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CONSIGNMENT STOCK POLICY IN AN INTEGRATED VENDOR-BUYER MODEL FOR DETERIORATING ITEM WITH STOCK DEPENDENT DEMAND UNDER BUYER'S SPACE LIMITATION

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Abstract. In this paper, a single-vendor single-buyer integrated inventory model for a deteriorating item with consignment stock policy is developed, assuming that the market demand is stock dependent and there is space limitation on the buyer's storage capacity. Both equal and unequal shipments from the vendor to the buyer are considered. The effects of the buyer's space capacity on the average cost, shipment size, and production batch are studied through numerical example. It is deduced that production rate is the key factor to determine whether to use equal or unequal shipment strategy. Sensitivity analysis is carried out to establish the robustness of the solutions of the models developed.

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1. Introduction

Warehouse space limitation of a channel member has significant impact on the average cost and optimal decisions of a supply chain. To overcome this problem, channel members often use rented warehouse. Ghiami *et al.* [19] studied inventory models with/without rented warehouse and showed that rented warehouse of the buyer enhances the total cost of inventory. In consignment stock (CS) policy, the vendor manufactures the product and stores it at the buyer's warehouse with an agreement that the buyer need not to pay to the vendor until the items are sold in the market. Furthermore, the vendor remains the owner of the product until it is consumed, the inventory holding cost being carried by the buyer although. According to Sarker [42], this business strategy is particularly effective where the customer demand depends largely on stock display or variety of the product (*e.g.* clothing and furniture stores, bookstores, sports equipments and musical instrument stores, etc.). In vendor managed inventory (VMI) policy, the vendor controls the buyer's inventory and takes decision to manage the replenishment policy. Gümüs *et al.* [22] proved that the combination of VMI policy and CS policy is better than VMI policy or CS policy alone for business houses to minimize the effective cost or maximize the profit. Under VMI-CS policy, the vendor has to bear the responsibilities of scrapping unsold or expired products, stock level management, and periodic inventory review; he has all the information related to customer demand,

Keywords. Consignment stock, vendor managed inventory, space constraint, stock dependent demand, deterioration.

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shipment schedule, production run time, etc. so that he may store the product at the buyer's warehouse as much as possible. However, joint determination of shipment size as well as scheduling time in an integrated vendor-buyer model is often complicated but worth studying since these decisions under VMI-CS policy can significantly reduce the average cost of the integrated inventory system.

Market demand has been one of the primary concerns of decision makers as well as practitioners over decades. Researchers have assumed different kinds of demand pattern to reflect real world scenario under different conditions. On-hand stock display is one among various parameters which affect market demand. Empirical evidence of the dependence of demand on inventory of specific products has been provided by Wolfe [51], Silver and Peterson [45], and Koschat [32]. Assuming that the presence of greater quantity of an item in stock tends to attract more customers, numerous theoretical models have been developed till today [59]. In practice, stock-dependent demand is observed for many items such as fashion apparel, electronic items, etc. in supermarkets and convenience stores.

Deterioration is broadly defined as decay, damage, spoilage, evaporation, obsolescence, pilferage, and loss of entity or loss of marginal value of a commodity that results in decreasing usefulness from the original [50]. The deterioration of goods is a common phenomenon. Therefore, study of deteriorating inventory systems is very important as resulting losses due to deterioration cannot be neglected. Deterioration can be broadly classified into two categories: (a) perishability or fixed life period – when an item may be retained in stock for a fixed period of time with no loss in utility, but it becomes obsolete thereafter, and (b) continuous decay – when an item continuously loses its quality over time, usually measured as a fixed fraction of the on-hand stock, resulting in exponentially reduced utility. Products such as medicine, cosmetics, packaged foods, etc. carry an expiry date, whereas vegetables, fruits, flowers, radioactive materials, fuels, volatile liquids such as alcohol and gasoline, etc. deteriorate day-by-day as time goes on [38]. As reported by Cohen [13], Zyl [60] was one of the early researchers who studied deteriorating inventory.

Sajadieh *et al.* [41] proposed an integrated vendor-buyer model, where the buyer receives the shipped batch in the warehouse and a smaller lot in the display area. They showed that the total system profit can be maximized by joint determination of the vendor's production batch size, the buyer's warehouse ordering lot size and the replenished quantity from the buyer's warehouse to the display area. The idea of such a three-layer inventory system is very much realistic, since it is indeed an usual practice by the buyers to have a display area apart from the warehouse. In this paper, we aim to extend the above model in a VMI-CS scenario. It would be interesting to decide optimal shipment strategy of the vendor in terms of shipment size (equal or unequal), or even would be better if an optimal shipment ratio is determined. We consider a single-vendor single-buyer integrated model in which the storage of the buyer consists of back-office storage along with separate display area. Assuming the market demand to be stock dependent, the proposed model is developed and analyzed aiming at minimizing the average total cost. Two scenarios depending on the shipment size (equal or unequal) from the vendor to the buyer are investigated, and numerical results are compared to determine possible superiority of one model over the other. The effects of the buyer's space capacity on the average cost, shipment size, and production batch are also studied.

Rest of the paper is organized as follows. Relevant literature is reviewed and contribution of the present work is highlighted in Section 2. Section 3 presents notations and assumptions used to develop the proposed models. Mathematical models are developed in Section 4. The models are illustrated with the help of a numerical example in Section 5. Finally, Section 6 draws the conclusion and indicates future research directions.

2. Literature review

In this section, we primarily focus on the review of inventory literature which deals with consignment stock policy. We study the relevant research works in the following domains: deterioration, stock dependent demand, and warehouse space limitation – which would pave the bridge between the existing literature and the present work.

Consignment stocking policy

Braglia and Zavanella [7] were the first to propose a consignment inventory model for solving the joint economic lot size (JELS) problem, followed by Zanoni and Grubbstrom [55]. Valentini and Zavanella [47] described the technique, underlining the CS policy's potential benefits and pitfalls. Wang *et al.* [48] studied the consignment stock policy with revenue sharing contract, and showed how the channel performance critically depends on demand price elasticity and the buyer's share of channel cost. Gümüs *et al.* [22] discussed the impacts of consignment inventory and vendor-managed inventory (VMI) for a two-level supply chain. Zavanella and Zanoni [58] showed that the consignment stock policy works better than uncoordinated optimization one for an industrial case of a single-vendor and multi-buyer productive situation. Li *et al.* [37] developed a supply chain model under a CS policy with revenue sharing contract, and analyzed how the system parameters impact the optimal supply chain decisions and supply chain performance. Chen *et al.* [12] formulated the profit maximization problem and carried out equilibrium analysis under cooperative and non-cooperative settings for consignment and VMI policies in a distribution system. Ru and Wang [40] addressed the issue of appropriate controller of inventory in the supply chain under consignment contract. Battini *et al.* [4] dealt with a multi-echelon inventory system in which a single vendor supplies an item to multiple buyers considering space constraints in clients' plant warehouse, stock-out risk due to the variability of consumption, and obsolescence risk for the materials stored. Adida and Ratisoontorn [1] investigated how competition among retailers influences the supply chain decisions and profits under different consignment arrangements. Zanoni *et al.* [57] considered VMI under consignment with learning and forgetting effects. Wang *et al.* [49] addressed a single-manufacturer single-buyer supply chain problem for a single deteriorating product under consignment stock policy. Braglia *et al.* [8] proposed a relationship between the vendor and the buyer in a win-win situation with fixed-batch manufacturing process. Bylka [10] proposed non cooperative consignment stock strategies in supply chain. A single-vendor multi-retailer supply chain operating under a VMI contract was investigated by Hariga *et al.* [23]. Jaber *et al.* [30] extended the production, remanufacturing and waste disposal two-level supply chain model with CS policy as a coordination mechanism. Hu *et al.* [28] showed how the cost of the supply chain is affected by return policy under consignment stock policy. Braglia *et al.* [9] discussed safety stock management in a single-vendor single-buyer supply chain model with VMI under consignment agreement. Lee and Cho [33] designed a VMI contract with CS and stock-out cost sharing between a supplier and a retailer in a (Q, r) inventory system. Khan *et al.* [31] discussed an integrated vendor-buyer model for determining the optimal inventory policy accounting for quality inspection errors at the buyer's end and learning in the production at the vendor's end. Bazan *et al.* [5] introduced various managerial decisions pertaining to imperfect items, specifically reworking items and applying minor set-ups for restoration. They showed that imperfect items with CS increases batch size and reduces the number of batch shipments per cycle. Zahran *et al.* [54] studied the payment schemes for a two-level consignment stock supply chain system, and established that frequent and equal payments is better. Sarker [42] presented an extensive review on CS inventory models. Hariga *et al.* [25] considered integrated economic and environmental models for a one-vendor one-buyer supply chain problem under vendor managed consignment inventory arrangement in order to study the impacts of two carbon reduction policies (carbon cap and carbon tax policies) on supply chain wide costs and carbon emissions. Hu *et al.* [27] investigated the supply chain coordination under VMI-CS contract with wholesale price constraint and fairness consideration. Ben-Daya *et al.* [6] considered the environment-friendly approach of remanufacturing, in the context of a two-stage closed-loop supply chain comprised of a single vendor and multi-buyers where all parties involved operate under a centralized consignment stock agreement. Gharaei *et al.* [15] addressed an integrated multi-product, multi-buyer supply chain having real stochastic constraints under penalty, green, and quality control policies, differentiating between the holding costs for financial and non-financial components. They developed an outer approximation with equality relaxation and augmented penalty algorithm to determine the optimal batch-sizing policy. Uthayakumar and Ganesh Kumar [46] used Genetic Algorithm to solve the constrained mixed integer non-linear programming problem developed while studying five different stock control policies in supply chain management. Related research in this field may be found in [14, 16–18].

Deterioration

Persona *et al.* [39] were the first to take into account the effect of obsolescence in a consignment inventory model. Battini *et al.* [3] made an extension of the model developed by Persona *et al.* [39] considering new critical factors in industrial environments such as stock-out risk due to demand variability and obsolescence risk for the perishable materials. The model proposed by Battini *et al.* [4] was another extension of the model developed by Persona *et al.* [39], which dealt with a multi-echelon inventory system in which one vendor supplies a particular item to multiple buyers. Wang *et al.* [49] studied the effect of warehouse space limitation on deteriorating inventory under consignment stock policy. Hemmati *et al.* [26] presented a single-vendor single-buyer coordinated model with VMI-CS agreement for a deteriorating item under stock- and price-dependent demand.

Stock dependent demand

Inventory model with stock dependent demand under consignment stock policy is scarce. Giri and Bardhan [20] were the first to incorporate the idea of stock dependent demand in a JELS model. Zanoni and Jaber [56] investigated and compared different policies for CS with stock-dependent demand. Lee *et al.* [36] compared forward and backward stocking policies under consignment stock and stock dependent demand. Recently Sen *et al.* [43] studied the impact of consignment stock policy on supply chain with stock dependent demand and warehouse space limitation on both the vendor and the buyer.

Space limitation

Lee and Wang [35] first considered warehouse space limitation of the buyer in a consignment stocking model, and studied the impact of space constraint on the manufacturer's total cost. Huang and Chen [29] made an attempt to solve two reduced models given in Braglia and Zavanella [7]. Battini *et al.* [3] extended the work of Braglia and Zavanella [7] by considering the buyer's space limitation as well as relaxing the assumption of deterministic demand. Yi and Sarker [52] provided a procedure to solve the generalized CS-k model using hybrid meta-heuristic algorithm. Yi and Sarkar [53] studied another model considering consignment stock policy, space limitation and controllable lead time. Hariga *et al.* [24] studied a vendor managed inventory model with unequal shipment frequencies having storage capacity limitation under different demand rates arriving at multiple retailers. Giri and Bardhan [20] developed a JELS model with stock dependent demand under space constraint of the buyer. Considering stock-out cost sharing, Lee *et al.* [34] examined supply chain coordination problem in VMI with constant demand under limited storage capacity. Shekarabi *et al.* [44] modeled a lot-sizing problem for an integrated multi-level multi-wholesaler supply chain under limited warehouse space.

Unequal shipment size

Bylka [10] proposed non-cooperative consignment stock strategies in supply chain, where two different sizes of shipment batch were considered. The similar idea was also applied by Bylka and Górny [11] who showed that a generalized CS performs better than the classical one. Giri *et al.* [21] developed an inventory model with unequal shipments under imperfect production process, and compared the results of two cases depending on whether rework is done or not.

Motivation of the current research

While surveying the existing literature, it has been found that research incorporating the idea of unequal shipments in VMI-CS policy is scarce. Although a few researchers did incorporate the concept of unequal shipments, they didn't derive the optimal shipment ratio which appears worth examining under some realistic business scenarios. In this paper, we extend the model proposed by Sajadieh *et al.* [41] in a VMI-CS scenario under unequal shipment policy. We also consider space limitation for both the vendor and the buyer. We prefer to study stock-dependent demand pattern, and include the consideration of deterioration of the on-hand inventory. Although Braglia and Zavanella [7] and Sajadieh [41] considered zero-inventory reordering level at the display area, we relax this constraint in this paper, following Zanoni and Jaber [56]. Till date, no study has been conducted to simultaneously consider stock dependent demand, consignment inventory policy, deteriorating effect, and warehouse space constraints in a three-level supply chain where the buyer has two

TABLE 1. Notations.

P	:	constant production rate of the vendor (units/year)
α_1	:	constant rate of deterioration at the vendor's warehouse
β	:	constant rate of deterioration at the buyer's warehouse
δ	:	constant deterioration rate at the buyer's display area
λ	:	scaling constant to determine the shipment size
h_v	:	unit holding cost at the vendor's warehouse (\$/unit/year)
h_b	:	unit holding cost at the buyer's warehouse (\$/unit/year)
h_d	:	unit holding cost at the buyer's display area (\$/unit/year)
q	:	replenishment batch size from the buyer's warehouse to the buyer's display area (units/replenishment)
x	:	constant buffer stock at the display area
$I_b(t)$	:	buyer's warehouse inventory level at time t
$I_v(t)$	:	vendor's inventory level at time t
$I_{bd}(t)$	:	buyer's display area inventory level at time t
S_v	:	setup cost of the vendor per cycle (\$/setup)
S_b	:	ordering cost of the buyer per cycle (\$/order)
c_d	:	per unit deterioration cost (\$/unit/year)
$W_{b_{\max}}$	:	maximum capacity at the buyer's warehouse (in units)
t_i	:	time length to produce $\lambda^{i-1}Q$ quantity of items
T	:	length of the business cycle (in years)
TCB	:	total cost of the buyer in a business cycle
TCV	:	total cost of the vendor in a business cycle
AC	:	average cost for the integrated system
n	:	total number of shipments in a cycle, a decision variable
m	:	number of unequal shipments
T_d	:	time to sell q quantity of the product in the display area

separate storage areas – the main warehouse and the display area. The contribution of the present work with respect to the existing literature is three-fold. Firstly, this is the first attempt towards finding optimal shipment ratio in an unequal shipment scenario under VMI-CS strategy. Secondly, this is also the first work considering space limitation for both the channel members under stock dependent demand scenario. Finally, deterioration of stored items is considered in each of the three inventories.

3. Notations and assumptions

The following notations are used throughout the paper while developing and analyzing the proposed model:

Assumptions

The following assumptions are made to develop the proposed model:

(1) A two-echelon supply chain problem with one vendor and one buyer (*i.e.* a bilateral monopoly) under stock dependent demand is considered. The demand rate is assumed to be of the form $D(t) = \alpha I^{\beta}(t),\ \ \alpha > 0$, $0 \le \beta < 1$, where $I(t)$ denotes the inventory level of the buyer at time t. Some of the advantages of this kind of demand pattern, as mentioned in Baker and Urban [2], are diminishing returns (marginal increase in demand rate will decrease for larger values of inventory level), richness (good approximate demand in many practical situations), and intrinsic linearity (linear regression can be used for parameter estimation after taking logarithm).
(2) The stock is subject to deterioration, the rates being different at different echelons.
(3) The production rate is higher than the demand rate.
(4) Shortages are not allowed.
(5) Lead time is zero.

4. The model

In this section, we develop two models M1 and M2. In model M1, the buyer receives the ordered quantity from the vendor in unequal shipments where consecutive shipment sizes bear a predetermined ratio λ. In model M2, the successive shipments from the vendor to the buyer's warehouse are equal. Clearly, M2 is a special case of M1 with $\lambda = 1$.

4.1. Model M1

A graphical representation of model M1 is shown in Figure 1.

We first calculate the total inventory of the vendor in a cycle, followed by the total cost of the vendor. The following differential equation governs the inventory level of the vendor at any time t.

$$\frac{\mathrm{d}I_v(t)}{\mathrm{d}t} = P - \alpha_1 I_v(t) \quad \text{with} \quad I_v(0) = 0.$$

The solution of the above differential equation is obtained as

$$I_v(t) = \frac{P}{\alpha_1}\left(1 - e^{-\alpha_1 t}\right), \quad 0 \le t \le t_1.$$

Since $I_v(t_1) = Q$, we obtain

$$Q = \frac{P}{\alpha_1}\left(1 - e^{-\alpha_1 t_1}\right). \tag{4.1}$$

According to our assumption, the vendor delivers to the buyer in unequal shipments. Let the first shipment size be Q and the successive shipment sizes be $Q,\ \lambda Q,\ \lambda^2 Q,\ \ldots,\ \lambda^{m-1}Q$, where m denotes the number of shipments within a business cycle T. Then, in view of the definition of t_i mentioned in Table 1, we may at once derive the following:

$$\begin{aligned} \lambda^{i-1}Q &= \frac{P}{\alpha_1}\left(1 - e^{-\alpha_1 t_i}\right), \quad i = 1, 2, 3, \ldots, m, \\ \text{or,} \quad t_i &= -\frac{1}{\alpha_1}\ln\left(1 - \frac{\alpha_1 \lambda^{i-1} Q}{P}\right). \end{aligned}$$

Using (4.1), we get each t_i as function of t_1 only:

$$t_i = -\frac{1}{\alpha_1}\ln\left(1 - \lambda^{i-1} + \lambda^{i-1}e^{-\alpha_1 t_1}\right). \tag{4.2}$$

From above, the total production run time is obtained as

$$t_1 + t_2 + t_3 + \cdots + t_m = t_1 - \frac{1}{\alpha_1}\sum_{i=1}^{m-1}\ln\left(1 - \lambda^i + \lambda^i e^{-\alpha_1 t_1}\right).$$

The total quantity shipped from the vendor to the buyer in a cycle is given by

$$Q + \lambda Q + \lambda^2 Q + \lambda^3 Q + \cdots + \lambda^{m-1}Q = \frac{P}{\alpha_1} - \frac{P}{\alpha_1}e^{-\alpha_1 t_1} + \frac{P}{\alpha_1} - \frac{P}{\alpha_1}e^{-\alpha_1 t_2} + \cdots + \frac{P}{\alpha_1} - \frac{P}{\alpha_1}e^{-\alpha_1 t_m}.$$

Simplifying, we get

$$e^{-\alpha_1 t_1} + e^{-\alpha_1 t_2} + e^{-\alpha_1 t_3} + \cdots + e^{-\alpha_1 t_m} = m - \frac{\alpha_1 Q}{P}\left(\frac{\lambda^m - 1}{\lambda - 1}\right). \tag{4.3}$$

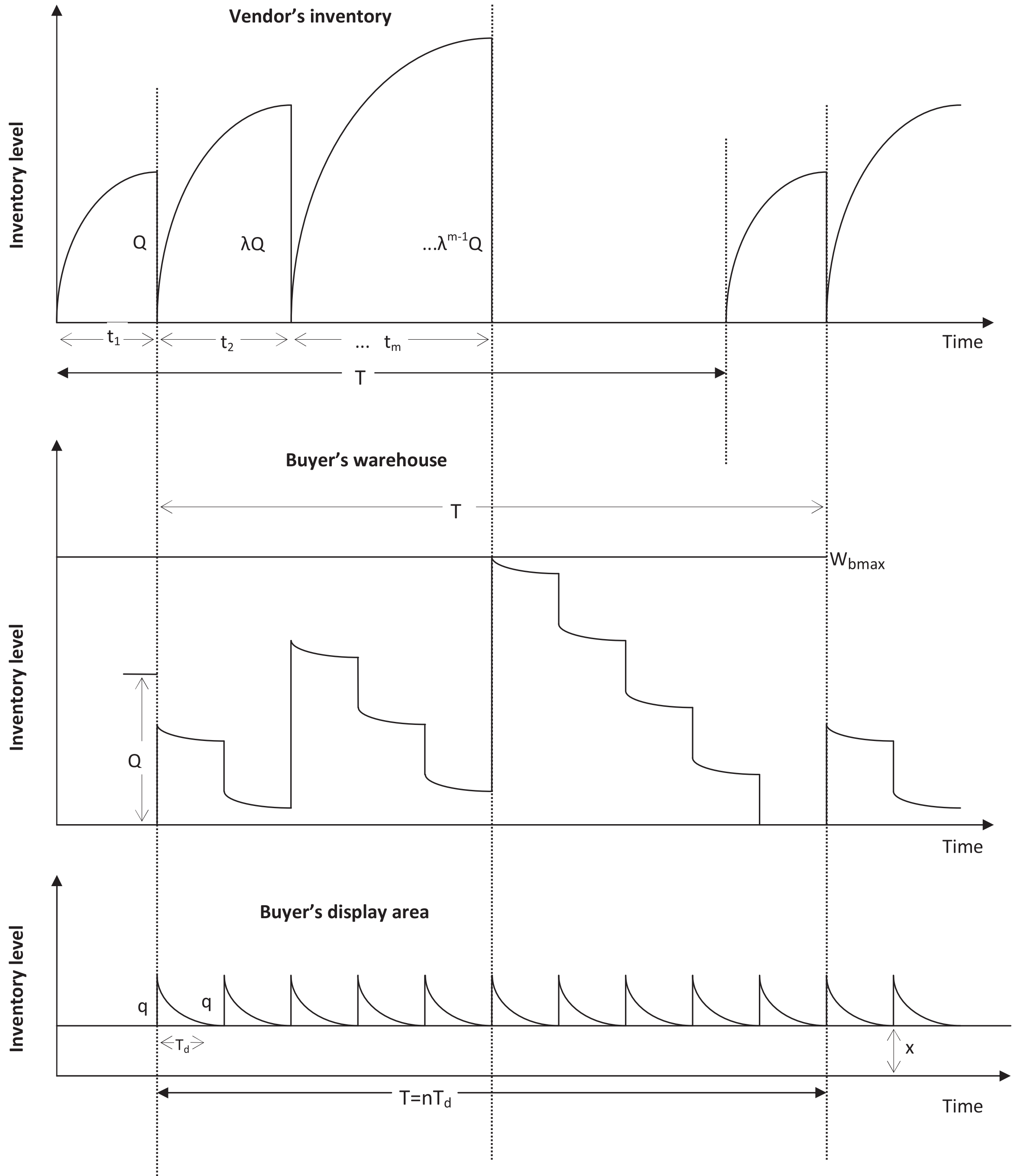


Figure 1. Graphical representation of model M1.

The total inventory held by the vendor in a cycle is obtained as (see Appendix A for detailed calculation)

$$P_1 = \frac{P}{\alpha_1^2}\left[m - \left(1 - e^{-\alpha_1 t_1}\right)\left(\frac{\lambda^m - 1}{\lambda - 1}\right)\right] + \frac{P}{\alpha_1}\left[t_1 - \frac{1}{\alpha_1}\sum_{i=1}^{m-1}\ln\left(1 - \lambda^i + \lambda^i e^{-\alpha_1 t_1}\right)\right] - \frac{mP}{\alpha_1^2}.$$

To calculate the total inventory cost associated with the buyer, we calculate costs associated with the warehouse and the display area separately. The inventory level of the buyer's warehouse at time t is governed by the differential equation $\frac{\mathrm{d}I_b(t)}{\mathrm{d}t} = -\beta I_b(t)$ with $I_b(0) = Q - q$, from which we get

$$I_b(t) = (Q - q)\, e^{-\beta t}, \quad 0 \le t \le T_d. \tag{4.4}$$

At $t = T_d$, the inventory level is $I_b(T_d) = (Q - q)\, e^{-\beta T_d} - q$.

Also, at the n_1^{th} shipment of q quantity from the buyer's warehouse to the buyer's display area, the vendor transfers λQ quantity to the buyer's warehouse. The inventory level in the interval $(n_1 - 1)\, T_d \le t \le n_1 T_d$ is $I_b(t) = \left(Q - \sum_{i=0}^{n_1 - 1} q e^{i\beta T_d}\right) e^{-\beta t}$. Also, we have

$$I_b(t) = \left(Q - \sum_{i=0}^{n_1} q e^{i\beta T_d} + \lambda Q e^{n_1 \beta T_d}\right) e^{-\beta t}, \quad n_1 T_d \le t \le (n_1 + 1)\, T_d.$$

In a similar way, shipment of $\lambda^2 Q$ quantity from the vendor to the buyer's warehouse takes place at $(n_1 + n_2)^{th}$ shipment of q quantity from the warehouse to the display area.

Generalizing the process, we deduce that the last shipment of size $\lambda^{(m-1)}Q$ is shipped by the vendor at $(n_1 + n_2 + \cdots + n_{m-1})^{th}$ shipment of q quantity to the display area. The inventory level for the time interval $\sum_{j=1}^{m-1} n_j T_d \le t \le \left(\sum_{j=1}^{m-1} n_j + 1\right) T_d$ is derived as

$$I_b(t) = \left(Q - \sum_{i=0}^{\sum_{j=1}^{m-1} n_j} q e^{i\beta T_d} + \sum_{j=1}^{m-1} \lambda^j e^{\sum_{k=1}^{j} n_k \beta T_d}\right) e^{-\beta t}.$$

Inventory level reaches its maximum storage capacity $W_{b_{\max}}$ at time point $\sum_{j=1}^{m-1} n_j T_d$ when the vendor stops the production, from which we have

$$I_b\left(\sum_{i=0}^{m-1} n_i T_d\right) = W_{b_{\max}} = \left(Q - \sum_{i=0}^{\sum_{j=1}^{m-1} n_j} q e^{i\beta T_d} + Q\sum_{j=1}^{m-1} \lambda^j e^{\sum_{k=1}^{j} n_k \beta T_d}\right) e^{-\beta \sum_{j=1}^{m-1} n_j T_d}. \tag{4.5}$$

Proposition 4.1. $n_i T_d = t_{i+1}$; n_i can be expressed as

$$n_i = -\frac{1}{\alpha_1 T_d}\ln\left(1 - \lambda^i + \lambda^i e^{-\alpha_1 t_1}\right) \quad i = 1, 2, 3 \ldots, m-1, \quad n_i \in \mathbb{N}.$$

Proposition 4.2. The value of n_i given in Proposition 4.1 is well defined for any value of λ.

Since the buyer ships the product in $\left(n - \sum_{i=1}^{m-1} n_i\right)$ number of batches from the warehouse to the display area after reaching the maximum storage capacity at his warehouse, the inventory level of the buyer's warehouse in the interval $iT_d \le t \le (i+1)T_d$ is given by

$$I_b(t) = \left(W_{b_{\max}} - q\sum_{j=1}^{i} e^{j\beta T_d}\right) e^{-\beta t}, \quad i = 0, 1, \ldots, \left(n - \sum_{j=1}^{m-1} n_j - 2\right).$$

Total inventory held at the buyer's warehouse $= W_1 = W_{11} + W_{12}$, where

$$W_{11} = \frac{1}{\beta}\left(Q - Qe^{-\sum_{j=1}^{m-1} n_j \beta T_d} - q\sum_{j=1}^{m-1} n_j + q\sum_{i=1}^{\sum_{j=1}^{m-1} n_j} e^{-i\beta T_d} + \lambda Q\left(\frac{\lambda^{m-2}-1}{\lambda-1}\right)\right) + \frac{1}{\beta}\left(\lambda Q e^{-\sum_{j=1}^{m-1} n_j \beta T_d}\sum_{l=1}^{m-2}\lambda^{l-1} e^{\sum_{j=1}^{l} n_j \beta T_d}\right),$$

which is the total inventory held during production time, and

$$W_{12} = \frac{1}{\beta}\left(W_{b_{\max}} - W_{b_{\max}} e^{-\left(n-\sum_{j=1}^{m-1} n_j - 1\right)\beta T_d} + qe^{-\beta T_d}\sum_{i=0}^{\left(n-\sum_{j=1}^{m-1} n_j - 3\right)} e^{-i\beta T_d} - \left(n - \sum_{j=1}^{m-1} n_j - 2\right) q\right),$$

which is the total inventory for the rest of the period. We refer the readers to Appendices B and C for detailed calculations.

Our next aim is to calculate the total inventory of the buyer's display area where the demand is stock dependent and items in inventory deteriorate with time. The inventory level at any time t in the interval $[0, T_d]$ is governed by the differential equation

$$\frac{\mathrm{d}I_{bd}(t)}{\mathrm{d}t} = -\alpha I_{bd}^{\beta_1}(t) - \delta I_{bd}(t), \quad \text{with} \quad I_{bd}(0) = q + x.$$

Solving, we get

$$I_{bd}(t) = \left[-\frac{\alpha}{\delta} + \left(\frac{\alpha}{\delta} + (q+x)^{1-\beta_1}\right) e^{-\delta(1-\beta_1)t}\right]^{\frac{1}{1-\beta_1}}, \ t \in [0, T_d]. \tag{4.6}$$

Further, using $I_{bd}(T_d) = x$, we get

$$T_d = \frac{1}{\delta(1-\beta_1)} \ln\left(\frac{\alpha + \delta(q+x)^{1-\beta_1}}{\alpha + \delta x^{1-\beta_1}}\right). \tag{4.7}$$

Using Maclaurin series and approximating, equation (4.6) takes the form

$$I_{bd}(t) = \left[-\frac{\alpha}{\delta} + \left(\frac{\alpha}{\delta} + (q+x)^{1-\beta_1}\right)(1 - \delta(1-\beta_1)t)\right]^{\frac{1}{1-\beta_1}}.$$

Total inventory P_2 in the display area is thus obtained as

$$\begin{aligned} P_2 &= n\int_0^{T_d} I_{bd}(t)\mathrm{d}t \\ &= n\int_0^{T_d}\left[-\frac{\alpha}{\delta} + \left(\frac{\alpha}{\delta} + (q+x)^{1-\beta_1}\right)(1-\delta(1-\beta_1)t)\right]^{\frac{1}{1-\beta_1}}\mathrm{d}t \\ &= \frac{n}{(2-\beta_1)\left(\alpha + \delta(q+x)^{1-\beta_1}\right)}\left[(q+x)^{2-\beta_1} - \left[-\frac{\alpha}{\delta} + \left(\frac{\alpha}{\delta} + (q+x)^{1-\beta_1}\right)(1-\delta(1-\beta_1)T_d)\right]^{\frac{2-\beta_1}{1-\beta_1}}\right]. \end{aligned}$$

We refer the readers to Appendix D for detailed calculation.

Since the deterioration is considered in the buyer's warehouse as well as in the display area, we need to calculate the amount of deteriorated items in order to determine the deterioration cost associated with the buyer. The amount of deteriorated items in the buyer's warehouse is $\left(\sum_{i=0}^{m-1}\lambda^i Q - W_1\right)$.

The amount of sold items P_3 in the display area is

$$P_3 = \frac{n\alpha}{\left[\alpha + \delta\left(q+x\right)^{1-\beta_1}\right]}\left[(q+x) - \left\{-\frac{\alpha}{\delta} + \left(\frac{\alpha}{\delta} + (q+x)^{1-\beta_1}\right)\left(1-\delta\left(1-\beta_1\right)T_d\right)\right\}^{\frac{1}{1-\beta_1}}\right],$$

so that the amount of deteriorated items at the buyer's custody is $\left[\left(\sum_{i=0}^{m-1}\lambda^i Q - W_1\right) + (nq - P_3)\right]$. We refer the readers to Appendix E for detailed calculation.

The total holding cost for the buyer is, therefore, $\text{HCB} = h_b W_1 + h_d P_2$.

The total cost at the buyer's side is obtained as the sum of ordering cost, holding cost and deterioration cost, *i.e.*, $\text{TCB} = S_b + h_b W_1 + h_d P_2 + c_d\left[\left(\sum_{i=0}^{m-1}\lambda^i Q - W_1\right) + (nq - P_3)\right]$.

The total holding cost on the vendor side is $h_v P_1$.

The amount of deteriorated items may be obtained from the difference between produced and delivered quantities, which is $\left[P\sum_{i=1}^{m} t_i - \sum_{i=0}^{m-1}\lambda^i Q\right]$.

The total cost of the vendor is the sum of set-up cost, holding cost, and deterioration cost, *i.e.* $\text{TCV} = S_v + h_v P_1 + c_d\left[P\sum_{i=1}^{m} t_i - \sum_{i=0}^{m-1}\lambda^i Q\right]$.

The average cost for the integrated system is thus obtained as $\text{AC}(t_1, m, n, T_d) = \frac{1}{T}(\text{TCB} + \text{TCV})$, where $T = nT_d$.

It is clear from equations (4.1) and (4.2), and Proposition 4.1 that Q, t_is and n_is depend solely on t_1. Also, equation (4.3) is an identity involving m and t_i, and equation (4.5) is a constraint on the objective function AC. The objective is then to Minimize $\text{AC}(t_1, m, n, T_d)$ subject to equations (4.1), (4.2), (4.3), (4.5), and (4.7); m and n both are positive integers, and $m \geq 2$.

Due to complicated form of the objective function, it is not possible to prove the convexity of the objective function analytically.

4.2. Model M2

The graphical representation of model M2 is shown in Figure 2.

Here a particular case of the previous model is exclusively studied where all the shipment sizes are equal, *i.e.* $\lambda = 1$. Consequently the time lengths t_i's are equal and we take $t_i = t_p$, for all i. Proceeding similarly as in M1, we obtain

$$I_v(t) = \frac{P}{\alpha_1} - \frac{P}{\alpha_1}e^{-\alpha_1 t}, \quad 0 \leq t \leq t_p$$
$$\text{and} \quad Q = \frac{P}{\alpha_1}\left(1 - e^{-\alpha_1 t_p}\right).$$

Since there are m regular shipments in M2, the total inventory $(P_1^{'})$ on the vendor side during one complete cycle is determined as

$$P_1^{'} = m\int_0^{t_p} I_v(t)\mathrm{d}t = m\left\{\frac{Pt_p}{\alpha_1} + \frac{P}{\alpha_1^2}\left(e^{-\alpha_1 t_p} - 1\right)\right\}.$$

Next, we calculate the total inventory $W_1^{'}$ held at the buyer's warehouse. The buyer's warehouse space is completely utilized as soon as he receives the mth shipment from the vendor. The next shipment of size q is received from the vendor at the time of n_1^{th} shipment from warehouse to the display area, and this is continued

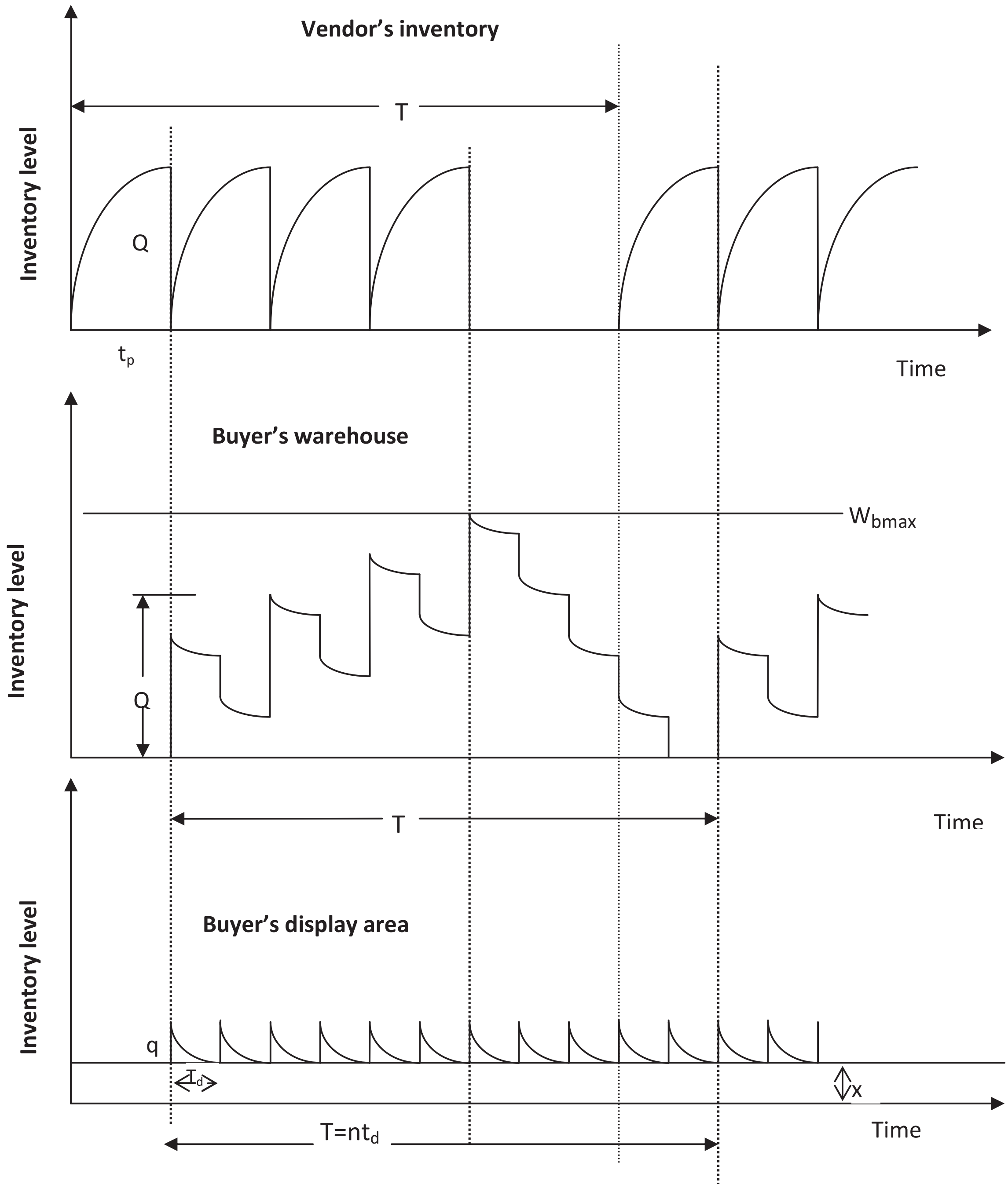


FIGURE 2. Graphical representation of model M2.

on regular basis until the warehouse is completely utilized. The buyer's inventory level at any time t is governed by the differential equation $\frac{\mathrm{d}I_b(t)}{\mathrm{d}t} = -\beta I_b(t)$ with $I_b(0) = Q - q$, from which we get

$$I_b(t) = (Q - q)\, e^{-\beta t}, \quad t \in [0, T_d]\,.$$

Let $I_b^-(iT_d)$ and $I_b^+(iT_d)$ be the inventory levels of the buyer's warehouse right before and after delivery of batch size q at time iT_d, $i = 1, 2, \ldots, n$, respectively. The inventory at iT_d can then be expressed as $I_b^+(iT_d) = I_b^-(iT_d) + q$.

The inventory level in the interval $[iT_d, (i+1)\,T_d]$, $i = 0, 1, 2, \ldots, n_1 - 1$ is $I_b(t) = \left(Q - q\sum_{j=0}^{i} e^{j\beta T_d}\right) e^{-\beta t}$.

The inventory level in $[n_1 T_d, (n_1+1)\,T_d]$ is thus $I_b(t) = \left(Q\left(1 + e^{n_1\beta T_d}\right) - q\sum_{i=0}^{n_1} e^{i\beta T_d}\right) e^{-\beta t}$.

Generalizing the above idea of regular shipment at a fixed time interval, the inventory level of the buyer's warehouse at any time t in the interval $[(m-1)n_1]\,T_d \le t \le [(m-1)\,n_1 + 1]\,T_d$ is obtained as

$$I_b(t) = \left[Q\sum_{i=0}^{m-1} e^{in_1\beta T_d} - q\sum_{i=0}^{(m-1)n_1} e^{i\beta T_d}\right] e^{-\beta t}.$$

Since the buyer's warehouse space capacity $W_{b_{\max}}$ is fully utilized at the time $(m-1)n_1T_d$, $W_{b_{\max}}$ can be expressed as

$$W_{b_{\max}} = \left[Q\sum_{i=0}^{m-1} e^{in_1\beta T_d} - q\sum_{i=0}^{(m-1)n_1} e^{i\beta T_d}\right] e^{-(m-1)n_1\beta T_d}. \tag{4.8}$$

To calculate the inventory level in the next part of the warehouse, we rename the time point $(m-1)n_1T_d$ as $t = 0$. In this part, the buyer ships q quantity to the display area for $n - (m-1)n_1$ times. The inventory level at any time t in the interval $[(i-1)T_d \le t \le iT_d]$ is given by

$$I_b(t) = \left[W_{b_{\max}} - q\sum_{j=1}^{i-1} e^{j\beta T_d}\right] e^{-\beta t}, \quad i = 1, 2, \ldots, (n - (m-1)n_1 - 1). \tag{4.9}$$

The inventory level at the time point $(n - (m-1)n_1 - 1)T_d$ is q. Using this, we get

$$W_{b_{\max}} = q\sum_{i=1}^{n-(m-1)n_1-1} e^{i\beta T_d}. \tag{4.10}$$

Next, we calculate the total inventory W_1' in the buyer's warehouse. W_1' comprises of two parts W_{11}' and W_{12}', where the first part denotes the total inventory in the interval $[0, (m-1)n_1T_d]$, and second part denotes the total inventory in the interval $[0, (n - (m-1)n_1)T_d]$. Explicit expressions for W_{11}' and W_{12}' are given by

$$W_{11}^{'} = \frac{1}{\beta}\left[(m-1)Q - Qe^{-(m-1)n_1\beta T_d}\sum_{i=o}^{m-2} e^{in_1\beta T-d} - (m-1)q + q\sum_{(i=1)}^{(m-1)n_1} e^{-i\beta T_d}\right], \quad \text{and}$$

$$W_{12}^{'} = \frac{1}{\beta}W_{b_{\max}}\left[1 - e^{-(n-(m-1)n_1-1)\beta T_d}\right] + \frac{1}{\beta}\left[q\sum_{i=1}^{n-(m-1)n_1-2} e^{i\beta T_d} - (n - (m-1)n_1 - 2)q\right].$$

The readers are referred to Appendices F and G for detailed calculations.

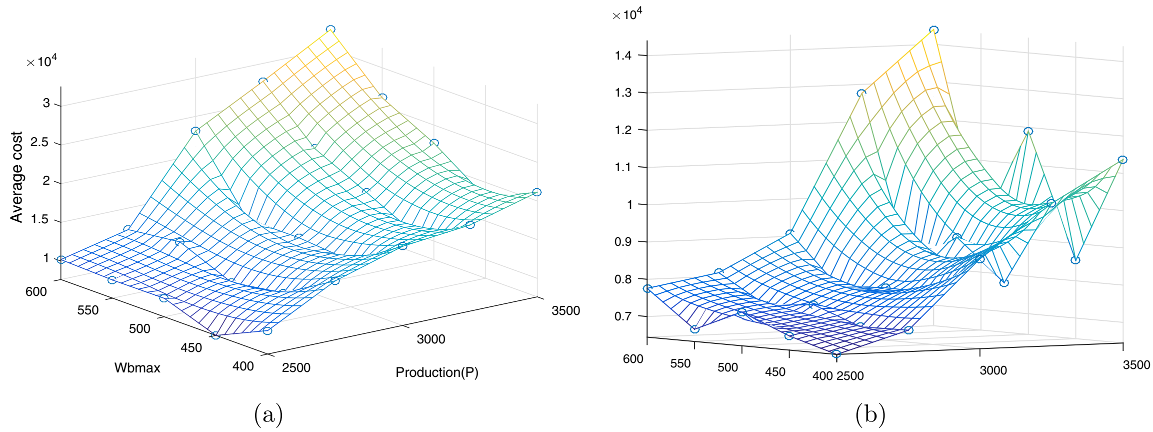


FIGURE 3. Changes in average costs with varying capacity of the buyer's warehouse and production rate. (a) Model M1. (b) Model M2.

The inventory level at the buyer's display area at any time t in the interval $[0, T_d]$ is governed by the differential equation

$$\frac{\mathrm{d}I_{bd}(t)}{\mathrm{d}t} = -\alpha I_{bd}^{\beta_1}(t) - \delta I_{bd}(t), \quad \text{with} \quad I_{bd}(0) = q + x.$$

The total cost on the buyer's side is obtained as the sum of ordering cost, holding cost and deterioration cost, *i.e.*, $\mathrm{TCB} = S_b + h_b W_1' + h_d P_2 + c_d \left[\left(mQ - W_1'\right) + (nq - P_3)\right]$.

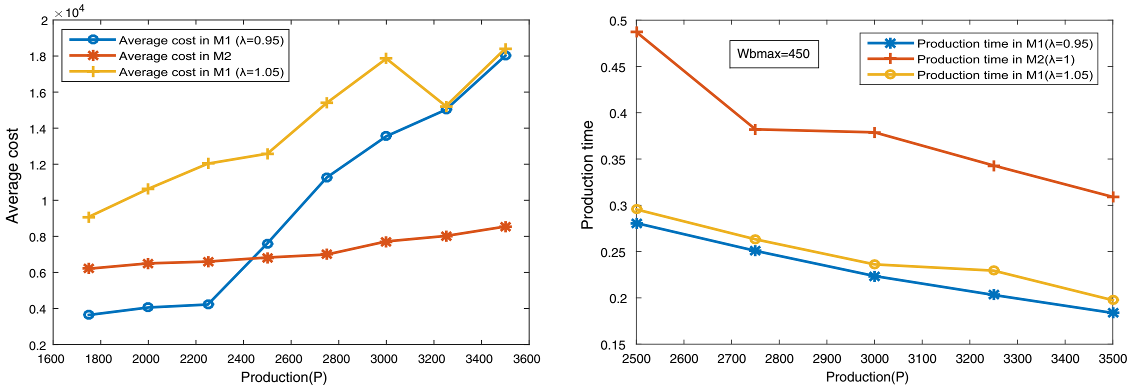
The total cost on the vendor side is the sum of set-up cost, holding cost, and deterioration cost, *i.e.* $\mathrm{TCV} = s_v + h_v P_1' + c_d \left[mPt_p - mQ\right]$.

The joint total cost of the vendor-buyer system is $TC = \mathrm{TCB} + \mathrm{TCV}$. Our objective is
Minimize $\mathrm{AC} = \frac{TC}{T}$, subject to $t_p > 0$, and $T_d > 0$, where n, m are integers, and $T = nT_d$.

5. Numerical illustration

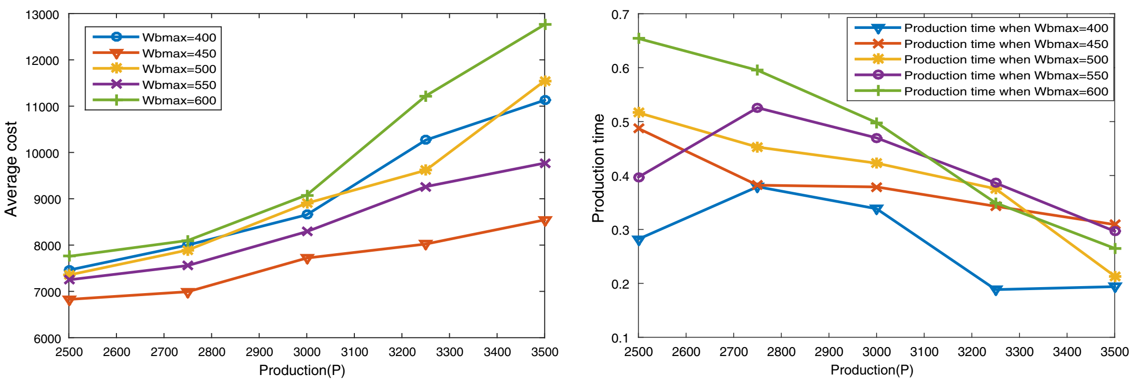
In this section, we demonstrate the proposed models and exhibit managerial insights through numerical illustration. The following parameter-values are considered: $P = 3500$, $\alpha_1 = 0.05$, $\beta = 0.05$, $\delta = 0.05$, $h_b = 5$, $h_d = 5$, $h_v = 5$, $x = 90$, $S_b = 25$, $S_v = 3000$, $c_d = 3$, $W_{b_{\max}} = 400$, $\alpha = 100$, $\beta_1 = 0.3$, in appropriate units. It is observed that, for a wide range of parameter-values, both the models continue to produce feasible optimal results, establishing the stability of the solution of the system under consideration. The sensitivity analysis provides important managerial insights while determining optimal decisions of the developed models for different inputs. The findings along with managerial insights drawn from the Figures 3–8 are summarized below.

– Figure 3 depicts that, for each pre-specified production rate, the cost function follows a convex pattern, establishing that there is an optimum space capacity to be used by the buyer for both equal and unequal shipment strategies. Keeping in mind that the model is developed utilizing the entire space available for the buyer, it may be concluded that managers should not blindly plan for utilizing capacity whatever is available to them. The proposed model is applicable only when the available space is lesser than the optimum one, for otherwise the storage of more finished products with higher total holding cost would increase the average cost. In general, managers should simulate production strategy for an initial pre-specified space limit to be used, and keep on increasing space limit as long as it results in lesser average cost, and stop when the average



(a) (b)

FIGURE 4. Changes in average cost and production run-time with varying production rate in M1 & M2 ($W_{b_{\max}} = 450$). (a) Changes in average costs. (b) Changes in production run-time.



(a) (b)

FIGURE 5. Changes in average cost and production run-time with varying production rate in model M2 for different values of warehouse capacity. (a) Changes in average costs. (b) Changes in production run-time.

cost starts increasing; if the average cost keeps on reducing, then only it would be wise to utilize the entire space.

– When the production rate is sufficiently low, unequal shipment strategy with diminishing shipment sizes (*i.e.* $\lambda < 1$) would be beneficial to business managers. However, if the rate goes beyond a threshold level ($P = 2450$ in Fig. 4a), equal shipment strategy should be adopted. Figure 6 exhibits that this observation is valid even for a wide range of space limitation.
– Figure 5a shows that a faster production rate compels managers to store more items for longer period of time thereby increasing the average cost. Under such a situation, production process has to be stopped after a comparatively shorter production run time (Fig. 5b). A larger amount of deteriorated items also plays partial role in enhancing the average cost. Managers should invest in reducing production rate if it is controllable at all. Figure 5a also allows us to determine the optimum space capacity as 450 under equal shipment strategy for specific parameter-values as mentioned. Also, a higher value of λ compels to run production for a longer

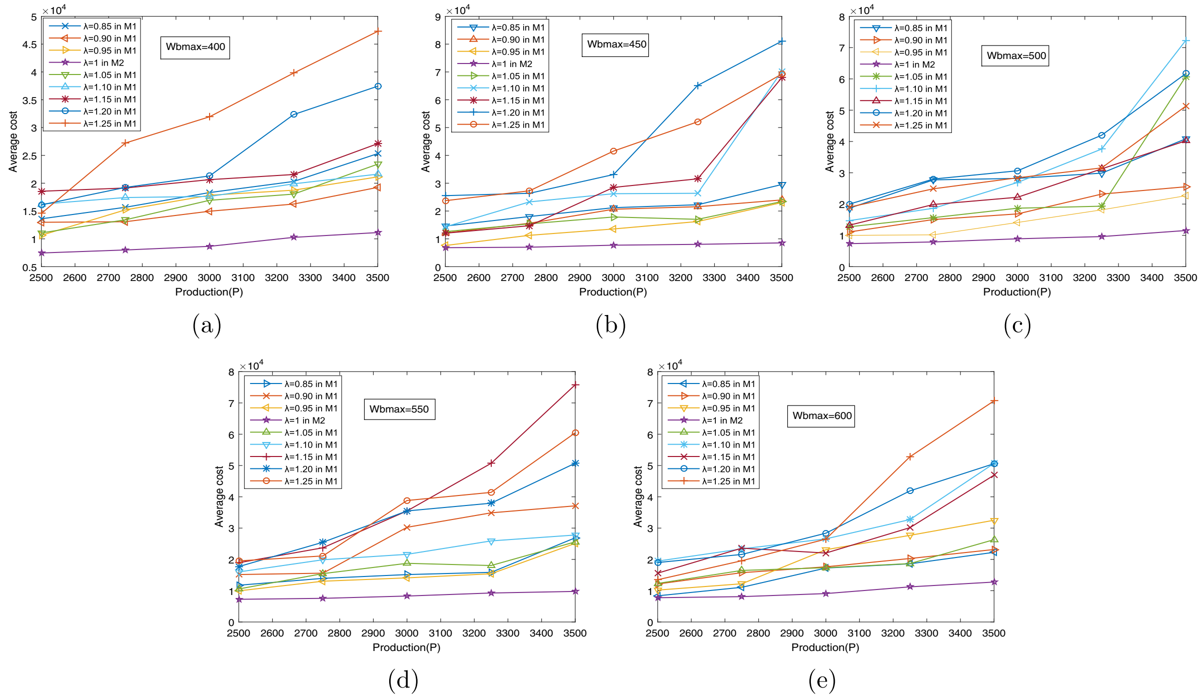


FIGURE 6. Changes in average cost with varying production rate for different values of the ratio λ in model M1. (a) when $W_{b_{\max}} = 400$. (b) when $W_{b_{\max}} = 450$. (c) when $W_{b_{\max}} = 500$. (d) when $W_{b_{\max}} = 550$. (e) when $W_{b_{\max}} = 600$.

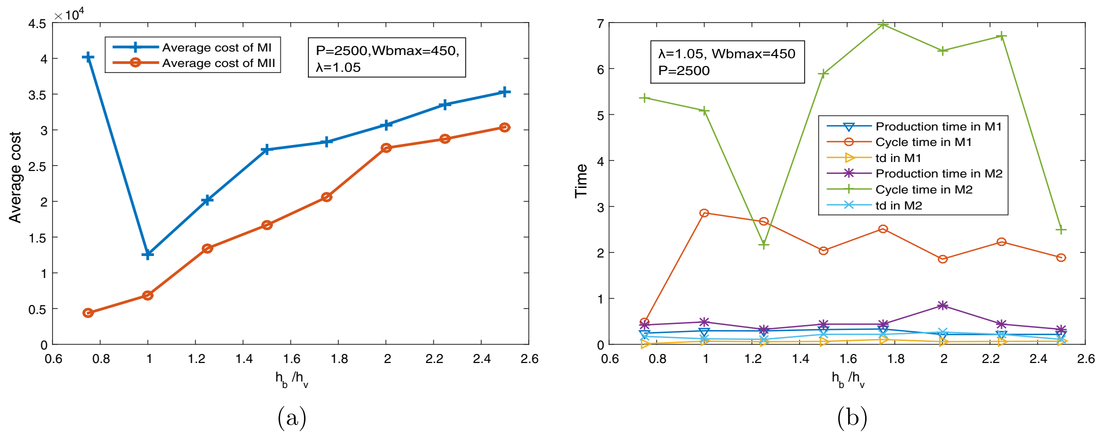
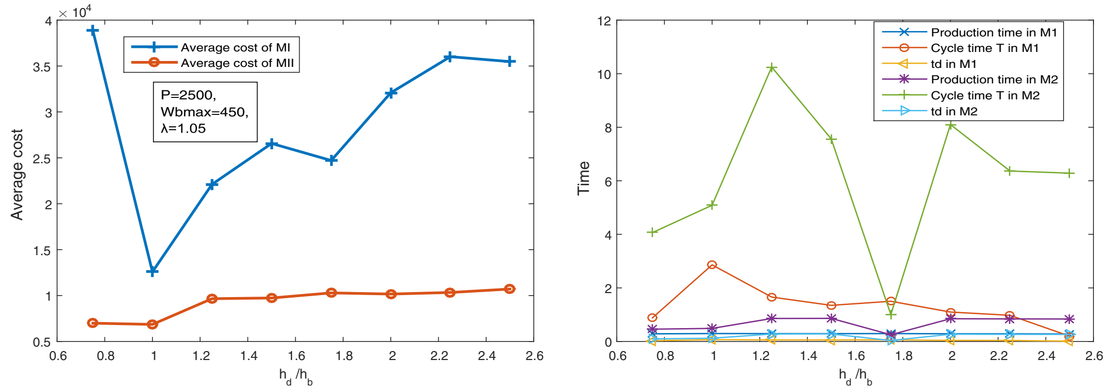


FIGURE 7. Changes in average cost, production run-time and cycle time with varying ratio h_b/h_v. (a) Changes in average costs. (b) Changes in production run-time, cycle time, and T_d.

time in model M1, and so does higher storage capacity. The findings from Figures 4b and 5b are thus quite obvious.

– Figure 7a exhibits that the ratio h_b/h_v has a positive effect on the average cost. Managers should realign their strategies to store the produced items in the vendor's inventory for longer period of time rather than



(a) (b)

FIGURE 8. Changes in average cost, production run-time and cycle time with varying ratio h_d/h_b. (a) Changes in average costs. (b) Changes in production run-time, cycle time, and T_d.

quickly delivering those to the buyer's inventory, so that the average inventory holding at the buyer's side is reduced. With higher holding cost ratio, managers should try to store lesser amount of product, resulting in shorter production run time, cycle time and replenishment time in general which is evident from Figure 7b. It is also observed that, with increasing shipment size, managers should reduce production run time to reduce cost. However, it needs more time to complete a business cycle as well as finish the stock at the display area with equal shipment strategy than an unequal one. Similar observations can be made from Figure 8 with an additional one being that the change in the ratio h_d/h_b has almost no effect on production run time under unequal shipment strategy.

6. Conclusion

This paper considers a vendor-buyer integrated inventory system for deteriorating items with consignment stocking where the buyer's inventory is extended over warehouse area and display area, and its warehouse is capacity constrained. Although the complexity of the model restrained us from obtaining any analytical result, sensitivity analysis through numerical simulation did enable us to exhibit the applicability of the model as well as draw a few valuable insights for the business managers. The present work corroborates with the observations made by Wang *et al.* [49] regarding utilization of space capacity available at the buyer. Further, we establish that the choice of consecutive shipment sizes (whether it should be equal or unequal) depends on the production rate. Till date, a few researchers like Bylka and Górny [14] and Giri *et al.* [27] have established the superiority of unequal shipment strategy under certain business scenario, but to the best of our knowledge, no attempt has been made to identify when one shipment strategy outperforms the other. In this direction, our finding is worth mentioning. However, the model has a few limitations too which could be overcome. The present work may be extended in a number of ways. One can consider the demand pattern to be dependent on some other parameters such as price, sales effort, etc. Under price dependent demand scenario, presence of multiple retailers and competition among them would be a challenging but interesting issue to study. Consideration of imperfect production process and possible rework may also be studied to derive condition when it would be indeed beneficial to go for rework. One may also consider the issue of controllable lead time on this model to represent more general situation. Investment in preservation technology to reduce spoilage is also an interesting issue in today's highly competitive market, which may also be addressed. The model may further be generalized by adding a stochastic component in the demand to represent real-life scenario, and maximizing the expected profit of the system. Consideration of a risk-averse retailer will further make the model complicated as well as interesting to be studied.

Appendix A.

$$
\begin{aligned}
P_1 &= \int_0^{t_1} I_v(t)\mathrm{d}t + \int_0^{t_2} I_v(t)\mathrm{d}t + \int_0^{t_3} I_v(t)\mathrm{d}t + \cdots + \int_0^{t_m} I_v(t)\mathrm{d}t \\
&= \int_0^{t_1} \left(\frac{P}{\alpha_1} - \frac{P}{\alpha_1}e^{-\alpha_1 t}\right)\mathrm{d}t + \int_0^{t_2} \left(\frac{P}{\alpha_1} - \frac{P}{\alpha_1}e^{-\alpha_1 t}\right)\mathrm{d}t + \cdots + \int_0^{t_m} \left(\frac{P}{\alpha_1} - \frac{P}{\alpha_1}e^{-\alpha_1 t}\right)\mathrm{d}t \\
&= \left[\frac{Pt}{\alpha_1} + \frac{P}{\alpha_1^2}e^{-\alpha_1 t}\right]_0^{t_1} + \left[\frac{Pt}{\alpha_1} + \frac{P}{\alpha_1^2}e^{-\alpha_1 t}\right]_0^{t_2} + \left[\frac{Pt}{\alpha_1} + \frac{P}{\alpha_1^2}e^{-\alpha_1 t}\right]_0^{t_3} + \cdots + \left[\frac{Pt}{\alpha_1} + \frac{P}{\alpha_1^2}e^{-\alpha_1 t}\right]_0^{t_m} \\
&= \frac{P}{\alpha_1^2}\left(e^{-\alpha_1 t_1} + e^{-\alpha_1 t_2} + e^{-\alpha_1 t_3} + \cdots + e^{-\alpha_1 t_m}\right) + \frac{P}{\alpha_1}\left(t_1 + t_2 + t_3 + \cdots + t_m\right) - \frac{mP}{\alpha_1^2} \\
&= \frac{P}{\alpha_1^2}\left[\left\{m - \frac{\alpha_1 Q}{P}\left(\frac{\lambda^m - 1}{\lambda - 1}\right)\right\} - \ln\left\{\left(1 - \frac{\alpha_1 Q}{P}\right)\left(1 - \frac{\lambda\alpha_1 Q}{P}\right)\cdots\left(1 - \frac{\lambda^{m-1}\alpha_1 Q}{P}\right)\right\}\right] - \frac{mP}{\alpha_1^2} \\
&= \frac{P}{\alpha_1^2}\left(m - \left(1 - e^{-\alpha_1 t_1}\right)\left(\frac{\lambda^m - 1}{\lambda - 1}\right)\right) + \frac{P}{\alpha_1}\left(t_1 - \frac{1}{\alpha_1}\sum_{i=1}^{m-1}\ln\left(1 - \lambda^i + \lambda^i e^{-\alpha_1 t_1}\right)\right) - \frac{mP}{\alpha_1^2}\cdot
\end{aligned}
$$

Appendix B.

$$
\begin{aligned}
W_{11} &= \int_0^{T_d} I_b(t)\mathrm{d}t + \int_{T_d}^{2T_d} I_b(t)\mathrm{d}t + \cdots + \int_{(n_1-1)T_d}^{n_1 T_d} I_b(t)\mathrm{d}t + \int_{n_1 T_d}^{(n_1+1)T_d} I_b(t)\mathrm{d}t + \cdots \\
&\quad + \int_{(n_1+n_2-1)T_d}^{(n_1+n_2)T_d} I_b(t)\mathrm{d}t + \int_{(n_1+n_2)T_d}^{(n_1+n_2+1)T_d} I_b(t)\mathrm{d}t + \cdots + \int_{\left(\sum_{j=1}^{m-1} n_j - 1\right)T_b}^{\sum_{j=1}^{m-1} n_j T_d} I_b(t)\mathrm{d}t \\
&= \int_0^{T_d} (Q - q)\, e^{-\beta t}\mathrm{d}t + \int_{T_d}^{2T_d} \left(Q - q - qe^{\beta T_d}\right) e^{-\beta t}\mathrm{d}t + \cdots + \int_{(n_1-1)T_d}^{n_1 T_d} \left(Q - \sum_{i=0}^{n_1-1} qe^{i\beta T_d}\right) e^{-\beta t}\mathrm{d}t \\
&\quad + \int_{n_1 T_d}^{(n_1+1)T_d} \left(Q - \sum_{i=0}^{n_1} qe^{i\beta T_d} + \lambda Q e^{n_1\beta T_d}\right) e^{-\beta t}\mathrm{d}t + \cdots \\
&\quad + \int_{(n_1+n_2-1)T_d}^{(n_1+n_2)T_d} \left(Q - \sum_{i=0}^{n_1+n_2-1} qe^{i\beta T_d} + \lambda Q e^{n_1\beta T_d}\right) e^{-\beta t}\mathrm{d}t \\
&\quad + \int_{(n_1+n_2)T_d}^{(n_1+n_2+1)T_d} \left(Q - \sum_{i=0}^{n_1+n_2} qe^{i\beta T_d} + \lambda Q e^{n_1\beta T_d} + \lambda^2 Q e^{(n_1+n_2)\beta T_d}\right) e^{-\beta t}\mathrm{d}t + \cdots \\
&\quad + \int_{\left(\sum_{j=1}^{m-1} n_j - 1\right)T_b}^{\sum_{j=1}^{m-1} n_j T_d} \left(Q - \sum_{i=0}^{\sum_{j=1}^{m-1} n_j - 1} qe^{i\beta T_d} + \lambda Q e^{n_1\beta T_d} + \cdots + \lambda^{m-2} Q e^{\sum_{j=1}^{m-2} n_j \beta T_d}\right) e^{-\beta t}\mathrm{d}t \\
&= -\frac{1}{\beta}\left[(Q - q)\, e^{-\beta t}\right]_0^{T_d} - \frac{1}{\beta}\left[\left(Q - q - qe^{\beta T_d}\right) e^{-\beta t}\right]_{T_d}^{2T_d} - \cdots - \frac{1}{\beta}\left[\left(Q - \sum_{i=0}^{n_1-1} qe^{i\beta T_d}\right) e^{-\beta t}\right]_{(n_1-1)T_d}^{n_1 T_d}
\end{aligned}
$$

$$-\frac{1}{\beta}\left[\left(Q-\sum_{i=0}^{n_1} qe^{i\beta T_d}+\lambda Qe^{n_1\beta T_d}\right)e^{-\beta t}\right]_{n_1T_d}^{(n_1+1)T_d}-\cdots$$
$$-\frac{1}{\beta}\left[\left(Q-\sum_{i=0}^{n_1+n_2-1} qe^{i\beta T_d}+\lambda Qe^{n_1\beta T_d}\right)e^{-\beta t}\right]_{(n_1+n_2-1)T_d}^{(n_1+n_2)T_d}$$
$$-\frac{1}{\beta}\left[\left(Q-\sum_{i=0}^{n_1+n_2} qe^{i\beta T_d}+\lambda Qe^{n_1\beta T_d}+\lambda^2 Qe^{(n_1+n_2)\beta T_d}\right)e^{-\beta t}\right]_{(n_1+n_2)T_d}^{(n_1+n_2+1)T_d}-\cdots$$

$$-\frac{1}{\beta}\left[\left(Q-\sum_{i=0}^{\sum_{j=1}^{m-1}n_j-1} qe^{i\beta T_d}+\lambda Qe^{n_1\beta T_d}+\cdots+\lambda^{m-2}e^{\sum_{j=1}^{m-2}n_j\beta T_d}\right)e^{-\beta t}\right]_{(\sum_{j=1}^{m-1}n_j-1)T_d}^{\sum_{j=1}^{m-1}n_jT_d}$$
$$=\frac{1}{\beta}\left(Q-Qe^{-\sum_{j=1}^{m-1}n_j\beta T_d}-q\sum_{j=1}^{m-1}n_j+q\sum_{i=1}^{\sum_{j=1}^{m-1}n_j}e^{-i\beta T_d}\right)$$
$$+\frac{1}{\beta}\left(\lambda Q\left(\frac{\lambda^{m-2}-1}{\lambda-1}\right)+\lambda Qe^{-\sum_{j=1}^{m-1}n_j\beta T_d}\sum_{l=1}^{m-2}\lambda^{l-1}e^{\sum_{j=1}^{l}n_j\beta T_d}\right).$$

Appendix C.

$$W_{12}=\int_0^{T_d}W_{b_{\max}}e^{-\beta t}\mathrm{d}t+\int_{T_d}^{2T_d}\left(W_{b_{\max}}-qe^{\beta T_d}\right)e^{-\beta t}\mathrm{d}t+\int_{2T_d}^{3T_d}\left(W_{b_{\max}}-qe^{\beta T_d}-qe^{2\beta T_d}\right)e^{-\beta t}\mathrm{d}t$$
$$+\int_{3T_d}^{4T_d}\left(W_{b_{\max}}-q\sum_{i=1}^{3}e^{i\beta T_d}\right)e^{-\beta t}\mathrm{d}t+\cdots+\int_{n-\sum_{j=1}^{m-1}n_j-2}^{n-\sum_{j=1}^{m-1}n_j-1}\left(W_{b_{\max}}-q\sum_{i=0}^{n-\sum_{j=1}^{m-2}-2}e^{i\beta T_d}\right)e^{-\beta t}\mathrm{d}t$$
$$=-\frac{1}{\beta}\left[W_{b_{\max}}e^{-\beta t}\right]_0^{T_d}-\frac{1}{\beta}\left[\left(W_{b_{\max}}-qe^{\beta T_d}\right)e^{-\beta t}\right]_{T_d}^{2T_d}-\frac{1}{\beta}\left[\left(W_{b_{\max}}-qe^{\beta T_d}-qe^{2\beta T_d}\right)e^{-\beta t}\right]_{2T_d}^{3T_d}$$
$$-\frac{1}{\beta}\left[\left(W_{b_{\max}}-\sum_{i=1}^{3}qe^{i\beta T_d}\right)e^{-\beta t}\right]_{3T_d}^{4T_d}-\cdots-\frac{1}{\beta}\left[\left(W_{b_{\max}}-\sum_{i=1}^{n-\sum_{j=1}^{m-2}n_j-2}qe^{i\beta T_d}\right)e^{-\beta t}\right]_{(n-\sum_{j=1}^{m-1}n_j-2)T_d}^{(n-\sum_{j=1}^{m-1}n_j-1)T_d}$$
$$=\frac{1}{\beta}\left(W_{b_{\max}}-W_{b_{\max}}e^{-\left(n-\sum_{j=1}^{m-1}n_j-1\right)\beta T_d}+qe^{-\beta T_d}\sum_{i=0}^{\left(n-\sum_{j=1}^{m-1}n_j-3\right)}e^{-i\beta T_d}-\left(n-\sum_{j=1}^{m-1}n_j-2\right)q\right).$$

Appendix D.

$$P_2=n\int_0^{T_d}\left[-\frac{\alpha}{\delta}+\left(\frac{\alpha}{\delta}+(q+x)^{1-\beta_1}\right)\left(1-\delta\left(1-\beta_1\right)t\right)\right]^{\frac{1}{1-\beta_1}}\mathrm{d}t.$$

Substituting $u = \left[-\frac{\alpha}{\delta} + \left(\frac{\alpha}{\delta} + (q+x)^{1-\beta_1}\right)(1-\delta(1-\beta_1)t)\right]^{\frac{1}{1-\beta_1}}$, we get

$$\begin{aligned} P_2 &= -n\int \frac{u^{1-\beta_1}}{\alpha+\delta(q+x)^{1-\beta_1}}\mathrm{d}u \\ &= -\frac{nu^{2-\beta_1}}{(2-\beta_1)\left(\alpha+\delta(q+x)^{1-\beta_1}\right)} \\ &= -\frac{n}{(2-\beta_1)\left(\alpha+\delta(q+x)^{1-\beta_1}\right)}\left[\left[-\frac{\alpha}{\delta}+\left(\frac{\alpha}{\delta}+(q+x)^{1-\beta_1}\right)(1-\delta(1-\beta_1)t)\right]^{\frac{2-\beta_1}{1-\beta_1}}\right]_0^{T_d} \\ &= \frac{n}{(2-\beta_1)\left(\alpha+\delta(q+x)^{1-\beta_1}\right)}\left[(q+x)^{2-\beta_1}-\left[-\frac{\alpha}{\delta}+\left(\frac{\alpha}{\delta}+(q+x)^{1-\beta_1}\right)(1-\delta(1-\beta_1)T_d)\right]^{\frac{2-\beta_1}{1-\beta_1}}\right]. \end{aligned}$$

Appendix E.

$P_3 = n\int_0^{T_d}\alpha I_{bd}^{\beta_1}(t)\mathrm{d}t = n\int_0^{T_d}\alpha\left[-\frac{\alpha}{\delta}+\left(\frac{\alpha}{\delta}+(q+x)^{1-\beta_1}\right)(1-\delta(1-\beta_1)t)\right]^{\frac{\beta_1}{1-\beta_1}}\mathrm{d}t.$

Substituting $u = \left[-\frac{\alpha}{\delta}+\left(\frac{\alpha}{\delta}+(q+x)^{1-\beta_1}\right)(1-\delta(1-\beta_1)t)\right]^{\frac{1}{1-\beta_1}}$, we get

$P_3 = \frac{n\alpha}{\left[\alpha+\delta(q+x)^{1-\beta_1}\right]}\left[(q+x)-\left\{-\frac{\alpha}{\delta}+\left(\frac{\alpha}{\delta}+(q+x)^{1-\beta_1}\right)(1-\delta(1-\beta_1)T_d)\right\}^{\frac{1}{1-\beta_1}}\right].$

Appendix F.

$$\begin{aligned} W_{11}^{'} &= \int_0^{T_d} I_b(t)\mathrm{d}t + \int_{T_d}^{2T_d} I_b(t)\mathrm{d}t + \cdots + \int_{(n_1-1)T_d}^{n_1T_d} I_b(t)\mathrm{d}t + \cdots + \int_{(m-2)n_1T_d}^{(m-1)n_1T_d} I_b(t)\mathrm{d}t \\ &= \int_0^{T_d}(Q-q)e^{-\beta t}\mathrm{d}t + \int_{T_d}^{2T_d}\left(Q-q-qe^{\beta T_d}\right)e^{-\beta t}\mathrm{d}t + \cdots \\ &\quad + \int_{(n_1-1)T_d}^{n_1T_d}\left(Q-q\sum_{i=0}^{n_1-1}e^{i\beta T_d}\right)e^{-\beta t}\mathrm{d}t + \int_{n_1T_d}^{(n_1+1)T_d}\left(Q(1+e^{n_1\beta T_d})-q\sum_{i=0}^{n_1}e^{i\beta T_d}\right)e^{-\beta t}\mathrm{d}t \\ &\quad + \cdots + \int_{(2n_1-1)T_d}^{2n_1T_d}\left(Q(1+e^{n_1\beta T_d})-q\sum_{i=0}^{2n_1-1}e^{i\beta T_d}\right)e^{-\beta t}\mathrm{d}t \\ &\quad + \int_{2n_1T_d}^{(2n_1+1)T_d}\left(Q\sum_{i=0}^{2}e^{in_1\beta T_d}-q\sum_{i=0}^{2n_1}e^{i\beta T_d}\right)e^{-\beta t}\mathrm{d}t \\ &\quad + \cdots + \int_{((m-1)n_1-1)T_d}^{(m-1)n_1T_d}\left(Q\sum_{i=0}^{m-2}e^{in_1\beta T_d}-q\sum_{i=0}^{(m-1)n_1-1}e^{i\beta T_d}\right)e^{-\beta t}\mathrm{d}t \\ &= -\frac{1}{\beta}\left[(Q-q)e^{-\beta t}\right]_0^{T_d} - \frac{1}{\beta}\left[\left(Q-q-qe^{\beta T_d}\right)e^{-\beta t}\right]_{T_d}^{2T_d} - \cdots \\ &\quad -\frac{1}{\beta}\left[\left(Q-q\sum_{i=0}^{n_1-1}e^{i\beta T_d}\right)e^{-\beta t}\right]_{(n_1-1)T_d}^{n_1T_d} - \frac{1}{\beta}\left[\left(Q(1+e^{n_1\beta T_d})-q\sum_{i=0}^{n_1}e^{i\beta T_d}\right)e^{-\beta t}\right]_{n_1T_d}^{(n_1+1)T_d} \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\beta}\left[\left(Q(1+e^{n_1\beta T_d})-q\sum_{i=0}^{2n_1-1}e^{i\beta T_d}\right)e^{-\beta t}\right]_{(2n_1-1)T_d}^{2n_1T_d} \\
&-\frac{1}{\beta}\left[\left(Q\sum_{i=0}^{2}e^{in_1\beta T_d}-q\sum_{i=0}^{2n_1}e^{i\beta T_d}\right)e^{-\beta t}\right]_{2n_1T_d}^{(2n_1+1)T_d}-\cdots \\
&-\frac{1}{\beta}\left[\left(Q\sum_{i=0}^{m-2}e^{in_1\beta T_d}-q\sum_{i=0}^{(m-1)n_1-1}e^{i\beta T_d}\right)e^{-\beta t}\right]_{((m-1)n_1-1)T_d}^{(m-1)n_1T_d} \\
=&\frac{1}{\beta}\left[(m-1)Q-Qe^{-(m-1)n_1\beta T_d}\sum_{i=o}^{m-2}e^{in_1\beta T-d}-(m-1)q+q\sum_{(i=1)}^{(m-1)n_1}e^{-i\beta T_d}\right].
\end{aligned}
$$

Appendix G.

$$
\begin{aligned}
W_{12}^{'}&=\int_0^{T_d}I_b(t)\mathrm{d}t+\int_{T_d}^{2T_d}I_b(t)\mathrm{d}t+\cdots+\int_{(n-(m-1)n_1-3)T_d}^{(n-(m-1)n_1-2)T_d}I_b(t)\mathrm{d}t+\int_{(n-(m-1)n_1-2)T_d}^{(n-(m-1)n_1-1)T_d}I_b(t)\mathrm{d}t \\
&=\int_0^{T_d}W_{b_{\max}}e^{-\beta t}\mathrm{d}t+\int_{T_d}^{2T_d}\left(W_{b_{\max}}-qe^{\beta T_d}\right)e^{-\beta t}\mathrm{d}t+\cdots \\
&\quad+\int_{(n-(m-1)n_1-2)T_d}^{(n-(m-1)n_1-1)T_d}\left(W_{b_{\max}}-q\sum_{i=1}^{n-(m-1)n_1-2}e^{i\beta T_d}\right)e^{-\beta t}\mathrm{d}t \\
&=-\frac{1}{\beta}\left[W_{b_{\max}}e^{-\beta t}\right]_0^{T_d}-\frac{1}{\beta}\left[\left(W_{b_{\max}}-qe^{\beta T_d}\right)e^{-\beta t}\right]_{T_d}^{2T_d}-\cdots \\
&\quad-\frac{1}{\beta}\left[\left(W_{b_{\max}}-q\sum_{i=1}^{n-(m-1)n_1-2}e^{i\beta T_d}\right)e^{-\beta t}\right]_{(n-(m-1)n_1-2)T_d}^{(n-(m-1)n_1-1)T_d} \\
&=\frac{1}{\beta}W_{b_{\max}}\left[1-e^{-(n-(m-1)n_1-1)\beta T_d}\right]+\frac{1}{\beta}\left[q\sum_{i=1}^{n-(m-1)n_1-2}e^{i\beta T_d}-(n-(m-1)n_1-2)q\right].
\end{aligned}
$$

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