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FLUID $M/M/1$ CATASTROPHIC QUEUE IN A RANDOM ENVIRONMENT

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Abstract. Our main objective in this study is to investigate the stationary behavior of a fluid catastrophic queue of $M/M/1$ in a random multi-phase environment. Occasionally, a queueing system experiences a catastrophic failure causing a loss of all current jobs. The system then goes into a process of repair. As soon as the system is repaired, it moves with probability $q_i \geq 0$ to phase i. In this study, the distribution of the buffer content is determined using the probability generating function. In addition, some numerical results are provided to illustrate the effect of various parameters on the distribution of the buffer content.

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1. Introduction

In recent years, studies on queueing systems in a random environment have become extremely important owing to their widespread application in telecommunication systems, advanced computer networks, and manufacturing systems. In addition, studies on fluid queueing systems are regarded as an important class of queueing theory; the interpretation of the behavior of such systems helps us understand and improve the behavior of many applications in our daily life.

A fluid queue is an input-output system where a continuous fluid enters and leaves a storage device called a buffer; the system is governed by an external stochastic environment at randomly varying rates. These models have been well established as a valuable mathematical modeling method and have long been used to estimate the performance of certain systems as telecommunication systems, transportation systems, computer networks, and production and inventory systems. Readers may refer to Anick *et al.* [3], Mitra [17], Elwalid and Mitra [8], Knessl and Morrison [14], Stern and Elwalid [23], Bekker and Mandjes [6], and Latouche and Taylor [16] for more details.

Interest in understanding the behaviors of these systems has resulted in an abundance of diverse studies. Studies on fluid queueing systems have mainly been divided into two types, *i.e.*, those under nonstationary and stationary states.

Keywords. $M/M/1$ queue, disasters, random environment, stationary probabilities, buffer content distribution.

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In addition, the emergence of certain concepts (vacations and disasters) have had a clear impact on the classification of such studies into two main categories: studies on classical fluid queueing systems and studies on fluid queueing systems with vacations, disasters, or both.

In this paper, we discuss previous studies pertaining to the current research topic. In [15], providing example studies on stationary behavior for classical fluid queueing systems, the author arranged the last studies on the fluid model driven by a stochastic environment with finite states and applied a general construction to analyze the system using a spectral analysis. In addition, the obtainable results and the solution methodology congruent to the system of the fluid queue driven by an external environment are outlined. In [29], Virtamo and Norros considered a fluid queue of $M/M/1$. The authors obtained a solution for the distribution of buffer content in a simple integral form, given the explicit expression of the associated individual vectors in terms of second-class Chebyshev polynomials. Nevertheless, numerous published papers on this subject, such as that by Adan and Resing [1], have represented the background process as a fluctuating restitution, transacting with successive idle and busy periods of an $M/M/1$ queue. In addition, Parthasarathy *et al.* [19] obtained complete solutions for a stationary buffer occupancy distribution as well as the buffer content using a continuous fraction method. Moreover, Barbot and Sericola [4] created an original analytical expression relating to the stationary buffer level solution and the $M/M/1$ queue status. Sericola and Tuffin [21] also developed an iteratively stable algorithm to address the distribution of the stationary buffer content of a fluid queue driven by a Markov queue as a $PH/PH/N/L$ queue for both $L < +\infty$ and $L = +\infty$.

In addition, there have been several papers published on a stationary fluid queueing system with disasters. Vijayalakshim and Thangaraj [24] analyzed the transient behavior of a fluid model operated by an $M/M/1$ queue with a disaster. A study on a stationary fluid queue driven by an $M/M/1$ queue subject to disasters and subsequent repair was conducted by Vijayashree and Anjuka [25]. Vijayashree and Anjuka [26] implemented a stationary fluid queue model driven by a disaster $M/M/1$ queue. In addition, Ammar [2] derived an explicit expression for the buffer content distribution in a fluid queue driven by an $M/M/1$ disaster queue for the stationary distribution function. Anjuka and Vijayashree [27] explored a stationary analysis of a queueing model driven by a disaster $M/M/1/N$ queue and subsequent repair. Xu *et al.* [30] recently examined the stationary behavior of a fluid $M/M/1$ queue with working vacations and policies for negative customers. Using the Laplace transform method, they obtained the mean of the buffer content and the probability of an empty buffer.

By contrast, to interpret the behavior of many aspects in our lives, there has been an increasing tendency to study queueing systems in a random environment. Excellent surveys on the infinite server queue in a random environment have been reported [5, 7, 9, 18].

Disaster queueing systems in a random environment have resulted in several studies, particularly a disaster analysis of single server queue behavior in a random environment. For instance, in a multi-phase random environment, Paz and Yechiali [20] considered an $M/M/1$ queue in which the system occasionally suffers from a disastrous failure causing the loss of all current jobs. They analyzed the behavior of this system in a steady state by finding the system probabilities and some performance measures. Using the supplementary variable technique, Jiang *et al.* [12] researched an $M/G/1$ queue in a multi-phase random environment with disasters. They obtained the distribution for a stationary queue and extracted the results of a cycle analysis, the distribution of the sojourn time, and the length of the working time during a service cycle. Vinodhini and Vidhya [28] proposed a dynamically changing traffic model in a multi-phase random system as an $M/M/1$ disaster queue. They obtained the steady state probabilities and some performance measures using a geometric matrix and function method generation. In addition, Jiang and Liu [10] studied a single $GI/M/1$ disaster queue in a multi-phase service environment. Using an analytical matrix approach and a semi-Markov process, they obtained a stationary queue length distribution at both the arrival and arbitrary times. Jiang and Liu [11] discussed the stationary behavior of a single server $GI/M/1$ queue in a multi-phase service environment with disasters and working breakdowns. They obtained the queue length distribution at both the arrival and arbitrary periods through an analytic matrix approach and a semi-Markov process. They also provided an elaborate analysis of some performance measures and the sojourn time distribution of an arbitrary customer. Jiang *et al.* [13] presented an N-policy $GI/M/1$ queue in a multi-phase service environment with disasters in which they derived the stationary queue length distribution; the distribution is then used for the computation of the Laplace Stieltjes transform of the sojourn time of an arbitrary customer and the server working time within a cycle. Sherman and Kharoufeh [22] investigated an $M/M/1$ retrial queue with an unreliable server. An exogenous random environment modulates the arrival, service, failure, repair, and retrial rates of the server. They were able to find the conditions for stability, the distribution of the orbit size, and the mean queueing performance measures described using an analytic matrix method.

It is clear from previous studies that there is a growing interest in studying both fluid queueing systems and queueing systems in a random environment, which have numerous applications in different fields.

The motivation of our model is derived from computer networking. As an example, we considered a storage area network (SAN), which is a mass storage setting used to provide improved capacity and high-speed network services. A SAN consists of one or more servers connected to storage devices, such as hubs, routers, and bridges, through switching devices. Because linking control between the capacity gadgets and customers is the key job of a SAN server, such servers are the main targets of distributed denial-of-service attacks (DDoS attacks). DDoS assaults result in the unavailability of all network infrastructure and information to the intended clients; destroyed data cannot be retrieved. Another practical motivation of this study stems from the application used in call centers. Recently, with the increasing advancement of the service industry, as a direct medium for engaging with consumers, customer service call centers are one of the main components of service businesses and can offer ticketing, consultancy, and telephone banking, among other services. Call centers can therefore be modeled as catastrophic fluid queueing systems in a random environment, where the workers in the call centers and calls reconcile with the servers and customers, respectively. For instance, an automatic call distributor instantly assigns an incoming call to the server in a call center if the server works upon arrival. Otherwise, no calls are received if the system goes down. An automatic call distributor provider distributes the incoming calls for processing to a server, which occurs with a certain possibility. Thus, all incoming calls are distributed for processing. In addition, the rate of call handling can be adjusted according to the number of calls to increase the quality of the service and decrease the online waiting times for customers. If the number of calls is less than a certain value after a processing period, the call center can reduce its processing rate (*e.g.*, decrease the number of online employees); otherwise, if the number of calls is greater than a certain value, customer service can increase the processing rate of the call center (*e.g.*, increase the number of online employees). Assuming the service area is divided into n different levels and that each level has a specific server during a period of time, the probability that the incoming call is in the ith level is $q_i, i = 1, 2, \ldots, n$. In addition, suppose that the arrival rate of incoming calls is λ_i and that the service rate of the processing is μ_i. Motivated by this application, we can structure an $M/M/1$ catastrophic fluid queue in a multi-phase Markovian environment. Finally, a catastrophe can be described as a machine breakdown that causes all processing work in the manufacturing systems to be decimated. In addition, it is taken from the field of transportation. In the case of an accident or damage to a car, for example, the vehicles are evacuated to another highway (in other words, they are lost) and new vehicles arriving on the road wait for the road to be repaired.

To the best of our knowledge, there have been no studies dealing with a fluid queueing system under a catastrophe in a random environment. The advantages and contributions of this study are as follows:

- System: a novel fluid catastrophe queue in a random environment is presented. This system is suitable for reflecting the characteristics of customer services and conducting a performance analysis for modern customer service call centers and computer networks.
- Methodology and results: we first adopt a generating function and Laplace transform to obtain the stationary probabilities, and then, based on the expressions of the stationary probabilities, we compute the buffer content distribution.
- Numerical illustrations: a numerical analysis is included in this study for illustrative purposes. Moreover, using some numerical examples, we discuss the effect of different parameters on the proposed system and show that they play an important role in changing the general behavior of the system under consideration.

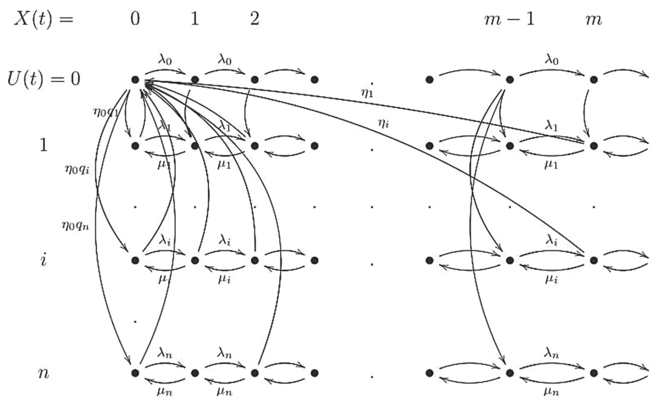

FIGURE 1. State transition diagram of an $M/M/1$ disaster queue in a random environment.

Using the generating function technique, the stationary probabilities are expressed in terms of a modified Bessel function of the first kind.

The structure of this study is as follows: We begin in Section 2 by describing the system under consideration. In Section 3, we present the stationary equations of the system under study. In Section 4, we obtain the stationary probabilities for the fluid queueing system under consideration. The results of a special case are then presented in Section 5. Section 6 describes a numerical illustration to clarify the general behavior of the fluid queueing system.

2. Model description

Consider an $M/M/1$ queueing system experiencing a catastrophe in a random multi-phase environment. Under environment $i, i = 1, 2, \ldots, n$, the rates of arrivals and services performed at the Poisson process arrival rate λ_i and the service times are distributed exponentially at μ_i. The period of time during which the system remains in phase i is exponentially distributed with mean η_i, that is, the catastrophe interval periods in phase i are distributed exponentially at a rate of $1/\eta_i, i = 1, 2, \ldots, n$. When the system operates in phase $i, i = 1, 2, \ldots, n$, the system often suffers a catastrophic failure that causes it to transfer to phase 0.

A catastrophic occurrence causes all current customers to leave the system. In the event of a disaster, the server abandons the service and the system will immediately undergo a repair process. The repair time is distributed at an exponential rate of η_0. During the 0 phase, the arrival rate of a Poisson process is λ_0.

After the system is repaired, it moves directly with probability $q_i, i = 1, 2, \ldots, n$ to the operating process i, where $\sum_{i=1}^{\infty} q_i = 1$. Thus, a transition among the operating phases is not allowed. When a catastrophe occurs, the system moves to phase 0 first, and then moves to phase i with a probability q_i. A state transition diagram of the background queueing model is shown in Figure 1.

3. Analysis of fluid queue

This section deals with the stationary analysis of a fluid queue modulated by an $M/M/1$ fluid catastrophe queue in a random multi-phase environment. Let $C(t)$ be the buffer content at time t. This is a non-negative random variable; the content of the buffer cannot decrease when the buffer is empty. That is,

$$\frac{\mathrm{d}C(t)}{\mathrm{d}t} = \begin{cases} r_0, & (U(t), X(t)) = (0,0), & C(t) > 0, \\ 0, & (U(t), X(t)) = (0,0), & C(t) = 0 \\ r, & (U(t), X(t)) = (0,m),\ (i,m), & i = 1, 2, \ldots, n, \quad m \geq 1. \end{cases}$$

This means that when the background queueing system is not empty and the system is within an operative phase and under a regular busy period, the buffer content linearly increases at a rate of $r > 0$. By contrast, the buffer content linearly decreases at a rate of $r_0 < 0$, when the system is in a failure phase or when it is empty. The stochastic process $\{U(t), X(t)\}$ under the aforementioned assumption is two-dimensional and describes the system at any time t as follows: Here, $U(t)$ denotes the system state at any time t ($0 =$ failure phase; $1, 2, \ldots, n =$ operating phase), and $X(t)$ denotes the number of customers in the system $(0, 1, 2, \ldots)$. The bivariate process $\{U(t), X(t); t \geq 0\}$ is a Markov chain with state space $\Omega = \{0, 1, 2, \ldots, n\} \times \mathrm{Z}_+$, where $\mathrm{Z}_+ = \{0, 1, 2, \ldots\}$.

Clearly, the three dimensional process $\{(U(t), X(t), C(t)), t \geq 0\}$ represents a fluid queue driven by an $M/M/1$ catastrophic queue in a random environment subject to a stability condition given by the following:

$$\rho_i = \lambda_i/\mu_i < 1, \quad i = 1, 2, \ldots, n, \quad d = r_0\pi_{00} + r\sum_{m=1}^{\infty}\pi_{0m} + r\sum_{i=1}^{n}\sum_{m=0}^{\infty}\pi_{im} < 0,$$

where π_{im} represents the steady state probability of a background queueing model to be in state (i, m). Readers may refer to Paz and Yechiali [20] for more details.

3.1. Stationary equations

For any $(i, m) \in \Omega$, we define $F_{im}(t, u) = P\{U(t) = i, C(t) \leq u, X(t) = m\}$. Then, $\{F_{im}(t, u), (i, m) \in \Omega\}$ are joint probability distribution functions of the Markov process $\{U(t), X(t), t \geq 0\}$ at time t. When the process $\{U(t), X(t), t \geq 0\}$ is stable, we write $F_{im}(u) = \lim_{t\to\infty} F_{im}(t, u)$, $((i, m) \in \Omega)$. Which are independent of the initial state of the process.

The stationary distribution function of the buffer content is given by

$$F(u) = \lim_{t\to\infty} P\{C(t) \leq u\} = \sum_{m=0}^{\infty} F_{0m}(u) + \sum_{i=1}^{n}\sum_{m=0}^{\infty} F_{im}(u).$$

Similar to the standard probability arguments (see, *e.g.*, [29]), it can be seen that the stable joint probability density sequence $\{F_{im}(u), (i, m) \in \Omega\}$ satisfies the following differential equation system:

For the failure phase $i = 0$

$$r_0\frac{\mathrm{d}F_{00}(u)}{\mathrm{d}u} = -(\lambda_0 + \eta_0)F_{00}(u) + \sum_{i=1}^{n}\eta_i\sum_{m=0}^{\infty}F_{im}(u), \tag{3.1}$$

$$r\frac{\mathrm{d}F_{0m}(u)}{\mathrm{d}u} = \lambda_0F_{0,m-1}(u) - (\lambda_0 + \eta_0)F_{0m}(u), \quad m \geq 1. \tag{3.2}$$

For $i = 1, 2, \ldots, n$

$$r\frac{\mathrm{d}F_{i0}(u)}{\mathrm{d}u} = -(\lambda_i + \eta_i)F_{i0}(u) + \mu_iF_{i1}(u) + \eta_0q_iF_{00}(u), \quad m = 0 \tag{3.3}$$

and

$$r\frac{\mathrm{d}F_{im}(u)}{\mathrm{d}u} = -(\lambda_i + \mu_i + \eta_i)F_{im}(u) + \lambda_iF_{i,m-1}(u) + \mu_iF_{i,m+1}(u) + \eta_0q_iF_{0m}(u), \quad m \geq 1 \tag{3.4}$$

subject to the boundary conditions.

$$F_{00}(0) = a, \quad F_{im}(0) = 0, \quad (i, m) \in \Omega/(0, 0). \tag{3.5}$$

To determine the constant a, which represents the $F_{00}(0)$, adding equations (3.1) to (3.4) yields

$$r_0\frac{\mathrm{d}F_{00}(u)}{\mathrm{d}u} + r\sum_{m=1}^{\infty}\frac{\mathrm{d}F_{0m}(u)}{\mathrm{d}u} + r\sum_{i=1}^{n}\sum_{m=0}^{\infty}\frac{\mathrm{d}F_{im}(u)}{\mathrm{d}u} = 0. \tag{3.6}$$

Integrating (3.6) from zero to infinity gives

$$r_0\left(F_{00}(\infty)-F_{00}(0)\right)+r\sum_{m=1}^{\infty}\left(F_{0m}(\infty)-F_{0m}(0)\right)+r\sum_{i=1}^{n}\sum_{m=0}^{\infty}\left(F_{im}(\infty)-F_{im}(0)\right)=0. \tag{3.7}$$

Note that

$$F_{i,m}(\infty)=\lim_{t\to\infty}P\left\{U(t)=i,\ X(t)=m,\ C(t)\leq\infty\right\}=P\left\{U=i,\ X=m\right\}=\pi_{im},\quad (i,m\in\Omega).$$

Using the boundary condition represented by (3.5), we obtain the following:

$$r_0\left(\pi_{00}-a\right)+r\sum_{m=1}^{\infty}\pi_{0m}+r\sum_{i=1}^{n}\sum_{m=0}^{\infty}\pi_{im}=0 \tag{3.8}$$

which upon simplification yields

$$a=\frac{r_0\pi_{00}+r\sum_{m=1}^{\infty}\pi_{0m}+r\sum_{i=1}^{n}\sum_{m=0}^{\infty}\pi_{im}}{r_0}=\frac{d}{r_0}. \tag{3.9}$$

Therefore, the constant a is explicitly given by

$$a=\frac{\left(r_0-r\right)\pi_{00}+r}{r_0}, \tag{3.10}$$

where π_{00} is given by Paz and Yechiali [20].

4. Stationary analysis

In this section, we address the stationary probabilities of the queueing system described in the previous section using the technique of generating functions and a Laplace transform.

4.1. Evaluation of $F_{im}(u)$

Define the probability generating function $G_i(s,u), i=1,2,\ldots,n$ for the stationary probabilities as

$$G_i(s,u)=\sum_{m=0}^{\infty}F_{im}(u)s^m,\quad |s|\leq 1. \tag{4.1}$$

Using the system of equations (3.3) and (3.4), we obtain a linear differential equation, *i.e.*,

$$\frac{\partial G_i(s,u)}{\partial u}=\left[\frac{\lambda_i s}{r}+\frac{\mu_i}{rs}-\left(\frac{\lambda_i+\mu_i+\eta_i}{r}\right)\right]G_i(s,u)+\frac{\mu_i}{r}\left(1-s^{-1}\right)F_{i0}(u)+\frac{\eta_0 q_0}{r}\sum_{m=0}^{\infty}F_{0m}(u)s^m$$

integrating

$$G_i(s,t)=\int_0^u\left\{\frac{\mu_i}{r}\left(1-\frac{1}{s}\right)F_{i0}(u)+\frac{\eta_0 q_i}{r}\sum_{m=0}^{\infty}F_{0m}(u)s^m\right\}\times e^{-(\lambda_i+\mu_i+\eta_i/r)(u-y)}e^{-(\lambda_i s/r+\mu_i/rs)(u-y)}\mathrm{d}y, \tag{4.2}$$

where $\omega_i=\frac{\lambda_i+\mu_i+\eta_i}{r}$.
It is well known that if $\alpha_i=\frac{2\sqrt{\lambda_i\mu_i}}{r}$ and $\beta_i=\sqrt{\lambda_i/\mu_i}$, then

$$\exp\left[\left(\frac{\lambda_i s}{r}+\frac{\mu_i}{rs}\right)u\right]=\sum_{m=-\infty}^{\infty}(\beta_i s)^m I_m(\alpha_i u), \tag{4.3}$$

where $I_m(.)$ is a modified Bessel function of the first kind. Comparing the coefficients of s^m on both sides of (4.2), we obtain $m \geq 1$ and $i = 1, 2, \ldots, n$.

$$F_{im}(u) = \frac{\mu_i \beta_i^m}{r} \int_0^u F_{i0}(y) e^{-\omega_i(u-y)} \left[I_m(\alpha_i(u-y)) - \beta_i I_{m+1}(\alpha_i(u-y)) \right] \mathrm{d}y + \frac{\eta_0 q_i}{r} \int_0^u \sum_{k=0}^{\infty} F_{0k}(y) \beta_i^{m-k} I_{m-k}(\alpha_i(u-y)) e^{-\omega_i(u-y)} \mathrm{d}y. \tag{4.4}$$

The aforementioned equation holds for $m = -1, -2, -3, \ldots,$ with the left-hand side replaced by a zero.

Using $I_{-m}(.) = I_m(.)$ for $m = 1, 2, 3, \ldots,$

$$0 = \frac{\mu_i \beta_i^{-m}}{r} \int_0^u F_{i0}(y) e^{-\omega_i(u-y)} \left[I_m(\alpha_i(u-y)) - \beta_i I_{m-1}(\alpha_i(u-y)) \right] \mathrm{d}y + \frac{\eta_0 q_i}{r} \int_0^u \sum_{k=0}^{\infty} F_{0k}(y) \beta_i^{-m-k} I_{m+k}(\alpha_i(u-y)) e^{-\omega_i(u-y)} \mathrm{d}y. \tag{4.5}$$

Equation (4.5) can be rewritten as follows:

$$\begin{aligned} &\frac{\mu_i \beta_i^m}{r} \int_0^u F_{i0}(y) e^{-\omega_i(u-y)} I_m(\alpha_i(u-y)) \mathrm{d}y \\ &\quad = \frac{\mu_i \beta_i^{m+1}}{r} \int_0^u F_{i0}(y) e^{-\omega_i(u-y)} I_{m-1}(\alpha_i(u-y)) \mathrm{d}y \\ &\qquad - \frac{\eta_0 q_i}{r} \int_0^u \sum_{k=0}^{\infty} F_{0k}(y) \beta_i^{m-k} I_{m+k}(\alpha_i(u-y)) e^{-\omega_i(u-y)} \mathrm{d}y. \end{aligned} \tag{4.6}$$

The use of (4.6) in (4.4) considerably simplifies the process and results in an elegant expression for $F_{im}(u)$, *i.e.*,

$$\begin{aligned} F_{im}(u) = {} & \frac{m \beta_i^m}{r} \int_0^u F_{i0}(y) e^{-\omega_i(t-u)} \frac{I_m(\alpha_i(u-y))}{(u-y)} \mathrm{d}y \\ & + \frac{\eta_0 q_i}{r} \int_0^u \sum_{k=1}^{\infty} F_{0k}(y) \beta_i^{m-k} \left[I_{m-k}(\alpha_i(u-y)) - I_{m+k}(\alpha_i(u-y)) \right] e^{-\omega_i(u-y)} \mathrm{d}y \\ & + \frac{\eta_0 q_i}{r} \int_0^u \sum_{k=m+1}^{\infty} F_{0k}(y) \beta_i^{m-k} \left[I_{k-m}(\alpha_i(u-y)) - I_{k+m}(\alpha_i(u-y)) \right] e^{-\omega_i(u-y)} \mathrm{d}y, \end{aligned} \tag{4.7}$$

for $m = 1, 2, 3, \ldots, i = 1, 2, \ldots, n$.

4.2. Evaluation of $F_{i0}(u)$

To obtain $F_{i0}(u)$, we use a Laplace transform. In the sequel, for any function $g(.)$, let $\hat{g}(z)$ denote its Laplace transform. For this, using (4.3) in (4.2) and comparing the coefficient of the constant terms on both sides, we have

$$F_{i0}(u) = \frac{\mu_i}{r} \int_0^u F_{i0}(y) e^{-\omega_i(u-y)} \left[I_0(\alpha_i(u-y)) - \beta_i I_1(\alpha_i(u-y)) \right] \mathrm{d}y + \frac{\eta_0 q_i}{r} \int_0^u \sum_{k=0}^{\infty} F_{0k}(y) \beta_i^{-k} I_k(\alpha_i(u-y)) e^{-\omega_i(u-y)} \mathrm{d}y. \tag{4.8}$$

The Laplace transform of (4.8) yields the following:

$$\hat{F}_{i0}(z) = \frac{\eta_0 q_i}{r} \sum_{k=0}^{\infty} \frac{\hat{F}_{0k}(z)}{(\alpha_i\beta_i)^k} \frac{\left[(z+\omega_i) - \sqrt{(z+\omega_i)^2 - \alpha_i^2}\right]^k}{\sqrt{(z+\omega_i)^2 - \alpha_i^2}} + \frac{\mu_i}{r}\hat{F}_{i0}(z)\left\{\frac{1}{\sqrt{(z+\omega_i)^2 - \alpha_i^2}} - \frac{\beta_i\left[z+\omega_i - \sqrt{(z+\omega_i)^2 - \alpha_i^2}\right]}{\alpha_i\sqrt{(z+\omega_i)^2 - \alpha_i^2}}\right\}.$$

Solving for $\hat{F}_{i0}(z)$, we obtain

$$\hat{F}_{i0}(z) = \frac{\frac{\eta_0 q_i}{r}\sum_{k=0}^{\infty}\frac{\hat{F}_{0k}(z)}{(\alpha_i\beta_i)^k}\left[(z+\omega_i) - \sqrt{(z+\omega_i)^2 - \alpha_i^2}\right]^k}{\left\{\frac{(z+\omega_i)+\sqrt{(z+\omega_i)^2-\alpha_i^2}}{2} - \frac{\mu_i}{r}\right\}}. \tag{4.9}$$

After some mathematical manipulations, (4.9) simplifies to

$$\hat{F}_{i0}(z) = \frac{\frac{\eta_0 q_i}{r}\left(\frac{2\beta_i}{\alpha_i}\right)\sum_{k=0}^{\infty}\hat{F}_{0k}(z)\left[\frac{(z+\omega_i)-\sqrt{(z+\omega_i)^2-\alpha_i^2}}{\alpha_i\beta_i}\right]^{k+1}}{\left\{1 - \left[\frac{(z+\omega_i)-\sqrt{(z+\omega_i)^2-\alpha_i^2}}{\alpha_i\beta_i}\right]\right\}}$$

and thus

$$\hat{F}_{i0}(z) = \frac{\eta_0 q_i}{\mu_i}\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\hat{F}_{0k}(z)\left[\frac{(z+\omega_i) - \sqrt{(z+\omega_i)^2 - \alpha_i^2}}{2\lambda_i/r}\right]^{l+k},$$

which upon inversion yields

$$F_{i0}(u) = \frac{\eta_0 q_i}{\mu_i}\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\int_0^u (l+k)\beta_i^{-(l+k)} e^{-\omega_i(u-y)}\frac{I_{l+k}(\alpha_i(u-y))}{(u-y)}F_{0k}(u)\mathrm{d}u. \tag{4.10}$$

4.3. Evaluation of $F_{0m}(u)$

To determine $F_{0m}(u)$, we will use the Laplace transform. By taking the Laplace transforms of (3.2), we obtain the expression

$$\hat{F}_{0m}(z) = \left(\frac{\lambda_0/r}{z+\eta_0/r+\lambda_0/r}\right)\hat{F}_{0m-1}(z).$$

Through an iteration, the aforementioned equation can be written as follows:

$$\hat{F}_{0m}(z) = \left(\frac{\lambda_0/r}{z+\eta_0/r+\lambda_0/r}\right)^m\hat{F}_{00}(z).$$

Upon an inversion, the aforementioned equation yields an expression for $F_{0,m}(u)$ as follows:

$$F_{0m}(u) = \frac{(\lambda_0/r)^m\, u^{m-1}}{(m-1)!}e^{-(\lambda_0/r+\eta_0/r)u} * F_{00}(u), \quad m \geq 1. \tag{4.11}$$

4.4. Evaluation of $F_{00}(u)$

To obtain the expression of $F_{00}(u)$, we will use the expression of $F_{im}(u)$. By taking the Laplace transforms of (3.1), we obtain

$$z\hat{F}_{00}(z) - a = -\left(\frac{\lambda_0 + \eta_0}{r_0}\right)\hat{F}_{00}(z) + \frac{1}{r_0}\sum_{i=1}^{n}\eta_i\sum_{m=0}^{\infty}\hat{F}_{im}(z). \tag{4.12}$$

In addition, the Laplace transform of (4.7) is as follows:

$$\begin{aligned}\hat{F}_{im}(z) = {} & \hat{F}_{i0}(z)\left[\frac{(z+\omega_i) - \sqrt{(z+\omega_i)^2 - \alpha_i^2}}{2\mu_i/r}\right]^m \\ & + \frac{\eta_0 q_i}{r}\sum_{k=1}^{\infty}\hat{F}_{0k}(z)\beta_i^{m-k}\left\{\frac{\left[(z+\omega_i) - \sqrt{(z+\omega_i)^2-\alpha_i^2}\right]^{m-k}}{\alpha_i^{m-k}\sqrt{(z+\omega_i)^2-\alpha_i^2}}\right. \\ & \left. - \frac{\left[(z+\omega_i) - \sqrt{(z+\omega_i)^2-\alpha_i^2}\right]^{m+k}}{\alpha_i^{m+k}\sqrt{(z+\omega_i)^2-\alpha_i^2}}\right\} \\ & + \frac{\eta_0 q_i}{r}\sum_{k=1}^{\infty}\hat{F}_{0k}(z)\beta_i^{m-k}\left\{\frac{\left[(z+\omega_i) - \sqrt{(z+\omega_i)^2-\alpha_i^2}\right]^{k-m}}{\alpha_i^{m-k}\sqrt{(z+\omega_i)^2-\alpha_i^2}}\right. \\ & \left. - \frac{\left[(z+\omega_i) - \sqrt{(z+\omega_i)^2-\alpha_i^2}\right]^{k+m}}{\alpha_i^{m+k}\sqrt{(z+\omega_i)^2-\alpha_i^2}}\right\}.\end{aligned} \tag{4.13}$$

Using (4.10), (4.11), and (4.13) together with (4.12), after some mathematical manipulations, we obtain the following:

$$\hat{F}_{00}(z) = \frac{a}{z + \lambda_0/r_0 + \eta_0/r_0\left[1 - \left(\frac{\hat{H}(z)}{z+\lambda_0/r_0+\eta_0/r_0}\right)\right]}, \tag{4.14}$$

where

$$\begin{aligned}\hat{H}(z) &= \hat{Y}(z) + \hat{A}(z) + \hat{B}(z),\\ \hat{Y}(z) &= \frac{\eta_0}{r_0}\sum_{i=1}^{n}\eta_i q_i\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{\lambda_i}{\mu_i}\right)^m\left(\frac{\lambda_0/r}{z+\eta_0/r+\lambda_0//r}\right)^k\left[\frac{(z+a_i) - \sqrt{(z+a_i)^2-\alpha_i^2}}{2\lambda_i}\right]^{m+l+k},\\ \hat{A}(z) &= \frac{\eta_0}{r}\sum_{i=1}^{n}\eta_i q_i\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\beta_i^{m-k}\left(\frac{\lambda_0/r}{z+\eta_0/r+\lambda_0/r}\right)^k\\ &\quad\times\left[\frac{\left[(z+a_i)-\sqrt{(z+a_i)^2-\alpha_i^2}\right]^{m-k}}{\alpha_i^{m-k}\sqrt{(z+a_i)^2-\alpha_i^2}} - \frac{\left[(z+a_i)-\sqrt{(z+a_i)^2-\alpha_i^2}\right]^{m+k}}{\alpha_i^{m+k}\sqrt{(z+a_i)^2-\alpha_i^2}}\right]\end{aligned}$$

and

$$\hat{B}(z) = \frac{\eta_0}{r_0}\sum_{i=1}^{n}\eta_i q_i \sum_{m=0}^{\infty}\sum_{k=m+1}^{\infty}\beta_i^{m-k}\left(\frac{\lambda_0/r}{z+\eta_0/r+\lambda_0/r}\right)^k$$
$$\times\left[\frac{\left[(z+a_i)-\sqrt{(z+a_i)^2-\alpha_i^2}\right]^{k-m}}{\alpha_i^{m-k}\sqrt{(z+a_i)^2-\alpha_i^2}} - \frac{\left[(z+a_i)-\sqrt{(z+a_i)^2-\alpha_i^2}\right]^{m+k}}{\alpha_i^{m+k}\sqrt{(z+a_i)^2-\alpha_i^2}}\right].$$

Equation (4.14) can be expressed as follows:

$$\hat{F}_{00}(z) = \frac{a}{z+\lambda_0/r_0+\eta_0/r_0}\times\sum_{j=0}^{\infty}\left[\left(\frac{1}{z+\eta_0/r_0+\lambda_0/r_0}\right)\hat{H}(z)\right]^j.$$

An inversion yields

$$F_{00}(u) = ae^{-(\lambda_0/r_0+\eta_0/r_0)u} * \sum_{j=0}^{\infty}\left(H(u) * e^{-(\lambda_0/r_0+\eta_0/r_0)u}\right)^{*j}, \tag{4.15}$$

where

$$H(u) = Y(u) + A(u) + B(u),$$
$$Y(u) = \frac{\eta_0}{r_0}\sum_{i=1}^{n}\eta_i q_i\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{\lambda_i}{\mu_i}\right)^m (m+l+k)\beta_i^{m+l+k}e^{-\omega_i u}\frac{I_{m+r+k}(\alpha_i u)}{u} * \frac{(\lambda_0/r)^m\, u^{m-1}}{(m-1)!}e^{-(\lambda_0/r+\eta_0/r)u},$$
$$A(u) = \frac{\eta_0}{r_0}\sum_{i=1}^{n}\eta_i q_i\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\beta_i^{m-k}\left[I_{m-k}(\alpha_i u) - I_{m+k}(\alpha_i u)\right]e^{-\omega_i u} * \frac{(\lambda_0/r)^m\, u^{m-1}}{(m-1)!}e^{-(\lambda_0/r+\eta_0/r)u},$$

and

$$B(u) = \frac{\eta_0}{r_0}\sum_{i=1}^{n}\eta_i q_i\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\beta_i^{m-k} * \left[I_{k-m}(\alpha_i(u)) - I_{k+m}(\alpha_i u)\right]e^{-\omega_i u} * \frac{(\lambda_0/r)^m\, u^{m-1}}{(m-1)!}e^{-(\lambda_0/r+\eta_0/r)u},$$

where $*$ denotes a convolution and $*j$ denotes a j-fold convolution. Thus, (4.7), (4.10), (4.11), and (4.15) give all stationary probabilities.

5. Special case

In this section, a particular case of the general model discussed in the previous sections is presented when $n = 1$. In this case, there is only one operative phase and one catastrophe phase. This means that the arrival, service, and catastrophe rates become equal, *i.e.*, $\lambda_0 = \lambda_i = \lambda$, $\mu_i = \mu$, and $\eta_i = \eta$. Therefore, our proposed system converts into an $M/1/1$ fluid catastrophe queue. Under this assumption, the aforementioned results are seen to coincide with those of [2] as follows:

$$F_{0m}(u) = \frac{(\lambda/r)^m\, u^{m-1}}{(m-1)!}e^{-(\lambda/r+\eta/r)u} * F_{00}(u), \quad m \geq 1.$$

In addition, the expressions for $F_{0,0}(u)$ and $F_{m,1}(u)$ agree with the results reported in [2]. Moreover, when there is no catastrophe, *i.e.*, $\eta_i \to 0$, the results obtained correspond with the results in [19].

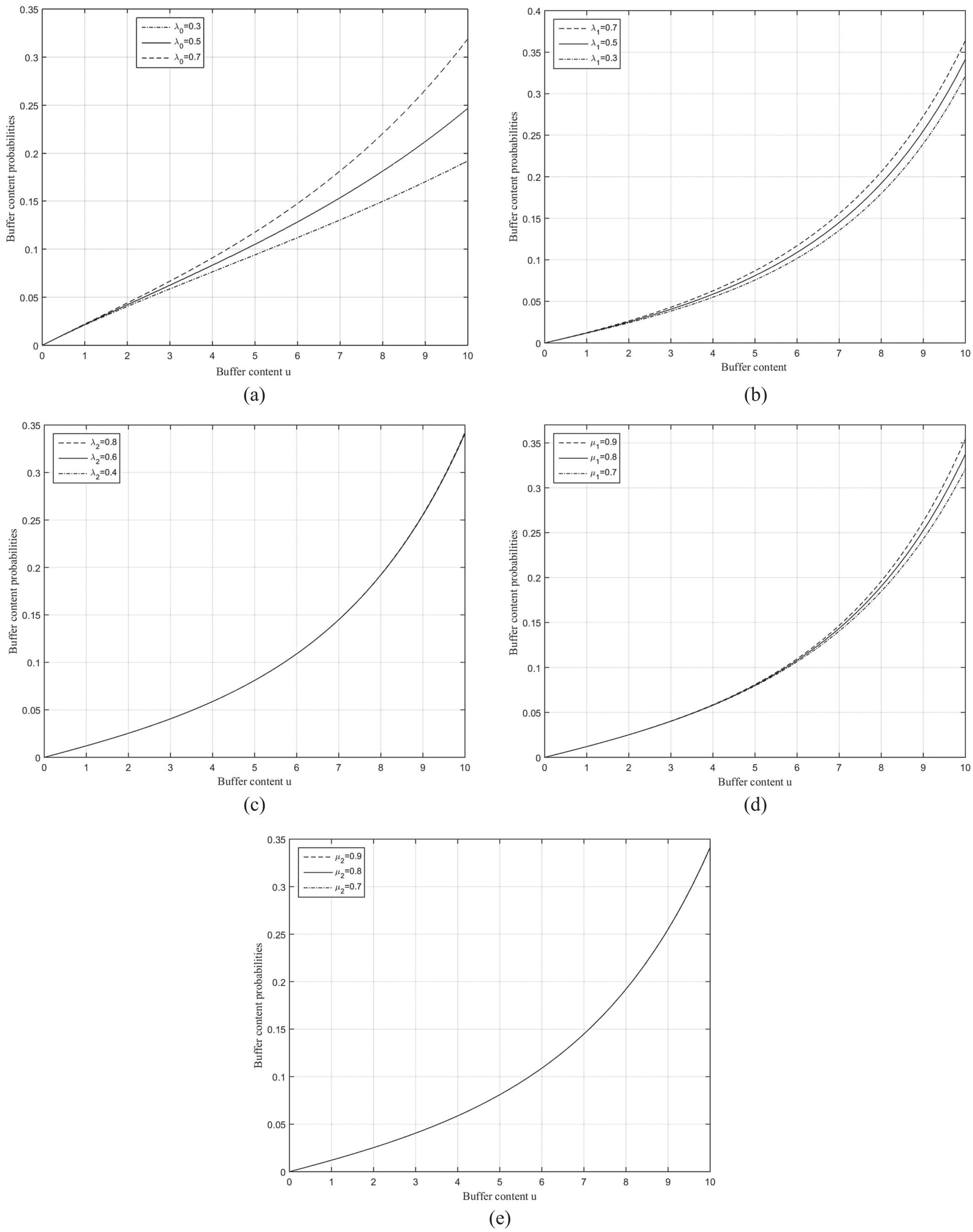

FIGURE 2. Effect of the arrival and service rates on $F(u)$. (a) $F(u)$ *versus* u for different λ_0 ($\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $\mu_1 = 0.7$, $\mu_2 = 0.8$). (b) $F(u)$ *versus* u for different λ_1 ($\lambda_0 = 0.3$, $\lambda_2 = 0.6$, $\mu_1 = 0.7$, $\mu_2 = 0.8$). (c) $F(u)$ *versus* u for different λ_2 ($\lambda_0 = 0.3$, $\lambda_1 = 0.6$, $\mu_1 = 0.7$, $\mu_2 = 0.8$). (d) $F(u)$ *versus* u for different μ_1 ($\lambda_0 = 0.3$, $\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $\mu_2 = 0.8$). (e) $F(u)$ *versus* u for different μ_2 ($\lambda_0 = 0.3$, $\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $\mu_1 = 0.7$).

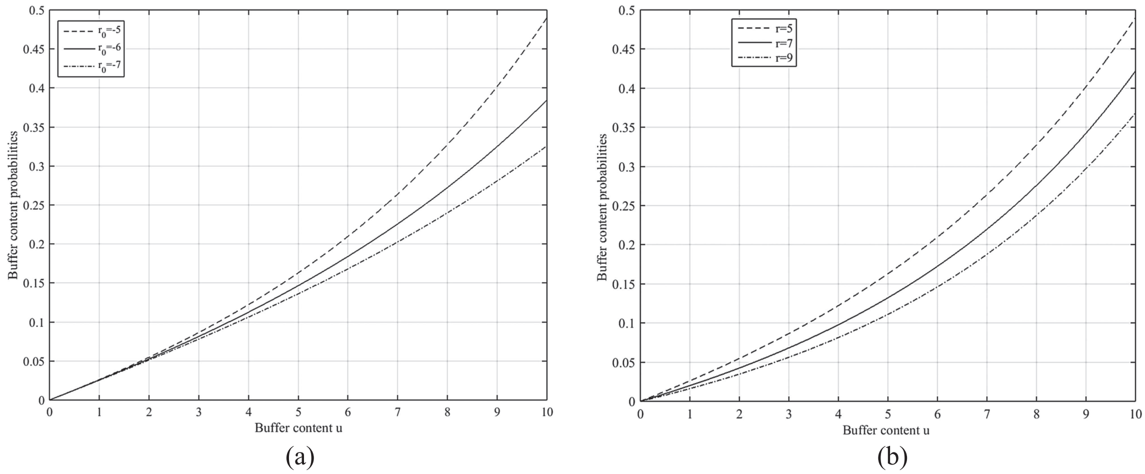

FIGURE 3. Impact of the net input rates on $F(u)$. (a) $F(u)$ *versus* u for different r_0 ($r_1 = 6$). (b) $F(u)$ *versus* u for different r_1 ($r_0 = -3$).

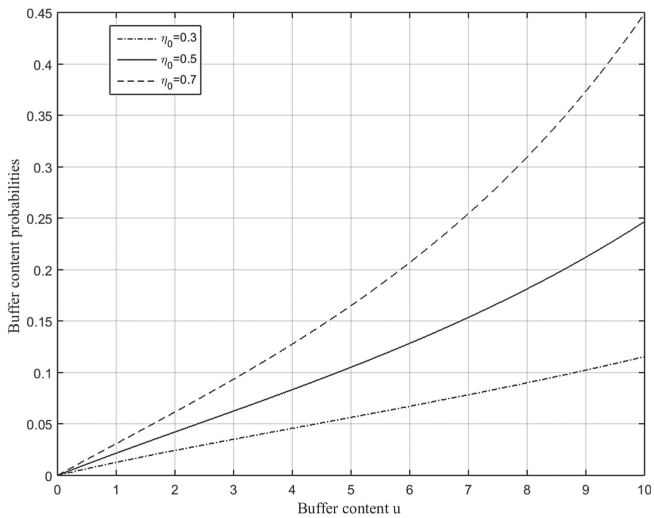

FIGURE 4. Influence of the disasters rate on $F(u)$.

6. Numerical illustrations

In this section, the analytical results described in the previous sections are discussed numerically. A numerical discussion is achieved by presenting some numerical examples illustrating the general behavior of the proposed system and the effect of various parameters on it. Without a lack of generality, we assume that $n = 2$, which means that the system will have two operating phases and a failure phase.

The convolutional term, expressed based on infinite sums, transforms, and Bessel functions involved in the aforementioned expressions is evaluated using the in-built command "quad" in MATLAB, which numerically evaluates the integral within an error of 10^{-6} using a recursive adaptive Simpson quadrature.

The influence of both the arrival rates and service rates on the buffer content distribution are illustrated in Figure 2. Thus, the plots of $F(u)$ *versus* the buffer content u are presented in Figure 2 for different arrival

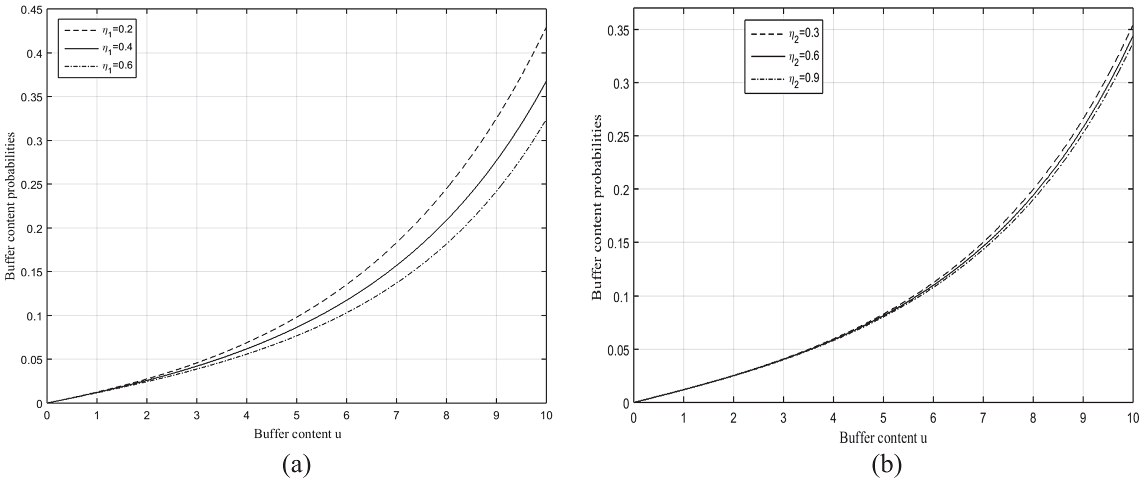

FIGURE 5. Effect of the repair rates on $F(u)$. (a) $F(u)$ *versus* u for different η_1 ($\eta_2 = 0.7$). (b) $F(u)$ *versus* u for different η_2 ($\eta_1 = 0.5$).

and service rates, respectively. The values of the other parameters are given by $r_0 = -3$, $r = 5$, $\eta_1 = 0.5$, $\eta_2 = 0.7$, and $p = 0.6$. It can be seen that variations in the arrival and services rates have a clear effect on $F(u)$, particularly in the failure and first operative phases, where the values of $F(u)$ increase with an increase in the values of u.

The effects of the net input rates r_0 and r on the behavior of the buffer content distribution are depicted in Figure 3, where $\lambda_0 = 0.3$, $\lambda_1 = 0.4$, $\lambda_2 = 0.5$, $\mu_1 = 0.6$, $\mu_2 = 0.8$, $\eta_0 = 0.6$, $\eta_1 = 0.5$, $\eta_2 = 0.7$, and $p = 0.6$. It can be seen from Figure 3 that increasing the values of the net input r and the absolute values of the net input rate r_0 lead to a considerable decrease in the values of the buffer content distribution.

In Figure 4, $\lambda_0 = 0.3$, $\lambda_1 = 0.4$, $\lambda_2 = 0.5$, $\mu_1 = 0.6$, $\mu_2 = 0.8$, $r_0 = -3$, $r = 5$, $\eta_1 = 0.5$, $\eta_2 = 0.7$, and $p = 0.6$. Figure 4 shows the influence of the disaster rate on $F(u)$ for different values of η_0. Interestingly, from Figure 4, we can see that $F(u)$ decreases with an increase in η_0.

The effect of the repair rates on the buffer content distribution *versus* the buffer content parameter has been presented in Figure 5, where the parameters have the following values: $\lambda_0 = 0.3$, $\lambda_1 = 0.4$, $\lambda_2 = 0.5$, $\mu_1 = 0.6$, $\mu_2 = 0.8$, $\eta_0 = 0.6$, and $p = 0.6$. It should be noted that repair rate variations have a significant impact on $F(u)$, where the increase in the value of the buffer content u contributes to a significant reduction in the buffer content distribution.

It is generally observed that the buffer content distribution monotonically increases when the values of the buffer content u goes on. In addition, the increase and decrease change depending on the type of parameter controlling the general behavior of $F(u)$. We observed that the buffer content distribution increases when both the arrival and services rates increase, whereas it decreases when the disaster and net input rates increase. This follows because, upon failure, all units are cleared from the system.

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