (single key) environment } (N = 3) \\ \text{TAP}_{\text{defuzz22}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy lock (single key) environment } (N = 4) \\ \text{TAP}_{\text{defuzz23}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy lock (single key) environment } (N = 1) \\ \text{TAP}_{\text{defuzz24}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy lock (single key) environment } (N = 2) \\ \text{TAP}_{\text{defuzz25}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy lock (single key) environment } (N = 3) \\ \text{TAP}_{\text{defuzz26}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy lock (single key) environment } (N = 4) \\ \text{TAP}_{\text{defuzz27}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy lock (single key) environment } (N = 1) \\ \text{TAP}_{\text{defuzz28}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy lock (single key) environment } (N = 2) \\ \text{TAP}_{\text{defuzz29}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy lock (single key) environment } (N = 3) \\ \text{TAP}_{\text{defuzz30}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy lock (single key) environment } (N = 4). \end{array} \right.$

**Step 19.** Apply dense fuzzy lock (double keys) environment on the selected crisp model. Input  $\sigma$ ,  $\rho$ ,  $K_1$  and  $K_2$ .

**Step 20.** For value  $N = 1$  to 4.

Calculate  $\left\{ \begin{array}{l} \text{TAP}_{\text{defuzz31}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy lock (double key) environment } (N = 1) \\ \text{TAP}_{\text{defuzz32}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy lock (double key) environment } (N = 2) \\ \text{TAP}_{\text{defuzz33}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy lock (double key) environment } (N = 3) \\ \text{TAP}_{\text{defuzz34}}^* = \text{TAP}_{\text{defuzz1}}^* \text{ in dense fuzzy lock (double key) environment } (N = 4) \\ \text{TAP}_{\text{defuzz35}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy lock (double key) environment } (N = 1) \\ \text{TAP}_{\text{defuzz36}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy lock (double key) environment } (N = 2) \\ \text{TAP}_{\text{defuzz37}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy lock (double key) environment } (N = 3) \\ \text{TAP}_{\text{defuzz38}}^* = \text{TAP}_{\text{defuzz2}}^* \text{ in dense fuzzy lock (double key) environment } (N = 4) \\ \text{TAP}_{\text{defuzz39}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy lock (double key) environment } (N = 1) \\ \text{TAP}_{\text{defuzz40}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy lock (double key) environment } (N = 2) \\ \text{TAP}_{\text{defuzz41}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy lock (double key) environment } (N = 3) \\ \text{TAP}_{\text{defuzz42}}^* = \text{TAP}_{\text{defuzz3}}^* \text{ in dense fuzzy lock (double key) environment } (N = 4). \end{array} \right.$

**Step 21.** Select  $\text{TAP}_{\text{defuzz}}^* = \max_{i=1 \text{ to } 42} \text{TAP}_{\text{defuzz}i}^*$ .

**Step 22.** End.

## 6.2. Memory sensitivity in crisp system

Let us consider the following values of the fundamental parameters as inputs:

$$a = 200; b = 0.5; p = 25; C = 600; h = 1.5.$$

TABLE 1. Sensitivity of optimal solution with respect to differential memory index.

Differential memory index ( $\alpha$ )	$T^*$	$Q_{\text{defuzz}}^*$	$\text{TAP}_{\text{defuzz}}^*$
1	2.066	387.298	4106.55
0.9	2.286	410.366	4133.51
0.8	2.601	432.511	4168.48
0.7	3.068	452.302	4212.58
0.6	3.802	467.619	4266.65
0.5	5.049	475.380	4330.97
0.4	7.427	471.268	4404.74
0.3	12.855	449.469	4485.24
0.2	29.803	402.652	4566.71
0.1	136.536	322.244	4639.16

TABLE 2. Sensitivity of optimal solution with respect to integral memory index.

Integral memory index ( $\beta$ )	$T^*$	$Q_{\text{defuzz}}^*$	$\text{TAP}_{\text{defuzz}}^*$
1	2.066	387.298	4106.55
0.9	0.982	184.148	4135.42
0.8	0.491	92.130	4504.31
0.7	0.253	47.439	5371.12
0.6	0.123	23.108	7218.78
0.5	0.051	9.631	11 632.79
0.4	0.016	2.921	25 644.59
0.3	0.002	0.454	106 255.90

Then, the optimum values of the objective function and the decision variables in the classical EOQ model are obtained as  $\text{TAP}_{\text{defuzz}} = 4106.55$ ;  $Q_{\text{defuzz}}^* = 387.298$  and  $T^* = 2.066$ .

Now, the impacts of memory on the classical EOQ model are examined letting the order of differentiation and integration to be constants. The Tables 1 and 2 represent the memory sensitivity individually with respect to the differential and integral memory index respectively. One memory index is fixed to the value 1 while measuring the impact of another memory index. The Table 3 shows the memory sensitivity of the result varying both memory indexes simultaneously.

The Figures 3, 4 and 5 present graphically the variance of the total average profit, lot size and total cycle time, respectively with respect to the memory indexes.

From the Figure 1, it is seen that as the memory index decreases, the graph of the optimal values of total average profit maintains a strictly increasing pattern. However, the influence of the integral memory index on the total average profit is far superior in comparison to the differential memory index. From the Figure 2, it is seen that as the differential memory index decreases, the graph of the optimal lot size increases initially and then ultimately decreases with a bell-shaped curve. However, for the case of integral memory index as well as the case of combined effects, the graphs maintain steady decreasing patterns. Also, in the Figure 3, it is noticed that the graph of the total cycle time follows an increasing pattern with respect to the decreasing trends of the differential memory index. Again, the scenarios for the case of the integral memory index and the combined case are totally reversed.

TABLE 3. Sensitivity of optimal solution with respect to both the memory indexes.

Differential memory index ( $\alpha$ )	Integral memory index ( $\beta$ )	$T^*$	$Q_{\text{defuzz}}^*$	$\text{TAP}_{\text{defuzz}}^*$
1	1	2.066	387.298	4106.55
0.9	0.9	1.006	195.942	4137.54
0.8	0.8	0.497	115.007	4497.14
0.7	0.7	0.254	79.105	5353.91
0.6	0.6	0.124	59.842	7188.58
0.5	0.5	0.051	48.000	11 579.93
0.4	0.4	0.016	40.026	25 534.66
0.3	0.3	0.002	34.307	105 884.9

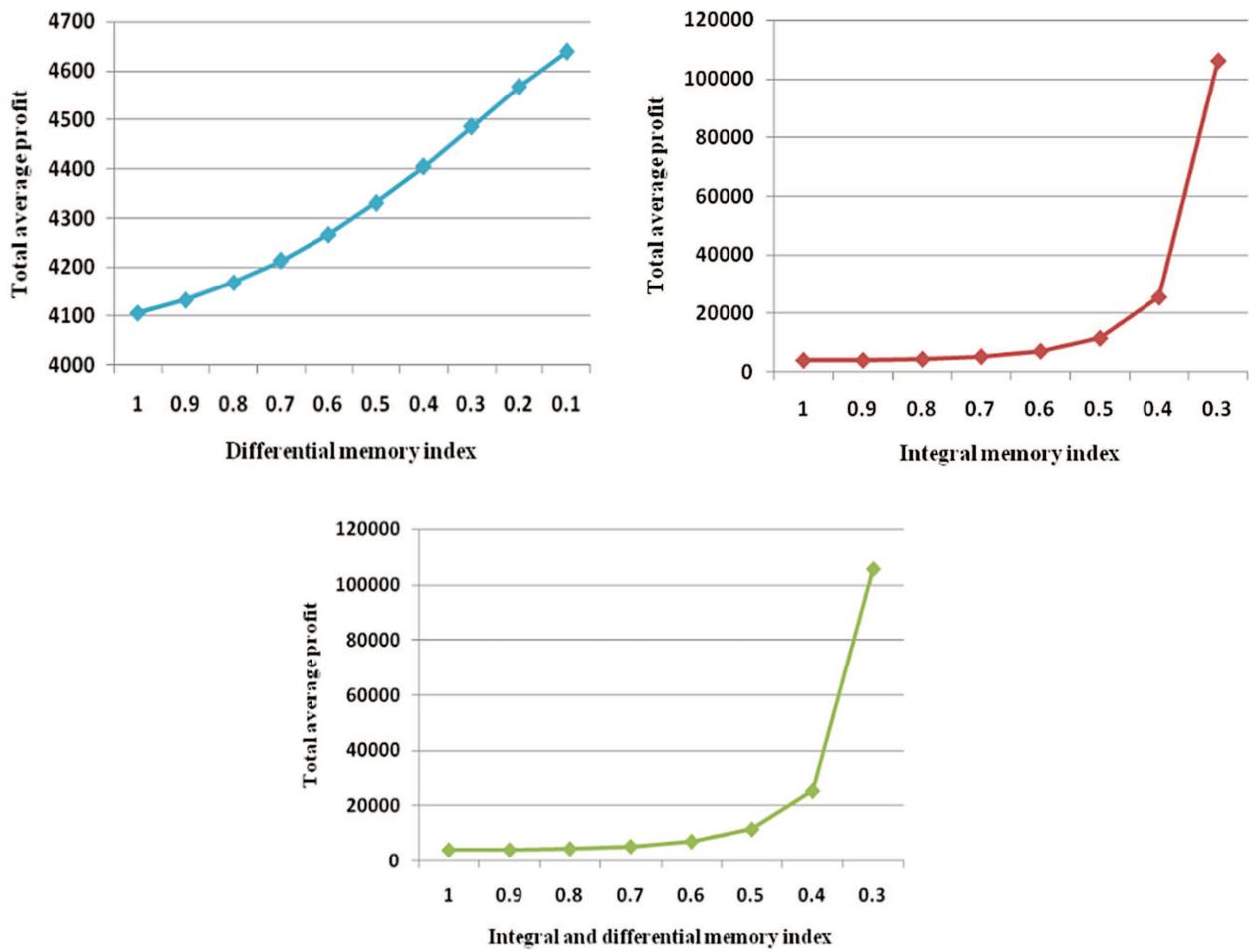


FIGURE 3. Total average profit for different memory index.

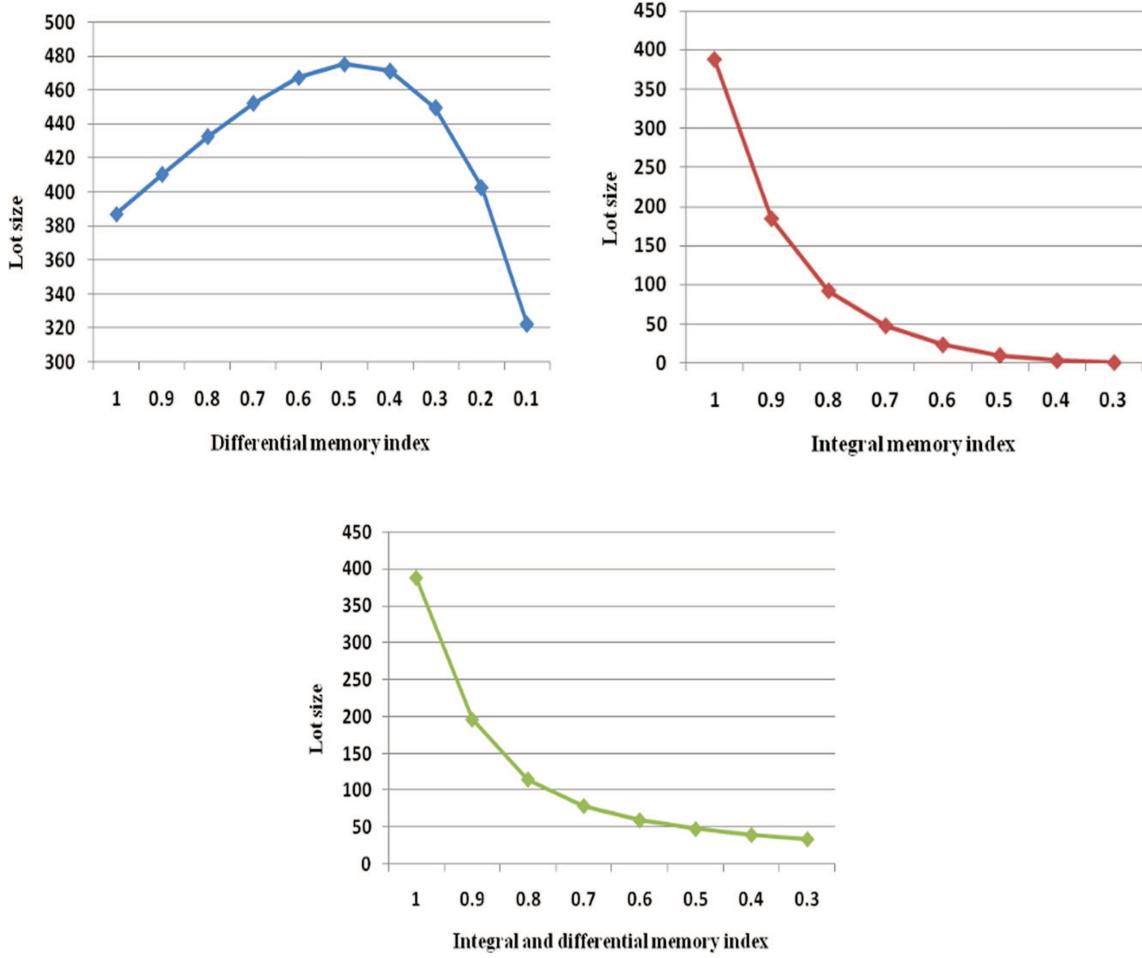


FIGURE 4. Lot size for different memory index.

### 6.3. Decision making under uncertainty

Apart from the memory less case (corresponding to  $\alpha = \beta = 1$ ), we select three different fractional cases on the basis of maximization objective of total average profit subject to the condition of feasible values of total time cycle and lot size from the analysis in the Section 6.2. The performance of the selected pairs of memory index in crisp and different fuzzy situations are displayed in the Table 4. Together with the non-fuzzy parameters, described in the Section 6.2, the below mentioned values of the fuzzy parameters are considered here:

- (i) For general fuzzy models we take  $\sigma = 0.35, \rho = 0.2$ .
- (ii) For dense fuzzy models we take  $\sigma = 0.35, \rho = 0.2$  and  $N = 1$  to 4.
- (iii) For dense fuzzy lock (single key) models we take  $\sigma = 0.35, \rho = 0.2, K = 0.5$  and  $N = 1$  to 4.
- (iv) For dense fuzzy lock (double key) models we take  $\sigma = 0.35, \rho = 0.2, K_1 = 0.5, K_2 = 0.45$  and  $N = 1$  to 4.

The bar diagram with fifty-six bars in the Figure 6 is regarded as a graphical correspondence of the results displayed by the Table 4.

The most preferable solution in the sense of profit maximization is therefore given by  $TAP_{\text{defuzz}}^* = 4836.45$ ,  $Q_{\text{defuzz}}^* = 469.628$  and  $T^* = 7.453$  corresponding to the inputs  $a = 200, b = 0.5, p = 25, C = 600, h = 1.5$ ,

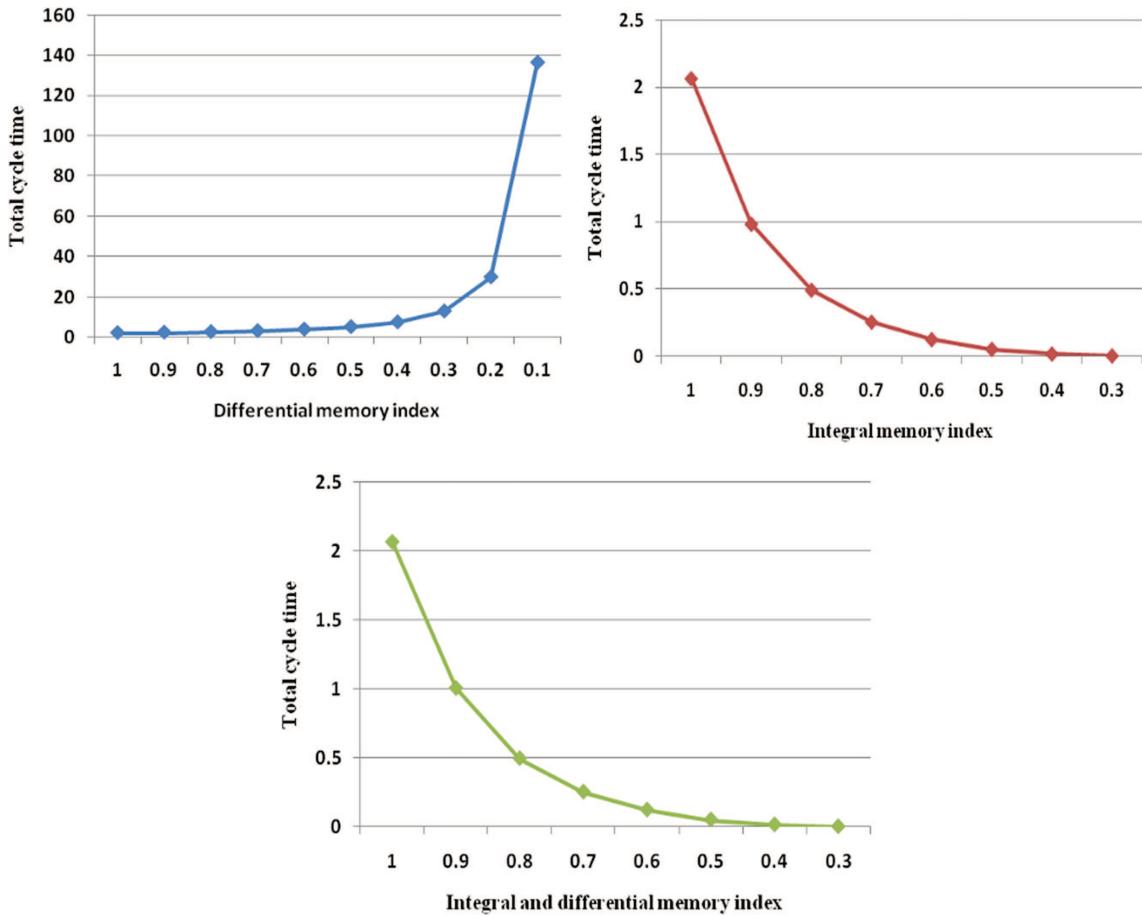


FIGURE 5. Total cycle time for different memory index.

$\sigma = 0.35$ ,  $\rho = 0.2$ ,  $K_1 = 0.5$ ,  $K_2 = 0.45$  and  $N = 4$ . The red coloured bar numbered at the serial 28 corresponds to the most preferable value of the total average profit in the Figure 6.

#### 6.4. Sensitivity analysis

The sensitivity of the most preferable solution with respect to the non-fuzzy parameters is discussed in this subsection. Changing the values of ordering cost ( $C$ ), the unit holding cost ( $h$ ), unit selling price ( $p$ ) from  $-50\%$  to  $+50\%$ , the sensitivity of the optimal solution is structured in the tabular and graphical forms by the Table 5 and the Figure 7 respectively.

The graph of total average profit preserves the expected facts that it is an increasing function with respect to  $p$  and is a decreasing function of  $C$  and  $h$ . Moreover, the total average profit is highly sensitive with respect to the parameter  $p$  compared to the parameters  $C$  and  $h$ . The sensitivity patterns are just reversed for the cases of the lot size and the total cycle time.

#### 6.5. Discussion and managerial insight

In this subsection, we have enlisted our observations that are obtained from the numerical analysis and enlighten the managerial strategies.

TABLE 4. Effects of memory and uncertainty on the EOQ models.

Memory index	Environment	Learning experience	$T^*$	$Q_{\text{defuzz}}^*$	$TAP_{\text{defuzz}}^*$
$\alpha = \beta = 1$	Crisp	—	2.066	387.298	4106.55
	General fuzzy ( $\sigma = 0.35, \rho = 0.2$ )	—	2.068	386.814	4278.63
	Dense fuzzy ( $\sigma = 0.35, \rho = 0.2$ )	1	2.064	387.540	4221.03
		2	2.066	387.258	4247.89
		3	2.066	387.150	4257.56
		4	2.067	387.090	4262.57
	Lock fuzzy dense (single key $K = 0.5, \sigma = 0.35, \rho = 0.2$ )	1	2.067	387.056	4414.99
		2	2.068	386.773	4436.78
		3	2.069	386.666	4445.35
		4	2.069	386.606	4449.93
$\alpha = 0.4, \beta = 1$	Lock fuzzy dense (double keys $K_1 = 0.5, K_2 = 0.45, \sigma = 0.35, \rho = 0.2$ )	1	2.068	386.805	4504.47
		2	2.070	386.522	4525.72
		3	2.070	386.414	4534.17
		4	2.071	386.354	4538.69
	Crisp	—	7.427	471.268	4404.74
	General fuzzy ( $\sigma = 0.35, \rho = 0.2$ )	—	2.863	543.287	4416.46
	Dense fuzzy ( $\sigma = 0.35, \rho = 0.2$ )	1	7.420	471.689	4519.33
		2	7.428	471.198	4546.05
		3	7.431	471.011	4555.68
		4	7.432	470.907	4560.66
$\alpha = 1, \beta = 0.9$	Lock fuzzy dense (single key $K = 0.5, \sigma = 0.35, \rho = 0.2$ )	1	7.433	470.847	4713.06
		2	7.441	470.356	4734.72
		3	7.444	470.169	4743.25
		4	7.446	470.065	4747.80
	Lock fuzzy dense (double keys $K_1 = 0.5, K_2 = 0.45, \sigma = 0.35, \rho = 0.2$ )	1	7.440	470.411	4802.44
		2	7.448	469.919	4823.54
		3	7.451	469.732	4831.94
		4	7.453	469.628	4836.45
	Crisp	—	0.982	184.148	4135.42
	General fuzzy ( $\sigma = 0.35, \rho = 0.2$ )	—	0.957	178.989	4314.47
$\alpha = 0.9, \beta = 0.9$	Dense fuzzy ( $\sigma = 0.35, \rho = 0.2$ )	1	0.965	181.103	4254.98
		2	0.961	180.155	4282.76
		3	0.960	179.811	4292.77
		4	0.959	179.630	4297.94
	Lock fuzzy dense (single key $K = 0.5, \sigma = 0.35, \rho = 0.2$ )	1	0.937	175.495	4457.24
		2	0.934	174.718	4479.79
		3	0.933	174.416	4488.67
		4	0.932	174.253	4493.41
	Lock fuzzy dense (double keys $K_1 = 0.5, K_2 = 0.45, \sigma = 0.35, \rho = 0.2$ )	1	0.925	172.973	4550.73
		2	0.922	172.225	4572.73
		3	0.921	171.931	4581.49
		4	0.921	171.772	4586.18

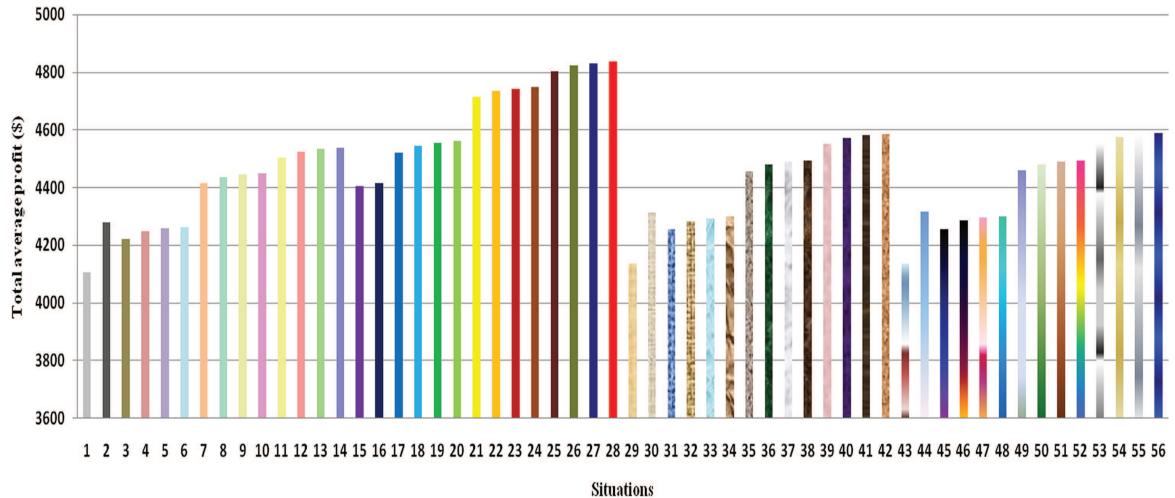


FIGURE 6. Total average profit in different decision-making situations.

#### 6.5.1. Major observations

Firstly, in the study of the fuzzy fractional differential equation under Caputo gH derivative, two kinds of equations appear corresponding to the  $C[(1) - \alpha]$  and  $C[(2) - \alpha]$  differentiability. Here, in both cases, the solutions converge into a single function. Secondly, it is to be noted that the lower values of  $\alpha$  and  $\beta$  (where  $\alpha$  and  $\beta$  indicate the memory indexes) represent the stronger senses of the system's memory. Thus, from the numerical values of Tables 1–3 and the Figure 3 of Section 6.2, we may conclude that stronger system's memory always favours towards the profit maximization goal. Thirdly, we get the following results on the optimality of the total average profit incorporating different kinds of uncertainties, learning experience and key facilities of dense fuzzy lock situations as described in Table 4:

- (i) Ordering with respect to memory index:  $(\alpha = \beta = 1) < (\alpha = 1, \beta = 0.9) < (\alpha = 0.9, \beta = 0.9) < (\alpha = 0.4, \beta = 1)$ .
- (ii) Ordering with respect to degree of uncertainty: Crisp < Dense fuzzy < General fuzzy < Lock fuzzy dense (single key) < Lock fuzzy dense (double keys).
- (iii) Ordering with respect to degree of experiences:

$$(N = 1) < (N = 2) < (N = 3) < (N = 4).$$

- (iv) Ordering with respect to number of keys:

$$\text{Single key} < \text{Double keys}.$$

From the above results, it appears that the stronger memory and TDPLS fuzzy set up with double keys facilities is the most desirable and generalized decision-making methodology among the different discussed phenomena.

#### 6.5.2. Managerial insights

It may be noted that in general, the demand rate of any product in the retailing process is not deterministic at all; rather lots of uncertainties are involved with it. Moreover, pricing is also a very sensitive decision to control demand pattern. In a developing country like India, common people pay concerns for the low-priced product even sacrificing the durability and quality of a product. In these circumstances, the demand can be viewed as a linear function of selling price. Now, to trace the demand pattern which is not precise to the

TABLE 5. Sensitivity of the optimal solution with respect to the non-fuzzy parameters.

Parameters	Changes (%)	$T^{*,\text{new}}$	$Q_{\text{defuzz}}^{*,\text{new}}$	$\text{TAP}_{\text{defuzz}}^{*,\text{new}}$	$\frac{\text{TAP}_{\text{defuzz}}^{*,\text{new}} - \text{TAP}_{\text{defuzz}}^{*}}{\text{TAP}_{\text{defuzz}}^{*}} \times 100\%$
$C$	+50	9.956	527.310	4801.84	-0.71561
	+40	9.477	517.017	4808.01	-0.58803
	+30	8.989	506.185	4814.51	-0.45364
	+20	8.489	494.740	4821.38	-0.31159
	+10	7.978	482.592	4828.67	-0.16086
	-10	6.912	455.701	4844.80	0.172647
	-20	6.355	440.621	4853.85	0.359768
	-30	5.776	424.127	4863.75	0.564464
	-40	5.174	405.853	4874.71	0.791076
	-50	4.542	385.252	4887.07	1.046635
$h$	+50	5.579	418.256	4741.79	-1.95722
	+40	5.860	426.583	4759.89	-1.58298
	+30	6.179	435.711	4778.37	-1.20088
	+20	6.543	445.790	4797.25	-0.81051
	+10	6.962	457.012	4816.59	-0.41063
	-10	8.035	483.980	4856.87	0.422211
	-20	8.740	500.544	4877.96	0.858274
	-30	9.615	520.010	4899.82	1.310259
	-40	10.734	543.425	4922.59	1.781058
	-50	12.227	572.483	4946.48	2.275016
$P$	+50	7.650	457.508	7149.05	47.81606
	+40	7.610	459.942	6700.06	38.5326
	+30	7.570	462.371	6244.30	29.10916
	+20	7.530	464.795	5781.78	19.54595
	+10	7.491	467.214	5312.50	9.842963
	-10	7.415	472.037	4353.63	-9.98294
	-20	7.377	474.441	3864.05	-20.1057
	-30	7.340	476.840	3367.71	-30.3681
	-40	7.303	479.234	2864.60	-40.7706
	-50	7.267	481.624	2354.73	-51.3128

retailer, the adjustment of the pricing will be a vague phenomenon. So, fuzzy decision-making technique is more desirable in this context. From the numerical simulation, it is perceived that the fuzzy decision making can favour the goal of the retailer. However, the retailer may have memory from the past dealing procedures. If there is any memory experience of the retailer, it can help to make a more accurate dealing policy to gain customer's faith which leads towards the ultimate gain of the organization. It is one of the major outcomes from the discussion of this article that stronger memory always helps to attain the highest gain of the retailer. However, the stronger memory sometimes may boost up the profit through lowering the values of the lot size and total cycle time in a very alarming measure which may not be cases of feasible phenomena in the practical sense. A smart decision maker must consider the system with moderate sense of memory for the feasibility of his/her decision in the real context. Another way to go for the accuracy in the optimal retailing policy is that the decision makers have to go through the self-learning procedure doing repeated deals in a particular retailing cycle. The demand pattern can be more precise to the decision-maker due to regular repetitions of the tasks. The numerical simulation in this study do explore the fact that the dense fuzzy frame carrying the sense of learning can give better result compared to the crisp and general fuzzy sense. The decision-making methodology can be further improved incorporating the sense of key and lock in the dense fuzzy decision-making process. The decision makers have better control on the system in terms of execution of desirable conditions when different

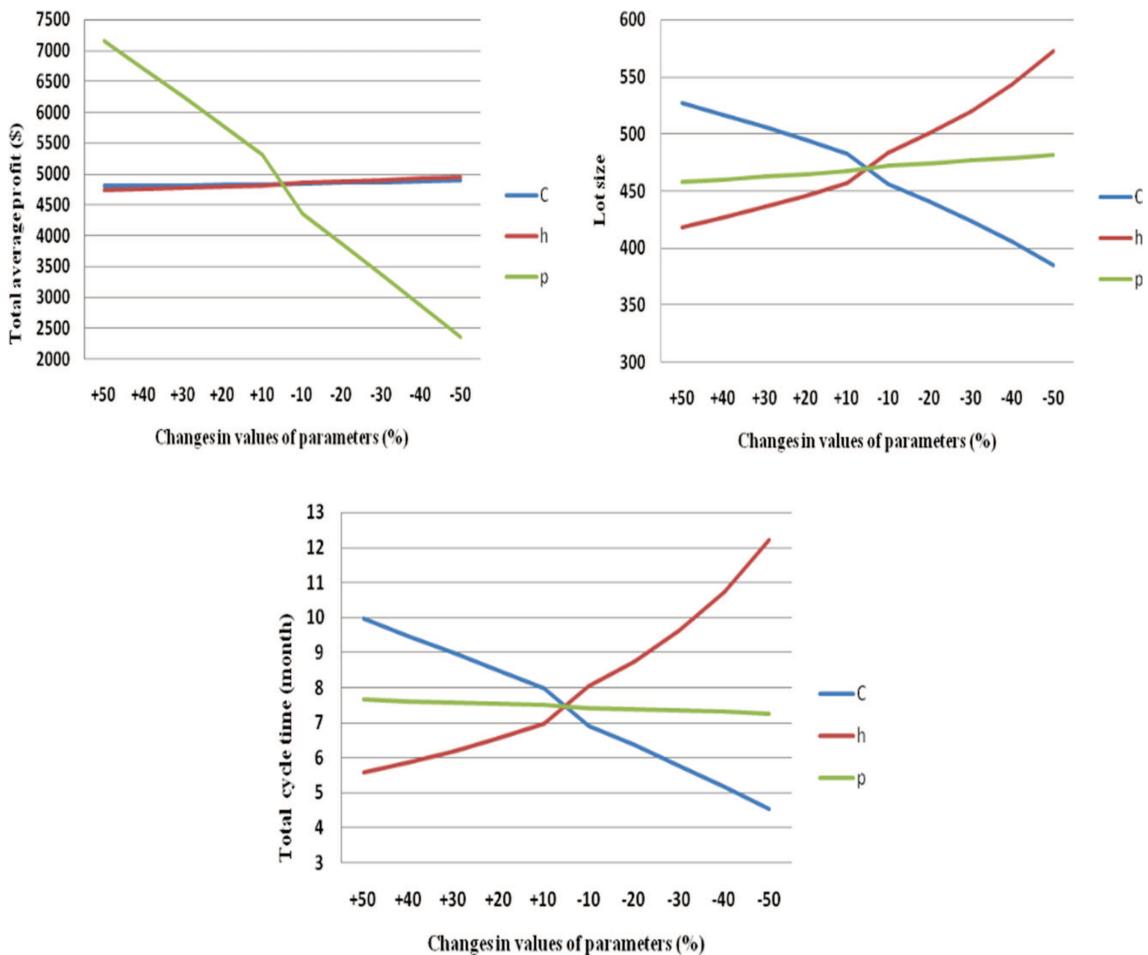


FIGURE 7. Sensitivity with respect to non-fuzzy parameters.

key features are available in the system. In this study, we have established that smart decisions (through the selection of suitable keys) of the experts can increase the profitability of the organization and several key options may contribute for the more accuracy on the decisions which also helps to the profit maximization. Actually, the mathematical concept of double keys represents two available alternatives of business strategies to be executed. Suppose, there are two experts engaged to make the decisions of their own. That is, two keys are available in the hand of the retailing organization to solve the puzzle due to impreciseness about the decision making for the optimal pricing to reach the maximum profit. Thus, optimal choice of the pricing through memory, learning experiences and key-lock technique can fulfil the retailer's objective of the profit maximization.

## 7. CONCLUSION

In this article, an EOQ model with price dependent demand is developed in a memory and learning motivated uncertain situation. Here, the fuzzy fractional differential equations of Caputo types are used to describe the proposed physical scenario. Also, the TDFLS decision making setup is utilized here to incorporate the experiences-based learning concept. The TDFLS decision making mechanism is designed for the optimal pricing which favours for increasing the total profit crystallizing the demand pattern. From the very discussion, it is

perceived that memory has a very positive impact on the profit maximization goal of the retailer. The second observation is that through more learning experience, the decision will be matured and thus it will lead the system towards more profits. The keys facility in the TDPLS decision making mechanism also boosts up the total profit regulating the uncertain parameters through smart decisions. Besides the mentioned intuitions and managerial insights, this present article can be viewed as an initiative for finding the applications of fuzzy fractional calculus in the domain of the operation research. Though the present article is engaging in the exploration of a very interesting and effective managerial insight, we also acknowledge the limitation of the present study. The modelling and the decision-making components in this article are very well connected with the real business scenarios. However, the numerical simulation is carried out on the hypothetical data. In future, this drawback can be overcome by collecting raw data from real world business sector to validate the proposed methodology of this present article. Searching for the analytical solution approach replacing the numerical simulation will be future challenges in this research direction.

## APPENDIX A.

Case II: when  $\tilde{q}(t)$  is  $C^[(2) - \alpha]$  differentiable.

Then from the equation (4.7),

$$\{-s^{\alpha-1}\tilde{q}(0)\} \ominus_{gH} \{-s^\alpha \mathcal{L}\{\tilde{q}(t); s\}\} = -\mathcal{L}\{\tilde{D}; s\}. \quad (\text{A.1})$$

In the parametric representation the equation (A.1) can be written as:

$$\begin{aligned} & \begin{cases} s^\alpha \ell\{q_L(t, r); s\} - s^{\alpha-1}q_L(0, r) = -\ell\{D_L(r); s\} \\ s^\alpha \ell\{q_R(t, r); s\} - s^{\alpha-1}q_R(0, r) = -\ell\{D_R(r); s\} \end{cases} \\ & \text{i.e., } \begin{cases} s^\alpha \ell\{q_L(t, r); s\} - s^{\alpha-1}Q_L(r) = -\frac{\{a-bp_R(r)\}}{s} \\ s^\alpha \ell\{q_R(t, r); s\} - s^{\alpha-1}Q_R(r) = -\frac{\{a-bp_L(r)\}}{s} \end{cases} \\ & \text{i.e., } \begin{cases} \ell\{q_L(t, r); s\} = \frac{Q_L(r)}{s} - \frac{\{a-bp_R(r)\}}{s^{\alpha+1}} \\ \ell\{q_R(t, r); s\} = \frac{Q_R(r)}{s} - \frac{\{a-bp_L(r)\}}{s^{\alpha+1}}. \end{cases} \end{aligned} \quad (\text{A.2})$$

Taking the inverse Laplace transformation of the system given by the equation (A.2), the following deduction is obtained:

$$\begin{cases} q_L(t, r) = Q_L(r) - \frac{\{a-bp_R(r)\}t^\alpha}{\Gamma(\alpha+1)} \\ q_R(t, r) = Q_R(r) - \frac{\{a-bp_L(r)\}t^\alpha}{\Gamma(\alpha+1)}. \end{cases} \quad (\text{A.3})$$

Using the initial conditions on the system given by the equation (A.3), the components of the parametric representation of the lot size are given as:

$$\begin{cases} Q_L(r) = \frac{\{a-bp_R(r)\}T^\alpha}{\Gamma(\alpha+1)} \\ Q_R(r) = \frac{\{a-bp_L(r)\}T^\alpha}{\Gamma(\alpha+1)}. \end{cases} \quad (\text{A.4})$$

Then, the equation (A.4) takes the form:

$$\begin{cases} q_L(t, r) = \frac{\{a-bp_R(r)\}(T^\alpha - t^\alpha)}{\Gamma(\alpha+1)} \\ q_R(t, r) = \frac{\{a-bp_L(r)\}(T^\alpha - t^\alpha)}{\Gamma(\alpha+1)}. \end{cases} \quad (\text{A.5})$$

Now, the parametric values of different costs and revenue are counted as follow:

Holding Cost,  $\text{HC}_{\alpha,\beta} = [\text{HC}_L(r), \text{HC}_R(r)]$ , where

$$\begin{aligned}
 \text{HC}_L(r) &= \frac{h}{\Gamma(\beta)} \int_0^T (T-t)^{\beta-1} q_L(t, r) dt \\
 &= \frac{h\{a - bp_R(r)\}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ T^\alpha \int_0^T (T-t)^{\beta-1} dt - \int_0^T t^\alpha (T-t)^{\beta-1} dt \right] \\
 &= \frac{h\{a - bp_R(r)\}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ \frac{T^{\alpha+\beta}}{\beta} - T^{\alpha+\beta} \int_0^1 t^\alpha (1-t)^{\beta-1} dt \right] \\
 &= \frac{h\{a - bp_R(r)\} T^{\alpha+\beta}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ \frac{1}{\beta} - B(\alpha+1, \beta) \right] \\
 &= h\{a - bp_R(r)\} T^{\alpha+\beta} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right]
 \end{aligned} \tag{A.6}$$

and

$$\begin{aligned}
 \text{HC}_R(r) &= \frac{h}{\Gamma(\beta)} \int_0^T (T-t)^{\beta-1} q_R(t, r) dt \\
 &= \frac{h\{a - bp_L(r)\}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ T^\alpha \int_0^T (T-t)^{\beta-1} dt - \int_0^T t^\alpha (T-t)^{\beta-1} dt \right] \\
 &= \frac{h\{a - bp_L(r)\}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ \frac{T^{\alpha+\beta}}{\beta} - T^{\alpha+\beta} \int_0^1 t^\alpha (1-t)^{\beta-1} dt \right] \\
 &= \frac{h\{a - bp_L(r)\} T^{\alpha+\beta}}{\Gamma(\beta)\Gamma(\alpha+1)} \left[ \frac{1}{\beta} - B(\alpha+1, \beta) \right] \\
 &= h\{a - bp_L(r)\} T^{\alpha+\beta} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right].
 \end{aligned} \tag{A.7}$$

So, total average profit is obtained as:

$$\text{TAP}_{\alpha,\beta} = [\text{TAP}_L, \text{TAP}_R] = \frac{1}{T} \{ \text{ER}_{\alpha,\beta} - (C + \text{HC}_{\alpha,\beta}) \}$$

This gives

$$\begin{aligned}
 \text{TAP}_L &= \frac{\text{ER}_L(r) - C - \text{HC}_R(r)}{T} \\
 &= \frac{p_L(r)\{a - bp_R(r)\} T^{\beta-1}}{\Gamma(\beta+1)} - \frac{C}{T} - h\{a - bp_R(r)\} T^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right]
 \end{aligned} \tag{A.8}$$

and

$$\begin{aligned}
 \text{TAP}_R &= \frac{\text{ER}_R(r) - C - \text{HC}_L(r)}{T} \\
 &= \frac{p_R(r)\{a - bp_L(r)\} T^{\beta-1}}{\Gamma(\beta+1)} - \frac{C}{T} - h\{a - bp_L(r)\} T^{\alpha+\beta-1} \left[ \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)} - \frac{1}{\Gamma(\alpha+\beta+1)} \right].
 \end{aligned} \tag{A.9}$$

## APPENDIX B.

From the equations (4.22) and (4.23), it is obtained that:

$$p_L(r) + p_R(r) = 2p + (1-r)p \left\{ \sigma \left( \frac{1}{K_2} - \frac{1}{n+1} \right) - \rho \left( \frac{1}{K_1} - \frac{1}{n+1} \right) \right\} \tag{B.1}$$

and

$$\begin{aligned} p_L(r)p_R(r) &= p^2 - (1-r)p^2\rho\left(\frac{1}{K_1} - \frac{1}{n+1}\right) + (1-r)p^2\sigma\left(\frac{1}{K_2} - \frac{1}{n+1}\right) \\ &\quad - (1-r)^2p^2\sigma\rho\left(\frac{1}{K_1} - \frac{1}{n+1}\right)\left(\frac{1}{K_2} - \frac{1}{n+1}\right). \end{aligned} \quad (\text{B.2})$$

Therefore,

$$\begin{aligned} \frac{\sum_{n=0}^N \int_0^1 \{p_L(r) + p_R(r)\} dr}{2N} &= \frac{\sum_{n=0}^N \left[ 2p + \frac{p}{2} \left\{ \sigma \left( \frac{1}{K_2} - \frac{1}{n+1} \right) - \rho \left( \frac{1}{K_1} - \frac{1}{n+1} \right) \right\} \right]}{2N} \\ &= p + \frac{p}{4} \left\{ \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \\ &= p \left\{ 1 + \frac{1}{4} \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma - \rho}{4N} \sum_{n=0}^N \frac{1}{n+1} \right\} \end{aligned} \quad (\text{B.3})$$

and

$$\begin{aligned} \frac{\sum_{n=0}^N \int_0^1 \{p_L(r)p_R(r)\} dr}{2N} &= \frac{1}{2N} \sum_{n=0}^N \left[ p^2 + \frac{p^2}{2} \left\{ \sigma \left( \frac{1}{K_2} - \frac{1}{n+1} \right) - \rho \left( \frac{1}{K_1} - \frac{1}{n+1} \right) \right\} - \frac{p^2\sigma\rho}{3} \left( \frac{1}{K_1} - \frac{1}{n+1} \right) \left( \frac{1}{K_2} - \frac{1}{n+1} \right) \right] \\ &= \frac{p^2}{2} + \frac{p^2}{4} \left\{ \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\} \\ &\quad - \frac{p^2\sigma\rho}{6} \left\{ \frac{1}{K_1K_2} - \left( \frac{1}{K_1} + \frac{1}{K_2} \right) \frac{1}{N} \sum_{n=0}^N \frac{1}{n+1} + \frac{1}{N} \sum_{n=0}^N \left( \frac{1}{n+1} \right)^2 \right\} \\ &= p^2 \left[ \frac{1}{2} + \frac{1}{4} \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma\rho}{6K_1K_2} - \left\{ \frac{\sigma - \rho}{4} + \frac{\sigma\rho}{6} \left( \frac{1}{K_1} + \frac{1}{K_2} \right) \right\} \frac{1}{N} \sum_{n=0}^N \frac{1}{n+1} - \frac{\sigma\rho}{6} \frac{1}{N} \sum_{n=0}^N \left( \frac{1}{n+1} \right)^2 \right]. \end{aligned} \quad (\text{B.4})$$

Using the equations (B.3) and (B.4) the following deduction can be done:

$$\begin{aligned} \frac{\sum_{n=0}^N \int_0^1 \{a(p_L(r) + p_R(r)) - 2bp_L(r)p_R(r)\} dr}{2N} &= a \frac{\sum_{n=0}^N \int_0^1 \{p_L(r) + p_R(r)\} dr}{2N} - 2b \frac{\sum_{n=0}^N \int_0^1 p_L(r)p_R(r) dr}{2N} \\ &= ap \left\{ 1 + \frac{1}{4} \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma - \rho}{4N} \sum_{n=0}^N \frac{1}{n+1} \right\} - 2bp^2 \left[ \frac{1}{2} + \frac{1}{4} \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma\rho}{6K_1K_2} \right. \\ &\quad \left. - \left\{ \frac{\sigma - \rho}{4} + \frac{\sigma\rho}{6} \left( \frac{1}{K_1} + \frac{1}{K_2} \right) \right\} \frac{1}{N} \sum_{n=0}^N \frac{1}{n+1} - \frac{\sigma\rho}{6} \frac{1}{N} \sum_{n=0}^N \left( \frac{1}{n+1} \right)^2 \right] \\ &= p \left[ (a - bp) + \left( \frac{a}{4} - \frac{bp}{2} \right) \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) + \frac{bp\sigma\rho}{3K_1K_2} - \left\{ \frac{a(\sigma - \rho)}{4} - \frac{bp(\sigma - \rho)}{2} - \frac{bp\sigma\rho}{3} \left( \frac{1}{K_1} + \frac{1}{K_2} \right) \right\} \right. \\ &\quad \left. \times \frac{1}{N} \sum_{n=0}^N \frac{1}{n+1} + \frac{bp\sigma\rho}{3} \frac{1}{N} \sum_{n=0}^N \left( \frac{1}{n+1} \right)^2 \right]. \end{aligned} \quad (\text{B.5})$$

Again

$$\begin{aligned}
 a - b \frac{\sum_{n=0}^N \int_0^1 (p_L(r) + p_R(r)) dr}{2N} &= a - bp \left\{ 1 + \frac{1}{4} \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma - \rho}{4N} \sum_{n=0}^N \frac{1}{n+1} \right\} \\
 &= (a - bp) - \frac{bp}{4} \left\{ \left( \frac{\sigma}{K_2} - \frac{\rho}{K_1} \right) - \frac{\sigma - \rho}{N} \sum_{n=0}^N \frac{1}{n+1} \right\}. \tag{B.6}
 \end{aligned}$$

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