

## SECOND-ORDER EFFICIENT OPTIMALITY CONDITIONS FOR SET-VALUED VECTOR OPTIMIZATION IN TERMS OF ASYMPTOTIC CONTINGENT EPIDERIVATIVES \*

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**Abstract.** We propose a generalized second-order asymptotic contingent epiderivative of a set-valued mapping, study its properties, as well as relations to some second-order contingent epiderivatives, and sufficient conditions for its existence. Then, using these epiderivatives, we investigate set-valued optimization problems with generalized inequality constraints. Both second-order necessary conditions and sufficient conditions for optimality of the Karush–Kuhn–Tucker type are established under the second-order constraint qualification. An application to Mond–Weir and Wolfe duality schemes is also presented. Some remarks and examples are provided to illustrate our results.

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### 1. INTRODUCTION

Optimality conditions play an important role in both theory and numerical methods in optimization. First-order conditions have been developed for many decades and now the research outcomes are very rich. Second-order optimality conditions have also attracted strong attention since they refine the first-order ones by second-order information and are much helpful to recognize optimal solutions as well as to design numerical algorithms for computing them. For second-order conditions for (scalar and vector) single-valued problems, see, *e.g.* [8, 13, 20, 21, 24, 26, 27, 35] and the references therein.

In the last several decades, set-valued optimization problems have become more attractive because they can be applicable for a wide range of practical situations and many advanced mathematical tools can be invoked for the study. Many generalized derivatives have been proposed and applied to investigate optimality conditions for nonsmooth problems. There are two main types of optimality conditions. Conditions of the primal form are expressed as the disjointedness of sets approximating the data of the optimization problem and sets of “better” directions of the objective and feasible directions of the constraints. Conditions of the dual form are represented by multiplier rules of the Fritz John or Karush–Kuhn–Tucker (KKT) type. For first-order considerations, Aubin [2] proposed a concept of a contingent derivative for set-valued maps, which was used to establish Fritz John multiplier rules in Corley [9] and followed by many papers. Jahn and Rauh [18] introduced

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a notion of a contingent epiderivative, which is an extension of the directional derivative, but it is a single map and its existence still remains a difficult problem. To overcome this difficulty, Chen and Jahn [6] proposed a generalized contingent epiderivative of a set-valued map and applied it to obtain primal-form optimality conditions in [6, 18] and KKT multiplier rules in [17]. Since minimization is more closely related to epigraphs than to graphs, these types of epiderivatives have advantages over the corresponding types of derivatives based on graphs. By using contingent epiderivatives, Götz and Jahn in [12] obtained a first-order Karush–Kuhn–Tucker (KKT) necessary optimality condition. With the well-known Dubovitski–Milutin approach, Issac and Khan in [15] established a multiplier rule for set-valued optimization with generalized inequality constraints. For the first-order optimality conditions, another fruitful approach in set-valued optimization is the dual space approach initiated by Mordukhovich (see [34, 45]).

The second-order optimality conditions for set-valued optimization take in the literature a place much smaller than that of the first-order ones. Let us mention the results related to the present paper. In [11], Durea employed second-order contingent derivatives to establish primal-form conditions. In [19], Jahn *et al.* proposed a notion of generalized second-order contingent epiderivative and applied it to obtain primal-form optimality conditions for problems subject to geometric constraints (set constraints). It is worth noting that a second-order contingent set is in general not a cone and nonconvex. Therefore the corresponding second-order contingent derivative and epiderivative are not convenient tools in some cases. To improve the situation, by using a second-order composed set, Li *et al.* proposed a second-order composed contingent epiderivative in [32] and a generalized second-order composed contingent derivative in [46] and applied them to problems with set constraints in [32] to get primal-form conditions and to problems with generalized inequality constraints in [46] to obtain KKT rules. By using the second-order contingent derivative, Khanh *et al.* [28] established some second-order KKK conditions for a set-valued optimization problem subject to generalized inequality constraints. In terms of second-order asymptotic contingent derivatives, in [25] Khan and Tammer proved second-order multiplier rules, following the Dubovitskii–Milutin approach. As far as we know, very few papers apply asymptotic derivatives in set-valued optimization.

Motivated by the above observations, in this paper by using second-order asymptotic contingent cones, we propose a concept of a generalized second-order asymptotic contingent epiderivative for a set-valued mapping. We discuss its properties as well as its relations to known generalized second-order contingent epiderivatives. Under standard assumptions, sufficient conditions are established for the existence of generalized second-order asymptotic contingent epiderivatives. Furthermore, by employing the free disposal condition, another existence theorem is also obtained. Using generalized second-order asymptotic contingent epiderivatives, we acquire second-order KKT multiplier rules under qualification conditions of the Kurcyusz–Robinson–Zowe (KRZ) type for a set-valued optimization problem subjected to a generalized inequality constraints, and compare these qualification conditions with some other existing ones. Note that dual-form optimality conditions expressed by multiplier rules are more useful than primal-form conditions, specially in applications and numerical methods. Moreover, KKT multiplier rules are more significant than Fritz John ones because they contain explicitly nonzero multipliers of the objectives and hence express optimality properties, not only extremal properties of the constraints as Fritz John rules do when the objective multiplier is zero. Note also that our KKT multiplier rules contain the envelope-like effect because the feasible set includes critical directions. This phenomenon was discovered by Kawasaki in [24], and developed in [8, 21, 35] for  $C^2$  scalar programming, in [13, 26] for nonsmooth multiobjective programming, in [27] for infinite dimensional nonsmooth optimization, and in [28–30] for set-valued optimization.

Beside optimality condition theory, duality theory also occupies an important place in the center of optimization theory. In [44], Weir and Mond proved duality results for weak minimizer of multiple objective optimization problems under pseudo-convexity and quasi-convexity assumptions. Sach *et al.* [37] used the codifferential of set-valued mappings to discuss Mond–Weir type and Wolfe type weak duality and strong duality results of set-valued optimization problems under generalized invexity. By using the higher-order contingent derivatives, Li *et al.* [31] proposed a higher-order Mond–Weir dual for a set-valued optimization problem and obtained their weak, strong, and converse duality properties. By virtue of higher order weak adjacent contingent epiderivatives

and Henig efficiency, Chen *et al.* [7] studied a higher order Mond–Weir type dual problem and a higher order Wolfe type dual problem for a constrained set-valued optimization problem and obtained the corresponding duality properties. Sun *et al.* [42] proposed a robust-type subdifferential constraint qualification for a uncertain convex optimization problem. Then they applied them to investigate Wolfe type robust duality between the uncertain convex optimization problem and its uncertain dual problem. Recently, in [39, 42], the authors studied some new characterizations of robust optimal solution sets for an uncertain optimization problems with a new Lagrangian dual approach. Inspired by the work of [1, 7, 31, 39, 42, 44], we apply these preceding results to study a Mond–Weir type dual problem and a Wolfe type dual problem for the set-valued optimization problem, and discuss the corresponding weak duality, strong duality and converse duality properties.

The organization of the paper is as follows. In Section 2, we collect definitions and preliminary facts for our use in the sequel. Section 3 contains some properties of generalized second-order asymptotic contingent epiderivatives. Section 4 is devoted to second-order necessary conditions and sufficient conditions in terms of KKT multipliers. Section 5 is an application of the results obtained in Sections 3 and 4 to duality.

## 2. PRELIMINARIES

Let  $X$ ,  $Y$  and  $Z$  be real Banach spaces,  $\mathbb{N}$ ,  $\mathbb{R}^n$  and  $\mathbb{R}_+^n$  be the set of the natural numbers, a  $n$ -dimensional Banach space and its nonnegative orthant, respectively (resp).  $B_X$  denotes the open unit ball of  $X$  and  $B_X(x, r)$  the open ball of center  $x$  and radius  $r > 0$ . For  $M \subseteq X$ ,  $\text{int } M$ ,  $\text{cl } M$ , and  $\text{bd } M$  stand for its interior, closure and boundary, resp, of  $M$ . The cone generated by  $M$  is  $\text{cone } M := \{\lambda x \mid \lambda \geq 0, x \in M\}$ . For a set-valued map  $\Phi : X \rightrightarrows Y$ , the domain, graph, and epigraph of  $\Phi$  are defined as

$$\text{dom } \Phi := \{x \in X \mid \Phi(x) \neq \emptyset\}, \quad \text{gph } \Phi := \{(x, y) \in X \times Y \mid y \in \Phi(x)\},$$

$$\text{epi } \Phi := \{(x, y) \in X \times Y \mid y \in \Phi(x) + C\}.$$

For a subset  $S \subseteq X$ ,  $\Phi(S) := \bigcup_{x \in S} \Phi(x)$  and the profile (or epigraphical) map  $\Phi_+ : X \rightrightarrows Y$  is defined by  $\Phi_+(x) := \Phi(x) + C$ .  $\Phi$  is said to be  $C$ -convex if, for all  $x_1, x_2 \in X$  and  $\lambda \in [0, 1]$ ,

$$\lambda\Phi(x_1) + (1 - \lambda)\Phi(x_2) \subseteq \Phi(\lambda x_1 + (1 - \lambda)x_2) + C.$$

It is easy to verify that  $\Phi$  is  $C$ -convex if and only if  $\text{epi } \Phi$  is a convex set in  $X \times Y$ .  $\Phi$  is said to be  $C$ -subadditive if, for all  $x_1, x_2 \in X$ ,  $\Phi(x_1) + \Phi(x_2) \subseteq \Phi(x_1 + x_2) + C$ .

**Definition 2.1** (see [16]). Let  $Y$  be partially ordered by a pointed, closed and convex cone  $C \subseteq Y$ ,  $\Delta$  be a nonempty subset of  $Y$ , and a point  $\bar{y} \in \Delta$ .

- (i)  $\bar{y}$  is called a minimal element of  $\Delta$  if  $(\{\bar{y}\} - C \setminus \{0\}) \cap \Delta = \emptyset$ .
- (ii) Supposing  $\text{int } C \neq \emptyset$ ,  $\bar{y}$  is called a weak minimal element of  $\Delta$  if  $(\{y\} - \text{int } C) \cap \Delta = \emptyset$ .

The set of all the minimal elements (resp, weak minimal elements) of  $\Delta$  is denoted by  $\text{Min}_C \Delta$  (resp,  $\text{WMin}_C \Delta$ ). It is obvious that  $\text{Min}_C \Delta \subseteq \text{WMin}_C \Delta$ .

**Definition 2.2** (see [16, 33]). Let  $Y$  be partially ordered by a pointed, closed and convex cone  $C \subseteq Y$ , and  $\Delta$  be a nonempty subset of  $Y$ .

- (i) A sequence  $\{y_n\} \subseteq Y$  is said to be  $C$ -decreasing if  $y_j \leq_C y_i$  for all  $i, j \in \mathbb{N}$  with  $i \leq j$ .
- (ii)  $\Delta$  is said to be  $C$ -lower bounded if there exists a  $y \in Y$ , such that  $\Delta \subseteq \{y\} + C$ .
- (iii)  $\Delta$  is said to have the domination property if  $\Delta \subseteq \text{Min}_C \Delta + C$ .
- (iv) The convex cone  $C$  is said to be Daniel if every  $C$ -decreasing and  $C$ -lower bounded sequence in  $Y$  converges to its infimum.

The following tangent cones and second-order tangent sets will be used in this paper.

**Definition 2.3** (see [8, 20, 35]). Let  $M \subseteq X$  and  $x_0, u \in X$ .

(i) The contingent cone (resp, adjacent cone and interior cone) of  $M$  at  $x_0$  is

$$\begin{aligned} T(M, x_0) &:= \{u \in X \mid \exists t_n \downarrow 0, \exists u_n \rightarrow u, \forall n \in \mathbb{N}, x_0 + t_n u_n \in M\} \\ (T^\flat(M, x_0)) &:= \{u \in X \mid \forall t_n \downarrow 0, \exists u_n \rightarrow u, \forall n \text{ large}, x_0 + t_n u_n \in M\}, \\ IT(M, x_0) &:= \{u \in X \mid \forall t_n \downarrow 0, \forall u_n \rightarrow u, \forall n \text{ large}, x_0 + t_n u_n \in M\}. \end{aligned}$$

(ii) The second-order contingent set (resp, adjacent set and interior set) of  $M$  at  $x_0$  in direction  $u$  is

$$\begin{aligned} T^2(M, x_0, u) &:= \left\{ w \in X \mid \exists t_n \downarrow 0, \exists w_n \rightarrow w, \forall n \in \mathbb{N}, x_0 + t_n u + \frac{1}{2} t_n^2 w_n \in M \right\} \\ (A^2(M, x_0, u)) &:= \left\{ w \in X \mid \forall t_n \downarrow 0, \exists w_n \rightarrow w, \forall n \text{ large}, x_0 + t_n u + \frac{1}{2} t_n^2 w_n \in M \right\}, \\ IT^2(M, x_0, u) &:= \left\{ w \in X \mid \forall t_n \downarrow 0, \forall w_n \rightarrow w, \forall n \text{ large}, x_0 + t_n u + \frac{1}{2} t_n^2 w_n \in M \right\}. \end{aligned}$$

(iii) The asymptotic second-order tangent cone (resp, adjacent cone and interior cone) of  $M$  at  $x_0$  in direction  $u$  is

$$\begin{aligned} T''(M, x_0, u) &:= \left\{ w \in X \mid \exists (t_n, r_n) \downarrow (0, 0) : t_n r_n^{-1} \downarrow 0, \exists w_n \rightarrow w, \forall n \in \mathbb{N}, x_0 + t_n u + \frac{1}{2} t_n r_n w_n \in M \right\} \\ (A''(M, x_0, u)) &:= \left\{ w \in X \mid \forall (t_n, r_n) \downarrow (0, 0) : t_n r_n^{-1} \downarrow 0, \exists w_n \rightarrow w, \forall n \text{ large}, x_0 + t_n u + \frac{1}{2} t_n r_n w_n \in M \right\}, \\ IT''(M, x_0, u) &:= \left\{ w \in X \mid \forall (t_n, r_n) \downarrow (0, 0) : t_n r_n^{-1} \downarrow 0, \forall w_n \rightarrow w, \forall n \text{ large}, x_0 + t_n u + \frac{1}{2} t_n r_n w_n \in M \right\}. \end{aligned}$$

$M \subseteq X$  is called second-order derivable (resp, asymptotic derivable) at  $(x_0, u)$  if

$$T^2(M, x_0, u) = A^2(M, x_0, u) \text{ (resp, } T''(M, x_0, u) = A''(M, x_0, u)).$$

Note that, if  $x_0 \notin \text{cl } M$ , then all the above tangent sets are empty; and if  $u \notin T(M, x_0)$ , then all the second-order tangent sets are empty. Hence, the conditions such as  $x_0 \in \text{cl } M$  and  $u \in T(M, x_0)$  are always assumed.  $T^2(M, x_0, u)$ ,  $T''(M, x_0, u)$  and  $T(T(M, x_0), u)$  are closed sets contained in  $\text{clcone}[\text{cone}(M - x_0) - u]$ .  $T(S, x_0)$ ,  $T(T(M, x_0), u)$ ,  $T''(M, x_0, u)$ ,  $A''(M, x_0, u)$  and  $IT''(M, x_0, u)$  are closed cones, and are convex if  $M$  is convex. The cones  $T''(M, x_0, u)$ ,  $A''(M, x_0, u)$  were proposed by Penot [35].  $T^2(M, x_0, u)$ ,  $A^2(M, x_0, u)$  and  $IT^2(M, x_0, u)$  are closed sets, but not necessarily cones. If  $M$  is convex, then  $A^2(M, x_0, u)$  is convex, while  $T^2(M, x_0, u)$  may not be convex.

In next proposition some known properties of second-order tangent sets are collected (see more details in [8, 13, 21, 24, 25, 35]).

**Proposition 2.4.** Let  $M \subseteq X$  and  $x_0, u \in X$ .

(i)  $T^2(M, x_0, 0) = T''(M, x_0, 0) = T(T(M, x_0), 0) = T(M, x_0)$ .

Let, in addition,  $M$  be convex and  $u \in T(M, x_0)$ . Then, the following assertions hold

(ii)  $T(T(M, x_0), u) = \text{clcone}(\text{cone}(M - x_0) - u)$  and  $T^2(M, x_0, u) \subseteq T(T(M, x_0), u)$ . Additionally, if  $0 \in T^2(M, x_0, u)$ , then  $T^2(M, x_0, u) = T(T(M, x_0), u)$ .

(iii) If  $T''(M, x_0, u) \neq \emptyset$ , then  $T''(M, x_0, u) = T(T(M, x_0), u)$  and  $T^2(M, x_0, u) \subseteq T''(M, x_0, u)$

(iv) If  $A^2(M, x_0, u) \neq \emptyset$ , then  $\text{cl } IT^2(M, x_0, u) = A^2(M, x_0, u)$  and

$$A^2(M, x_0, u) + T(T(M, x_0), u) \subseteq A^2(M, x_0, u).$$

(v) If  $A''(M, x_0, u) \neq \emptyset$ , then  $\text{cl } IT''(M, x_0, u) = A''(M, x_0, u)$  and

$$A''(M, x_0, u) + T(T(M, x_0), u) \subseteq A''(M, x_0, u).$$

According to Proposition 2.4,  $T^2(M, x_0, u)$  may be properly contained in  $T''(M, x_0, u)$  and  $T(T(M, x_0), u)$ . Whenever  $T''(M, x_0, u)$  is nonempty and  $M$  is convex,  $T''(M, x_0, u) = T(T(M, x_0), u)$ . In general, these sets are different as shown by the following example.

**Example 2.5.** Let  $M = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = |x_1|^{\frac{5}{4}}\}$  and  $x_0 = (0, 0)$ . By directed calculations, one has  $T(M, x_0) = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_2 = 0\}$ . Take  $u = (1, 0)$ ,  $T^2(M, x_0, u) = \emptyset$ ,  $T(T(M, x_0), u) = \mathbb{R} \times \{0\}$ , and  $T''(M, x_0, u) = \mathbb{R} \times \mathbb{R}_+$ . Thus, we have  $T^2(M, x_0, u) \subsetneq T(T(M, x_0), u) \subsetneq T''(M, x_0, u)$ .

### 3. GENERALIZED SECOND-ORDER ASYMPTOTIC CONTINGENT EPIDERIVATIVE

**Definition 3.1.** Let  $\Phi : X \rightrightarrows Y$ ,  $(x_0, y_0) \in \text{gph } \Phi$ , and  $(u, v) \in X \times Y$ .

- (i) ([18]). The contingent epiderivative of  $\Phi$  at  $(x_0, y_0)$  is a single-valued map  $D\Phi(x_0, y_0) : X \rightarrow Y$  defined by  $\text{epi } D\Phi(x_0, y_0) := T(\text{epi } \Phi, (x_0, y_0))$ .
- (ii) ([6]). The generalized contingent epiderivative of  $\Phi$  at  $(x_0, y_0)$  is a set-valued map  $D_g\Phi(x_0, y_0) : X \rightrightarrows Y$  defined by  $D_g\Phi(x_0, y_0)(x) := \text{Min}_C \{y \in Y \mid (x, y) \in T(\text{epi } \Phi, (x_0, y_0))\}$ .
- (iii) ([19]). The generalized second-order contingent epiderivative of  $\Phi$  at  $(x_0, y_0)$  in direction  $(u, v)$  is a set-valued map  $D_g^2\Phi(x_0, y_0, u, v) : X \rightrightarrows Y$  defined by

$$D_g^2\Phi(x_0, y_0, u, v)(x) := \text{Min}_C \{y \in Y \mid (x, y) \in T^2(\text{epi } \Phi, (x_0, y_0), (u, v))\}.$$

- (iv) ([32]). The generalized composed second-order contingent epiderivative of  $\Phi$  at  $(x_0, y_0)$  in direction  $(u, v)$  is a set-valued map  $D_g^{c(2)}\Phi(x_0, y_0, u, v) : X \rightrightarrows Y$  defined by

$$D_g^{c(2)}\Phi(x_0, y_0, u, v)(x) := \text{Min}_C \{y \in Y \mid (x, y) \in T(T(\text{epi } \Phi, (x_0, y_0)), (u, v))\}.$$

In general, the second-order contingent set, composed second-order contingent set and asymptotic second-order tangent cone are different (see Example 2.5). Since the composed second-order contingent set  $T(T(M, x_0), u)$  is the tangent cone of the tangent cone of the set  $M$  at  $x_0$  at direction  $u$ , then it does not contain some information concerning the directions of  $M$ . We propose a generalized asymptotic second-order contingent epiderivative of a set-valued map as follows.

**Definition 3.2.** Let  $\Phi : X \rightrightarrows Y$ ,  $(x_0, y_0) \in \text{gph } \Phi$ , and  $(u, v) \in X \times Y$ . The generalized asymptotic second-order contingent epiderivative of  $\Phi$  at  $(x_0, y_0)$  in direction  $(u, v)$  is a set-valued map  $D_g''\Phi(x_0, y_0, u, v) : X \rightrightarrows Y$  defined by

$$D_g''\Phi(x_0, y_0, u, v)(x) := \text{Min}_C \{y \in Y \mid (x, y) \in T''(\text{epi } \Phi, (x_0, y_0), (u, v))\}.$$

In the next example, the generalized second-order epiderivatives mentioned in Definitions 3.1 and 3.2 are different, and the second-order contingent epiderivative defined in Definition 3.1 does not exist.

**Example 3.3.** Let  $C = \mathbb{R}_+ \times \{0\}$ ,  $\Phi : \mathbb{R} \rightrightarrows \mathbb{R}^2$  be defined by  $\Phi(x) := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq x^2, y_2 = |x|^{\frac{3}{2}}\}$ ,  $x_0 = 0$ , and  $y_0 = (0, 0)$ . Thus, one gets  $T(\text{epi } \Phi, (x_0, y_0)) = \{(u, v) \in \mathbb{R} \times \mathbb{R}^2 \mid u \in \mathbb{R}, v_1 \geq 0, v_2 = 0\}$ . Take  $(u, v) = (1, (0, 0))$ ,  $T^2(\text{epi } \Phi, (x_0, y_0), (u, v)) = \emptyset$ ,  $T(T(\text{epi } \Phi, (x_0, y_0)), (u, v)) = \{(x, y) \in \mathbb{R} \times \mathbb{R}^2 \mid x \in \mathbb{R}, y_1 \geq 0, y_2 = 0\}$ , and  $T''(\text{epi } \Phi, (x_0, y_0), (u, v)) = \{(x, y) \in \mathbb{R} \times \mathbb{R}^2 \mid x \in \mathbb{R}, y_1 \geq 0, y_2 \geq 0\}$ . From the Definitions 3.1 and 3.2, we have  $D_g^2\Phi(x_0, y_0, u, v)(x) = \emptyset$ ,  $D_g^{c(2)}\Phi(x_0, y_0, u, v)(x) = \{(0, 0)\}$ , and  $D_g''\Phi(x_0, y_0, u, v)(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 0, y_2 \geq 0\}$ . Therefore,

$$D_g^2\Phi(x_0, y_0, u, v)(x) \subsetneq D_g^{c(2)}\Phi(x_0, y_0, u, v)(x) \subsetneq D_g''\Phi(x_0, y_0, u, v)(x).$$

For  $(x_0, y_0) \in \text{epi } \Phi$  and  $(u, v) \in X \times Y$ , we will use the following notations

$$E^2(x) := \{y \in Y \mid (x, y) \in T^2(\text{epi } \Phi, (x_0, y_0), (u, v))\},$$

$$E''(x) := \{y \in Y \mid (x, y) \in T''(\text{epi } \Phi, (x_0, y_0), (u, v))\}.$$

The following proposition gives relationships between the epigraph of the second-order contingent epiderivative of a set-valued mapping and the second-order tangent set of the epigraph of this mapping. The proof is similar to that of Proposition 2.4 in [6, 19, 32] and therefore is omitted.

**Proposition 3.4.** *Let  $\Phi : X \rightrightarrows Y$ ,  $(x_0, y_0) \in \text{gph } \Phi$  and  $(u, v) \in X \times Y$ . If  $D_g''\Phi(x_0, y_0, u, v)(x)$  exists and  $E''(x)$  has the domination property for all  $x \in X$ , then  $\text{epi } D_g''\Phi(x_0, y_0, u, v) = T''(\text{epi } \Phi, (x_0, y_0), (u, v))$ .*

The next statement presents relationships between the generalized contingent epiderivative mentioned in Definitions 3.1 and 3.2.

**Proposition 3.5.** *Let  $\Phi : X \rightrightarrows Y$ ,  $(x_0, y_0) \in \text{gph } \Phi$ ,  $(u, v) \in X \times Y$ , and  $x \in X$ .*

- (i)  $D_g^2\Phi(x_0, y_0, 0, 0)(x) = D_g^{c(2)}\Phi(x_0, y_0, 0, 0)(x) = D_g''\Phi(x_0, y_0, 0, 0)(x) = D_g\Phi(x_0, y_0)(x)$ .  
Let, in addition,  $\Phi$  be  $C$ -convex and  $T''(\text{epi } \Phi, (x_0, y_0), (u, v)) \neq \emptyset$ , then
- (ii)  $D_g^{c(2)}\Phi(x_0, y_0, u, v)(x) = D_g''\Phi(x_0, y_0, u, v)(x)$ . Additionally, if  $(0, 0) \in T^2(\text{epi } \Phi, (x_0, y_0), (u, v))$ , then  $D_g^2\Phi(x_0, y_0, u, v)(x) = D_g^{c(2)}\Phi(x_0, y_0, u, v)(x) = D_g''\Phi(x_0, y_0, u, v)(x)$ .
- (iii) If the sets  $E^2(x)$  and  $E''(x)$  enjoy the domination property, then

$$D_g^2\Phi(x_0, y_0, u, v)(x) \subseteq D_g''\Phi(x_0, y_0, u, v)(x) + C.$$

*Proof.* (i) This follows from Proposition 2.4(i) and Definitions 3.1 and 3.2.

Next let  $\Phi$  be  $C$ -convex. Then,  $\text{epi } \Phi$  is a convex set.

- (ii) By Proposition 2.4(iii), one has  $T''(\text{epi } \Phi, (x_0, y_0), (u, v)) = T(T(\text{epi } \Phi, (x_0, y_0)), (u, v))$ . From Definition 3.1, one has  $D_g''\Phi(x_0, y_0, u, v)(x) = D_g^{c(2)}\Phi(x_0, y_0, u, v)(x)$ . Furthermore, if  $(0, 0) \in T^2(\text{epi } \Phi, (x_0, y_0), (u, v))$ , by Proposition 2.4,  $T^2(\text{epi } \Phi, (x_0, y_0), (u, v)) = T(T(\text{epi } \Phi, (x_0, y_0)), (u, v)) = T''(\text{epi } \Phi, (x_0, y_0), (u, v))$ . Thus, we have  $D_g^2\Phi(x_0, y_0, u, v)(x) = D_g^{c(2)}\Phi(x_0, y_0, u, v)(x) = D_g''\Phi(x_0, y_0, u, v)(x)$ .
- (iii) By Proposition 2.4(iii),  $T^2(\text{epi } \Phi, (x_0, y_0), (u, v)) \subseteq T''(\text{epi } \Phi, (x_0, y_0), (u, v))$ . Since  $E^2(x)$  and  $E''(x)$  possess the domination property, by Proposition 2.4,  $\text{epi } D_g^2\Phi(x_0, y_0, u, v) \subseteq \text{epi } D_g''\Phi(x_0, y_0, u, v)$ . This implies that  $D_g^2\Phi(x_0, y_0, u, v)(x) \subseteq D_g''\Phi(x_0, y_0, u, v)(x) + C$ . □

The following proposition indicates some existence criteria for generalized second-order asymptotic contingent epiderivatives, similar to those in Theorem 2 in [6] and Theorem 3.1 in [32].

**Proposition 3.6.** *Let  $\Phi : X \rightrightarrows Y$ ,  $(x_0, y_0) \in \text{gph } \Phi$ ,  $(u, v) \in X \times Y$ , and  $Y$  be partially ordered by a Daniel pointed, closed and convex cone  $C$ . For every  $x \in X$ , the  $C$ -lower boundedness of  $E''(x)$  implies the existence of  $D_g''\Phi(x_0, y_0, u, v)(x)$ .*

*Proof.* Since the asymptotic second-order tangent cone is always a closed cone and  $E''(x)$  is  $C$ -lower bounded for every  $x \in X$ , by the existence theorem for minimal elements in Luc [33],  $\text{Min } E''(x)$  is nonempty, i.e.,  $D_g''\Phi(x_0, y_0, u, v)(x)$  exists. □

Next, we give another existence results for generalized second-order asymptotic contingent epiderivatives by using a free disposal hypothesis. We recall (see [5, 22, 23]) that a closed set  $M \subseteq Y$  satisfies the free disposal hypothesis with respect to a closed set  $Z$  containing 0, if  $M - Z \subseteq M$ .

**Lemma 3.7.** Suppose that  $M \subseteq Y$  satisfies the free disposal hypothesis with respect to a closed cone  $Z$ . Then,  $Z \subseteq -IT(M, y)$  for all  $y \in M$ .

*Proof.* Let  $z, z_n \in Z$  and  $t_n > 0$  such that  $z_n \rightarrow z$  and  $t_n \rightarrow 0^+$ . As  $Z$  is a cone,  $t_n z_n \in Z$ . By the free disposal hypothesis, one gets  $y + t_n(-z_n) \in M$  for  $y \in M$ . This implies that  $-z \in IT(M, y)$ .  $\square$

**Proposition 3.8.** Let  $\Phi : X \rightrightarrows Y$ ,  $(x_0, y_0) \in \text{gph } \Phi$ ,  $(u, v) \in X \times Y$ , and  $Y$  be partially ordered by a closed and convex cone  $C$ . Suppose that  $E''(x)$  satisfies the free disposal hypothesis with respect to  $-C$  and either of the following tangential relations holds at  $y \in E''(x)$ :  $T^\flat(E''(x), y) \cap (-C) = \{0\}$  or  $IT(E''(x), y)$  does not contain any line. Then,  $y \in D''_g \Phi(x_0, y_0, u, v)(x)$ .

*Proof.* We need to prove that  $y \in \text{Min}_C E''(x)$ , meaning that  $(E''(x) - y) \cap (-C) = \{0\}$ . Suppose there is a nonzero  $z \in (E''(x) - y) \cap (-C)$ . As  $E''(x)$  satisfies the free disposal hypothesis with respect to  $-C$ , for all  $t \in [0, 1]$ ,  $y + z - tz \in E''(x) - (-C) \subseteq E''(x)$ . So, one has  $y + tz \in E''(x)$  for all  $t \in [0, 1]$ . Thus,  $z \in IT(E''(x), y) \subseteq T^\flat(E''(x), y)$  and then  $z \in T^\flat(E''(x), y) \cap (-C)$ . If  $T^\flat(E''(x), y) \cap (-C) = \{0\}$ , then  $z = 0$ , a contradiction. If this intersection properly contains 0, by Lemma 3.7, one has  $C \subseteq IT(E''(x), y)$  and hence  $-z \in IT(E''(x), y)$ . Because  $IT(E''(x), y)$  is a cone and does not contain any line, we obtain again the contradiction  $z = 0$ .  $\square$

Next is an example to illustrate Proposition 3.8.

**Example 3.9.** Consider the set-valued map  $\Phi : \mathbb{R} \rightrightarrows \mathbb{R}^2$  defined by

$$\Phi(x) = \left\{ (y_1, y_2) \in \mathbb{R}^2 \mid y_1 = x \sin(\sqrt{|x|}), y_1 + y_2 \geq x \right\},$$

$C = \mathbb{R}_+^2$ , and  $x_0 = 0$ ,  $y_0 = (0, 0)$ . Direct computations give  $T(\text{epi } \Phi, (x_0, y_0)) = \{(u, (v_1, v_2)) \in \mathbb{R} \times \mathbb{R}^2 \mid v_1 \geq 0, v_1 + v_2 \geq u\}$ . Take  $(u, (v_1, v_2)) = (1, (0, 1)) \in T(\text{epi } \Phi, (x_0, y_0))$ , one gets

$$T''(\text{epi } \Phi, (x_0, y_0), (u, v)) = \{(x, (y_1, y_2)) \in \mathbb{R} \times \mathbb{R}^2 \mid y_1 \geq 0, y_1 + y_2 \geq x\}.$$

Then for every  $x \in \mathbb{R}$ ,  $E''(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0, y_1 + y_2 \geq x\}$ . It follows from Definition 3.2 that

$$D''_g \Phi(x_0, y_0, u, v)(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0, y_1 + y_2 = x\}.$$

Now we apply Proposition 3.8 to check the above argument. Firstly, we see that, for every  $x \in \mathbb{R}$ ,  $E''(x)$  satisfies the free disposal hypothesis with respect to  $-C$ .

Next, we consider two cases for the choice of  $(\bar{y}_1, \bar{y}_2) \in \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0, y_1 + y_2 = x\}$ .

– First case:  $\bar{y}_1 > 0$ , by directly calculating, one has  $T^\flat(E''(x), (\bar{y}_1, \bar{y}_2)) = \{(w_1, w_2) \in \mathbb{R}^2 \mid w_1 + w_2 \geq 0\}$ , hence  $T^\flat(E''(x), (\bar{y}_1, \bar{y}_2)) \cap (-C) = \{0\}$ . By Proposition 3.8,

$$\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 > 0, y_1 + y_2 = x\} \subseteq D''_g \Phi(x_0, y_0, u, v)(x).$$

– Second case:  $\bar{y}_1 = 0$ , by Definition 2.3(i),  $IT(E''(x), (\bar{y}_1, \bar{y}_2)) = \{(w_1, w_2) \in \mathbb{R}^2 \mid w_1 > 0, w_1 + w_2 > 0\}$ , and it does not contain any line. According to Proposition 3.8,

$$\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 0, y_2 = x\} \subseteq D''_g \Phi(x_0, y_0, u, v)(x).$$

Some properties of generalized second-order asymptotic contingent epiderivatives are collected in the following.

**Proposition 3.10.** Let  $\Phi : X \rightrightarrows Y$ ,  $(x_0, y_0) \in \text{gph } \Phi$ ,  $(u, v) \in X \times Y$ , and  $x \in X$ . Then,

- (i)  $D_g''\Phi(x_0, y_0, u, v)(\cdot)$  is strictly positively homogeneous.  
Moreover, if  $\Phi$  is  $C$ -convex, and  $E''(x)$  has the domination property, then
- (ii)  $D_g''\Phi(x_0, y_0, u, v)$  is subadditive;
- (iii) for  $v \in \Phi(u) + C$ ,  $\Phi(x) - y_0 \subseteq D_g''\Phi(x_0, y_0, u - x_0, v - y_0)(x - x_0) + C$ .

*Proof.* The proof of (i) and (ii) are similar to that of Theorem 3.2 in [32].

- (iii) For any  $y \in \Phi(x)$  and sequence  $(t_n, r_n) \downarrow (0, 0)$  such that  $t_n r_n^{-1} \downarrow 0$ , since  $F$  is  $C$ -convex, one has

$$\begin{aligned} y_0 + 2t_n(v - y_0) &= 2t_n v + (1 - 2t_n)y_0 \in 2t_n\Phi(u) + (1 - 2t_n)\Phi(x_0) + C \subseteq \Phi(2t_n u + (1 - 2t_n)x_0) + C, \\ y_0 + t_n r_n(y - y_0) &= t_n r_n y + (1 - t_n r_n)y_0 \in t_n r_n\Phi(x) + (1 - t_n r_n)\Phi(x_0) + C \subseteq \Phi(t_n r_n x + (1 - t_n r_n)x_0) + C. \end{aligned}$$

Therefore,

$$\begin{aligned} y_0 + t_n(v - y_0) + \frac{1}{2}t_n r_n(y - y_0) &\in \frac{1}{2}\Phi(2t_n u + (1 - 2t_n)x_0) + \frac{1}{2}\Phi(t_n r_n x + (1 - t_n r_n)x_0) + C \\ &\subseteq \Phi(x_0 + t_n(u - x_0) + \frac{1}{2}t_n r_n(x - x_0)) + C. \end{aligned}$$

This implies that  $(x - x_0, y - y_0) \in T''(\text{epi } \Phi, (x_0, y_0), (u - x_0, v - y_0))$ . Moreover, as  $E''(x - x_0)$  has the domination property, by Proposition 3.4(iii), one has  $\text{epi } D_g''\Phi(x_0, y_0, u - x_0, v - y_0) = T''(\text{epi } \Phi, (x_0, y_0), (u - x_0, v - y_0))$ , and hence  $(x - x_0, y - y_0) \in \text{epi } D_g''\Phi(x_0, y_0, u - x_0, v - y_0)$ . Therefore,

$$y - y_0 \in D_g''\Phi(x_0, y_0, u - x_0, v - y_0)(x - x_0) + C.$$

The proof is complete.  $\square$

#### 4. SECOND-ORDER KKT MULTIPLIERS

In this section, let  $C \subseteq Y$  be a pointed, closed and convex cone with nonempty interior, which defines a partial order on  $Y$ . Let  $D$  be a closed convex cone with nonempty interior in  $Z$ . Our set-valued vector optimization problem is

$$(P) \quad \text{Min}_C F(x) \quad \text{s.t.} \quad G(x) \cap (-D) \neq \emptyset,$$

where  $F : X \rightrightarrows Y$ ,  $G : X \rightrightarrows Z$  are nonempty-valued. Let  $\Omega := \{x \in X \mid G(x) \cap (-D) \neq \emptyset\}$  denote the feasible set. We denote  $D(z_0) := \text{cone}(D + z_0)$  and  $(F, G)(x) = F(x) \times G(x)$ .

For a cone  $C \subseteq Y$  (resp,  $D \subseteq Z$ ), the dual cone and its quasi-interior are, resp,

$$\begin{aligned} C^* &= \{y^* \in Y^* \mid \langle y^*, c \rangle \geq 0, \forall c \in C\} \text{ (resp, } D^*), \\ C^{*i} &= \{y^* \in Y^* \mid \langle y^*, c \rangle > 0, \forall c \in C \setminus \{0\}\}. \end{aligned}$$

Then, it is not hard to check that, for  $z_0 \in -D$ ,  $[D(z_0)]^* = N(-D, z_0)$ , the normal cone of  $-D$  at  $z_0$ . Note that, if  $D$  is a convex cone, then  $N(-D, z_0) = \{d^* \in D^* \mid \langle d^*, z_0 \rangle = 0\}$ .

The following optimality notions of set-valued vector optimization are discussed in our works.

**Definition 4.1.** Let  $x_0 \in \Omega$  and a pair  $(x_0, y_0) \in \text{gph } F$ .

- (i)  $(x_0, y_0)$  is said to be a local minimizer of (P) if there exists a neighborhood  $U$  of  $x_0$  such that

$$(F(\Omega \cap U) - y_0) \cap (-C \setminus \{0\}) = \emptyset.$$

- (ii) Supposing  $\text{int } C \neq \emptyset$ ,  $(x_0, y_0)$  is said to be a local weak minimizer of (P) if there exists a neighborhood  $U$  of  $x_0$  such that

$$(F(\Omega \cap U) - y_0) \cap (-\text{int } C) = \emptyset.$$

(iii) ([4]).  $(x_0, y_0)$  is said to be a local Benson-proper minimizer of (P) if there exist a neighborhood  $U$  of  $x_0$  such that

$$\text{clcone}(F(\Omega \cap U) + C - y_0) \cap (-C) = \{0\}.$$

(iv) ([14]).  $(x_0, y_0) \in \Omega$  is said to be a local Henig-proper minimizer of (P) if there exist a neighborhood  $U$  of  $x_0$  and a convex cone  $K$  such that  $C \setminus \{0\} \subseteq \text{int } K$  and

$$(F(\Omega \cap U) - y_0) \cap (-\text{int } K) = \emptyset.$$

If  $U = X$ , then we have the corresponding global solutions. For  $(x_0, y_0) \in \text{gph } F$  and  $z_0 \in G(x_0) \cap (-D)$ , a triple  $(u, v, w) \in X \times Y \times Z$  is termed a critical direction, denoted by  $(u, v, w) \in \mathcal{C}(x_0, y_0, z_0)$ , if

$$(u, (v, w)) \in T(\text{epi}(F, G), (x_0, (y_0, z_0))) \text{ with } v \in -\text{bd } C \text{ and } w \in -\text{cl } D(z_0).$$

Now, we establish a KKT second-order necessary condition for a local weak minimizer.

**Theorem 4.2.** *Let  $(x_0, y_0) \in \text{gph } F$  be a local weak minimizer of (P) and  $z_0 \in G(x_0) \cap (-D)$ . For  $(u, v, w) \in \mathcal{C}(x_0, y_0, z_0)$  such that  $D_g''(F, G)(x_0, (y_0, z_0), u, (v, w))(X)$  is a convex set, there exist multipliers  $(c^*, d^*) \in C^* \times N(-D, z_0) \setminus \{(0, 0)\}$  such that  $\langle c^*, v \rangle = \langle d^*, w \rangle = 0$  and*

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq 0$$

for all  $(y, z) \in D_g''(F, G)(x_0, (y_0, z_0), u, (v, w))(X)$ . Moreover, if the KRZ qualification condition

$$\{z \in Z \mid (y, z) \in D_g''(F, G)(x_0, (y_0, z_0), u, (v, w))(X)\} + D(z_0) = Z$$

is satisfied, then  $c^* \neq 0$ .

*Proof.* We claim that

$$D_g''(F, G)(x_0, (y_0, z_0), u, (v, w))(X) \bigcap (IT(-C, v) \times IT''(-D, z_0, w)) = \emptyset. \quad (4.1)$$

Suppose to the contrary the existence of  $x \in X$  and  $(y, z) \in Y \times Z$  such that

$$(y, z) \in D_g''(F, G)(x_0, (y_0, z_0), u, (v, w))(x) \bigcap (IT(-C, v) \times IT''(-D, z_0, w)).$$

As  $(y, z) \in D_g''(F, G)(x_0, (y_0, z_0), u, (v, w))(x)$ , according to Definition 3.2, one gets  $(x, (y, z)) \in T''(\text{epi}(F, G), (x_0, (y_0, z_0), u, (v, w)))$ . It follows from the definition of asymptotic second-order tangent cone that there exist sequences  $(t_n, r_n) \downarrow (0, 0) : t_n r_n^{-1} \downarrow 0$ ,  $x_n \rightarrow x$ , and  $z_n \rightarrow z$  such that

$$\begin{aligned} y_0 + t_n v + \frac{1}{2} t_n r_n y_n &\in F\left(x_0 + t_n u + \frac{1}{2} t_n r_n x_n\right) + C, \\ z_0 + t_n w + \frac{1}{2} t_n r_n z_n &\in G\left(x_0 + t_n u + \frac{1}{2} t_n r_n x_n\right) + D. \end{aligned}$$

Since  $z \in IT''(-D, z_0, w)$ ,  $z_0 + t_n w + \frac{1}{2} t_n r_n z_n \in -D$ , for sufficiently large  $n$ . Thus, one has

$$\left(G\left(x_0 + t_n u + \frac{1}{2} t_n r_n x_n\right) + D\right) \cap (-D) \neq \emptyset.$$

Moreover  $D$  is a convex cone,  $G\left(x_0 + t_n u + \frac{1}{2} t_n r_n x_n\right) \cap (-D) \neq \emptyset$ , i.e.,  $x_0 + t_n u + \frac{1}{2} t_n r_n x_n \in \Omega$ .

On the other hand, because  $y \in IT(-C, v) = IT(-\text{int } C, v)$  (see [20], Prop. 2.3), for sufficiently large  $n$ ,  $t_n v + \frac{1}{2} t_n r_n y_n \in -\text{int } C$ . Therefore,

$$\left( F \left( x_0 + t_n u + \frac{1}{2} t_n r_n x_n \right) - y_0 \right) \cap (-\text{int } C) \neq \emptyset,$$

which contradicts the weak efficiency of  $(x_0, y_0)$ .

From the equality (4.1) and the convexity assumption, the standard separation theorem is employed to obtain  $(c^*, d^*) \in Y^* \times Z^* \setminus \{(0, 0)\}$  such that

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq \langle c^*, c \rangle + \langle d^*, d \rangle \quad (4.2)$$

for all  $(y, z) \in D_g''(F, G)(x_0, (y_0, z_0), u, (v, w))(X)$  and  $c \in IT(-C, v)$ ,  $d \in IT''(-D, z_0, w)$ . Because  $C$  is a convex cone,  $IT(-C, v) = \text{int}(\text{cone}(-C - v))$  (see [20], Prop. 2.3). It follows from (4.2) that  $\langle c^*, c \rangle \leq 0$  for all  $c \in \text{cone}(-C - v)$ , and so  $c^* \in [\text{cone}(C + v)]^*$ . As  $v \in -\text{bd } C$ , one has  $c^* \in C^*$  and  $\langle c^*, v \rangle = 0$ . As  $\text{cl } IT''(-D, z_0, w) = A''(-D, z_0, w)$ , (4.2) becomes, for all  $d \in A''(-D, z_0, w)$ ,

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq \langle d^*, d \rangle.$$

Since  $D$  is convex,  $A''(-D, z_0, w) = T(T(-D, z_0), w)$ , which is a cone. Hence,  $d^* \in [T(T(-D, z_0), w)]^*$ , i.e.,  $d^* \in N(-D, z_0)$  and  $\langle d^*, w \rangle = 0$ . Because  $A''(-D, z_0, w)$  is a cone, we have  $\langle c^*, y \rangle + \langle d^*, z \rangle \geq 0$ .

The next step is to prove that  $c^* \neq 0$  under the qualification condition. Supposing  $c^* = 0$ , one gets  $\langle d^*, z \rangle \geq 0$  for every  $(y, z) \in D_g''(F, G)(x_0, (y_0, z_0), u, (v, w))(X)$ . Take arbitrarily  $\bar{z} \in Z$ . By the qualification condition, there exist  $t \geq 0$ ,  $z \in \{z' \in Z \mid (y', z') \in D_g''(F, G)(x_0, (y_0, z_0), u, (v, w))(X)\}$ , and  $d \in D$  such that  $\bar{z} = z + t(d + z_0)$ . Since  $d^* \in D^*$  and  $\langle d^*, z_0 \rangle = 0$ , one has

$$\langle d^*, \bar{z} \rangle = \langle d^*, z \rangle + t \langle d^*, d + z_0 \rangle \geq 0.$$

Thus,  $d^* = 0$ , a contradiction because  $(c^*, d^*) \neq (0, 0)$ . Hence, this completes the proof.  $\square$

In most known necessary conditions,  $(F, G)$  and its derivatives are used, for example in [11, 13, 26, 27, 46]. Inspired by the idea in [28, 30], the Aubin property is employed to obtain a sharper second-order necessary conditions involving separately derivatives of  $F$  and  $G$ . From there, constraint qualifications of the Kurcyusz–Robinson–Zowe type, not qualification condition in terms of  $(F, G)$ , can be invoked to get Karush–Kuhn–Tucker multiplier rules for problem (P).

Recall that  $F$  is said to be  $C$ -Aubin at  $(x_0, y_0) \in \text{gph } F$  if there exist the neighborhoods  $U$  of  $x_0$ ,  $V$  of  $y_0$ , and  $L > 0$  such that

$$F(x) \cap V \subseteq F(x') + L\|x - x'\| \text{cl } B_Y + C, \quad \forall x, x' \in U.$$

If  $C = \{0\}$ , this is the well-known Aubin property (known also as the pseudo-Lipschitz property or Lipschitzlike property) in [3].

**Theorem 4.3.** *Let  $(x_0, y_0) \in \text{gph } F$  be a local weak minimizer of (P),  $z_0 \in G(x_0) \cap (-D)$ , and  $F_+$  be  $C$ -Aubin at  $(x_0, y_0)$ . For  $(u, v, w) \in \mathcal{C}(x_0, y_0, z_0)$  such that  $\text{epi } F$  is second-order asymptotic derivable at  $((x_0, y_0), (u, v))$  and  $(D_g''F(x_0, y_0, u, v), D_g''G(x_0, z_0, u, w))(X)$  is convex, there exist multipliers  $(c^*, d^*) \in C^* \times N(-D, z_0) \setminus \{(0, 0)\}$  such that  $\langle c^*, v \rangle = \langle d^*, w \rangle = 0$  and*

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq 0 \quad (4.3)$$

for all  $(y, z) \in (D_g''F(x_0, y_0, u, v), D_g''G(x_0, z_0, u, w))(X)$ . Moreover, if the KRZ constraint qualification (KRZCQ)

$$D_g''G(x_0, z_0, u, w)(X) + D(z_0) = Z \quad (4.4)$$

is fulfilled, then  $c^* \neq 0$ .

*Proof.* We prove that, for all  $x \in X$ ,

$$D_g''F(x_0, y_0, u, v)(x) \times D_g''G(x_0, z_0, u, w)(x) \cap (IT(-C, v) \times IT''(-D, z_0, w)) = \emptyset. \quad (4.5)$$

Suppose to the contrary the existence of  $x \in X$  and  $(y, z) \in Y \times Z$  such that  $(y, z)$  lies on the left-hand side of (4.5). As  $z \in D_g''G(x_0, z_0, u, w)(x)$ ,  $(x, z) \in T''(\text{epi } G, (x_0, z_0), (u, w))$ . Then, there exist  $(t_n, r_n) \downarrow (0, 0) : t_n r_n^{-1} \downarrow 0$ ,  $x_n \rightarrow x$ , and  $z_n \rightarrow z$  such that  $z_0 + t_n w + \frac{1}{2} t_n r_n z_n \in G(x_0 + t_n u + \frac{1}{2} t_n r_n x_n) + D$ . As  $z \in IT''(-D, z_0, w)$ ,  $z_0 + t_n w + \frac{1}{2} t_n r_n z_n \in -D$  for sufficiently large  $n$ . This implies that

$$\left( G\left(x_0 + t_n u + \frac{1}{2} t_n r_n x_n\right) + D \right) \cap (-D) \neq \emptyset.$$

Since  $D$  is a convex cone,  $G(x_0 + t_n u + \frac{1}{2} t_n r_n x_n) \cap (-D) \neq \emptyset$ , i.e.,  $x_0 + t_n u + \frac{1}{2} t_n r_n x_n \in \Omega$ .

On the other hand, as  $y \in D_g''F(x_0, y_0, u, v)(x)$ ,  $(x, y) \in T''(\text{epi } F, (x_0, y_0), (u, v))$ . As  $\text{epi } F$  is second-order asymptotic derivable at  $((x_0, y_0), (u, v))$ , one has  $(x, y) \in A''(\text{epi } F, (x_0, y_0), (u, v))$ . Hence, there exist  $x'_n \rightarrow x$  and  $y_n \rightarrow y$  such that  $y_0 + t_n v + \frac{1}{2} t_n r_n y_n \in F(x_0 + t_n u + \frac{1}{2} t_n r_n x'_n) + C$ . By employing the Aubin property of  $F_+$  at  $(x_0, y_0)$ , there exist a neighborhood  $V$  of  $y_0$  and  $L_F > 0$  such that, for sufficiently large  $n$ ,

$$F_+\left(x_0 + t_n u + \frac{1}{2} t_n r_n x'_n\right) \cap V \subseteq F\left(x_0 + t_n u + \frac{1}{2} t_n r_n x_n\right) + \frac{1}{2} L_F t_n r_n \|x'_n - x_n\| B_Y + C.$$

Thus, for some  $b_n \in B_Y$ ,

$$y_0 + t_n v + \frac{1}{2} t_n r_n (y_n - L_F \|x_n - x'_n\| b_n) \in F\left(x_0 + t_n u + \frac{1}{2} t_n r_n x_n\right) + C.$$

As  $y_n - L_F \|x_n - x'_n\| b_n \rightarrow y \in IT(-C, v)$  and  $IT(-C, v) = IT(-\text{int } C, v)$ , we have that, for large  $n$ ,  $t_n v + \frac{1}{2} t_n r_n (y_n - L_F \|x_n - x'_n\| b_n) \in -\text{int } C$ . Therefore,  $(F(x_0 + t_n u + \frac{1}{2} t_n r_n x_n) - y_0) \cap (-\text{int } C) \neq \emptyset$ . This however contradicts the weak optimality of  $(x_0, y_0)$ .

Argue similarly as for Theorem 4.2, by (4.5) one gets multipliers  $(c^*, d^*) \in C^* \times N(-D, z_0) \setminus \{(0, 0)\}$  such that  $\langle c^*, v \rangle = \langle d^*, w \rangle = 0$  and  $\langle c^*, y \rangle + \langle d^*, z \rangle \geq 0$  for all  $(y, z) \in (D_g''F(x_0, y_0, u, v), D_g''G(x_0, z_0, u, w))(X)$ . Under the (KRZCQ), similar to the proof process of Theorem 4.2, one obtains that  $c^* \neq 0$  and the proof is completed.  $\square$

**Remark 4.4.** (i) In Theorem 4.3, the assumptions that  $F_+$  is  $C$ -Aubin at  $(x_0, y_0)$  and  $\text{epi } F$  is second-order asymptotic derivable at  $((x_0, y_0), (u, v))$  can be replaced by “ $G_+$  is  $D$ -Aubin at  $(x_0, z_0)$  and  $\text{epi } G$  is second-order asymptotic derivable at  $((x_0, z_0), (u, w))$ ”. Indeed, suppose that the relation (4.5) does not hold. Then, there are  $x \in X$  and  $(y, z) \in Y \times Z$  such that

$$(y, z) \in (D_g''F(x_0, y_0, u, v)(x) \times D_g''G(x_0, z_0, u, w)(x)) \cap (IT(-C, v) \times IT''(-D, z_0, w)).$$

As  $y \in D_g''F(x_0, y_0, u, v)(x)$ , the are  $(t_n, r_n) \downarrow (0, 0) : t_n r_n^{-1} \downarrow 0$ ,  $x_n \rightarrow x$  and  $y_n \rightarrow y$  such that

$$y_0 + t_n v + \frac{1}{2} t_n r_n y_n \in F\left(x_0 + t_n u + \frac{1}{2} t_n r_n x_n\right) + C.$$

Because  $z \in D_g''G(x_0, z_0, u, w)(x)$ ,  $(x, z) \in T''(\text{epi } G, (x_0, z_0), (u, w))$ . Moreover, as  $\text{epi } G$  is second-order asymptotic derivable at  $((x_0, z_0), (u, v))$ , one gets  $(x, z) \in A''(\text{epi } G, (x_0, z_0), (u, w))$ . Then, there exist  $x'_n \rightarrow x$  and  $z_n \rightarrow z$  such that  $z_0 + t_n w + \frac{1}{2} t_n r_n z_n \in G(x_0 + t_n u + \frac{1}{2} t_n r_n x'_n) + D$ . Since  $G_+$  is  $D$ -Aubin at  $(x_0, z_0)$ , there exist a neighborhood  $V$  of  $z_0$ , and  $L_G > 0$  such that, for sufficiently large  $n$ ,

$$G_+\left(x_0 + t_n u + \frac{1}{2} t_n r_n x'_n\right) \cap V \subseteq G\left(x_0 + t_n u + \frac{1}{2} t_n r_n x_n\right) + \frac{1}{2} L_G t_n r_n \|x'_n - x_n\| B_Z + D.$$

Consequently,  $z_0 + t_n w + \frac{1}{2} t_n r_n z_n \in G(x_0 + t_n u + \frac{1}{2} t_n r_n x_n) + \frac{1}{2} L_G t_n r_n \|x_n - x'_n\| B_Z + D$ . Thus, there exists  $\bar{b}_n \in B_Z$  such that  $G(x_0 + t_n u + \frac{1}{2} t_n r_n x_n) \cap (z_0 + t_n w + \frac{1}{2} t_n r_n (z_n - L_G \|x_n - x'_n\| \bar{b}_n) - D) \neq \emptyset$ . It follows from  $z \in IT''(-D, z_0, w)$  and  $z_n - L_G \|x_n - x'_n\| \bar{b}_n \rightarrow z$  that

$$z_0 + t_n w + \frac{1}{2} t_n r_n (z_n - L_G \|x_n - x'_n\| \bar{b}_n) \in -D,$$

for sufficiently large  $n$ . Therefore,  $G(x_0 + t_n w + \frac{1}{2} t_n r_n x_n) \cap (-D) \neq \emptyset$ , i.e.,  $x_0 + t_n w + \frac{1}{2} t_n r_n x_n \in \Omega$ . The rest of the proof is similar to that of Theorem 4.3.

- (ii) When  $F$  is  $C$ -convex and  $G$  is  $D$ -convex, according to Proposition 3.5, the generalized composed contingent epiderivative coincides with the generalized asymptotic contingent epiderivative. In this case, Theorem 4.3 improves the corresponding result of Theorem 4.1 in [46], since the authors of [46] used the derivatives of a disjunction map, composed from the objective and the constraints, and the regularity assumptions involving these maps.
- (iii) Note that, the authors in [28] use the cone-Aubin properties of both the objective and the constraint maps to separate the derivatives of them. In our works, we only assume this property for only the objective map or the constraint map.
- (iv) For  $(u, (v, w)) = (0, (0, 0))$ , by Proposition 3.5,  $D_g''F(x_0, y_0, 0, 0)(x) = D_gF(x_0, y_0)(x)$  and  $D_g''G(x_0, z_0, 0, 0)(x) = D_gG(x_0, z_0)(x)$ . Obviously, as a direct consequence of Theorem 4.3 with  $(u, (v, w)) = (0, (0, 0))$ , the first-order optimality condition is immediately acquired. This condition collapses to Theorem 2.7 in [17].

Theorem 4.3 is illustrated by the following examples.

**Example 4.5.** Let  $X = Z = \mathbb{R}^2$ ,  $Y = \mathbb{R}$ ,  $C = \mathbb{R}_+$ ,  $D = \mathbb{R}_+^2$ ,  $x_0 = (0, 0)$ ,  $y_0 = (0, 0)$ ,  $z_0 = 0$ , and

$$\begin{aligned} F(x) &= \begin{cases} \{y \in \mathbb{R} \mid y \geq x_1^2 + |x_2|^{\frac{5}{4}}\} & \text{if } x_1 x_2 \geq 0, \\ \emptyset & \text{if } x_1 x_2 < 0, \end{cases} \\ G(x) &= \begin{cases} \{(x_1^2, x_2^2), (-|x_1|, -|x_2|)\} & \text{if } x_1 x_2 \geq 0, \\ \emptyset & \text{if } x_1 x_2 < 0. \end{cases} \end{aligned}$$

Then,  $F$  is  $C$ -Aubin at  $(x_0, y_0)$ . Direct calculations give  $T(\text{epi } F, (x_0, y_0)) = \{(u, v) \in \mathbb{R}^2 \times \mathbb{R} \mid v \geq 0\}$  and  $T(\text{epi } G, (x_0, z_0)) = \{(u, w) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid w_1 \geq -|u_1|, w_2 \geq -|u_2|\}$ . Take  $(u, v) = ((1, 1), 0) \in T(\text{epi } F, (x_0, y_0))$  and  $(u, w) = ((1, 1), (-1, -1)) \in T(\text{epi } G, (x_0, z_0))$ , one has  $v \in -\text{bd } C$ ,  $w \in -\text{cl } D(z_0)$ , hence  $(u, v, w) \in \mathcal{C}(x_0, y_0, z_0)$ . By Definition 2.3, we get

$$\begin{aligned} A''(\text{epi } F, (x_0, y_0), (u, v)) &= T''(\text{epi } F, (x_0, y_0), (u, v)) = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R} \mid y \geq 0\}, \\ T''(\text{epi } G, (x_0, y_0), (u, w)) &= \{(x, z) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid z_1 \geq -x_1, z_2 \geq -x_2\}. \end{aligned}$$

Thus,  $\text{epi } F$  is second-order asymptotic derivable at  $((x_0, y_0), (u, v))$  and

$$D_g''F(x_0, y_0, u, v)(x) = \{0\}, \quad D_g''G(x_0, z_0, u, w)(x) = \{(-x_1, -x_2)\}.$$

As  $D_g''G(x_0, z_0, u, w)(X) + D(z_0) = \mathbb{R}^2$ , the (KRZCQ) is satisfied.

We can verify that  $(x_0, y_0) = ((0, 0), 0)$  is a local weak minimizer of (P) and all the assumptions of Theorem 4.3 hold. By taking  $c^* = 1$  and  $d^* = (0, 0) \in D^*$ , one has  $\langle c^*, v \rangle = \langle d^*, w \rangle = 0$  and

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq 0$$

for all  $(y, z) \in (D_g''F(x_0, y_0, u, v), D_g''G(x_0, z_0, u, w))(X)$ . Therefore, the conclusions of Theorem 4.3 hold.

We can check that the second-order contingent derivative and epiderivative of  $F$  at  $(x_0, y_0)$  does not exist, then Theorem 3.1 in [19] and Theorems 3.1 and 3.2 in [28] cannot be employed. Furthermore,  $F$  is not  $C$ -convex, hence Theorem 4.1 in [46] cannot also be used.

In the next example, Theorem 4.3 rejects a candidate for a local weak minimizer of problem (P).

**Example 4.6.** Let  $X = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $Z = \mathbb{R}$ ,  $C = \mathbb{R}_+^2$ ,  $D = \mathbb{R}_+$ , and

$$F(x) := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 + y_2 \geq -|x|\}, \quad G(x) := \{z \in \mathbb{R} \mid z \geq x^3 - 2x\}.$$

Consider  $x_0 = 0$ ,  $y_0 = (0, 0)$ , and  $z_0 = 0$ . Then,  $F$  is  $C$ -Aubin at  $(x_0, y_0)$ . By direct calculations, we have

$$T(\text{epi } F, (x_0, y_0)) = \{(u, v) \in \mathbb{R} \times \mathbb{R}^2 \mid v_1 + v_2 \geq -|u|\}, \quad T(\text{epi } G, (x_0, z_0)) = \{(u, w) \in \mathbb{R} \times \mathbb{R} \mid w \geq -2u\}.$$

Take  $(u, v) = (1, (0, -1)) \in T(\text{epi } F, (x_0, y_0))$  and  $(u, w) = (1, -2) \in T(\text{epi } G, (x_0, z_0))$ , one gets  $v \in -\text{bd } C$ ,  $w \in -\text{cl } D(z_0)$ , hence  $(u, v, w) \in \mathcal{C}(x_0, y_0, z_0)$ . Direct calculations yield

$$\begin{aligned} A''(\text{epi } F, (x_0, y_0), (u, v)) &= T''(\text{epi } F, (x_0, y_0), (u, v)) = \{(x, y) \in \mathbb{R} \times \mathbb{R}^2 \mid y_1 + y_2 \geq -x\}, \\ T''(\text{epi } G, (x_0, y_0), (u, w)) &= \{(x, z) \in \mathbb{R} \times \mathbb{R} \mid z \geq -2x\}, \end{aligned}$$

Consequently,  $\text{epi } F$  is second-order asymptotic derivable at  $((x_0, y_0), (u, v))$  and

$$D''_g F(x_0, y_0, u, v)(x) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 + y_2 = -x\}, \quad D''_g G(x_0, z_0, u, w)(x) = \{z \in \mathbb{R} \mid z = -2x\}.$$

Then,  $(D''_g F(x_0, y_0, u, v), D''_g G(x_0, z_0, u, w))(X)$  is a convex set and (KRZCQ) is fulfilled.

Take  $x = 1 \in X$ ,  $y = (-1, 0) \in D''_g F(x_0, y_0, u, v)(x)$ , and  $z = -2 \in D''_g G(x_0, z_0, u, w)(x)$ . To check the necessary condition given in this theorem, we discuss all  $c^* = (c_1, c_2) \in \mathbb{R}_+^2 \setminus \{(0, 0)\}$  and  $d^* \in N(-D, z_0)$  with  $\langle c^*, v \rangle = 0$  and  $\langle d^*, w \rangle = 0$ . One has  $c_1 > 0$ ,  $c_2 = 0$ , and  $d^* = 0$ . Then, for any  $c^* = (c_1, 0) \in \mathbb{R}_+^2$  with  $c_1 > 0$  and  $d^* = 0$ ,

$$\langle c^*, y \rangle + \langle d^*, z \rangle = -c_1 < 0.$$

Theorem 4.3 ensures that  $(x_0, y_0)$  is not a local weak minimizer of problem (P).

Our constraint qualifications in Theorems 4.2 and 4.3 are form of Kurcyusz–Robinson–Zowe condition, see more details in [12, 28, 30, 46]. With the help of Robinson–Ursescu open mapping theorem, a sufficient condition for the qualification (4.4) is studied. This qualification is also compared with the following constraint qualifications.

(MFCQ)<sub>1</sub> (Mangasarian–Fromovitz constraint qualification). There is  $\bar{x} \in X$  such that

$$D''_g(x_0, z_0, u, w)(\bar{x}) \cap (-\text{int } D) \neq \emptyset.$$

(MFCQ)<sub>2</sub> (Relaxed Mangasarian–Fromovitz constraint qualification). There is  $\bar{x} \in X$  such that

$$D''_g(x_0, z_0, u, w)(\bar{x}) \cap IT(-D, z_0) \neq \emptyset.$$

**Proposition 4.7.** (i) *If the graph of  $D''_g(x_0, z_0, u, w)$  is closed and convex;  $0 \in \text{core } D''_g(x_0, z_0, u, w)(X)$  ( $\text{core}(\cdot)$  stands for the algebraic interior of a set  $(\cdot)$ ), then the (KRZCQ) is satisfied.*

(ii)  $(\text{MFCQ})_1 \Rightarrow (\text{MFCQ})_2 \Rightarrow (\text{KRZCQ})$ .

*Proof.* (i) By assumption, the graph of the map  $\Phi$  defined by  $\Phi(x) := D''_g G(x_0, z_0, u, w)(x) + D(z_0)$  is closed and convex,  $0 \in \text{core } \Phi(X)$ , and  $\Phi(X)$  is a convex set. By the Robinson–Ursescu open mapping theorem (see [10, 36, 43]), for  $\bar{x} \in X$  with  $0 \in \Phi(\bar{x})$ , there exists  $\epsilon > 0$  such that  $\epsilon B(0, 1) \subset \Phi(\bar{x} + B(0, 1))$ . Therefore,  $\epsilon B(0, 1) \subset D''_g G(x_0, z_0, u, w)(X) + D(z_0)$ . As  $D''_g G(x_0, z_0, u, w)$  is strictly positively homogeneous and  $D(z_0)$  is a cone, one gets (4.4).

(ii) It is easy to verify that  $-\text{int } D \subseteq \text{IT}(-D, z_0)$  and then  $(\text{CQ})_1 \Rightarrow (\text{CQ})_2$ . Next, we claim that  $(\text{CQ})_2 \Rightarrow (4.4)$ .

By  $(\text{CQ})_2$ , for  $\bar{z} \in D_g''G(x_0, z_0, u, w)(\bar{x}) \cap \text{IT}(-D, z_0)$ ,  $z \in Z$ , and large  $n \in \mathbb{N}$ , one has  $-z_0 - \frac{1}{n} \left( \bar{z} - \frac{1}{n} z \right) \in D$ . By the strictly positively homogeneous property of  $D_g''G(x_0, z_0, u, w)$ ,  $n\bar{z} \in D_g''G(x_0, z_0, u, w)(n\bar{x})$ . Thus

$$z = n\bar{z} + n^2 \left( -z_0 - \frac{1}{n} \bar{z} + \frac{1}{n^2} z + z_0 \right) \in D_g''G(x_0, z_0, u, w)(X) + D(z_0).$$

Because  $z \in Z$  is arbitrary, the conditions (4) is fulfilled.  $\square$

We explain Proposition 4.7(ii) by the following examples.

**Example 4.8.** (a) Let  $C, D, G$ , and  $(x_0, z_0)$  be as in Example 4.6. By choosing  $\bar{x} = 1$ , one has  $D_g''(x_0, z_0, u, w)(\bar{x}) = \{-2\} \in -\text{int } D$ , hence  $(\text{MFCQ})_1$  holds. By Proposition 4.7(ii),  $(\text{KRZCQ})$  also holds. Indeed, by directed calculations, we have  $D_g''G(x_0, z_0, u, w)(\mathbb{R}) + D(z_0) = \mathbb{R}$ .

(b) (the converse of Prop. 4.7(ii) maybe not true). Let  $C = \mathbb{R}_+ \times \mathbb{R}$ ,  $D = \{(z_1, z_2) \in \mathbb{R} \mid z_2 = 0\}$ , and  $G : \mathbb{R} \rightrightarrows \mathbb{R}^2$  be defined by  $G(x) := \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \geq x^2\}$ ,  $x_0 = 0$ , and  $z_0 = (0, 0)$ . Direct computations yield  $\text{epi } G, (x_0, z_0)) = \{(u, v) \in \mathbb{R} \times \mathbb{R}^2 \mid u \in \mathbb{R}, v_1 \geq 0\}$ . For  $(u, v) = (1, (0, 0))$ , calculations yield  $T''(\text{epi } G, (x_0, z_0), (u, v)) = \{(x, z) \in \mathbb{R} \times \mathbb{R}^2 \mid x \in \mathbb{R}, z_1 \geq 0\}$ , hence  $D_g''G(x_0, z_0, u, v)(x) = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 = 0\}$ . One can verify that  $D_g''G(x_0, z_0, u, w)(\mathbb{R}) + D(z_0) = \mathbb{R}^2$  and  $(\text{KRZCQ})$  is fulfilled. However, neither  $(\text{MFCQ})_1$  nor  $(\text{MFCQ})_2$  is satisfied because  $\text{int } D = \text{IT}(-D, z_0) = \emptyset$ .

In [19], Jahn *et al.* proposed a second-order generalized contingent epiderivative and applied it to establish second-order necessary conditions in a primal form for a set-valued optimization problem with an abstract feasible set. In this paper, we present a KKT second-order rule for local weak minimizers using this second-order generalized contingent epiderivative.

**Theorem 4.9.** Let  $(x_0, y_0) \in \text{gph } F$  be a local weak minimizer of (P),  $z_0 \in G(x_0) \cap (-D)$ , and  $F$  be  $C$ -Aubin at  $(x_0, y_0)$ . For  $(u, v, w) \in \mathcal{C}(x_0, y_0, z_0)$  such that  $\text{epi } F$  is second-order derivable at  $((x_0, y_0), (u, v))$  and  $(D_g^2F(x_0, y_0, u, v), D_g^2G_+(x_0, z_0, u, w))(X)$  is convex, there exist  $(c^*, d^*) \in C^* \times N(-D, z_0) \setminus \{(0, 0)\}$  such that  $\langle c^*, v \rangle = \langle d^*, w \rangle = 0$  and

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq \sup_{d \in A^2(-D, z_0, w)} \langle d^*, d \rangle$$

for all  $(y, z) \in (D_g^2F(x_0, y_0, u, v), D_g^2G(x_0, z_0, u, w))(X)$ . Moreover, if the KRZ constraint qualification

$$D_g^2G(x_0, z_0, u, w)(X) - A^2(-D, z_0, w) + D(z_0) = Z \quad (4.6)$$

is fulfilled, then  $c^* \neq 0$ .

*Proof.* Similar to the first part of the proof of Theorem 4.3, we also get, for all  $x \in X$ ,

$$D_g^2F(x_0, y_0, u, v)(x) \times D_g^2G(x_0, z_0, u, w)(x) \bigcap (\text{IT}(-C, v) \times \text{IT}^2(-D, z_0, w)) = \emptyset.$$

Hence, by the convexity assumption, the standard separation theorem gives  $(c^*, d^*) \in Y^* \times Z^* \setminus \{(0, 0, 0)\}$  such that

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq \langle c^*, c \rangle + \langle d^*, d \rangle \quad (4.7)$$

for all  $(y, z) \in (D_g^2F(x_0, y_0, u, v), D_g^2G_+(x_0, z_0, u, w))(X)$  and  $c \in \text{IT}(-C, v)$ ,  $d \in \text{IT}^2(-D, z_0, w)$ . Since  $\text{IT}(-C, v)$  is a cone, (4.7) yields that  $c^* \in C^*$  and  $\langle c^*, v \rangle = 0$ . According to Proposition 2.4,  $\text{cl } \text{IT}^2(-D, z_0, w) = A^2(-D, z_0, w)$  and  $A^2(-D, z_0, w) + T(T(-D, z_0), w) \subseteq A^2(-D, z_0, w)$ . Then, taking  $c = 0$  in (4.7) one has, for all  $d \in A^2(-D, z_0, w)$  and  $d' \in T(T(-D, z_0), w)$ ,

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq \langle d^*, d \rangle + \langle d^*, d' \rangle.$$

Because  $T(T(-D, z_0), w)$  is a cone,  $d^* \in -[T(T(-D, z_0), w)]^* = \{d^* \in N(-D, z_0) \mid \langle d^*, w \rangle = 0\}$ . By letting  $d' = 0$ , the following inequality is acquired

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq \sup_{d \in A^2(-D, z_0, w)} \langle d^*, d \rangle.$$

Next, we verify that  $c^* \neq 0$  under the constraint qualification. Supposing  $c^* = 0$ , one obtains

$$\langle d^*, z \rangle \geq \sup_{d \in A^2(-D, z_0, w)} \langle d^*, d \rangle$$

for every  $z \in D_g^2 G_+(x_0, z_0, u, w)(X)$ . Take arbitrarily  $\bar{z} \in Z$ . By the constraint qualification, there are  $t \geq 0$ ,  $z \in D_g^2 G_+(x_0, z_0, u, w)(X)$ ,  $\bar{d} \in A^2(-D, z_0, w)$ , and  $d \in D$  such that  $\bar{z} = z - \bar{d} + t(d + z_0)$ . AS  $d^* \in D^*$  and  $\langle d^*, z_0 \rangle = 0$ , one has

$$\langle d^*, \bar{z} \rangle = \langle d^*, z - \bar{d} \rangle + t_2 \langle d^*, d + z_0 \rangle \geq \sup_{d \in A^2(-D, z_0, w)} \langle d^*, d \rangle - \langle d^*, \bar{d} \rangle \geq 0.$$

By the arbitrariness of  $\bar{z} \in Z$ , the results is a contradiction that  $d^* = 0$ .  $\square$

**Remark 4.10.** (i) Since  $A^2(-D, z_0, w) \subseteq \text{cl}[\text{cone}[\text{cone}(-D - z_0) - w]]$  and

$$d^* \in -[T(T(-D, z_0), w)]^* = -[\text{cl}[\text{cone}[\text{cone}(-D - z_0) - w]]]^*,$$

we have

$$\sup_{d \in A^2(-D, z_0, w)} \langle d^*, d \rangle \leq 0.$$

It can be strictly negative, *i.e.*, the envelope-like effect occurs. Of course, this supremum vanishes if  $0 \in A^2(-D, z_0, w)$ . So, for direction  $w$  satisfying this, the multiplier rule in Theorem 4.9 takes the classical form. For example, if  $w \in -D(z_0)$ , then  $0 \in A^2(-D, z_0, w)$ . However, Theorem 4.9 also considers critical directions  $w \in -\text{cl } D(z_0)$ . For  $w \in -(\text{cl } D(z_0) \setminus D(z_0))$ , the envelope-like effect can occur. Since  $A''(-D, z_0, w)$  is a cone,  $0 \in A''(-D, z_0, w)$  and hence in Theorems 4.2 and 4.3, this phenomenon does not occur.

(ii) If  $w \in -D(z_0)$ ,  $0 \in A^2(-D, z_0, w)$  and (4.6) becomes  $D_g^2 G(x_0, z_0, u, w)(X) + D(z_0) = Z$ . Hence, the constraint qualifications (4.4) and (4.6) are of the same type, but in terms of different kinds of derivatives.

Later, we move on to sufficient conditions for some global minimizers of problem (P).

**Theorem 4.11.** For (P), let  $x_0 \in \Omega$ ,  $y_0 \in F(x_0)$ ,  $z_0 \in G(x_0) \cap (-D)$ , and  $(v, w) \in (F, G)(u) + C \times D$ . Assume that  $(F, G)$  is  $C \times D$ -convex and  $E''_{(F, G)}(x - x_0)$  fulfills the domination property for all  $x \in \Omega$ . If there exist  $c^* \in C^* \setminus \{0\}$  and  $d^* \in D^*$  such that  $\langle d^*, z_0 \rangle = 0$  and

$$\langle c^*, y \rangle + \langle d^*, z \rangle \geq 0$$

for all  $x \in \Omega$  and  $(y, z) \in D_g''(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(x - x_0)$ , then  $(x_0, y_0)$  is a global weak minimizer of (P).

*Proof.* Suppose that  $(x_0, y_0)$  is not a global weak minimizer of (P), then there are  $\bar{x} \in \Omega$ ,  $\bar{y} \in F(\bar{x})$ , and  $\bar{z} \in Z$  such that

$$\bar{y} - y_0 \in -\text{int } C \quad \text{and} \quad \bar{z} \in G(\bar{x}) \cap (-D).$$

It follows from Proposition 3.10 that, for  $\bar{x} \in \Omega$ ,

$$(F, G)(\bar{x}) - (y_0, z_0) \subseteq D_g''(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(\bar{x} - x_0) + C \times D.$$

Thus, there exist  $(y', z') \in D_g''(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(\bar{x} - x_0)$  and  $(\bar{c}, \bar{d}) \in C \times D$  such that  $\bar{y} - y_0 = y' + c$  and  $\bar{z} - z_0 = z' + d$ . For  $c^* \in C^* \setminus \{0\}$  and  $d^* \in D^*$  with  $\langle d^*, z_0 \rangle = 0$ , one has

$$\langle c^*, y' \rangle + \langle d^*, z' \rangle = \langle c^*, \bar{y} - y_0 - c \rangle + \langle d^*, \bar{z} - z_0 - d \rangle.$$

Because  $\bar{y} - y_0 - c \in -\text{int } C - C \subseteq -\text{int } C$ ,  $\bar{z} - d \in -D - D \subseteq -D$ , and  $\langle d^*, z_0 \rangle = 0$ , we have  $\langle c^*, \bar{y} - y_0 - c \rangle < 0$  and  $\langle d^*, \bar{z} - z_0 - d \rangle \leq 0$ . Therefore,

$$\langle c^*, y' \rangle + \langle d^*, z' \rangle < 0,$$

which is a contradiction. Consequently,  $(x_0, y_0)$  is a global weak minimizer of (P).  $\square$

**Theorem 4.12.** *For (P), let  $x_0 \in \Omega$ ,  $y_0 \in F(x_0)$ ,  $z_0 \in G(x_0) \cap (-D)$ , and  $(v, w) \in (F, G)(u) + C \times D$ . Assume that  $(F, G)$  is  $C \times D$ -convex and  $E''_{(F, G)}(x - x_0)$  fulfills the domination property for all  $x \in \Omega$ . If there exist  $c^* \in C^{*i}$  and  $d^* \in D^*$  such that  $\langle d^*, z_0 \rangle = 0$  and*

$$\langle q^*, y \rangle + \langle d^*, z \rangle \geq 0$$

for all  $x \in \Omega$  and  $(y, z) \in D''_g(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(x - x_0)$ , then  $(x_0, y_0)$  is both global Henig-proper and Benson-proper minimizer of (P).

*Proof.* (i) *Henig-proper minimizer.* According to Proposition 3.10, for all  $x \in \Omega$ ,

$$(F, G)(x) - (y_0, z_0) \subseteq D''_g(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(x - x_0) + C \times D.$$

For  $x \in \Omega$  and  $(y, z) \in (F, G)(x)$ , there exist  $(y', z') \in D''_g(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(x - x_0)$ , and  $(c, d) \in C \times D$  such that  $y - y_0 = y' + q$  and  $z - z_0 = z' + d$ . Given  $c^* \in C^{*i}$  and  $d^* \in D^*$  satisfying the assumptions of the theorem, one gets

$$\begin{aligned} \langle c^*, y \rangle + \langle d^*, z \rangle &= \langle c^*, y_0 \rangle + \langle d^*, z_0 \rangle + \langle c^*, y' \rangle + \langle d^*, z' \rangle + \langle c^*, c \rangle + \langle d^*, d \rangle, \\ \langle c^*, y - y_0 \rangle + \langle d^*, z \rangle &= \langle d^*, z_0 \rangle + \langle c^*, y' \rangle + \langle d^*, z' \rangle + \langle c^*, c \rangle + \langle d^*, d \rangle. \end{aligned}$$

By assumption, one has  $\langle d^*, z_0 \rangle = 0$  and  $\langle c^*, y' \rangle + \langle d^*, z' \rangle + \langle c^*, c \rangle + \langle d^*, d \rangle \geq 0$ . So, for all  $x \in \Omega$  and  $(y, z) \in (F, G)(x)$ , one gets

$$\langle c^*, y - y_0 \rangle + \langle d^*, z \rangle \geq 0.$$

Assume that  $x_0$  is not a global Henig-proper solution, i.e., for any pointed convex cone  $H$  with  $C \setminus \{0\} \subseteq \text{int } H$ , one can find  $x_H \in \Omega$  such that  $(F(x_H) - y_0) \cap (-H \setminus \{0\}) \neq \emptyset$ . Setting a set  $\mathcal{H} := \{y \in Y \mid \langle c^*, y - y_0 \rangle > 0\} \cup \{0\}$ , we see that  $\mathcal{H}$  is a convex cone with  $C \setminus \{0\} \subseteq \text{int } \mathcal{H}$ . Thus, one has  $x_{\mathcal{H}} \in \Omega$  satisfying  $(F(x_{\mathcal{H}}) - y_0) \cap (-\mathcal{H} \setminus \{0\}) \neq \emptyset$ . Consequently, there exist  $y' - y_0 \in (F(x_{\mathcal{H}}) - y_0) \cap (-\mathcal{H} \setminus \{0\})$  and  $z' \in G(x_{\mathcal{H}}) \cap (-D)$  such that

$$\langle c^*, y' - y_0 \rangle + \langle d^*, z' \rangle < 0,$$

a contradiction.

(ii) *Benson-proper minimizer.* Similar to (i), one has  $\langle c^*, y - y_0 \rangle + \langle d^*, z \rangle \geq 0$  for all  $x \in \Omega$  and  $(y, z) \in (F, G)(x)$ . If  $x_0$  is not a global Benson-proper solution, then there exists

$$y \in (-C \setminus \{0\}) \cap \text{clcone}(F(\Omega) - y_0 + C).$$

Therefore, there exist  $x_n \in \Omega$ ,  $y_n \in F(x_n)$ ,  $z_n \in G(x_n) \cap (-D)$ ,  $c_n \in C$ , and  $t_n > 0$  such that  $\lim_{n \rightarrow \infty} t_n(y_n - y_0 + c_n) = y$ . Hence,  $\lim_{n \rightarrow \infty} \langle c^*, t_n(y_n - y_0 + c_n) \rangle = \langle c^*, y \rangle$ . As  $c^* \in C^{*i}$ ,  $y \in -C \setminus \{0\}$ , one has  $\langle c^*, y \rangle < 0$ . Moreover, as  $\langle c^*, c_n \rangle \geq 0$ , one has  $\langle c^*, y_n - y_0 \rangle < 0$  for sufficiently large  $n$ . Thus,

$$\langle c^*, y_n - y_0 \rangle + \langle d^*, z_n \rangle < 0.$$

This is a contradiction to the assumption. Hence, this completes the proof.  $\square$

## 5. APPLICATIONS TO DUALITY

Motivated by the work reported in [1, 7, 41, 42], we employ some results obtained in Section 4 to consider some duality schemes for global weak minimizer of the problem (P). In this section, we always consider  $x_0 \in \Omega$ ,  $y_0 \in F(x_0)$ ,  $z_0 \in G(x_0) \cap (-D)$ ,  $u \in X$ ,  $(v, w) \in (F, G)(u) + C \times D$ , and assume that  $(F, G)$  is  $C \times D$ -convex and  $E''_{(F, G)}(x - x_0)$  fulfills the domination property for all  $x \in \Omega$ .

### 5.1. Mond–Weir duality

We consider a Mond–Weir type dual problem, denoted by  $(D_{MW})$ , of (P) as follows

maximize  $y_0$

$$\text{s.t. } \langle c^*, y' \rangle + \langle d^*, z' \rangle \geq 0 \text{ for all } (y', z') \in D''_g(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(x - x_0), x \in \Omega \quad (5.1)$$

$$x_0 \in \Omega, \quad y_0 \in F(x_0), \quad z_0 \in G(x_0) \cap (-D), \quad (5.2)$$

$$c^* \in C^* \setminus \{0\}, \quad d^* \in D^*, \quad \text{and} \quad \langle d^*, z_0 \rangle = 0. \quad (5.3)$$

A point  $(c^*, d^*, x_0, y_0, z_0)$  is called a feasible solution of  $(D_{MW})$  if it satisfies (5.1)–(5.3). Let

$$\Delta := \{y \in F(x) \mid \text{there exist } c^*, d^*, z \text{ such that } (q^*, d^*, x, y, z) \text{ satisfies conditions (5.1)–(5.3)}\}.$$

A feasible solution  $(c^*, d^*, x_0, y_0, z_0)$  is a weak maximizer of  $\Delta$  if

$$(\Delta - y_0) \cap \text{int } C = \emptyset.$$

**Proposition 5.1** (Weak duality). *If  $(\bar{x}, \bar{y})$  is a feasible solution of (P) and  $(c^*, d^*, x_0, y_0, z_0)$  is a feasible solution of  $(D_{MW})$ , then  $\langle c^*, \bar{y} \rangle \geq \langle c^*, y_0 \rangle$ .*

*Proof.* Because  $(\bar{x}, \bar{y})$  is a feasible solution of (P), then  $\bar{x} \in \Omega$ ,  $\bar{y} \in F(\bar{x})$  and there is  $\bar{z} \in G(\bar{x}) \cap (-D)$ . It follows from Proposition 3.10 that

$$(F, G)(\bar{x}) - (y_0, z_0) \subseteq D''_g(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(\bar{x} - x_0) + C \times D.$$

Thus, there exist  $(y', z') \in D''_g(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(\bar{x} - x_0)$  and  $(c, d) \in C \times D$ , such that  $\bar{y} - y_0 = y' + c$  and  $\bar{z} - z_0 = z' + d$ . Moreover,  $(c^*, d^*, x_0, y_0, z_0)$  is a feasible solution of  $(D_{MW})$ , one gets

$$\langle c^*, \bar{y} - y_0 \rangle + \langle d^*, \bar{z} - z_0 \rangle = \langle c^*, y' + c \rangle + \langle d^*, z' + d \rangle = \langle c^*, y' \rangle + \langle d^*, z' \rangle + \langle c^*, c \rangle + \langle d^*, d \rangle \geq 0. \quad (5.4)$$

As  $\bar{z} \in -D$  and  $\langle d^*, z_0 \rangle = 0$ , one has  $\langle d^*, \bar{z} - z_0 \rangle \leq 0$ . It follows from this inequality and (5.3) that  $\langle c^*, \bar{y} - y_0 \rangle \geq 0$ , and hence one obtain  $\langle c^*, \bar{y} \rangle \geq \langle c^*, y_0 \rangle$ . Thus, the proof is complete.  $\square$

**Proposition 5.2** (Strong duality I). *Assume that  $(\bar{x}, \bar{y})$  is a feasible solution of (P),  $(c^*, d^*, x_0, y_0, z_0)$  is a feasible solution of  $(D_{MW})$ , and  $\langle c^*, \bar{y} \rangle = \langle c^*, y_0 \rangle$ , then  $(\bar{x}, \bar{y})$  is a global weak minimizer of (P) and  $(c^*, d^*, x_0, y_0, z_0)$  is a global weak maximizer of  $(D_{MW})$*

*Proof.* – Suppose that  $(\bar{x}, \bar{y})$  is not a global weak minimizer of (P), there are  $\hat{x} \in \Omega$  and  $\hat{y} \in F(\hat{x})$  such that  $\hat{y} - \bar{y} \in -\text{int } C$ . By  $c^* \in C^* \setminus \{0\}$ ,  $\langle c^*, \hat{y} \rangle < \langle c^*, \bar{y} \rangle = \langle c^*, y_0 \rangle$ , a contradiction to Proposition 5.1. Consequently,  $(\bar{x}, \bar{y})$  is a global weak minimizer.

– Now, we assume that  $(c^*, d^*, x_0, y_0, z_0)$  is not a global weak maximizer of  $(D_{MW})$ , there exist a feasible solution  $(\hat{c}^*, \hat{d}^*, x', y', z')$  of the problem  $(D_{MW})$  such that  $y' - y_0 \in \text{int } C$ . As  $c^* \in C^* \setminus \{0\}$ , we have  $\langle \hat{c}^*, y' \rangle > \langle c^*, y_0 \rangle$ . On the other hand,  $(x_0, y_0)$  is feasible solution of (P) and  $(\hat{c}^*, \hat{d}^*, x', y', z')$  is feasible solution of the problem  $(D_{MW})$ , according to Proposition 5.1,  $\langle \hat{c}^*, y_0 \rangle \geq \langle c^*, y' \rangle$ , a contradiction. Consequently,  $(c^*, d^*, x_0, y_0, z_0)$  is a global weak maximizer of  $(D_{MW})$ .  $\square$

**Proposition 5.3** (Strong duality II). *If  $(x_0, y_0)$  is a global weak minimizer of (P) and the (KRZCQ) in Theorem 4.2 holds, then there are  $c^* \in C^* \setminus \{0\}$  and  $d^* \in D^*$  such that  $(c^*, d^*, x_0, y_0, z_0)$  is a global weak maximizer of  $(D_{MW})$ .*

*Proof.* It follows from Theorem 4.2 that there are  $c^* \in C^* \setminus \{0\}$  and  $d^* \in D^*$  such that  $(c^*, d^*, x_0, y_0, z_0)$  is a feasible solution of  $(D_{MW})$ . By the same arguments as in Proposition 5.3, we conclude that  $(c^*, d^*, x_0, y_0, z_0)$  is a global weak maximizer of  $(D_{MW})$  and this completes the proof.  $\square$

Next we employ Theorem 4.11 to write down the following proposition.

**Proposition 5.4.** (Converse duality) *If  $(c^*, d^*, x_0, y_0, z_0)$  is a feasible solution of  $(D_{MW})$  then  $(x_0, y_0)$  is a global weak minimizer of (P).*

## 5.2. Wolfe duality

We define the Wolfe dual problem  $(D_W)$  as follows

$$\text{maximize } \psi(c^*, d^*, x_0, y_0, z_0) = \langle q^*, y_0 \rangle + \langle d^*, z_0 \rangle \quad (5.5)$$

$$\text{s.t. } \langle c^*, y' \rangle + \langle d^*, z' \rangle \geq 0 \text{ for all } (y', z') \in D_g''(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(x - x_0), x \in \Omega \quad (5.5)$$

$$x_0 \in \Omega, \quad y_0 \in F(x_0), \quad z_0 \in G(x_0) \cap (-D), \quad (5.6)$$

$$c^* \in C^* \setminus \{0\}, \quad d^* \in D^*. \quad (5.7)$$

A point  $(c^*, d^*, x_0, y_0, z_0)$  is a feasible solution of  $(D)_W$  if it satisfies (5.5)–(5.7).  $(c^*, d^*, x_0, y_0, z_0)$  is an optimal solution of  $(D)_W$  if for any feasible solution  $(c'^*, d'^*, x', y', z')$ ,

$$\psi(c^*, d^*, x_0, y_0, z_0) \geq \psi(c'^*, d'^*, x', y', z').$$

**Proposition 5.5** (Weak duality). *If  $(\bar{x}, \bar{y})$  is a feasible solution of (P) and  $(c^*, d^*, x_0, y_0, z_0)$  is a feasible solution of  $(D_W)$ , then  $\langle c^*, \bar{y} \rangle \geq \psi(c^*, d^*, x_0, y_0, z_0)$ .*

*Proof.* By virtue of Proposition 3.10, one has

$$(F, G)(\bar{x}) - (y_0, z_0) \subseteq D_g''(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(\bar{x} - x_0) + C \times D.$$

Hence, there exist  $(y', z') \in D_g''(F, G)(x_0, (y_0, z_0), u - x_0, (v - y_0, w - z_0))(\bar{x} - x_0)$  and  $(c, d) \in C \times D$ , such that  $\bar{y} - y_0 = y' + c$  and  $\bar{z} - z_0 = z' + d$ . Since  $(c^*, d^*, x_0, y_0, z_0)$  is a feasible solution of  $(D_W)$ , one gets

$$\langle c^*, \bar{y} - y_0 \rangle + \langle d^*, \bar{z} - z_0 \rangle = \langle c^*, y' + c \rangle + \langle d^*, z' + d \rangle = \langle c^*, y' \rangle + \langle d^*, z' \rangle + \langle c^*, c \rangle + \langle d^*, d \rangle \geq 0.$$

As  $\bar{z} \in -D$ ,  $\langle c^*, \bar{y} \rangle \geq \langle q^*, y_0 \rangle + \langle d^*, z_0 \rangle - \langle d^*, d \rangle \geq \langle q^*, y_0 \rangle + \langle d^*, z_0 \rangle$ . Hence, we get that

$$\langle c^*, \bar{y} \rangle \geq \psi(c^*, d^*, x_0, y_0, z_0).$$

$\square$

**Proposition 5.6** (Strong duality). *If  $(x_0, y_0)$  is a global weak minimizer of (P), where  $y_0 = 0$ , and the (KRZCQ) in Theorem 4.2 holds, then there are  $c^* \in C^* \setminus \{0\}$  and  $d^* \in D^*$  such that  $(c^*, d^*, x_0, y_0, z_0)$  is an optimal solution of  $(D_W)$ .*

*Proof.* By Theorem 4.2, there are  $c^* \in C^* \setminus \{0\}$  and  $d^* \in D^*$  with  $\langle d^*, z_0 \rangle = 0$  such that  $(c^*, d^*, x_0, y_0, z_0)$  is a feasible solution of  $(D_W)$ . Assume that  $(c^*, d^*, x_0, y_0, z_0)$  is not an optimal solution of  $(D_W)$ . Then, there exists a feasible solution  $(c'^*, d'^*, x', y', z')$  of  $(D_W)$  such that

$$\psi(c^*, d^*, x_0, y_0, z_0) < \psi(c'^*, d'^*, x', y', z').$$

As  $(x_0, y_0)$  is a feasible of (P), by Proposition 5.5, one has  $\psi(c'^*, d'^*, x', y', z') \leq \langle c'^*, y_0 \rangle$ . Moreover,  $\psi(c^*, d^*, x_0, y_0, z_0) = \langle c^*, y_0 \rangle$ , then  $\langle c^*, y_0 \rangle < \langle c'^*, y_0 \rangle$ , meaning that  $\langle c^* - c'^*, 0 \rangle < 0$ , a contradiction to  $y_0 = 0$ . The proof is complete.  $\square$

Finally we apply Theorem 4.11 to obtain the following result of converse duality.

**Proposition 5.7** (Converse duality). *If  $(c^*, d^*, x_0, y_0, z_0)$  is a feasible solution of  $(D_W)$  then  $(x_0, y_0)$  is a global weak minimizer of  $(P)$ .*

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