

## A CONJECTURE ON A CONTINUOUS OPTIMIZATION MODEL FOR THE GOLOMB RULER PROBLEM

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**Abstract.** A Golomb Ruler (GR) is a set of integer marks along an imaginary ruler such that all the distances of the marks are different. Computing a GR of minimum length is associated to many applications (from astronomy to information theory). Although not yet demonstrated to be NP-hard, the problem is computationally very challenging. This brief note proposes a new continuous optimization model for the problem and, based on a given theoretical result and some computational experiments, we conjecture that an optimal solution of this model is also a solution to an associated GR of minimum length.

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### 1. INTRODUCTION

Given a positive integer  $n$ , the *Golomb Ruler Problem* (GRP) asks for locating  $n$  integer marks along an imaginary ruler such that all distances between pairs of marks are distinct and the length of the ruler is minimum [3,6]. The GRP has many applications, including astronomy [4], communications [1], and information theory [18].

Historically, the GRP first appeared related to Sidon sets [21], *i.e.* sets of integers so that all pairwise sums of elements of the set are distinct. Lower bounds for the GRP and connections to number theory are discussed, respectively, in [7, 14, 22, 23].

We say that we have a *Golomb Ruler* (GR), given  $n$  marks, when the associated distances are all different, and an *Optimal Golomb Ruler* (OGR), when a GR has the smallest length.

For example, the OGR for  $n = 4$  is  $\{0, 1, 4, 6\}$ , and for  $n = 5$ , there are two solutions:  $\{0, 1, 4, 9, 11\}$  and  $\{0, 2, 7, 8, 11\}$ .

Although not yet demonstrated to be NP-hard, solving the GRP exactly proved to be very difficult [5, 19]. Solutions for all  $n \leq 27$  are available on Wikipedia (“Golomb Ruler”, accessed on February 15th, 2021).

In [13], a linear programming model is proposed for the GRP and, more recently, authors in [8] compare different optimization approaches to certify optimality to the GRP.

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*Keywords.* Nonlinear programming, Golomb Ruler Problem, continuous models.

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We present a new model to the GRP based on a continuous optimization formulation (Sect. 2.1). In Section 2.2, we give a theoretical result and conjecture that a solution of this model is also a solution to the associated GRP. Computational experiments are given in Section 3. Section 4 concludes the paper with new research directions.

## 2. MODELS, A THEORETICAL RESULT AND A CONJECTURE FOR THE GRP

Formally, a GR consists of a set of integers  $R = \{r_1, \dots, r_n\}$ , with  $r_1 < r_2 < \dots < r_n$ , such that for each positive integer  $d$ , there is at most one pair  $\{r_i, r_j\} \subset R$ ,  $i < j$ , that satisfies

$$d = r_j - r_i.$$

The set  $R$  represents positions of  $n$  marks on the ruler with length  $L = r_n - r_1$ . Without loss of generality, we fix  $r_1 = 0$ .

### 2.1. A new model for the GRP

In order to present the first model to the GRP (first described in [8]), we need an upper bound  $L_u$  for the length of the ruler, given also as input of the problem, in addition to  $n$  marks on the ruler:

$$\begin{aligned} \min_{t \geq 0, x_i \in \{0,1\}} \quad & t \\ \text{s.t.} \quad & \\ & ix_i \leq t, \quad i = 1, \dots, L_u, \\ & \sum_{i=1}^{L_u} x_i = n - 1, \\ & x_j + \sum_{i=1}^{L_u-j} (x_i x_{i+j}) \leq 1, \quad j = 1, \dots, L_u - 1, \\ & x_i \in \{0, 1\}, \quad i = 1, \dots, L_u. \end{aligned} \tag{2.1}$$

Since we are minimizing  $t$ , inequalities

$$ix_i \leq t,$$

for  $i = 1, \dots, L_u$ , “force” the length of the ruler to be minimum. At the same time, equation

$$\sum_{i=1}^{L_u} x_i = n - 1$$

requires that  $n - 1$  of the variables  $x_i$  are equal to 1, and inequalities

$$x_j + \sum_{i=1}^{L_u-j} (x_i x_{i+j}) \leq 1,$$

for  $j = 1, \dots, L_u - 1$ , guarantee that all distances between pairs of marks are distinct.

A solution  $(x_1, \dots, x_{L_u})$  of model (2.1) provides a GRP solution given by  $(ix_i)$ , for  $i = 1, \dots, L_u$ , whose minimum length of the associated ruler is the optimal value found for  $t$ .

In the literature, there are many heuristics for calculating upper bounds for the GRP (e.g. see [8]). We propose a simple greedy heuristic whose pseudocode is given below<sup>1</sup>:

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**Algorithm 1:** Greedy Heuristic.

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**Input:** number of marks  $n$

Set  $d = [0, 0, \dots]$ ,  $mark = [0, 0, \dots]$ ,  $I = \{0, 1, 2, 3, \dots\}$ ,  $J = \emptyset$ ,  $L = 0$ ;

Put a mark in position 0, set  $nmark = 1$  and  $mark[0] = 1$ ;

**while**  $nmark < n$  **do**

    Define  $k = \min_{i \in I - J} \{i\}$  such that  $d[i] = 0$  and set  $L_0 = L + k$ ;

**if**  $d[L_0 - i] = 0$  for all  $i$  such that  $mark[i] = 1$  **then**

        Set  $L = L_0$ ,  $mark[k] = 1$  and  $d[L_0 - i] = 1$  for all  $i$  such that  $mark[i] = 1$  ;

**else**

$J = J \cup \{i\}$  and  $nmark = nmark - 1$ ;

**end**

$nmark = nmark + 1$

**end**

**Output:** upper bound  $L_0$  and vector  $mark$

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To solve larger instances of the GRP, compared to model (2.1), we propose below a continuous version of this model, with a simple but fundamental modification on the first constraint:

$$\begin{aligned}
 & \min_{t \geq 0, x_i \in [0,1]} \quad t \\
 & \text{s.t.} \\
 & \quad ix_i \leq tx_i, \quad i = 1, \dots, L_u, \\
 & \quad \sum_{k=1}^{L_u} x_i = n - 1, \\
 & \quad x_j + \sum_{i=1}^{L_u-j} (x_i x_{i+j}) \leq 1, \quad j = 1, \dots, L_u - 1, \\
 & \quad 0 \leq x_i \leq 1, \quad i = 1, \dots, L_u. \tag{2.2}
 \end{aligned}$$

## 2.2. A theorem and a conjecture for the GRP

The next result establishes that, if the optimal solution  $(x_1, \dots, x_{L_u})$  of model (2.2) is such that  $x_i \in \{0, 1\}$ , for  $i = 1, \dots, L_u$ , in fact,  $(x_1, \dots, x_{L_u})$  is also a solution for the associated GRP.

**Theorem 2.1.** *Given an upper bound  $L_u$  for model (2.2) and let  $L$  be the optimal value associated to an optimal solution  $x = (x_1, \dots, x_{L_u})$ . If  $x_i \in \{0, 1\}$ , for  $i = 1, \dots, L_u$ , then  $x = (x_1, \dots, x_{L_u})$  is an optimal solution for model (2.1) with optimal value equal to  $L$ .*

*Proof.* If  $x = (x_1, \dots, x_{L_u})$  is an optimal solution to problem (2.2) with  $x_i \in \{0, 1\}$ , for  $i = 1, \dots, L_u$ , then  $x = (x_1, \dots, x_{L_u})$  is a feasible solution to problem (2.1). Suppose that there exists another feasible solution  $y = (y_1, \dots, y_{L_u})$  to problem (2.1) such that  $t = \max_{i=1, \dots, L_u} \{iy_i\} < L$ . Since  $y$  is a feasible solution to problem (2.2),  $L$  cannot be the optimal value, which is a contradiction.  $\square$

Computational experiments provided in the next section indicate that the opposite direction of the above result is also true, which we state as a conjecture, given as follows.

**Conjecture 2.2.** *Given an upper bound  $L_u$  for model (2.1), if  $L$  is the optimal value associated to an optimal solution  $x = (x_1, \dots, x_{L_u})$ , then  $x = (x_1, \dots, x_{L_u})$  is an optimal solution for model (2.2) with optimal value equal to  $L$ .*

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<sup>1</sup>The source code is available on GitHub at <https://github.com/luizelduino/golombruler/blob/master/heuristic>.

TABLE 1. Upper bounds.

$n$	Heuristic	$n^2$	$n^2 - n$
5	12	25	20
6	20	36	30
7	30	49	42
8	44	64	56
9	59	81	72
10	75	100	90
11	96	121	120
12	118	144	132
13	143	169	156
14	169	196	182
15	197	225	210
16	230	256	240
17	264	289	272
18	299	324	306
19	335	361	342
20	373	400	380
21	413	441	420
22	455	484	462
23	501	529	506
24	549	576	552
25	598	625	600
26	648	676	650
27	701	729	702
28	758	784	756

### 3. COMPUTATIONAL RESULTS

For  $n = 5, \dots, 28$ , the values for  $L_u$  were obtained using the greedy heuristic as described in the previous section. Table 1 gives such values, comparing to two conjectured upper bounds for the GRP [8].

For  $n = 5, \dots, 11$ , Table 2 shows the computational time, in seconds, required to find the associated optimal values (also given) considering models (2.1) and (2.2).

Model (2.2) was solved using the software “Knitro 12.3.0”, *via* Neos Server Version 6.0<sup>2</sup>, and “Cplex 12.8.0” was used to solve model (2.1), since it is better than Knitro to combinatorial problems. All the instances run on a Lenovo laptop with 6 GB RAM, intel celeron 1.6 GHz and 64 bit windows 10.

The limit time was set to 120 000 s when the solver Cplex was used. For the solver Knitro, we used 5000 random initial solutions.

From Table 2, we see that the computational time required for solving model (2.1) increases exponentially and no solution is found for  $n > 7$ .

### 4. CONCLUSIONS

Using a very modest computer, small-size instances of the GRP were solved, with related computational time increasing sub-exponentially (differently from the exponential cost of the combinatorial model), which may suggest promising research directions based on this new approach. Indeed, it is possible to develop an ad hoc nonlinear optimization algorithm for the mathematical model proposed, in order to improve the computational

<sup>2</sup><https://neos-server.org/neos/solvers/cp:Knitro/AMPL.html>.

TABLE 2. Computational time in seconds and optimal values.

$n$	$L_u$	Optimal Value	Model (2.1) – Cplex	Model (2.2) – Knitro
5	12	11	0.125	1.66
6	20	17	0.266	4.30
7	30	25	395.265	11.37
8	44	34	—	18.78
9	59	44	—	299.14
10	75	55	—	579.84
11	96	72	—	589.30

results. This possibility can be considered the main advantage of our proposal, compared to the other approaches given in literature.

Compared to two conjectured upper bounds for the GRP, the proposed greedy heuristic also presented good results as observed in Table 1.

Our approach is based on a continuous optimization model and we conjecture that an optimal solution of this model is also a solution to the related GRP.

We emphasize that GRP solutions for  $n = 24, 25, 26, 27$  have been obtained by a massively parallel computational system ([www.distributed.net](http://www.distributed.net)) and the problem for  $n = 28$  is currently under way, whose search started in 2014.

Motivated by the application of the GRP on wireless localization [17] and Partial Digest Problem [15], the connection with Distance Geometry [2, 9–12, 16] should be exploited, also considering the variants of the GRP in 2D or higher dimensions [20].

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