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ERGODICITY AND PERTURBATION BOUNDS FOR $M_t/M_t/1$ QUEUE WITH BALKING, CATASTROPHES, SERVER FAILURES AND REPAIRS

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Abstract. In this paper, we display methods for the computation of convergence and perturbation bounds for $M_t/M_t/1$ system with balking, catastrophes, server failures and repairs. Based on the logarithmic norm of linear operators, the bounds on the rate of convergence, perturbation bounds, and the main limiting characteristics of the queue-length process are obtained. Finally, we consider the application of all obtained estimates to a specific model.

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1. Introduction

Recently, there has been a noticeable interest from researchers to study nonstationary queueing systems because they represent the actual reality of many applications in our life. Nevertheless, we find few works around these systems, as studying these systems needs new, unrecognized mechanisms to analyze their behavior.

In the current paper, we deal with nonstationary $M/M/1$ queue with balking, catastrophes, server failures and repairs. We investigate the bounds on the rate of convergence, and the perturbation bounds for the corresponding queue-length process. Such kinds of bounds give us the possibility for finding the limiting bounds for the class of close to this queue Markovian models. We apply the approach based on the notion of logarithmic norm of an operator function, see detailed description in our recent papers [2, 20]. The motivation of the proposed system comes from having wide and many applications and contributions in many fields, one of them is the field of communication network systems. The applicability of this model can be seen in communication network systems. If there are numerous packets lined up in the system, local packets are always accepted and remote packets are less than a threshold value packets, waiting in the node for process. Then a new arrival either decides not to join the system or departs after joining the system. If this network was infected with a virus, this could lead to the loss of some packets as a result of restarting the network again, or transferring theses packets to

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another network. Also, in computer systems where there are several clients (data) lined up in the system until a certain threshold value, a new arrival may decide not to enter the system after that value. Additionally, if a virus-infected data will annihilate or transfer it to other processors. These systems can be described as queueing models with catastrophes and balking. These systems can be represented as proposed queueing model.

Much of the literature on this topic is devoted exclusively to the study of stationary behavior, although this behavior is a special case, as well as the parameters in many applications are varying with time in our daily life. An example of some works that has discussed queueing systems related to the subject of this paper, we find that in [1] the author discussed the stationary behavior of a two-processor heterogeneous system with catastrophes, server failures and repairs. Kumar *et al.* [9] analyzed the stationary behavior for an $M/M/1$ queue with catastrophes, server failures and repairs. In [14] the author has extend work which has been done by Kumar *et al.* [9] for an $M/M/1$ queue with balking, catastrophes, server failures and repairs where balking occurs if and only if the system size equals or exceeds a threshold value k. Also, Suranga *et al.* [13] considered an $M/M/1$ queue with reneging, catastrophes, server failures and repairs, they obtained the explicit expression for the stationary probabilities. While in [4] Crescenzo *et al.* studied the stationary behavior of a double-ended queue with catastrophes and repairs.

On the other hand, we find some of the works that discussed the nonstationary behavior as in [2] Ammar *et al.* explored the nonstationary of a two-processor heterogeneous system with catastrophes, server failures and repairs. In [5] Di Crescenzo *et al.* construed the a time-non-homogeneous for double-ended queue subject to catastrophes and repairs, as this is an extension of their previous work in [4]. Some other nonstationary models were studied by a number of authors, see for instance [6–8, 10, 11, 23].

We note from the previous literature that no paper has discussed the behavior of the proposed model and based on this observation, in this paper we examine convergence bounds for a non-stationary behavior of the proposed system. In case of stat constant parameters, our results are consistent with those found by Tarabia [14].

The paper is organized as follows. In Section 2, description of the model and basic notions are introduced. In Sections 3 and 4, general theorems on the rate of convergence and perturbation bounds are considered, respectively. Finally, in Section 5, a specific queueing example is studied.

2. Model description and basic notions

The proposed system in the current paper is an $M/M/1$ catastrophic queue involving balking, server failure and repairs. The arrival process of customers' is a Poisson process with mean λ arrival rate during times that the server is running. Suppose that the discipline of customers is served on first-come, first-served with the service time following an exponential appropriation with mean $1/\mu$. When the customer arrives the system, his joining to the system depends on threshold value of k. If the number of customers is a fewer the threshold value of k, they join the queue with probability one. Also, if the number of customers is more than or equal the threshold value of k they join the queue with probability β and may balk with probability $1-\beta$. System capacity is infinite. At the point when the system is inactive or busy, catastrophes happen at the service station as indicated by Poisson process of rate γ. If a failure happens on the busy server, all the system's customers are automatically pulverized and the server is inactivated, *i.e.* the server fails and needs to repair it. Failed server repair times are *i.i.d*, based on an exponential distribution with η parameter. After the server has been repaired, the system is available to provide the service of new customers. Let $r(t)$ be the probability of the server at instant t with $r(0) = 0$ is under repair.

Unlike previous studies (see [9, 13, 14]), we consider in the paper the non-stationary case, that is, we suppose that all possible transition intensities $\lambda(t)$, $\mu(t)$, $\beta(t)$, $\gamma_i(t)$, $\eta(t)$, are non-random functions of time, which are nonnegative and locally integrable on $[0, \infty)$.

According to the above assumptions, the system can be described by Markov process $X(t), t > 0$ where $X(t)$ refers to the number of the system customers t (queue-length process) at the time. Denote by $p_n(t) = P(X(t) =$

n), $n = 0, 1, 2, 3, \ldots$. Assuming that $\mathbf{p}\,(t)$ represents the state vector of probabilities at the moment t, where $\mathbf{p}\,(t) = (r(t), p_0\,(t)\,, p_1\,(t)\,, \ldots)^T$.

From the previous presumptions, the resulting conduct of the state probabilities are described by forward Kolmogorov system as:

$$r'(t) = -\eta(t)r(t) + \sum_{i=0}^{\infty} \gamma_i(t)p_i(t) \tag{2.1}$$

$$p_0'(t) = \mu(t)p_1(t) - (\lambda(t) + \gamma_0(t))p_0(t) + \eta(t)r(t) \tag{2.2}$$

$$p_n'(t) = \lambda(t)p_{n-1}(t) - (\lambda(t) + \gamma_n(t) + \mu(t))p_n(t) + \mu(t)p_{n+1}(t), \qquad 1 \le n \le k-1 \tag{2.3}$$

$$p_k'(t) = \lambda(t)p_{k-1}(t) - (\lambda(t)\beta(t) + \gamma_k(t) + \mu(t))p_k(t) + \mu(t)p_{k+1}(t), \qquad n = k \tag{2.4}$$

$$p_n'(t) = \lambda(t)\beta(t)p_{n-1}(t) - (\lambda(t)\beta(t) + \gamma_n(t) + \mu(t))p_n(t) + \mu(t)p_{n+1}(t), \qquad n > k. \tag{2.5}$$

Put $a_{ij}\,(t) = q_{ji}\,(t)$ for $j \ne i$ and $a_{ii}\,(t) = -\sum_{j\ne i} a_{ji}\,(t) = -\sum_{j \ne i} q_{ij}\,(t)\,.$

We will suppose that $|a_{ii}\,(t)\,| \le L < \infty$, for any i and almost all $t \ge 0$.

We symbolize of the l_1-norm of vector by $\|\cdot\|$, $\|x\| = \sum |x_i|$, $\|B\| = \sup_j \sum_i |b_{ij}|$, if $B = (b_{ij})_{i,j=0}^{\infty}$, and set of all vectors with non-negative coordinates and unit norm from l_1 by Ω. We have $\|A(t)\| = 2\sup_k |a_{kk}\,(t)| \le 2L$ for almost all $t \ge 0$.

Therefore, in the space of sequences l_1, we can rewrite the forward Kolmogorov system (2.1)–(2.5) as a differential equation

$$\frac{\mathrm{d}\mathbf{p}\,(t)}{\mathrm{d}t} = A\,(t)\,\mathbf{p}\,(t)\,, \tag{2.6}$$

where $A\,(t)$ is a bounded for almost all $t \ge 0$ linear operator from l_1 to itself (see related theory in [3]) and it is the respective transposed intensity matrix is generated as:

$$\begin{pmatrix}
-\eta(t) & \gamma_0(t) & \gamma_1(t) & \cdots & \gamma_{k-1}(t) & \gamma_k(t) & \cdots & \cdots & \cdots \\
\eta(t) & -(\lambda(t)+\gamma_0(t)) & \mu(t) & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & \lambda(t) & -(\lambda(t)+\gamma_1(t)+\mu(t)) & \cdots & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
0 & 0 & 0 & \cdots & \lambda(t) & -(\lambda(t)\beta(t)+\gamma_k(t)+\mu(t)) & \mu(t) & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & \lambda(t)\beta(t) & -(\lambda(t)\beta(t)+\gamma_{k+1}(t)+\mu(t)) & \mu(t) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}. \tag{2.7}$$

The mathematical expectation (the mean) of $X(t)$ is symbolized by $E(t,k) = E\left\{X(t)\,|X(0) = k\right\}$, if $X(0) = k$ at the moment t.

Remember that $X\,(t)$ is called *a weakly ergodic Markov chain*, if $\lim_{t\to\infty} \left\|\mathbf{p}^1\,(t) - \mathbf{p}^2\,(t)\right\| = 0$ for any initial conditions $\mathbf{p}^1\,(0) = \mathbf{p}^1 \in \Omega$, $\mathbf{p}^2\,(0) = \mathbf{p}^2 \in \Omega$; and it has the limiting mean $\phi(t)$, if $|E(t;k) - \phi(t)| \to 0$ as $t \to \infty$ for any k.

We use in the paper the notion of the logarithmic norm of operator function from l_1 to itself, it is calculated by the formula

$$\gamma\,(B\,(t))_1 = \sup_i \left(b_{ii}\,(t) + \sum_{j \ne i} |b_{ji}\,(t)\,|\right). \tag{2.8}$$

Moreover, the following bound holds

$$\|U\,(t,s)\,\| \le e^{\int_s^t \gamma(B(\tau))\,\mathrm{d}\tau}, \tag{2.9}$$

where $U\,(t,s)$ is the Cauchy operator of the corresponding differential equation $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = B(t)\mathbf{x}$.

3. Bounds on the rate of convergence

As we noted earlier, our method based on the notion of logarithmic norm and the corresponding bounds for the Cauchy operator. Moreover, for the considered model we can use the both approaches of [2, 20]. Describe briefly these approaches.

First approach (see [2])**.** Denote by $\gamma^*(t) = \inf_n \gamma_n(t)$, by $\mathbf{g}\,(t) = \left(\gamma^*\,(t)\,, 0, 0, \dots\right)^T$. Put

$$a^*_{ij}\,(t) = \begin{cases} a_{0j}\,(t) - \gamma^*\,(t)\,, & \text{if } i = 0, \\ a_{ij}\,(t)\,, & \text{otherwise.} \end{cases} \tag{3.1}$$

Let $A^*\,(t) = \left(a^*_{ij}\,(t)\right)_{i,j=0}^{\infty}$. Then we can consider the equation

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = A^*\,(t)\,\mathbf{p} + \mathbf{g}\,(t)\,, \quad t \geq 0, \tag{3.2}$$

instead of (2.6), where

$$A^*\,(t) =$$
$$\begin{pmatrix}
-(\eta(t)+\gamma^*(t)) & \gamma_0(t)-\gamma^*(t) & \gamma_1(t)-\gamma^*(t) & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\eta(t) & -(\lambda(t)+\gamma_0(t)) & \mu(t) & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & \lambda(t) & -(\lambda(t)+\gamma_1(t)+\mu(t)) & \cdots & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
0 & 0 & 0 & \cdots & \lambda(t) & -(\lambda(t)\beta(t)+\gamma_k(t)+\mu(t)) & \mu(t) & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & \lambda(t)\beta(t) & -(\lambda(t)\beta(t)+\gamma_{k+1}(t)+\mu(t)) & \mu(t) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}. \tag{3.3}$$

If we denote by $U^*\,(t,s)$ the Cauchy operator of the corresponding homogeneous equation $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A^*\,(t)\,\mathbf{x}$, then the equation (3.2) can be solved by the formula

$$\mathbf{p}\,(t) = U^*\,(t,0)\,\mathbf{p}\,(0) + \int_0^t U^*\,(t,\tau)\,\mathbf{g}\,(\tau)\;\mathrm{d}\tau. \tag{3.4}$$

Let $1 = d_0 \leq d_1 \leq \dots$ be positive numbers. Denote by $\mathsf{D} = \mathrm{diag}\,(d_0, d_1, d_2, \dots)$ the corresponding diagonal matrix. Consider the auxiliary space of sequences $l_{1\mathsf{D}} = \{\mathbf{p}/\|\mathbf{p}\|_{1\mathsf{D}} = \|\mathsf{D}\mathbf{p}\|_1 < \infty\}$, and put

$$\gamma_{**}(t) = \inf_i \left(|a^*_{ii}(t)| - \sum_{j \neq i} \frac{d_j}{d_i} a^*_{ji}(t) \right). \tag{3.5}$$

Then, using the arguments as in [2], we obtain the following statements.

Theorem 3.1. *In the situation of sufficiently large catastrophe rate (i.e. the following equality holds):*

$$\int_0^\infty \gamma^*\,(t)\;\mathrm{d}t = +\infty, \tag{3.6}$$

the corresponding queue-length process $X\,(t)$ is weakly ergodic in the uniform operator topology. Moreover, for any different initial conditions $\mathbf{p}^*\,(0)\,, \mathbf{p}^{**}\,(0)$ and any $t \geq 0$ we have

$$\|\mathbf{p}^*\,(t) - \mathbf{p}^{**}\,(t)\| \leq e^{-\int\limits_0^t \gamma^*(\tau)\,\mathrm{d}\tau} \|\mathbf{p}^*\,(0) - \mathbf{p}^{**}\,(0)\| \leq 2e^{-\int\limits_0^t \gamma^*(\tau)\,\mathrm{d}\tau}. \tag{3.7}$$

Theorem 3.2. *Let there exist a positive sequence $\{d_i\}$, $1 = d_0 \le d_1 \le \dots$ such that,*

$$\int_0^\infty \gamma_{**}(t)\,\mathrm{d}t = +\infty. \tag{3.8}$$

Then $X(t)$ is weakly ergodic and the following bound on the rate of convergence holds:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \le e^{-\int\limits_0^t \gamma_{**}(\tau)\,\mathrm{d}\tau}\,\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}, \tag{3.9}$$

for any initial conditions $\mathbf{p}^*(0), \mathbf{p}^{**}(0)$ and any $t \ge 0$.

Let $l_{1E} = \{\mathbf{p} = (r, p_0, p_1, p_2, \dots)\}$ be a space of sequences such that $\|\mathbf{p}\|_{1E} = \sum_{k\ge 0} k|p_k| < \infty$. $\|\mathbf{p}\|_{1D} = \|D\mathbf{p}\| = \|(d_0 r, d_1 p_0, d_2 p_1, \dots)^T\| = d_0 r + \sum_{k\ge 0} d_{k+1} p_k \ge \sum_{k\ge 1} k\frac{d_{k+1}}{k} p_k$. Put $W = \inf_{k\ge 1}\frac{d_{k+1}}{k}$. Then $W\|\mathbf{p}\|_{1E} \le \|\mathbf{p}\|_{1D}$.

Corollary 3.3. *Let, under assumptions of Theorem 3.2, in addition inequality $W > 0$, holds. Then there exists the limiting mathematical expectation, say $\phi(t) = E(t, 0)$, and the inequality*

$$|E(t,j) - E(t,0)| \le \frac{d_{j+1}}{W} e^{-\int\limits_0^t \gamma_{**}(\tau)\,\mathrm{d}\tau}, \tag{3.10}$$

gives us the corresponding speed of convergence to zero, for any j and any $t \ge 0$.

Put now $d_0 = 1$ and $d_{n+1} = (1+\varepsilon)d_n$ for $n \ge 0$, for a small positive ε.

Then also similar to [2], we can obtain the following explicit bounds.

Proposition 3.4. *Let there exist $\varepsilon > 0$ such that*

$$\int_0^\infty (\gamma^*(t) - \varepsilon\upsilon(t))\,\mathrm{d}t = +\infty, \tag{3.11}$$

where $\upsilon(t) = \max(\eta(t), \lambda(t))$. Then:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1D} \le e^{-\int\limits_0^t (\gamma^*(\tau) - \varepsilon\upsilon(\tau))\,\mathrm{d}\tau}\,\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1D}, \tag{3.12}$$

and

$$|E(t,j) - E(t,0)| \le \frac{d_{j+1}}{W} e^{-\int\limits_0^t (\gamma^*(\tau) - \varepsilon\upsilon(\tau))\,\mathrm{d}\tau}, \tag{3.13}$$

for the corresponding l_{1D} and W.

Second approach (see also [20])**.**

Consider firstly the particular case of the same catastrophe rates, namely, suppose that all $\gamma_n(t) = \gamma^*(t)$. In this situation the equation (2.1) will look like this $r'(t) = -\eta(t)r(t) + \gamma^*(t)$, hence one can solve it:

$$r(t) = \int\limits_0^t e^{-\int\limits_\tau^t \eta(u)\,du}\,\gamma^*(\tau)\mathrm{d}\tau, \tag{3.14}$$

because $r(0) = 0$.

Consider now the reduced forward Kolmogorov system (3.2) in the form

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = B(t)\,\mathbf{z} + \mathbf{f}(t), \quad t \ge 0, \tag{3.15}$$

where $\mathbf{f}(t) = (\eta(t) r(t), 0, 0, \dots)^T$, $\mathbf{z}(t) = (p_0(t), p_1(t), \dots)^T$, and

$$B(t) = \begin{pmatrix} -(\lambda(t)+\gamma^*(t)) & \mu(t) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \lambda(t) & -(\lambda(t)+\gamma^*(t)+\mu(t)) & \mu(t) & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda(t) & -(\lambda(t)\beta(t)+\gamma^*(t)+\mu(t)) & \mu(t) & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda(t)\beta(t) & -(\lambda(t)\beta(t)+\gamma^*(t)+\mu(t)) & \mu(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The solution of equation (3.15) can be written in the form

$$\mathbf{z}(t) = U_B(t,0)\,\mathbf{z}(0) + \int_0^t U_B(t,\tau)\,\mathbf{f}(\tau)\,\mathrm{d}\tau, \tag{3.16}$$

where $U_B(t,s)$ is the Cauchy operator of the corresponding homogeneous equation

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = B(t)\,\mathbf{v}. \tag{3.17}$$

Note that the uniform estimate is completely analogous to Theorem 3.1, only with the replacement on the left side of $\mathbf{p}$ by $\mathbf{z}$.

A significantly different situation with this approach arises when we would like to consider general case, and even more to obtain weighted estimates.

Now we cannot find $r(t)$ in the closed form as in (3.14). Instead of this put $r(t) = 1 - \sum_{i\ge 0} p_i(t)$. Then again we get the equation (3.15) with another $B(t)$,

$$B(t) = \begin{pmatrix} -(\lambda(t)+\gamma_0(t)+\eta(t)) & \mu(t)-\eta(t) & -\eta(t) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda(t) & -(\lambda(t)+\gamma_1(t)+\mu(t)) & \mu(t) & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda(t) & -(\lambda(t)\beta(t)+\gamma_k(t)+\mu(t)) & \mu(t) & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda(t)\beta(t) & -(\lambda(t)\beta(t)+\gamma_{k+1}(t)+\mu(t)) & \mu(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Moreover, now $\mathbf{f}(t) = (\eta(t), 0, 0, \dots)^T$, while $\mathbf{z}(t) = (p_0(t), p_1(t), \dots)^T$.

Let

$$\mathfrak{D} = \begin{pmatrix} d_0 & d_0 & d_0 & \cdots \\ 0 & d_1 & d_1 & \cdots \\ 0 & 0 & d_2 & \cdots \\ & \ddots & \ddots & \ddots \end{pmatrix} \tag{3.18}$$

and

$$B^*(t) = \mathfrak{D}B(t)\mathfrak{D}^{-1}(t) =$$
$$\begin{pmatrix} -(\eta(t)+\gamma_0(t)) & \frac{d_0}{d_1}(\gamma_0(t)-\gamma_1(t)) & \frac{d_0}{d_2}(\gamma_1(t)-\gamma_2(t)) & \cdots & \cdots & \cdots \\ \frac{d_1}{d_0}\lambda(t) & -(\lambda(t)+\gamma_1(t)+\mu(t)) & \frac{d_1}{d_2}(\mu(t)+\gamma_1(t)-\gamma_2(t)) & \frac{d_1}{d_3}(\gamma_2(t)-\gamma_3(t)) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{3.19}$$

Put

$$\gamma_B(t) = \inf_i \left(|b^*_{ii}(t)| - \sum_{j\neq i} b^*_{ji}(t) \right). \tag{3.20}$$

We have

$$\begin{aligned} \|B^*(t)\| = \|B(t)\|_{1\mathfrak{D}} = \|\mathfrak{D}B(t)\mathfrak{D}^{-1}\| &= \sup_i \left(|b^*_{ii}(t)| + \sum_{j\neq i} b^*_{ji}(t) \right) \\ &= \sup_i \left(2|b^*_{ii}(t)| + \sum_{j\neq i} b^*_{ji}(t) - |b^*_{ii}(t)| \right) \le 2\sup_i |b^*_{ii}(t)| - \gamma_B(t) \le 2L - \gamma_B(t), \end{aligned} \tag{3.21}$$

hence the operator function $B(t)$ is bounded on the space $l_{1\mathfrak{D}}$. Therefore we can apply the same approach to equation (3.15) in the space $l_{1\mathfrak{D}}$. Now the equality

$$\gamma\left(B(t)\right)_{1\mathfrak{D}} = \gamma\left(\mathfrak{D}B(t)\mathfrak{D}^{-1}\right) = \sup_i \left(b^*_{ii}(t) + \sum_{j\neq i} b^*_{ji}(t) \right) = -\gamma_B(t), \tag{3.22}$$

implies the following statement.

Theorem 3.5. *Let*

$$\int_0^\infty \gamma_B(t)\,\mathrm{d}t = +\infty, \tag{3.23}$$

for some $\{d_i\}$, $1 = d_0 \le d_1 \le \dots$. *Then* $X(t)$ *is weakly ergodic and*

$$\|z^*(t) - z^{**}(t)\|_{1D} \le e^{-\int\limits_0^t \gamma_B(\tau)\,\mathrm{d}\tau} \|z^*(0) - z^{**}(0)\|_{1D}, \tag{3.24}$$

for any initial conditions $z^*(0), z^{**}(0)$ *and any* $t \ge 0$.

Put now $W = \inf_{k\ge1} \frac{d_k}{k}$. *Then* $W\|\mathbf{z}\|_{1E} \le \|\mathbf{z}\|_{1D}$.

Corollary 3.6. *Let under assumptions of Theorem 3.5, in addition* $W > 0$. *Then the following bound holds*

$$|E(t,j) - E(t,0)| \le \frac{1+d_j}{W} e^{-\int\limits_0^t \gamma_B(\tau)\,\mathrm{d}\tau}, \tag{3.25}$$

for any j *and any* $t \ge 0$.

Remark. One can put $d_0 = 1$, $d_1 = \epsilon$, $d_{k+1} = (1+\epsilon)d_k$ for $k \geq 1$, and obtain the analogue of Proposition 3.4 for the second approach.

Remark. In all our statements, we can replace the condition of monotonicity of the sequence $\{d_k\}$ by condition $d = \inf_k d_k > 0$, with the corresponding change in the estimates; see, *e.g.* [22].

4. Perturbation bounds

Consider here the application of general perturbation bounding (see the recent review in [21]) for the models under study.

Consider a "perturbed" queue-length process $\bar{X}(t), t \geq 0$ with the corresponding transposed intensity matrix $\bar{A}(t)$, where the "perturbation" matrix $\hat{A}(t) = A(t) - \bar{A}(t)$ is small in a sense. Namely, we assume that the perturbed queue is of the same nature as the original one. Hence, the perturbed intensity matrix also has the same structure, with the corresponding perturbed intensities $\bar{\eta}(t)$, $\bar{\gamma}_n(t)$, $\bar{\lambda}(t)$, $\bar{\mu}(t)$, $\bar{\beta}(t)$.

Let

$$
\begin{aligned}
&|\eta(t) - \bar{\eta}(t)| = |\hat{\eta}(t)| \leq \hat{\epsilon}, \quad |\gamma_n(t) - \bar{\gamma}_n(t)| = |\hat{\gamma}_n(t)| \leq \hat{\epsilon},\\
&|\lambda(t) - \bar{\lambda}(t)| = |\hat{\lambda}(t)| \leq \hat{\epsilon}, \quad |\mu(t) - \bar{\mu}(t)| = |\hat{\mu}(t)| \leq \hat{\epsilon}, \quad |\beta(t) - \bar{\beta}(t)| = |\hat{\beta}(t)| \leq \hat{\epsilon}.
\end{aligned} \tag{4.1}
$$

Hence

$$
|\lambda(t)\beta(t) - \bar{\lambda}(t)\bar{\beta}(t)| \leq \lambda(t)|\hat{\beta}(t)| + \bar{\beta}(t)|\hat{\lambda}(t)| \leq (L+1)\hat{\epsilon}. \tag{4.2}
$$

Then we obtain from (2.7) the following bound

$$
\begin{aligned}
\|\hat{A}(t)\| &= 2\sup_k |\hat{a}_{kk}(t)| = 2\max\Big(|\hat{\eta}(t)|, |\hat{\lambda}(t)| + |\hat{\gamma}(t)|, |\hat{\lambda}(t)| + |\hat{\gamma}_n(t)| + |\hat{\mu}(t)|, (L+1)\hat{\epsilon} + |\hat{\gamma}(t)| + |\hat{\mu}(t)|\Big)\\
&\leq (2L+6)\,\hat{\epsilon}.
\end{aligned} \tag{4.3}
$$

Firstly we formulate the perturbation bounds for the vector of state probability in the situation of Theorem 3.1.

The next statement follows immediately from Theorem 3.1 [21] (see also the first corresponding homogeneous result in [12] and for inhomogeneous situation in [15]).

Theorem 4.1. *Let under assumption of Theorem 3.1 the catastrophe intensity $\gamma(t)$ be such that*

$$
e^{-\int_s^t \gamma^*(\tau)\,\mathrm{d}\tau} \leq N e^{-\gamma_0(t-s)}, \tag{4.4}
$$

for some positive N, γ_0. Then the following perturbation bound holds:

$$
\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \leq \frac{\hat{\epsilon}\,(2L+6)\,(1+\log{(N/2)})}{\gamma_0}, \tag{4.5}
$$

for any perturbed queue with the respectively closed intensities satisfying to (4.1).

Stability bound from Theorem 3.2 is based on results of [17,18].

Note that (3.3), (4.1) and (4.3) imply the inequality:

$$
\|\hat{A}^*(t)\| = 2\sup_k |\hat{a}^*_{kk}(t)| \leq \|\hat{A}(t)\| \leq (2L+6)\,\hat{\epsilon}. \tag{4.6}
$$

On the other hand, we have $\|\mathbf{g}(t)\|_{1D} = \gamma^*(t) \leq L$ for almost all $t \geq 0$.

Then, Theorem 4 from [18] implies the next statement.

Theorem 4.2. *Let, under assumptions of Theorem 3.2, additionally the following bounds hold:*

$$e^{-\int\limits_s^t \gamma_{**}(\tau)\,d\tau} \le N^{**} e^{-\gamma_0^{**}(t-s)}, \tag{4.7}$$

for some positive N^{**}, γ_0^{**}, and

$$H = \sup_{|i-j|=1} \frac{d_i}{d_j} < \infty. \tag{4.8}$$

Then,

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_{1D} \le \frac{(4L+12)\,\hat{\epsilon} H L (N^{**})^2}{\gamma_0^{**}\left(\gamma_0^{**} - (4L+12)\,\hat{\epsilon} H\right)}. \tag{4.9}$$

Moreover, if $W = \inf_{k\ge 1} \frac{d_{k+1}}{k} > 0$, then

$$\limsup_{t\to\infty} |E(t,0) - \bar{E}(t,0)| \le \frac{(4L+12)\,\hat{\epsilon} H L (N^{**})^2}{W\gamma_0^{**}\left(\gamma_0^{**} - (4L+12)\,\hat{\epsilon} H\right)}. \tag{4.10}$$

Finally, we obtain perturbation bounds based on the ergodicity estimates of Theorem 3.5.

Theorem 4.3. *Let the assumptions of Theorem 3, and the following bounds hold:*

$$e^{-\int\limits_s^t \gamma_{B}(\tau)\,d\tau} \le N^{B} e^{-\gamma_0^{B}(t-s)}, \tag{4.11}$$

for some positive N^B, γ_0^B, and

$$H = \sup_{|i-j|=1} \frac{d_i}{d_j} < \infty. \tag{4.12}$$

Then

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_{1D} \le \frac{\hat{\epsilon} N^B (L+1)\left(6HLN^B + \gamma_0^B\right)}{\gamma_0^B\left(\gamma_0^B - 12\hat{\epsilon} H N^B (L+1)\right)}. \tag{4.13}$$

Moreover, if $W = \inf_{k\ge 1} \frac{d_k}{k} > 0$, then

$$\limsup_{t\to\infty} |E(t,0) - \bar{E}(t,0)| \le \frac{\hat{\epsilon} N^B (L+1)\left(6HLN^B + \gamma_0^B\right)}{W\gamma_0^B\left(\gamma_0^B - 12\hat{\epsilon} H N^B (L+1)\right)}. \tag{4.14}$$

Proof. It is sufficient to note that

$$\|B(t) - \bar{B}(t)\|_{1\mathsf{D}} \le H\|B(t) - \bar{B}(t)\|_1 \le H\|A(t) - \bar{A}(t)\|_1 \le (2L+6)\,H\hat{\epsilon} \tag{4.15}$$

and

$$\|\mathbf{f}(t) - \bar{\mathbf{f}}(t)\|_{1\mathsf{D}} = ||\eta(t) r(t) - \bar{\eta}(t)\bar{r}(t)| \le (L+1)\,\hat{\epsilon}. \tag{4.16}$$

Then our claim follows from Theorem 2 of [21]. ☐

Remark. Applying estimates (21), (22) from [21], one can also obtain estimates for time-dependent perturbations $\|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_{1\mathsf{D}}$, in the case of coinciding the corresponding initial conditions $X(0) = \bar{X}(0)$. In addition, using such estimates, we can make the necessary corrections to the inaccurately known intensities values by changing them accordingly.

5. Numerical examples

In this section, we will review two numerical examples to support the results obtained as well as clarify the nature of the behavior of the proposed system. Where through these examples, we will apply the analytical results obtained in the theories corollaries in the previous sections. For these examples, we would consider the threshold value equal to 100 ($k = 100$).

Example 5.1. Let our queueing model have the following rates: $\eta(t) = 3 + \sin 2\pi t$, $\gamma_n(t) = 2 + 0.5\cos 2\pi t$, for any n, $\lambda(t) = 10 + 10\sin 2\pi t$, $\mu(t) = 2 + \cos 2\pi t$, $\beta(t) = 0.7$.

Apply all our bounds for this specific situation.

For Theorem 3.1 and the respective "stability" Theorem 4.1 we need L, N and γ_0. Obviously we have $L \leq 25.5$, and $\gamma^*(t) = 2 + 0.5\cos 2\pi t$. Consider now

$$e^{-\int\limits_s^t \gamma^*(\tau)\,\mathrm{d}\tau} = e^{-2(t-s)-\frac{\sin 2\pi t - \sin 2\pi s}{4\pi}} \leq e^{-2(t-s)+\frac{1}{2\pi}} \leq 2e^{-2(t-s)},$$

hence one can put $N = 2$ and $\gamma_0 = 2$ in (4.4).

For applying of Theorems 3.2, 4.2 we put $\varepsilon = 0.05$, $d_0 = 1$ and $d_{k+1} = (1+\varepsilon)d_k$ for $k \geq 0$. Then have $H = 1 + \varepsilon < 2$, $\upsilon(t) = \max(\eta(t), \lambda(t)) = 10 + 10\sin 2\pi t$, and $\gamma_{**}(t) = \gamma_\varepsilon(t) = 1.5 + 0.5\cos 2\pi t - 0.5\sin 2\pi t$.

$$e^{-\int\limits_s^t \gamma_{**}(\tau)\,\mathrm{d}\tau} \leq e^{-1.5(t-s)+\frac{1}{4\pi}} \leq 2e^{-1.5(t-s)},$$

therefore one can get $N^{**} = 2$ and $\gamma_0^{**} = 1.5$ in (4.7).

Finally, for applying of Theorems 3.2, 4.2 we put $\varepsilon = 0.05$, $d_0 = 1$, $d_1 = \varepsilon$, and $d_{k+1} = (1+\varepsilon)d_k$ for $k \geq 1$. Then we have $H = \frac{1}{\varepsilon}$, and $\gamma_B(t) = \gamma_{**}(t) = \gamma_\varepsilon(t)$, hence one can put $N^B = 2$, $\gamma_0^B = 1.5$ in (4.11).

Now we have from Theorem 3.1:

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\| \leq 4e^{-2t}\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|; \tag{5.1}$$

and

$$\|\mathbf{p}^*(t) - \mathbf{p}^{**}(t)\|_{1\mathsf{D}} \leq 2e^{-1.5t}\|\mathbf{p}^*(0) - \mathbf{p}^{**}(0)\|_{1\mathsf{D}}, \tag{5.2}$$

$$|E(t,j) - E(t,0)| \leq \frac{2\cdot 1.05^{j+1}}{W}e^{-1.5t}, \tag{5.3}$$

from Theorem 3.2 and Corollary 3.3, and almost the same from Theorem 3.5 and Corollary 3.6.

The corresponding perturbation bounds are:

$$\limsup_{t\to\infty}\|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\| \leq 30\hat{\epsilon}, \tag{5.4}$$

from Theorem 4.1;

$$\limsup_{t\to\infty}\|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_{1D} \leq \frac{4\times 10^5\hat{\epsilon}}{1 - 4\times 10^3\hat{\epsilon}}, \tag{5.5}$$

and

$$\limsup_{t\to\infty}|E(t,0) - \bar{E}(t,0)| \leq \frac{4\times 10^5\hat{\epsilon}}{W(1 - 4\times 10^3\hat{\epsilon})}, \tag{5.6}$$

from Theorem 4.2; and bounds of Theorem 4.3 are much worse.

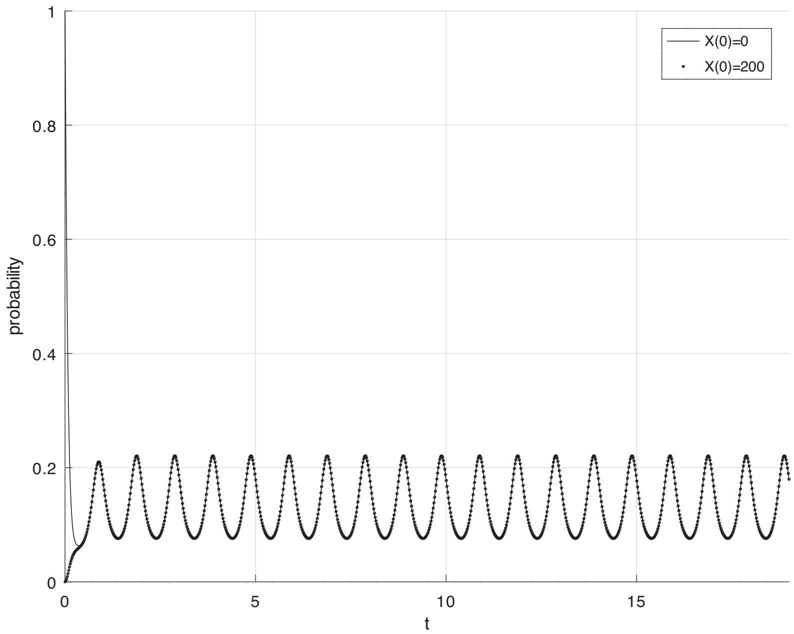

FIGURE 1. Example 5.1. Probability of the empty queue for $t \in [0, 19]$.

One can note that for this model bounds from Theorems 3.5 and 4.3 are worse because of the matrices $B(t)$ and $B^*(t)$ have very special structure.

At the same time, a general approach is used to build graphs, namely: first, similarly to [16, 19], we find out which dimension of the truncated process (200) is enough to take then, using the convergence rate estimates, we find the interval $[0, 20]$, at the right end of which the initial conditions are "forgotten", which means, in the end, on the interval $[19, 20]$, we will find all the limiting characteristics with the required accuracy. Further, we solve the forward Kolmogorov system with the corresponding initial conditions for the truncated process by the Runge-Kutta method on the interval $[0, 20]$.

Now Figures 1–4 shows us the behavior of the probability of the empty queue and the mean respectively. In Figures 5 and 6 one can see the perturbation bounds for the corresponding limiting characteristics with $\hat{\epsilon} = 10^{-3}$ for bound (5.4) and $\hat{\epsilon} = 10^{-6}$ for (5.6) (different values of $\hat{\epsilon}$ are selected for the better pictures).

Example 5.2. Consider now the model with the following rates: $\eta(t) = 3 + \sin 2\pi t$, $\gamma_0(t) = 2 + 0.5 \cos 2\pi t$, $\gamma_k(t) = 0$ if $k \geq 1$, $\lambda(t) = 1 + \sin 2\pi t$, $\mu(t) = 5 + \cos 2\pi t$, $\beta(t) = 0.7$.

Here we have $\gamma^* = 0$ and the first approach is not applicable.

For using Theorems 3.5 and 4.3 we firstly put $d_0 = 1$, $d_1 = 2.5$, and $d_{k+1} = (1.5)d_k$ for $k \geq 1$.

Then we have:

$$|b^*_{00}(t)| - \sum_{j \neq 0} b^*_{j0}(t) = \eta(t) + \gamma_0(t) - \frac{5}{2}\lambda(t) \geq 0.5,$$

$$|b^*_{11}(t)| - \sum_{j \neq 1} b^*_{j1}(t) = \mu(t) - 0.5\lambda(t) - \frac{2}{5}\gamma_0(t) \geq 2,$$

$$|b^*_{ii}(t)| - \sum_{j \neq i} b^*_{ji}(t) = \frac{0.5}{1.5}\mu(t) - 0.5\lambda(t) \geq \frac{1}{3}, \qquad 2 \leq i \leq k-2,$$

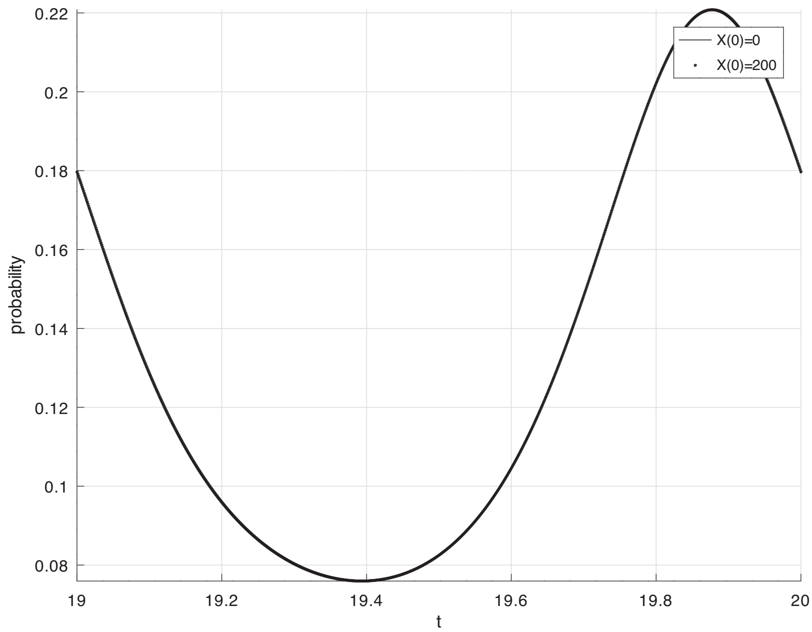

FIGURE 2. Example 5.1. Approximation of the limiting probability of empty queue for $t \in [19, 20]$.

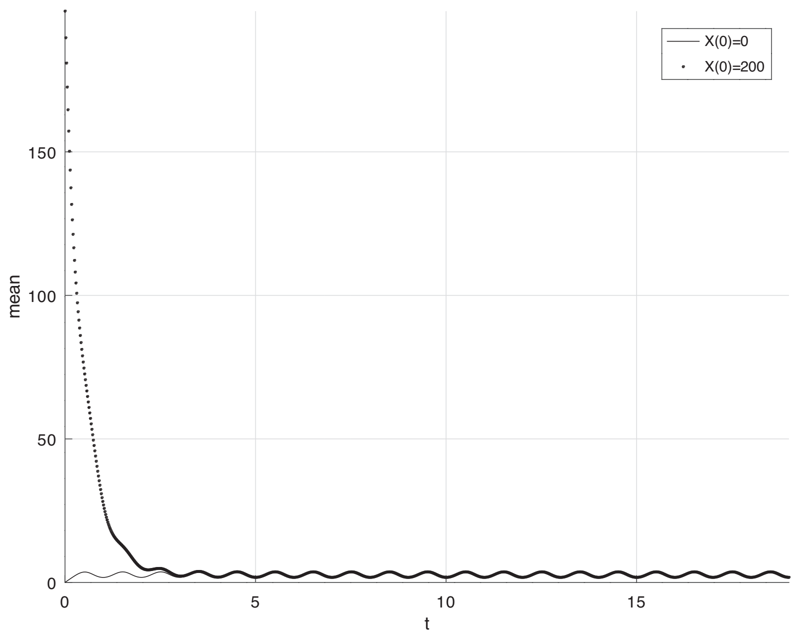

FIGURE 3. Example 5.1. The mean $E(t, k)$ for $t \in [0, 19]$.

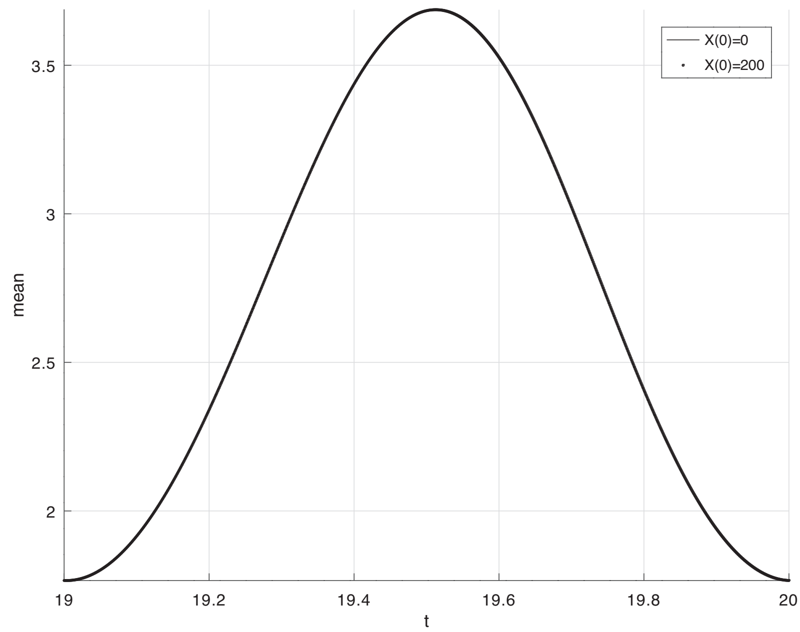

FIGURE 4. Example 5.1. Approximation of the limiting mean $E(t,k)$ for $t \in [19, 20]$.

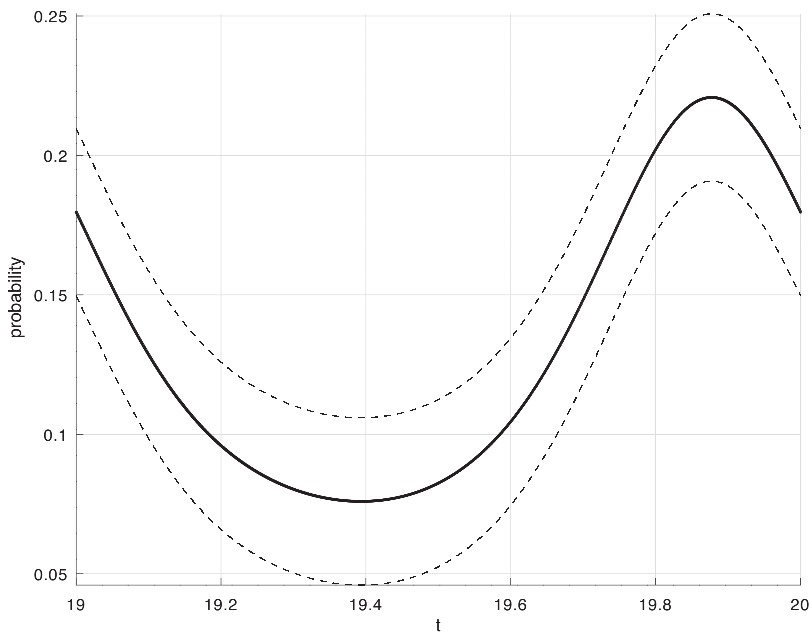

FIGURE 5. Example 5.1. Perturbation bounds for the limiting probability of empty queue for $t \in [19, 20]$.

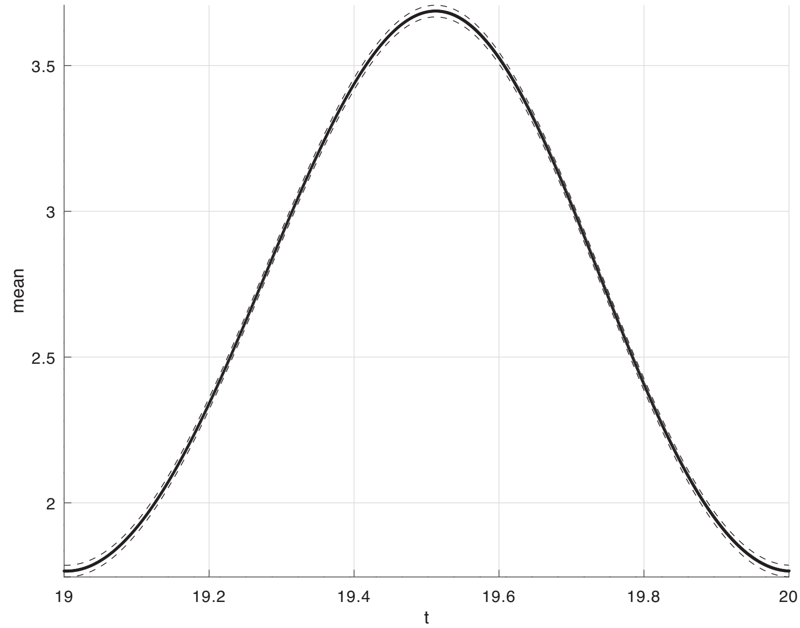

FIGURE 6. Example 5.1. Perturbation bounds for the limiting mean $E(t,0)$ for $t \in [19, 20]$.

$$|b^*_{ii}(t)| - \sum_{j\neq i} b^*_{ji}(t) = \frac{0.5}{1.5}\mu(t) + \lambda(t)(1-\beta(t)) - 0.5\lambda(t)\beta(t) \geq \frac{1}{3}, \qquad i = k-1,$$

$$|b^*_{ii}(t)| - \sum_{j\neq i} b^*_{ji}(t) = \frac{0.5}{1.5}\mu(t) - 0.5\lambda(t)\beta(t) \geq \frac{1}{3}, \qquad i \geq k.$$

Hence, in (3.22) we get

$$\gamma_B(t) = \inf_i \left(|b^*_{ii}(t)| - \sum_{j\neq i} b^*_{ji}(t) \right) \geq \frac{1}{3},$$

and we obtain instead of (3.24) and (3.25) the following bounds:

$$\|z^*(t) - z^{**}(t)\|_{1\mathrm{D}} \leq e^{-\frac{t}{3}} \|z^*(0) - z^{**}(0)\|_{1\mathrm{D}}, \tag{5.7}$$

and

$$|E(t,j) - E(t,0)| \leq \frac{1+d_j}{W} e^{-\frac{t}{3}}. \tag{5.8}$$

Moreover, Theorem 4.3 gives us the corresponding perturbation bounds, namely, we have $L = 8$, $H = 2$, $N^B = 1$, $\gamma^B_0 = \frac{1}{3}$, and instead of (4.13) and (4.14) the following inequalities hold: Then

$$\limsup_{t\to\infty} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_{1\mathrm{D}} \leq 10^4 \cdot \hat{\epsilon}, \tag{5.9}$$

and

$$\limsup_{t\to\infty} |E(t,0) - \bar{E}(t,0)| \leq 2 \times 10^4 \cdot \hat{\epsilon}, \tag{5.10}$$

for sufficiently small $\hat{\epsilon} > 0$.

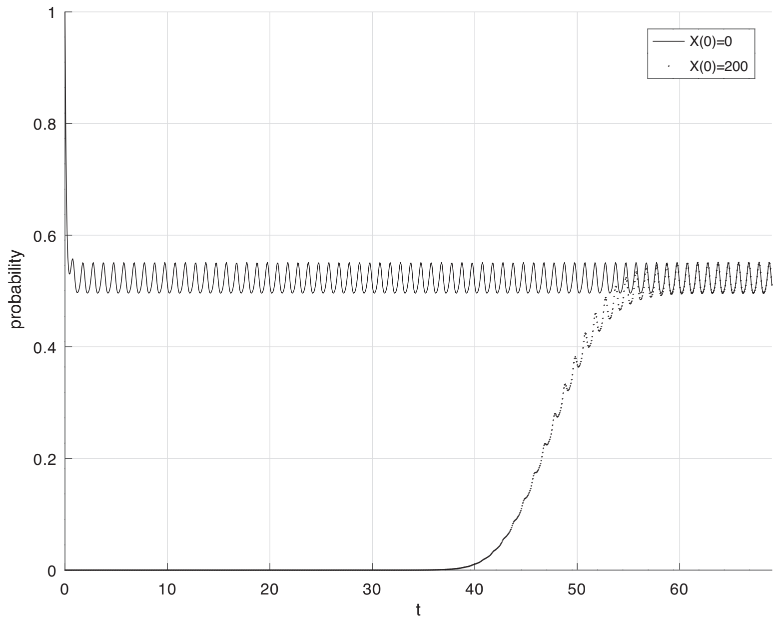

FIGURE 7. Example 5.2. Probability of the empty queue for $t \in [0, 69]$.

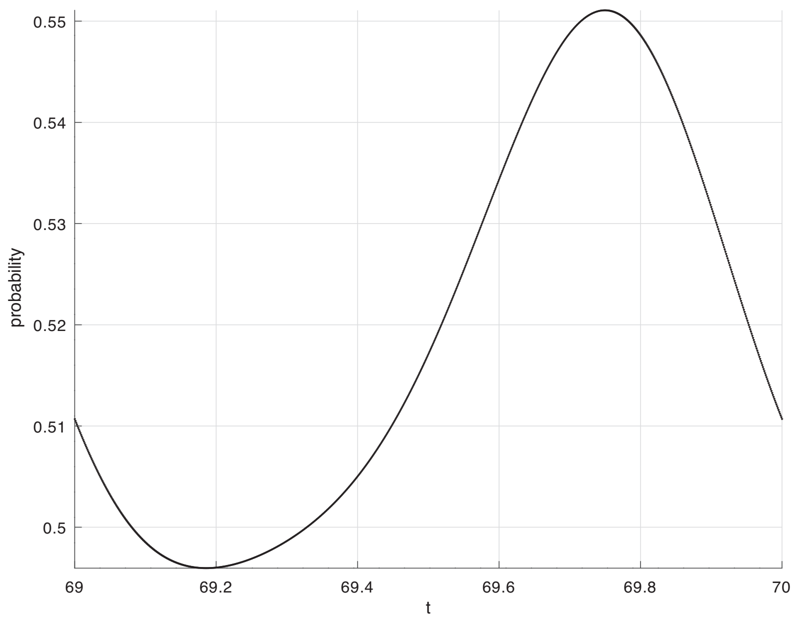

FIGURE 8. Example 5.2. Approximation of the limiting probability of empty queue for $t \in [69, 70]$.

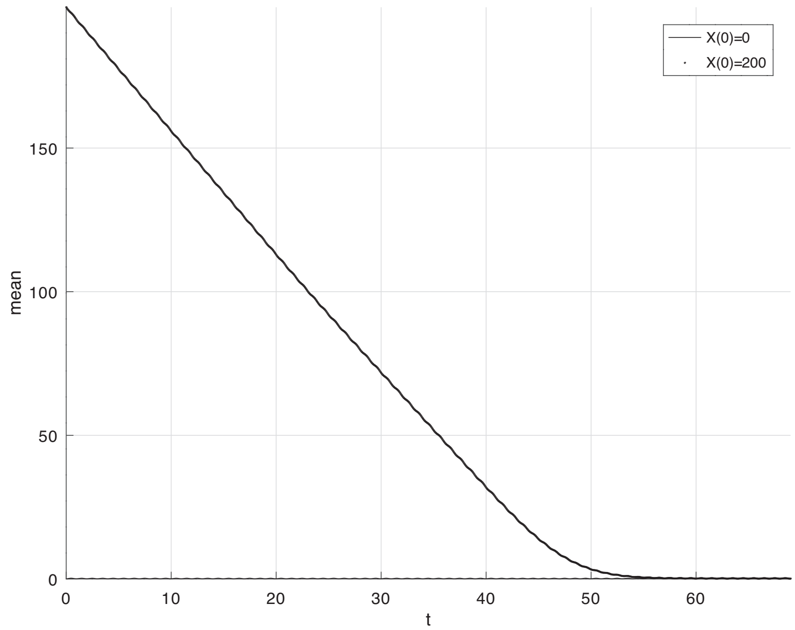

FIGURE 9. Example 5.2. The mean $E(t, k)$ for $t \in [0, 69]$.

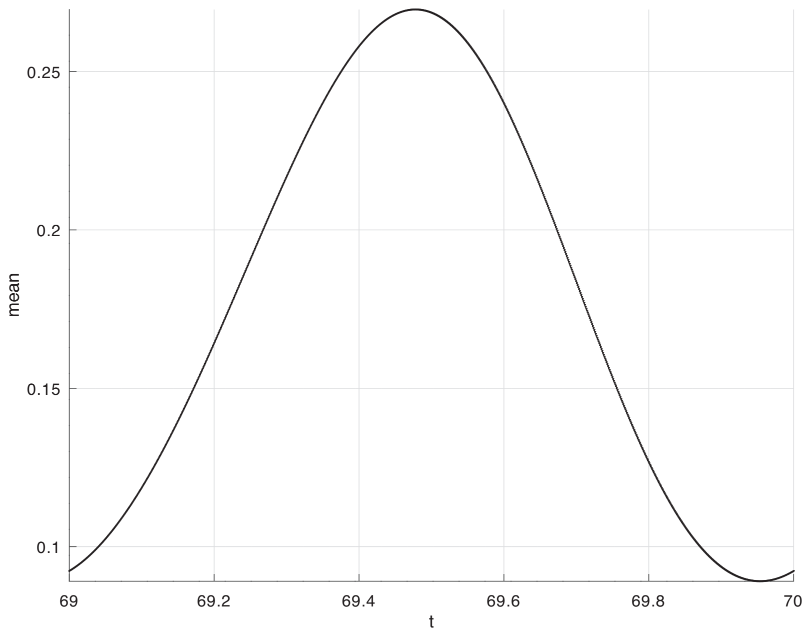

FIGURE 10. Example 5.2. Approximation of the limiting mean $E(t, k)$ for $t \in [69, 70]$.

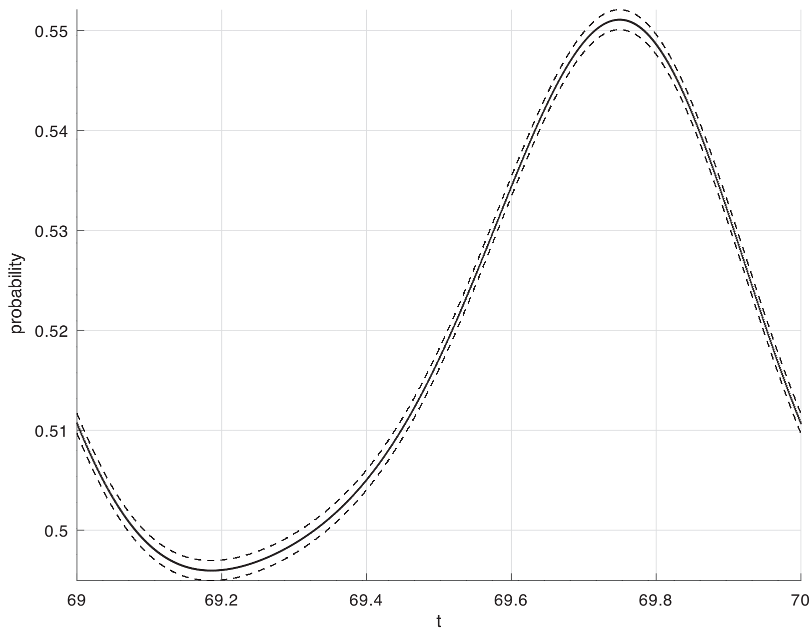

FIGURE 11. Example 5.2. Perturbation bounds for the limiting probability of empty queue for $t \in [69, 70]$.

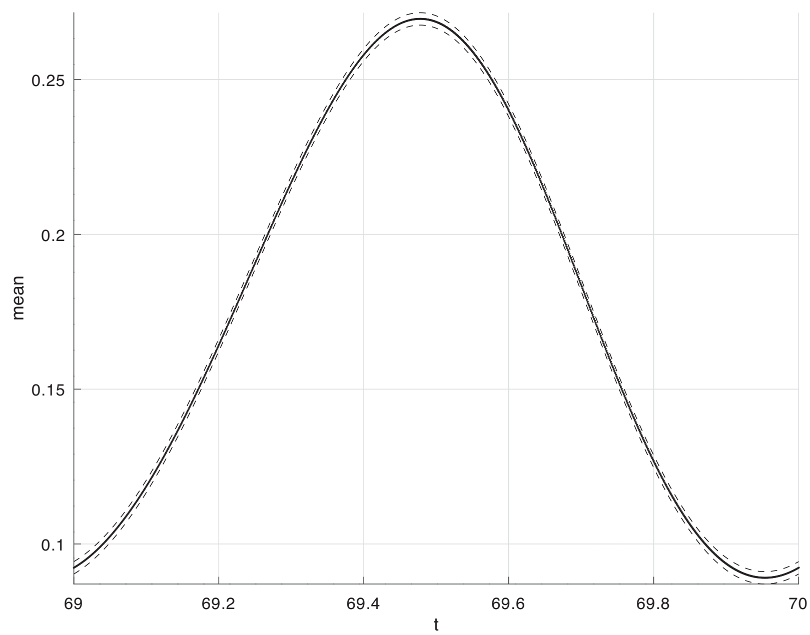

FIGURE 12. Example 5.2. Perturbation bounds for the limiting mean $E(t, 0)$ for $t \in [69, 70]$.

Further we compute the corresponding limiting characteristics. Figures 7–10 shows us the behavior of the probability of the empty queue and the mean respectively, and in Figures 11 and 12 one can see the perturbation bounds for the corresponding limiting characteristics with $\hat{\epsilon} = 10^{-4}$ for bound (5.9) and $\hat{\epsilon} = 5 \times 10^{-5}$ for (5.10).

Remark. On the other hand, having incomplete information about the process, but knowing its limiting characteristics, it is possible to restore the rest of the system parameters, *e.g.* the threshold value k. And if we know the boundaries for the limiting characteristics for the initial or disturbed process, then we can control the corresponding boundaries for the parameters of the system.

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