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ALGORITHMS FOR FOUR-MACHINE FLOWSHOP SCHEDULING PROBLEM WITH UNCERTAIN PROCESSING TIMES TO MINIMIZE MAKESPAN

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Abstract. We consider the four-machine flowshop scheduling problem to minimize makespan where processing times are uncertain. The processing times are within some intervals, where the only available information is the lower and upper bounds of job processing times. Some dominance relations are developed, and twelve algorithms are proposed. The proposed algorithms first convert the four-machine problem into two stages, then, use the well-known Johnson's algorithm, known to yield the optimal solution for the two-stage problem. The algorithms also use the developed dominance relations. The proposed algorithms are extensively evaluated through randomly generated data for different numbers of jobs and different gaps between the lower and upper bounds of processing times. Computational experiments indicate that the proposed algorithms perform well. Moreover, the computational experiments reveal that one of the proposed algorithms, *Algorithm A7*, performs significantly better than the other eleven algorithms for all possible combinations of the number of jobs and the gaps between the lower and upper bounds. More specifically, error percentages of the other eleven algorithms range from 2.3 to 27.7 times that of *Algorithm A7*. The results have been confirmed by constructing 99% confidence intervals and tests of hypotheses using a significance level of 0.01.

Mathematics Subject Classification. 20-XX.

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1. Introduction

Fuchigami and Rangel [13] provide a recent survey of practical research and case studies in production scheduling. They consider research papers addressing different industries including food, electronics, furniture, pharmaceuticals, and chemistry. They highlight that more than two thirds of the considered real life case studies were in the flowshop scheduling area including four-machine flowshop, which includes the focus of this paper, namely four machines flowshop. For example, Stefansson [21] study a real-world scheduling problem from a pharmaceutical company where the objective is to minimize total weighted tardiness. The scheduling problem consists of four production stages (machines) where each of the four operations of granulation, compression, coating, and packing is performed sequentially.

Keywords. Flowshop scheduling, makespan, uncertain processing times, algorithm.

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The objective to minimize is makespan in this paper. Minimizing makespan is the main objective considered by companies. It considerably increases the revenue by decreasing the number of late deliveries and reducing inventories. Furthermore, a shorter makespan decreases production costs.

In most of the scheduling literature, it is assumed that processing times are deterministic, Seidgar *et al.* [19] and Keshavarz and Salmasi [16]. Nevertheless, Gonzalez-Neira *et al.* [15] and Wang and Choi [23] point out that real world manufacturing systems are often subject to a wide range of uncertainties. Hence, it is not realistic to assume deterministic processing times for some real-life manufacturing systems since there are several reasons causing uncertainty in processing times. Examples include untested processing technology, the condition of tools, machine operator fatigue, condition of auxiliary devices for holding the job at the appropriate position, and disruptions in manufacturing systems, *e.g.*, Tayanithi *et al.* [22]. Furthermore, the originally estimated time may not be precise as there may be absent workers, late materials, and so on. Thus, uncertainty should be incorporated into the planning phase and production control must consider these in advance to avoid any possible inconvenience. Also, past data may not be available for new jobs, hence making it difficult to predict their exact duration.

For some manufacturing systems, processing times may be uncertain as a result of changes in the dynamic scheduling environments. In such a manufacturing system, processing times may be assumed to be random variables with a specific probability distribution. Such an assumption may not be valid for some cases. For example, we may not have enough prior information to describe the probability distribution. Kouvelis and Yu [17] state that distributional assumption is unsuitable for some manufacturing systems where factors such as tool conditions and worker skill levels control the uncertainty of processing times. For instance, the tool condition might affect the processing times of the jobs. In general, obtaining upper and lower bounds on processing times are easier, even in cases where the probability distribution may not be known beforehand.

In an uncertain model, the lower and upper bounds of processing times may be close to each other if we have some perturbations for some jobs. However, we can set the lower and upper bounds far from each other for some other jobs for which no perturbations of processing times are available. Yet for some other jobs when processing times are known, the lower and upper bounds can be set to the same value.

The processing times are random variables with unidentified probability distributions. Let $t_{j,k}$ denote processing time of job $j\,(j \in J = \{1, 2, \ldots, n\})$ on machine $k\,(k \in K = \{1, 2, \ldots, m\})$ The lower bound $Lt_{j,k} \geq 0$ and the upper bound $Ut_{j,k} \geq Lt_{j,k}$ of $t_{j,k}$ is the only known information. Therefore, $t_{j,k}$ satisfies $Lt_{j,k} \leq t_{j,k} \leq Ut_{j,k}$.

Allahverdi and Sotskov [7] provide some dominance relations while Allahverdi and Aydilek [5] provide several polynomial time algorithms for the problem of $F2|Lt_{j,k} \leq t_{j,k} \leq Ut_{j,k}|C_{\max}$. They show that one of their proposed algorithms yields a solution that is close to the optimal solution. On the other hand, Allahverdi and Aydilek [6] propose fourteen heuristics for the $F2|Lt_{j,k} \leq t_{j,k} \leq Ut_{j,k}|L_{\max}$ problem and experimentally demonstrate that one of their proposed heuristics performs best. Moreover, Sotskov *et al.* [20] provide some dominance relations while Aydilek and Allahverdi [8] present eleven heuristics, utilizing the lower and upper bounds on job processing times, for the $F2|Lt_{j,k} \leq t_{j,k} \leq Ut_{j,k}|\sum C_i$ problem. Through the computational experiments, they show that one of their heuristics performs best for different distributions of processing times. Sotskov *et al.* [20] also provide some dominance relations for the $F3|Lt_{j,k} \leq t_{j,k} \leq Ut_{j,k}|\sum C_i$ problem.

When job processing times are uncertain within lower and upper bounds, some other scheduling problems have been addressed in the literature for other scheduling environments, *e.g.*, Aydilek *et al.* [9–11], Allahverdi and Allahverdi [3, 4].

In this paper, the problem of $F4|\text{prmu},\, Lt_{j,k} \leq t_{j,k} \leq Ut_{j,k}|C_{\max}$ is addressed for the first time. Dominance relations are established, and twelve algorithms are proposed, and the algorithms are compared with each other. Problem formulation and dominance relations are described in the following section. The algorithms are defined in the next section and computational experiments are provided in Section 4. Lastly, concluding remarks are provided the last section.

2. Problem formulation and dominance relations

We consider a four-machine flowshop scheduling problem where there are four machines numbered from 1 to 4. There are n jobs available for scheduling. Each job has to be processed first by machine 1, then by machine 2, next by machine 3, and finally by machine 4. Therefore, a job can start its process on machine k only after its processing is finished on machine $k-1$. We consider permutation flowshop where the processing order of the jobs on all the machines is the same. The objective is to minimize makespan, completion time of the last job in the set of n jobs.

Let $t_{i,k}$ denote the processing time of job i on machine k and $t_{[j,k]}$ denote the processing time of the job in position j on machine k. Let $\mathrm{TP}_{[j,k]}$ represent the total processing time of jobs in positions $1, 2, \ldots, j$ on machine k, *i.e.*, $\mathrm{TP}_{[j,k]} = t_{[1,k]} + \ldots + t_{[j,k]}$. Also let $\Delta_{[j,k]}$ be the total idle time on the kth machine up to the completion time of the job in position j. It should be noted that there will be no idle time on the first machine, and hence, $\Delta_{[j,1]} = 0$ for $j = 1, \ldots, n$.

Let $\phi_{[j,2]} = \mathrm{TP}_{[j,1]} - TP_{[j-1,2]}$ where $\mathrm{TP}_{[0,2]} = 0$. Then, the total idle time until the job in position j is completed on the second machine is given by Allahverdi [1],

$$\Delta_{[j,2]} = \max\left\{\phi_{[1,2]}, \phi_{[2,2]}, \ldots, \phi_{[j,2]}\right\}. \tag{2.1}$$

Let $\phi_{[j,3]} = \Delta_{[j,2]} + \mathrm{TP}_{[j,2]} - \mathrm{TP}_{[j-1,3]}$ where $P_{[0,3]} = 0$. It is known that [2]

$$\Delta_{[j,3]} = \max\left\{\phi_{[1,3]}, \phi_{[2,3]}, \ldots, \phi_{[j,3]}\right\}. \tag{2.2}$$

Let $\phi_{[1,4]} = \Delta_{[j,3]} + TP_{[j,3]} - TP_{[j-1,4]}$ where $P_{[0,4]} = 0$, it can be shown that

$$\Delta_{[j,4]} = \max\left\{\phi_{[1,4]}, \phi_{[2,4]}, \ldots, \phi_{[j,4]}\right\}. \tag{2.3}$$

Therefore, $C_{[j,k]} = \mathrm{TP}_{[j,k]} + \Delta_{[j,k]}$ for $j = 1, 2, \ldots, n$ and $k = 1, 2, 3, 4$ where $C_{[j,k]}$ denotes the completion time of the job in position j on machine k. Hence, the makespan ($C_{\max}$) for a four-machine flowshop is given by

$$C_{\max} = \mathrm{TP}_{[n,4]} + \Delta_{[n,4]}. \tag{2.4}$$

Since the value of the term $\mathrm{TP}_{[n,4]}$ is constant regardless of the sequence, minimizing $C_{\max}$ is equivalent to minimizing $\Delta_{[n,4]}$.

Consider a job sequence π_1 such that job h is in arbitrary position α and job g is in position $\alpha + 1$ in sequence π_1. Let sequence π_2 be derived from sequence π_1 by only interchanging jobs h and g. Let σ_1 denote the subsequence containing the jobs in positions $1, 2, \ldots, \alpha - 1$, and σ_2 denote the subsequence containing the jobs in positions $\alpha + 2, \ldots, n$ where n denotes the number of jobs. Therefore, sequences π_1 and π_2 can be written as $\pi_1 = \{\sigma_1, h, g, \sigma_2\}$ and $\pi_2 = \{\sigma_1, g, h, \sigma_2\}$.

Theorem 2.1. *If one of the following seven conditions (C1-1 to C1-7) hold,*

$$\begin{aligned}
&\text{C1-1: } Ut_{g,1} \le Lt_{h,1} \quad \text{and} \quad Ut_{h,2} \le Lt_{g,2},\\
&\text{C1-2: } Ut_{g,2} \le Lt_{h,1} \quad \text{and} \quad Ut_{h,2} \le Lt_{g,2},\\
&\text{C1-3: } Ut_{g,1} \le Lt_{h,1} \quad \text{and} \quad Ut_{g,1} \le Lt_{g,2},\\
&\text{C1-4: } Ut_{h,2} \le Lt_{h,1} \quad \text{and} \quad Ut_{h,2} \le Lt_{g,2},\\
&\text{C1-5: } Ut_{g,2} \le Lt_{h,1} \quad \text{and} \quad Ut_{g,1} \le Lt_{g,2},\\
&\text{C1-6: } Ut_{h,1} \le Lt_{g,2} \quad \text{and} \quad Ut_{g,1} \le Lt_{h,1},\\
&\text{C1-7: } Ut_{h,1} \le Lt_{g,2} \quad \text{and} \quad Ut_{h,2} \le Lt_{h,1},
\end{aligned}$$

then,

$$\max\{\phi_{[\alpha,2]}\left(\pi_2\right), \phi_{[\alpha+1,2]}\left(\pi_2\right)\} \le \max\{\phi_{[\alpha,2]}\left(\pi_1\right), \phi_{[\alpha+1,2]}\left(\pi_1\right)\}.$$

In other words, the contribution to the idle time on the second machine from jobs in positions α and $\alpha + 1$ in the sequence π_2 is less than or equal to that of the jobs in positions α and $\alpha + 1$ in the sequence π_1.

Proof. The proof is given in the Appendix A. ☐

Theorem 2.2. *If one of the following six conditions (C2-1 to C2-6) hold,*

$$\begin{aligned}
&\text{C2-1: } Ut_{g,1} \leq Lt_{h,1} \quad \text{and} \quad Ut_{h,2} \leq Lt_{g,2} \quad \text{and} \quad Ut_{g,3} \leq Lt_{h,2} \quad \text{and} \quad Ut_{h,3} \leq Lt_{g,3},\\
&\text{C2-2: } Ut_{g,1} \leq Lt_{h,1} \quad \text{and} \quad Ut_{g,1} \leq Lt_{g,2} \quad \text{and} \quad Ut_{g,2} \leq Lt_{h,2} \quad \text{and} \quad Ut_{g,2} \leq Lt_{g,3},\\
&\text{C2-3: } Ut_{g,1} \leq Lt_{h,1} \quad \text{and} \quad Ut_{h,2} \leq Lt_{g,2} \quad \text{and} \quad Ut_{h,3} \leq Lt_{h,2} \quad \text{and} \quad Ut_{h,3} \leq Lt_{g,3},\\
&\text{C2-4: } Ut_{g,1} \leq Lt_{h,1} \quad \text{and} \quad Ut_{g,1} \leq Lt_{g,2} \quad \text{and} \quad Ut_{g,2} \leq Lt_{g,3} \quad \text{and} \quad Ut_{g,3} \leq Lt_{h,2},\\
&\text{C2-5: } Ut_{g,1} \leq Lt_{h,1} \quad \text{and} \quad Ut_{h,1} \leq Lt_{g,2} \quad \text{and} \quad Ut_{g,2} \leq Lt_{h,2} \quad \text{and} \quad Ut_{h,2} \leq Lt_{g,3},\\
&\text{C2-6: } Ut_{g,1} \leq Lt_{h,1} \quad \text{and} \quad Ut_{h,1} \leq Lt_{g,2} \quad \text{and} \quad Ut_{h,2} \leq Lt_{g,3} \quad \text{and} \quad Ut_{h,3} \leq Lt_{h,2},
\end{aligned}$$

then,

$$\max\{\phi_{[\alpha,3]}(\pi_2), \phi_{[\alpha+1,3]}(\pi_2)\} \leq \max\{\phi_{[\alpha,3]}(\pi_1), \phi_{[\alpha+1,3]}(\pi_1)\}.$$

In other words, contribution to the idle time on the third machine from jobs in positions α and $\alpha+1$ in the sequence π_2 is less than or equal to that of the jobs in positions α and $\alpha+1$ in the sequence π_1.

Proof. The proof is provided in the Appendix A. ☐

Theorem 2.3. *If one of the following seven conditions (C3-1 to C3-7) hold,*

C3-1: $Ut_{g,r} \leq Lt_{h,r}$ for $r = 1, 2, 3$ and $Ut_{h,1} \leq Lt_{g,2}$ and $Ut_{h,2} \leq Lt_{g,3}$ and $Ut_{h,4} \leq Lt_{g,4}$,
C3-2: $Ut_{h,1} \leq Lt_{g,2}$ and $Ut_{h,2} \leq Lt_{g,3}$ and $Ut_{h,4} \leq Lt_{g,4}$ and $Ut_{g,4} \leq Lt_{h,3}$,
C3-3: $Ut_{g,r} \leq Lt_{h,r}$ for $r = 1, 2, 3$ and $Ut_{g,r} \leq Lt_{g,r+1}$ for $r = 1, 2, 3$,
C3-4: $Ut_{g,r} \leq Lt_{h,r}$ for $r = 1, 2$ and $Ut_{h,1} \leq Lt_{g,2}$ and $Ut_{h,2} \leq Lt_{g,3}$ and $Ut_{h,4} \leq Lt_{g,4}$ and $Ut_{h,4} \leq Lt_{h,3}$,
C3-5: $Ut_{g,1} \leq Lt_{h,1}$ and $Ut_{h,1} \leq Lt_{g,2}$ and $Ut_{g,2} \leq Lt_{h,2}$ and $Ut_{h,2} \leq Lt_{g,3}$ and $Ut_{g,3} \leq Lt_{g,4}$, and $Ut_{g,4} \leq Lt_{h,3}$,
C3-6: $Ut_{g,r} \leq Lt_{h,r}$ for $r = 1, 2, 3$ and $Ut_{h,r} \leq Lt_{g,r+1}$ for $r = 1, 2, 3$,
C3-7: $Ut_{h,r} \leq Lt_{g,r+1}$ for $r = 1, 2, 3,$ and $Ut_{g,r} \leq Lt_{h,r}$ for $r = 1, 2$ and $Ut_{h,4} \leq Lt_{h,3}$,

then,

$$\max\{\phi_{[\alpha,4]}(\pi_2), \phi_{[\alpha+1,4]}(\pi_2)\} \leq \max\{\phi_{[\alpha,4]}(\pi_1), \phi_{[\alpha+1,4]}(\pi_1)\}.$$

which means that the contribution to the idle time on the fourth machine from jobs in positions α and $\alpha+1$ in the sequence π_2 is less than or equal to that of the jobs in positions α and $\alpha+1$ in the sequence π_1.

Proof. The proof is given in the Appendix A. ☐

Lemma 2.4. *For $r = 1, 2, \ldots, \alpha - 1$,*

$$\begin{aligned}
\phi_{[r,2]}(\pi_2) &= \phi_{[r,2]}(\pi_1),\\
\phi_{[r,3]}(\pi_2) &= \phi_{[r,3]}(\pi_1),\\
\phi_{[r,4]}(\pi_2) &= \phi_{[r,4]}(\pi_1).
\end{aligned}$$

Proof. It follows from the fact that both sequences π_1 and π_2 have the same jobs in position $1, 2, \ldots, \alpha - 1$. ☐

Lemma 2.5. $\phi_{[r,2]}(\pi_2) = \phi_{[r,2]}(\pi_1)$ *for $r = \alpha + 2, \ldots, n$.*

Proof. The proof is provided in the Appendix A. ☐

Lemma 2.6. *$\phi_{[r,3]}(\pi_2) \leq \phi_{[r,3]}(\pi_1)$ for $r = \alpha + 2, \ldots, n$ if*

$$\max\{\phi_{[\alpha,2]}(\pi_2), \phi_{[\alpha+1,2]}(\pi_2)\} \leq \max\{\phi_{[\alpha,2]}(\pi_1), \phi_{[\alpha+1,2]}(\pi_1)\}.$$

Proof. The proof is given in the Appendix A. □

Lemma 2.7. *$\phi_{[r,4]}(\pi_2) \leq \phi_{[r,4]}(\pi_1)$ for $r = \alpha + 2, \ldots, n$ if*

$$\max\{\phi_{[\alpha,3]}(\pi_2), \phi_{[\alpha+1,3]}(\pi_2)\} \leq \max\{\phi_{[\alpha,3]}(\pi_1), \phi_{[\alpha+1,3]}(\pi_1)\}.$$

Proof. The proof is provided in the Appendix A. □

Theorem 2.8. *If any of the following conditions (C4-1 to C4-5) hold,*

C4-1: $Ut_{g,r} \leq Lt_{h,r}$ for $r = 1, 2, 3$ and $Ut_{h,1} \leq Lt_{g,2}$ and $Ut_{h,2} \leq Lt_{g,3}$ and $Ut_{h,4} \leq Lt_{g,4}$,
C4-2: $Ut_{g,r} \leq Lt_{h,r}$ for $r = 1, 2, 3$ and $Ut_{g,r} \leq Lt_{g,r+1}$ for $r = 1, 2, 3$,
C4-3: $Ut_{g,r} \leq Lt_{h,r}$ for $r = 1, 2$ and $Ut_{h,1} \leq Lt_{g,2}$ and $Ut_{h,2} \leq Lt_{g,3}$ and $Ut_{h,4} \leq Lt_{g,4}$ and $Ut_{h,4} \leq Lt_{h,3}$,
C4-4: $Ut_{g,r} \leq Lt_{h,r}$ for $r = 1, 2, 3$ and $Ut_{h,r} \leq Lt_{g,r+1}$ for $r = 1, 2, 3$,
C4-5: $Ut_{h,r} \leq Lt_{g,r+1}$ for $r = 1, 2, 3,$ and $Ut_{g,r} \leq Lt_{h,r}$ for $r = 1, 2$ and $Ut_{h,4} \leq Lt_{h,3}$,

then, $C_{\max}(\pi_2) \leq C_{\max}(\pi_1)$. Therefore, job g should precede job h whenever they are adjacent in order to minimize makespan.

Proof. The proof is given in the Appendix A. □

3. Proposed algorithms

It is known that the addressed problem is NP-hard since even when $Ut_{i,k} = Lt_{i,k}$ for all $i = 1, 2, \ldots, n$ and all machines k, the problem is NP-hard for $k = 3$ [14, 18]. In general, $Ut_{i,k} \neq Lt_{i,k}$ for at least some jobs. Furthermore, the precise realization of the processing time $t_{j,k}$ is not known until job j finishes its processing on machine k. On the other hand, a decision on the position of job j in a sequence on machine k has to be made before the exact realization of $t_{j,k}$. This implies that a decision on position of job j on machine k in a sequence has to be made only based on the available information, which is $Ut_{i,k}$ and $Lt_{i,k}$.

It should be noted that advanced algorithms such as meta-heuristics cannot be used since processing times ($t_{j,k's}$) are not known, Allahverdi and Aydilek [5]. Therefore, the proposed algorithms should utilize $Ut_{i,k}$ and $Lt_{i,k}$ instead of $t_{j,k}$.

Johnson's algorithm finds the optimal solution for the two-machine flowshop scheduling problem for minimizing makespan when $Ut_{i,k} = Lt_{i,k}$ for all i and k. Therefore, we propose an algorithm which is based on Johnson's algorithm by reducing the four-machine problem to two stages and by utilizing $Ut_{i,k}$ and $Lt_{i,k}$. Depending on the weights assigned to the machines, the algorithm results in different algorithms. Below are steps of the algorithms.

Statements of algorithms

Given n, and the values $Lt_{i,k}$ and $Ut_{i,k}$ for $i = 1, \ldots, n$ and $k = 1, \ldots, 4$.
Choose $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (see Tab. 1)
  For algorithm $A1, \ldots, A9$ or $M23$
    Let $a_i = \lambda_1((Lt_{i,1} + Ut_{i,1})/2) + \lambda_2((Lt_{i,2} + Ut_{i,2})/2)$ for $i = 1, \ldots, n$
    Let $b_i = \lambda_3((Lt_{i,3} + Ut_{i,3})/2) + \lambda_4((Lt_{i,4} + Ut_{i,4})/2)$ for $i = 1, \ldots, n$
  For algorithm $M12$
    Let $a_i = (Lt_{i,1} + Ut_{i,1})/2$ for $i = 1, \ldots, n$

TABLE 1. Description of algorithms $A1, \ldots, A9$ and $M23$.

Algorithm	λ_1	λ_2	λ_3	λ_4
$A1$	0.25	0.75	0.25	0.75
$A2$	0.25	0.75	0.50	0.50
$A3$	0.25	0.75	0.75	0.25
$A4$	0.50	0.50	0.25	0.75
$A5$	0.50	0.50	0.50	0.50
$A6$	0.50	0.50	0.75	0.25
$A7$	0.75	0.25	0.25	0.75
$A8$	0.75	0.25	0.50	0.50
$A9$	0.75	0.25	0.75	0.25
$M23$	0.00	1.00	1.00	0.00

Let $b_i = (Lt_{i,2} + Ut_{i,2})/2$ for $i = 1, \ldots, n$
For algorithm $M34$
Let $a_i = (Lt_{i,3} + Ut_{i,3})/2$ for $i = 1, \ldots, n$
Let $b_i = (Lt_{i,4} + Ut_{i,4})/2$ for $i = 1, \ldots, n$
End For
Let $\pi_1 = \{1, 2, \ldots, n\}$ and $\pi_s = \phi$
Let $d_1 = 1$, and $d_2 = d_3 = 0$
While $d_3 < n$,
Let $a_m = \min\{a_r\}$ and $b_m = \min\{b_r\}$ where $r \in \pi$
If $a_m \leq b_m$, place that job in position d_1 of π_s and let $d_1 = d_1 + 1$
Else place that job in position $n - d_2$ of π_s and let $d_2 = d_2 + 1$
Remove that job from the sequence π
Set $d_3 = d_3 + 1$
End While
Assign the last job in π to the only remaining position of π_s
Set $p = 1$
While $p < n$,
Set $t_{h,k} = t_{[p,k]}(\pi_s)$ for $k = 1, \ldots, 4$ and $t_{g,k} = t_{[p+1,k]}(\pi_s)$ for $k = 1, \ldots, 4$
If $t_{h,k}$ and $t_{g,k}$ satisfy any of the conditions C4-1 to C4-5 of Theorem 2.8, swap the jobs in positions p and $p + 1$ of the sequence π_s
Else Set $p = p + 1$
End While

The sequence π_s is the solution of the algorithm.

For a given problem the only known information is the number of jobs (n), and the lower and upper bounds of processing times, *i.e.*, $Lt_{i,k}$ and $Ut_{i,k}$ for $i = 1, \ldots, n$ and $k = 1, \ldots, 4$. The algorithms $A1$–$A9$ and $M23$ have four parameters of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ where λ_k denotes the weight given to the lower and upper bounds on processing times on the kth machine. We consider different values for λ_k parameters, which result in 10 algorithms ($A1-A9$, and $M23$) defined in Table 1. Please note that the other two algorithms of $M12$ and $M34$ do not have the λ_k parameters, and hence are not included in the table. It should be noted that different combinations for λ_k parameters have been investigated for different problems and it was observed that those results are not significantly different than the results with the combinations of the parameters given in Table 1.

The idea beyond algorithm $M12$ is to minimize the total idle time on the second machine by applying the Johnson's algorithm where weighted average of processing times of lower and upper bounds is used. Similarly, the objectives for algorithm *M23* and *M34* are to minimize total idle time on the third machine and on the fourth machine, respectively.

4. Algorithm evaluations

The proposed twelve algorithms of $A1-A9$, $M12$, $M23$, and $M34$, are evaluated utilizing randomly generated data. $Ut_{i,k}$ is generated from $U(D+1, 100)$, which is commonly used in the scheduling literature for generating processing times. $Lt_{i,k}$ is also generated from a uniform distribution between 1 and $Ut_{i,k}-D$, *i.e.*, $U(1, Ut_{i,k}-D)$, where D is set at four values of 10, 20, 30, and 40. D denotes the gap between the upper and lower bounds. This is how Aydilek *et al.* [11] also generated data for their scheduling problem with bounded processing times. After $Lt_{i,k}$ and $Ut_{i,k}$ are generated, instances of processing times are generated from different distributions.

Even though the distribution of processing times is not known, instances of processing times have to be generated for computational purposes and it would not be appropriate to only generate instances between the lower and upper bounds following the uniform distribution between the lower and upper bounds. Therefore, we also consider positive linear, negative linear, and normal distributions for generating processing times between $Lt_{i,k}$ and $Ut_{i,k}$. Normal and uniform distributions are examples of symmetric distributions while positive and negative linear distributions are examples of extreme distributions. Therefore, the considered distributions are representatives of many distributions [11].

We now explain the four distributions for generating processing times. Essentially, $t_{i,k}$ is generated from $U(Lt_{i,k}, Ut_{i,k})$ for the uniform distribution. On the other hand, for the normal distribution, the mean μ is set to $(Lt_{i,k} + Ut_{i,k})/2$ and the standard deviation μ is set to $(Ut_{i,k} - Lt_{i,k})/6$. Notice that with a small probability the generated value of $t_{i,k}$ will be outside $Lt_{i,k}$ and $Ut_{i,k}$. If this happens, we regenerate $t_{i,k}$ so that this value will fall between $Lt_{i,k}$ and $Ut_{i,k}$. In other words, the normal distributions for generating $t_{i,k}$ is truncated. For negative linear and positive linear distributions, the probability density functions are $f_n(t_{i,k}) = 2(Ut_{i,k} - t_{i,k})/x(Ut_{i,k} - Lt_{i,k})^2$ and $f_p(t_{i,k}) = 2(t_{i,k} - Lt_{i,k})/x(Ut_{i,k} - Lt_{i,k})^2$ for $t_{i,k} \in (Lt_{i,k}, Ut_{i,k})$, respectively. Figure 1 shows all the four distributions for generating processing times between the lower and upper bounds.

We compare the performance of the algorithms by using two measures of performance; average error percentage (Error) and standard deviation (Std). The error percentage of Algorithm X is computed as 100∗ ($C_{\max}$(Algorithm X)-minimum makespan of all the algorithms)/minimum of makespan of all the algorithms. This error is in fact a relative error since the solutions of the algorithms are compared with each other rather than compared with the optimal solution, which is unknown.

We consider five different values for n, which are 100, 200, 300, 400, and 500. Moreover, we consider four values of 10, 20, 30, and 40 for D and four distributions (positive linear, negative linear, uniform, and normal). This results in 80 (5∗ 4∗ 4) combinations. For each combination, 1000 replications were generated. This results in a total of 80 000 problems generated.

A random sequence was initially included in the evaluations in addition to the considered algorithms. The computational experiments indicated that the error of the random sequence was about ten times that of the worst performing algorithms. This shows that even the worst algorithm performs well. Therefore, the random sequence was excluded from computations and the algorithms $A1-A9$, $M12, M23, M34$ were compared with each other for the rest of the analysis.

The computational results are given in Tables 3–6 for the uniform distribution, normal distribution, positive linear distribution, and negative linear distribution, respectively. The first column in the tables denotes the algorithm, the second column shows D (the gap between the lower and upper bounds of processing times), the next five columns denote the average error percentages of the algorithms for $n = 100, 200, 300, 400, 500$, respectively, and the final column indicates the average error percentage over n. The last row in the tables illustrates the average error percentage over D.

The results in Tables 3–6 are summarized in Figure 2 where the average error percentages of all the algorithms over n and D values are illustrated with respect to the considered four distributions for generating the

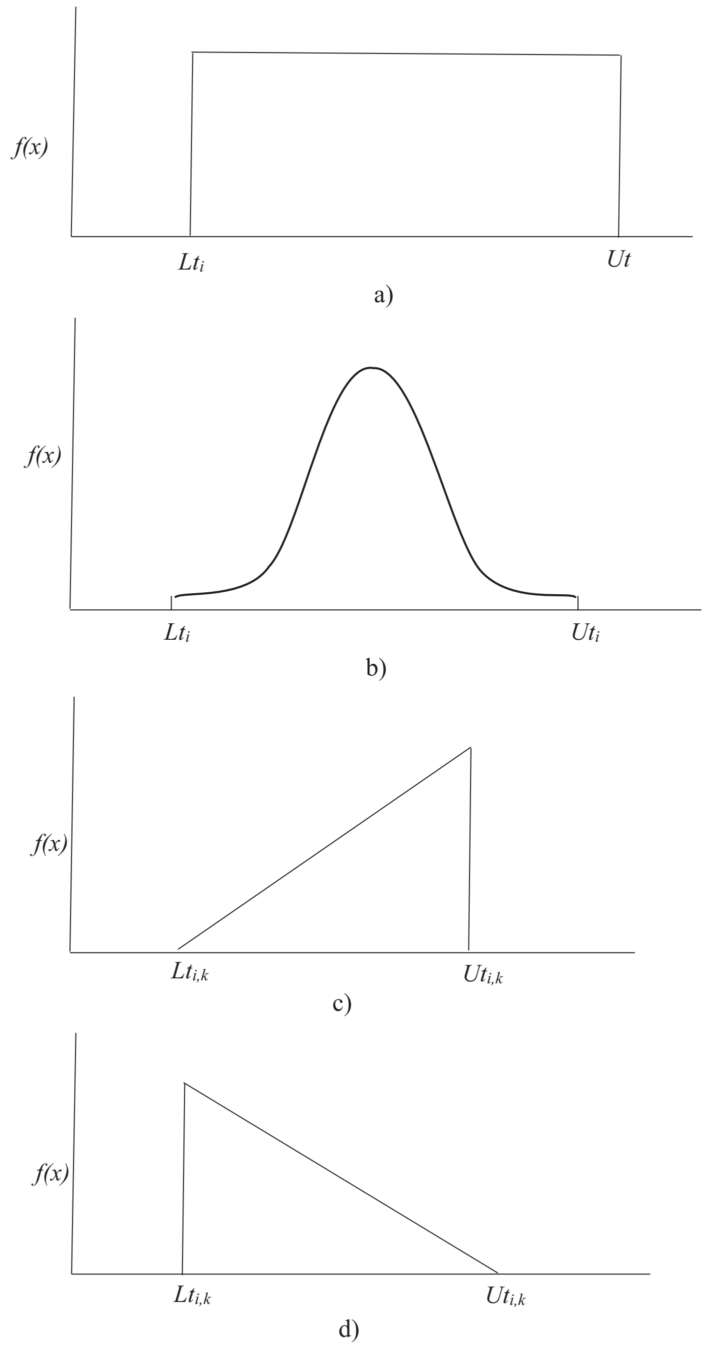

FIGURE 1. The four distributions used for generating $t_{i,k}$ between $Lt_{i,k}$ and $Ut_{i,k}$. (a) Uniform, (b) Normal, (c) Positive linear, and (d) Negative linear.

TABLE 2. Confidence intervals.

Algorithm	Avg. error	95% Confidence interval on the Avg. error	99% Confidence interval on the Avg. error
$A1$	5.74	(5.48–5.99)	(5.39–6.07)
$A2$	6.08	(5.80–6.35)	(5.72–6.44)
$A3$	7.74	(7.41–8.06)	(7.31–8.17)
$A4$	1.15	(1.12–1.19)	(1.11–1.21)
$A5$	1.71	(1.65–1.77)	(1.64–1.79)
$A6$	6.08	(5.81–6.34)	(5.72–6.44)
$A7$	0.50	(0.48–0.52)	(0.47–0.53)
$A8$	1.17	(1.12–1.20)	(1.11–1.22)
$A9$	5.73	(5.48–5.98)	(5.40–6.06)
$M12$	12.78	(12.21–13.33)	(12.04–13.51)
$M23$	13.87	(13.24–14.49)	(13.04–14.70)
$M34$	3.54	(3.41–3.65)	(3.38–3.69)

processing times between the lower and upper bounds. Figure 3 shows the average standard deviations of the error percentages of the algorithms, again over n and D values, for the four distributions. The average error percentages and average standard deviations are proportional as can be seen from Figures 2 and 3. In other words, in general, the algorithms with large error percentages have large standard deviations while those algorithms with small error percentages have small standard deviations. This is due to the large number of replications (1000). As can be seen from Figures 2 and 3, the algorithms $A4$, $A5$, $A7$, and $A8$ perform better than the rest of the algorithms. Figures 4 and 5 compare the average error percentages and standard deviations only for these four algorithms, respectively.

The average error percentages of the all algorithms, over n and the four distributions, are summarized in Figure 6 with respect to D. As can be seen from the figure, the performance of the well performing four algorithms ($A4$, $A5$, $A7$, and $A8$) do not seem to depend on the D value. In other words, these four algorithms perform well regardless of the gap between the lower and upper bounds of the processing times.

Figure 7 illustrates the average error percentages of the algorithms, over D and the four distributions, with respect to the number of jobs, n. The performance of all the algorithms do not seem to depend on the number of jobs.

The effect of Theorem 2.8 in Step 17 of the algorithms was also investigated. It was observed that the improvement on the objective value, on the average, was less than 10%.

The overall average errors of all the algorithms along with 95% and 99% confidence intervals are given in Table 2.

Table 2 confirms that the best performing algorithms are $A4$, $A5$, $A7$, and $A8$ among the considered twelve algorithms. It should be noted that algorithms $A1$–$A9$ take into account the processing times of the jobs on all the four machines while $M12, M23$, and $M34$ only take into account the processing times of the jobs on only two machines. Perhaps that is the reason why algorithms $A1$–$A9$ perform better than the rest. However, in general, it is interesting to observe that the algorithm $M34$ performs better than algorithms $A1, A2, A3, A6$, and $A9$. The reason could be that the objective in $M34$ is to minimize the total idle time on the fourth machine which indirectly minimizes makespan. The good performance of algorithms $A4, A5, A7$, and $A8$ indicates that giving more weight to the processing times of jobs on machine one than on those on machine two along with giving more weights to processing times of jobs on machine four (last machine) than those on machine three yield better results.

As can be seen from Table 2, the algorithm $A7$ is the best among the well performing four algorithms $A4, A5, A7$, and $A8$. This is confirmed by conducting the following test of hypotheses.

TABLE 3. Results for uniform distribution.

Algorithm	D	n 100	200	300	400	500	Avg.
$A1$	10	7.39	7.22	7.31	7.14	7.22	7.25
$A2$		7.79	7.71	7.79	7.65	7.64	7.72
$A3$		9.52	9.48	9.83	9.61	9.51	9.59
$A4$		1.44	1.30	1.34	1.32	1.36	1.35
$A5$		2.06	1.98	2.13	2.03	2.06	2.05
$A6$		7.45	7.43	7.76	7.52	7.57	7.54
$A7$		0.52	0.56	0.51	0.53	0.46	0.51
$A8$		1.34	1.39	1.37	1.43	1.32	1.37
$A9$		7.07	6.94	7.32	6.98	7.16	7.09
$M12$		15.98	16.02	15.99	15.89	16.07	15.99
$M23$		16.97	17.26	17.56	17.14	17.37	17.26
$M34$		4.28	4.31	4.21	4.21	4.28	4.26
$A1$	20	6.14	5.97	6.07	6.06	6.03	6.06
$A2$		6.54	6.42	6.44	6.52	6.39	6.46
$A3$		8.44	8.30	8.34	8.31	8.27	8.33
$A4$		1.25	1.29	1.23	1.19	1.20	1.23
$A5$		1.94	1.83	1.77	1.86	1.84	1.85
$A6$		6.59	6.48	6.60	6.50	6.44	6.52
$A7$		0.51	0.52	0.51	0.52	0.51	0.51
$A8$		1.28	1.27	1.20	1.28	1.22	1.25
$A9$		6.17	6.08	6.16	6.15	5.98	6.11
$M12$		13.81	13.75	13.51	13.57	13.49	13.63
$M23$		14.92	14.96	14.76	14.82	14.79	14.85
$M34$		3.81	3.70	3.79	3.73	3.80	3.76
$A1$	30	5.12	5.09	4.78	5.08	5.12	5.04
$A2$		5.45	5.33	5.11	5.28	5.33	5.30
$A3$		7.11	6.94	7.03	6.71	6.97	6.95
$A4$		1.18	1.25	1.11	1.18	1.15	1.17
$A5$		1.75	1.74	1.69	1.62	1.62	1.69
$A6$		5.40	5.32	5.66	5.28	5.39	5.41
$A7$		0.57	0.60	0.57	0.62	0.58	0.59
$A8$		1.25	1.20	1.22	1.19	1.15	1.20
$A9$		5.01	4.96	5.37	5.06	5.08	5.10
$M12$		11.66	11.34	11.13	11.45	11.54	11.42
$M23$		12.36	12.04	12.45	11.95	12.58	12.28
$M34$		3.39	3.25	3.49	3.42	3.36	3.38
$A1$	40	4.16	4.30	4.23	4.27	4.16	4.22
$A2$		4.39	4.56	4.43	4.53	4.40	4.46
$A3$		5.85	5.96	5.82	5.97	5.83	5.89
$A4$		1.20	1.17	1.20	1.17	1.22	1.19
$A5$		1.56	1.54	1.61	1.58	1.64	1.59
$A6$		4.35	4.42	4.36	4.39	4.40	4.38
$A7$		0.69	0.66	0.66	0.67	0.72	0.68
$A8$		1.23	1.18	1.16	1.21	1.26	1.21
$A9$		4.21	4.27	4.14	4.17	4.25	4.21
$M12$		9.37	9.56	9.38	9.41	9.24	9.39
$M23$		9.97	9.91	9.77	9.84	9.86	9.87
$M34$		3.12	3.10	3.00	2.98	3.26	3.09
	Avg.	5.49	5.46	5.48	5.44	5.46	

TABLE 4. Results for normal distribution.

		n					
Algorithm	D	100	200	300	400	500	Avg.
$A1$	10	7.19	7.53	7.30	7.35	7.30	7.33
$A2$		7.65	7.88	7.74	7.77	7.79	7.77
$A3$		9.39	9.71	9.64	9.64	9.64	9.60
$A4$		1.23	1.27	1.20	1.28	1.24	1.24
$A5$		1.87	2.01	1.94	1.98	2.03	1.96
$A6$		7.65	7.86	7.76	7.89	7.79	7.79
$A7$		0.43	0.42	0.47	0.44	0.46	0.44
$A8$		1.17	1.21	1.26	1.27	1.24	1.23
$A9$		7.27	7.42	7.32	7.40	7.33	7.35
$M12$		15.80	16.22	16.05	15.96	16.21	16.05
$M23$		17.47	17.55	17.67	17.72	17.74	17.63
$M34$		3.99	4.20	4.00	4.10	4.20	4.10
$A1$	20	6.24	6.23	6.41	6.05	6.27	6.24
$A2$		6.63	6.57	6.73	6.47	6.72	6.62
$A3$		8.29	8.23	8.26	8.24	8.31	8.27
$A4$		1.13	1.08	1.09	1.13	1.06	1.10
$A5$		1.67	1.66	1.70	1.71	1.69	1.69
$A6$		6.65	6.56	6.47	6.63	6.74	6.61
$A7$		0.43	0.38	0.44	0.41	0.39	0.41
$A8$		1.01	1.04	1.03	1.12	1.07	1.05
$A9$		6.19	6.15	6.04	6.19	6.31	6.18
$M12$		13.83	13.87	13.84	13.66	13.73	13.79
$M23$		15.18	15.16	14.94	15.05	15.19	15.10
$M34$		3.60	3.56	3.61	3.50	3.56	3.57
$A1$	30	5.27	5.31	5.30	5.31	5.14	5.27
$A2$		5.67	5.68	5.59	5.68	5.50	5.62
$A3$		7.18	7.05	6.99	7.14	6.90	7.05
$A4$		0.98	1.01	0.99	0.99	0.95	0.98
$A5$		1.47	1.46	1.48	1.50	1.44	1.47
$A6$		5.82	5.61	5.54	5.61	5.57	5.63
$A7$		0.37	0.39	0.40	0.42	0.35	0.39
$A8$		1.02	0.98	1.01	1.01	0.94	0.99
$A9$		5.53	5.34	5.25	5.27	5.31	5.34
$M12$		11.78	11.78	11.77	11.65	11.53	11.70
$M23$		13.04	12.85	12.68	12.84	12.76	12.84
$M34$		3.22	3.12	3.19	3.12	3.00	3.13
$A1$		4.37	4.33	4.32	4.28	4.29	4.32
$A2$	40	4.57	4.57	4.57	4.59	4.58	4.57
$A3$		5.70	5.89	5.77	5.86	5.93	5.83
$A4$		0.94	0.89	0.88	0.89	0.89	0.90
$A5$		1.27	1.32	1.29	1.30	1.28	1.29
$A6$		4.36	4.54	4.52	4.56	4.55	4.51
$A7$		0.42	0.35	0.37	0.38	0.39	0.38
$A8$		0.83	0.83	0.84	0.87	0.87	0.85
$A9$		4.11	4.27	4.32	4.22	4.30	4.25
$M12$		9.61	9.73	9.75	9.68	9.52	9.66
$M23$		10.18	10.48	10.36	10.40	10.37	10.36
$M34$		2.67	2.71	2.75	2.66	2.70	2.70
	Avg.	5.46	5.51	5.48	5.48	5.48	

Table 5. Results for positive linear distribution.

Algorithm	D	n 100	200	300	400	500	Avg.
$A1$	10	7.29	7.01	6.97	7.04	7.22	7.11
$A2$		7.75	7.35	7.44	7.40	7.56	7.50
$A3$		9.38	9.09	9.20	9.23	9.23	9.23
$A4$		1.23	1.17	1.18	1.22	1.18	1.20
$A5$		1.85	1.77	1.78	1.88	1.84	1.82
$A6$		7.37	7.33	7.34	7.51	7.35	7.38
$A7$		0.41	0.43	0.42	0.44	0.45	0.43
$A8$		1.10	1.10	1.19	1.24	1.15	1.16
$A9$		6.87	6.88	6.92	7.00	7.03	6.94
$M12$		15.43	15.29	15.25	15.21	15.45	15.33
$M23$		16.66	16.73	16.69	16.65	16.69	16.68
$M34$		3.88	3.79	3.88	3.87	3.93	3.87
$A1$	20	6.02	5.88	5.96	6.10	5.80	5.95
$A2$		6.29	6.22	6.28	6.42	6.17	6.28
$A3$		7.71	7.81	7.82	7.95	7.82	7.82
$A4$		1.11	1.03	1.06	1.11	1.00	1.06
$A5$		1.59	1.59	1.61	1.68	1.66	1.63
$A6$		6.27	6.13	6.17	6.23	6.44	6.25
$A7$		0.41	0.43	0.41	0.44	0.41	0.42
$A8$		1.04	1.07	1.08	1.10	1.10	1.08
$A9$		5.96	5.76	5.90	5.88	6.02	5.91
$M12$		12.82	12.71	12.85	12.95	12.85	12.84
$M23$		13.91	13.87	13.92	14.09	14.06	13.97
$M34$		3.37	3.36	3.38	3.43	3.45	3.40
$A1$	30	4.70	4.77	4.74	4.76	4.78	4.75
$A2$		4.90	5.02	5.03	5.04	5.06	5.01
$A3$		6.46	6.39	6.43	6.46	6.44	6.44
$A4$		0.94	0.95	0.94	0.89	0.93	0.93
$A5$		1.39	1.41	1.42	1.40	1.38	1.40
$A6$		5.14	5.06	5.17	5.11	5.16	5.13
$A7$		0.43	0.42	0.40	0.40	0.42	0.41
$A8$		1.02	1.00	0.97	0.94	0.97	0.98
$A9$		4.95	4.89	4.85	4.79	4.90	4.88
$M12$		10.50	10.54	10.73	10.53	10.55	10.57
$M23$		11.60	11.34	11.52	11.43	11.58	11.50
$M34$		3.00	2.95	2.90	2.91	3.02	2.95
$A1$	40	4.02	3.93	3.77	3.99	4.00	3.94
$A2$		4.13	4.12	3.92	4.19	4.16	4.10
$A3$		5.37	5.37	5.34	5.45	5.26	5.36
$A4$		0.94	0.89	0.80	0.89	0.91	0.89
$A5$		1.27	1.29	1.20	1.26	1.26	1.26
$A6$		4.21	4.13	4.20	4.23	4.11	4.18
$A7$		0.44	0.46	0.38	0.40	0.43	0.43
$A8$		0.88	0.91	0.92	0.87	0.91	0.90
$A9$		3.95	3.93	3.97	4.07	3.92	3.97
$M12$		8.77	8.60	8.53	8.78	8.74	8.68
$M23$		9.26	9.27	9.35	9.43	9.18	9.30
$M34$		2.63	2.56	2.52	2.59	2.59	2.58
	Avg.	5.14	5.08	5.10	5.14	5.14	

Table 6. Results for negative linear distribution.

Algorithm	D	n					Avg.
		100	200	300	400	500	
$A1$	10	7.17	7.54	7.74	7.19	7.42	7.41
$A2$		7.67	8.09	8.26	7.71	7.95	7.94
$A3$		9.94	10.26	10.29	9.81	10.11	10.08
$A4$		1.36	1.41	1.49	1.38	1.45	1.42
$A5$		2.11	2.09	2.29	2.09	2.20	2.16
$A6$		7.97	8.06	8.02	7.67	7.81	7.90
$A7$		0.61	0.53	0.56	0.63	0.57	0.58
$A8$		1.53	1.42	1.51	1.39	1.38	1.44
$A9$		7.46	7.62	7.50	7.18	7.32	7.41
$M12$		16.41	16.86	17.26	16.64	16.64	16.76
$M23$		18.36	18.61	18.40	18.28	18.55	18.44
$M34$		4.40	4.46	4.55	4.45	4.57	4.49
$A1$	20	6.48	6.51	6.30	6.46	6.46	6.44
$A2$		6.99	6.93	6.77	6.88	6.88	6.89
$A3$		9.08	8.92	8.64	8.85	8.99	8.90
$A4$		1.34	1.29	1.30	1.29	1.32	1.31
$A5$		2.03	1.90	2.01	1.90	2.06	1.98
$A6$		7.04	6.88	6.85	6.87	7.19	6.97
$A7$		0.56	0.61	0.59	0.54	0.56	0.57
$A8$		1.40	1.34	1.40	1.25	1.40	1.36
$A9$		6.62	6.45	6.49	6.34	6.76	6.53
$M12$		15.06	14.83	14.70	14.63	14.76	14.80
$M23$		16.43	16.29	16.00	16.09	16.29	16.22
$M34$		4.18	4.00	3.92	4.09	4.24	4.09
$A1$	30	5.39	5.83	5.50	5.81	5.76	5.66
$A2$		5.79	6.15	5.87	6.08	5.97	5.97
$A3$		7.73	7.83	7.71	8.03	7.80	7.82
$A4$		1.27	1.32	1.22	1.32	1.25	1.28
$A5$		1.91	1.87	1.75	1.91	1.75	1.84
$A6$		6.11	5.93	5.86	6.17	5.90	5.99
$A7$		0.67	0.61	0.60	0.65	0.60	0.63
$A8$		1.39	1.30	1.25	1.40	1.24	1.32
$A9$		5.84	5.60	5.48	5.88	5.62	5.68
$M12$		12.77	13.08	12.91	12.95	12.86	12.91
$M23$		14.08	13.90	13.84	14.07	13.98	13.97
$M34$		3.67	3.84	3.80	3.85	3.76	3.78
$A1$		4.70	4.73	4.84	4.88	4.77	4.78
$A2$	40	4.98	4.96	5.16	5.16	5.05	5.06
$A3$		6.70	6.64	6.69	6.75	6.69	6.69
$A4$		1.30	1.27	1.32	1.30	1.28	1.29
$A5$		1.79	1.75	1.77	1.72	1.78	1.76
$A6$		5.11	5.11	4.96	5.03	5.15	5.07
$A7$		0.68	0.68	0.61	0.68	0.73	0.68
$A8$		1.29	1.26	1.23	1.25	1.30	1.27
$A9$		4.77	4.76	4.78	4.81	4.86	4.80
$M12$		10.74	10.85	10.98	11.04	10.82	10.89
$M23$		11.67	11.66	11.61	11.72	11.70	11.67
$M34$		3.32	3.34	3.48	3.55	3.62	3.46
	Avg.	5.96	5.98	5.96	5.95	5.98	

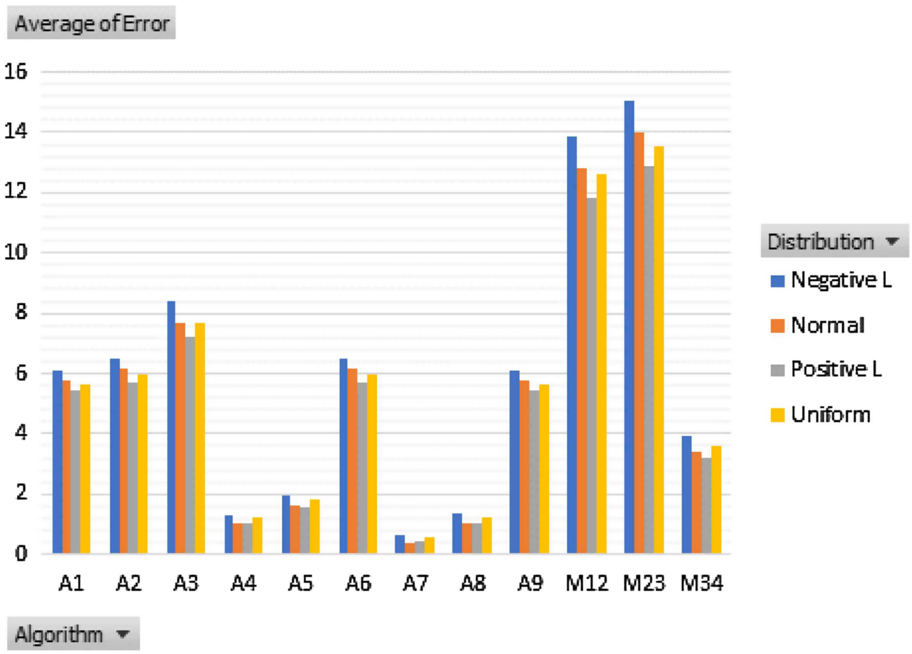

FIGURE 2. Avg. error of all algorithms for different distributions.

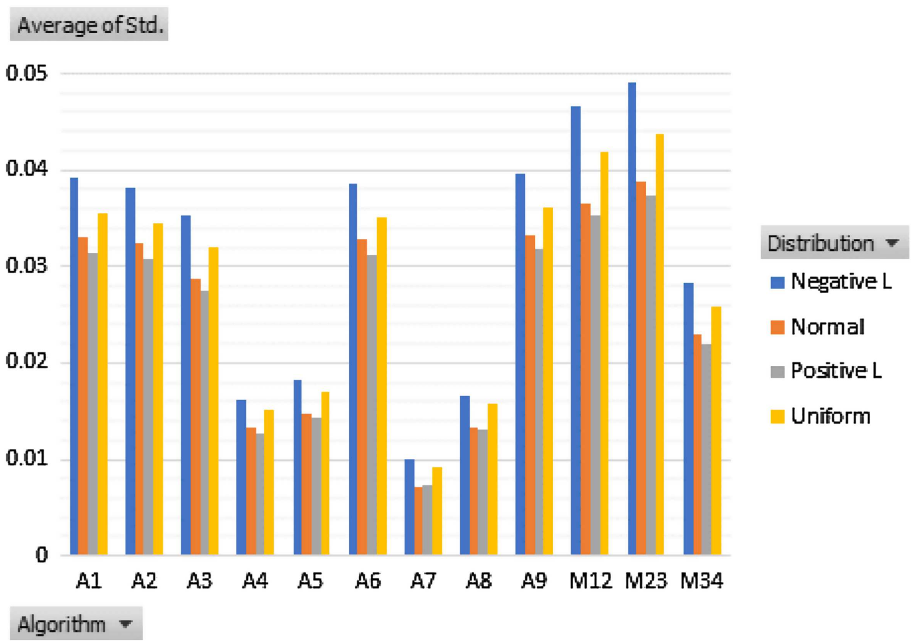

FIGURE 3. Avg. Std. of all algorithms for different distributions.

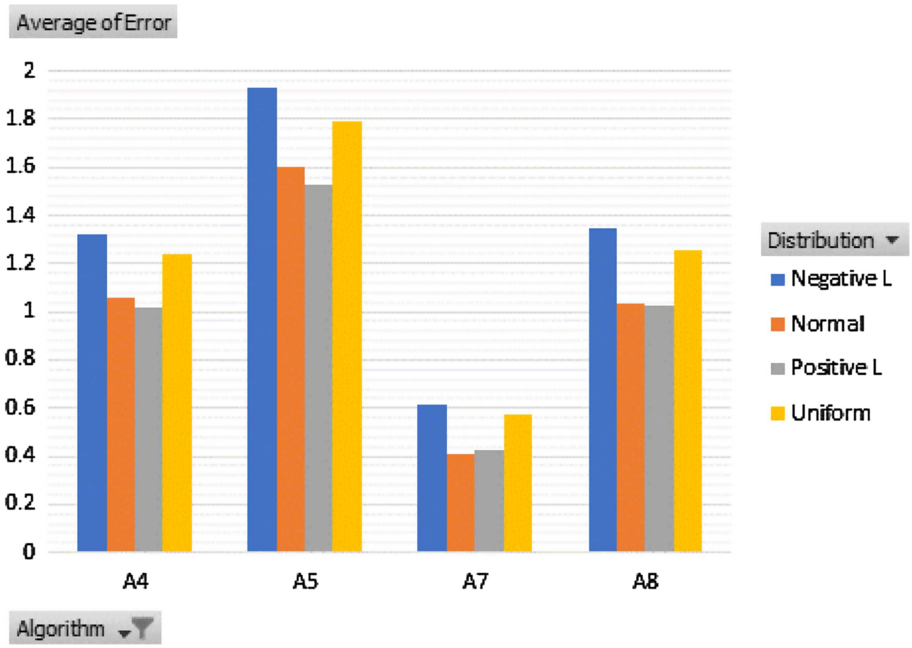

Figure 4. Avg. error of the four algorithms for different distributions.

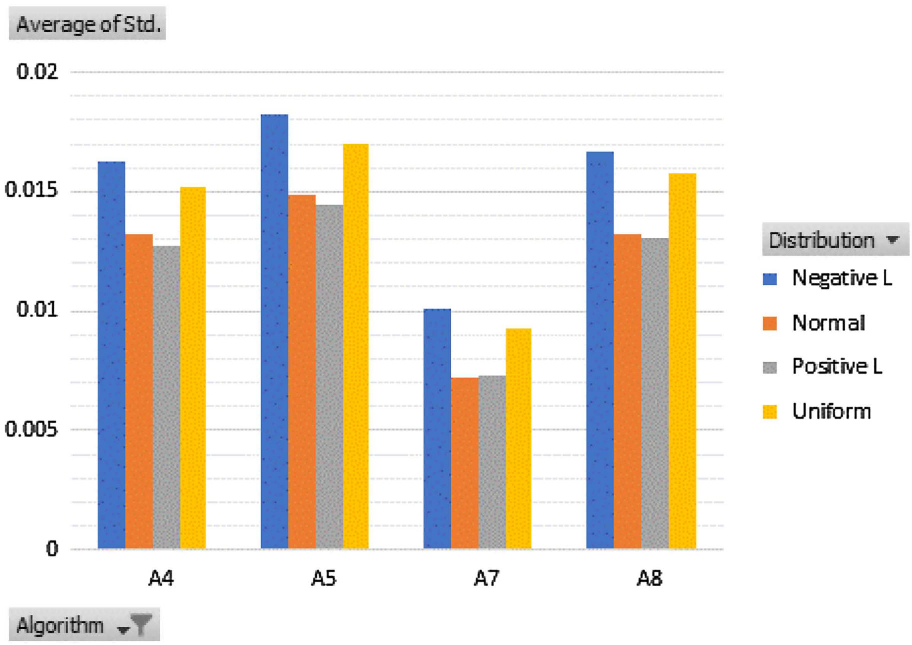

Figure 5. Avg. Std. of the four algorithms for different distributions.

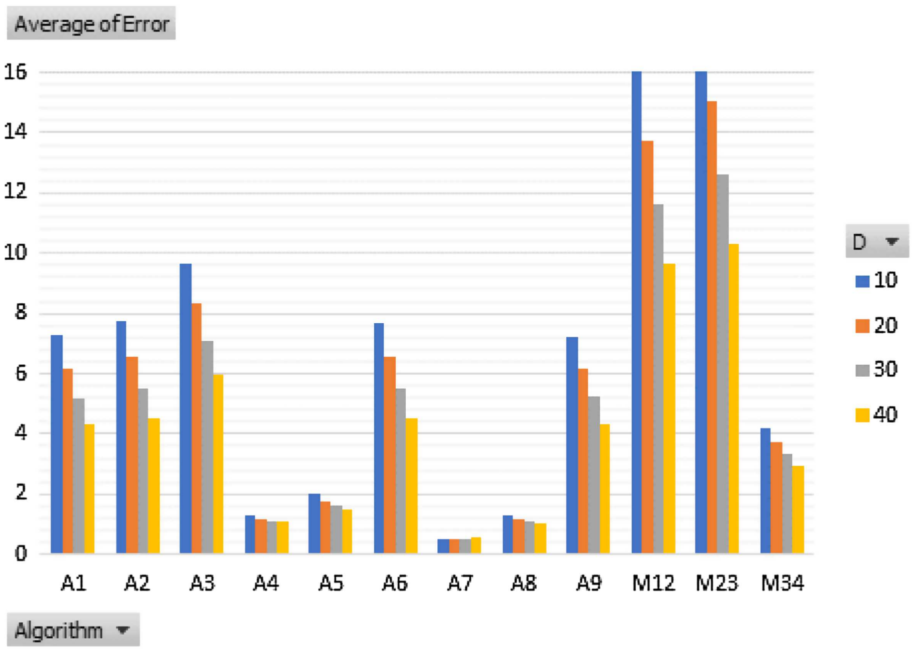

FIGURE 6. Avg. error of all algorithms with respect to D.

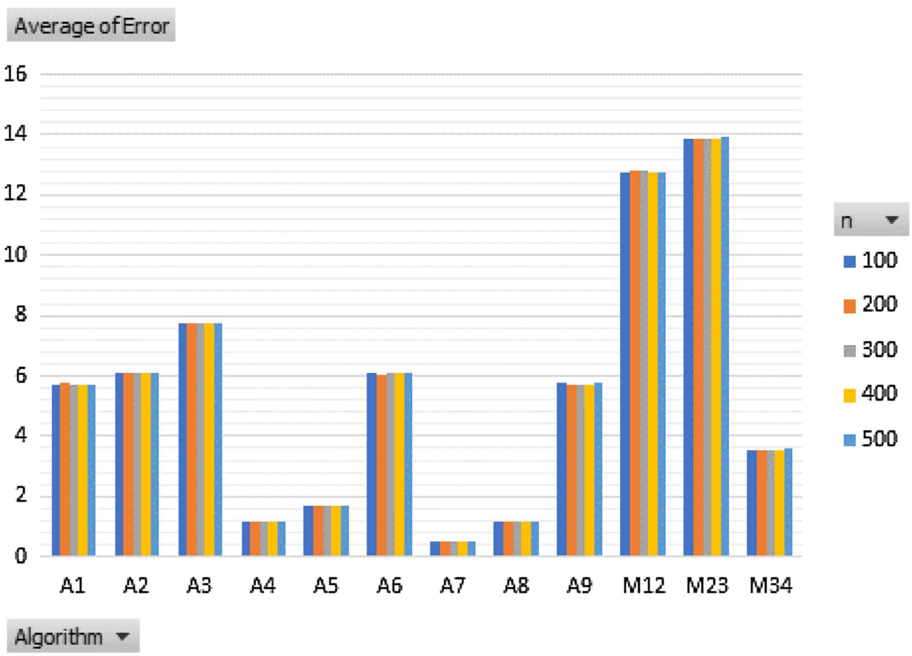

FIGURE 7. Avg. error of all algorithms with respect to n.

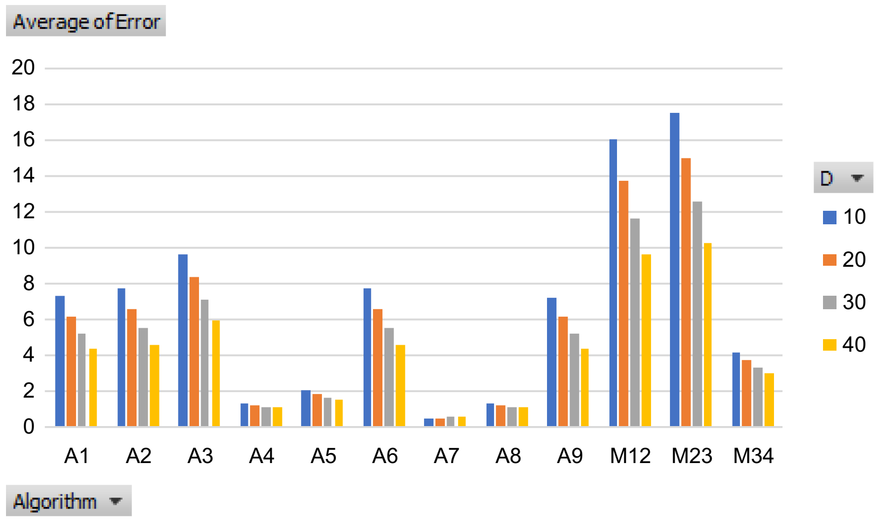

FIGURE 8. 50% Negative distribution and 50% Positive distribution.

The performance of $A7$ and $A4$ were statistically verified by using a two-sample t test. The following null and alternative hypotheses testing were conducted for comparing the performance of $A7$ and $A4$ statistically.

$$H_0 : \mu(A7) = \mu(A4)$$
$$H_1 : \mu(A7) < \mu(A4)$$

where $\mu(Ah)$ denotes the overall average error of the algorithm Ah. The null hypothesis was rejected at a significance level of $\alpha = 0.01$. Similarly, the following hypothesis testing

$$H_0 : \mu(A7) = \mu(A5)$$
$$H_1 : \mu(A7) < \mu(A5)$$

and the hypothesis testing

$$H_0 : \mu(A7) = \mu(A8)$$
$$H_1 : \mu(A7) < \mu(A8)$$

were conducted for comparing the performance of algorithms $A7$ and $A5$, and $A7$ and $A8$, respectively. For both cases, the null hypotheses were rejected at a significance level of $\alpha = 0.01$.

The intuition behind why *Algorithm* $A7$ performs well can be described as following. Recall that Johnson's algorithm is designed for the two-stage (two-machine) flowshop, where the objective is to minimize the total idle time on the (second) last machine by considering job processing times on both machines. For our four-machine flowshop, the first stage consists of the first two machines while the second stage consists of the third and fourth machines. However, in *Algorithm* $A7$, more weight is given to the first machine (in the first stage) while more weight is given to the fourth machine (in the second stage). More specifically, in *Algorithm* $A7$, a 75% weight is given to the processing times of the jobs on the first machine while a 25% weight is given to the processing times of the jobs on the second machine. Moreover, a 25% weight is given to the processing times of the jobs on the third machine while a 75% weight is given to the processing time of the jobs on the fourth machine.

As stated before, the above results were obtained by using only uniform, or positive linear, or negative linear, or normal distributions for generating processing times between $Lt_{i,k}$ and $Ut_{i,k}$. We also mixed the stated distributions for generating processing times and we observed that the results in general were similar. For example,

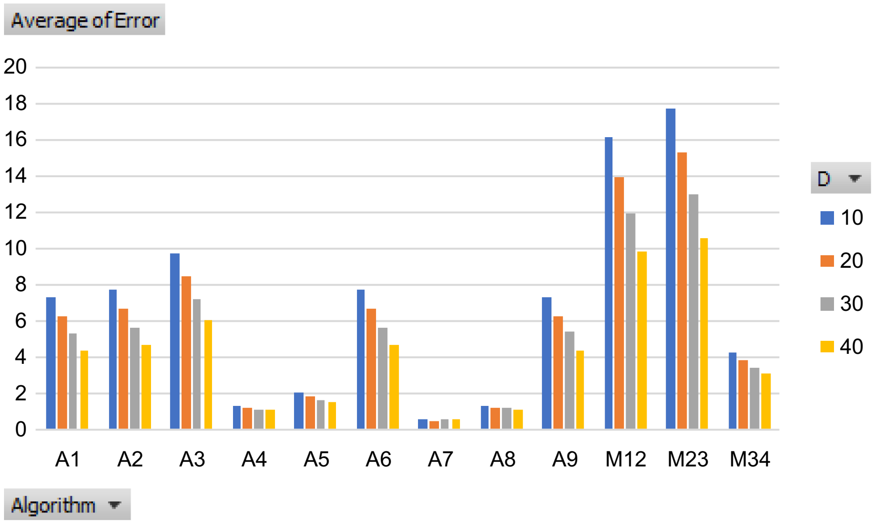

FIGURE 9. 80% Negative distribution and 20% positive distribution.

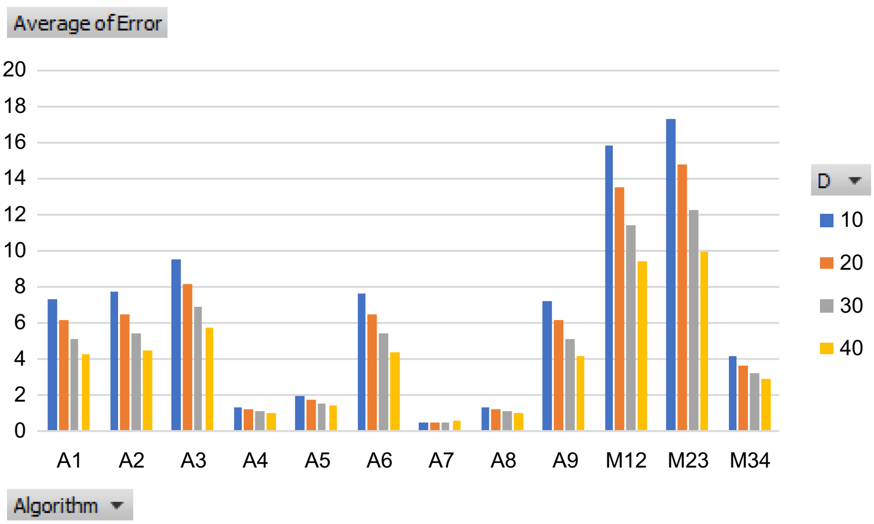

FIGURE 10. 20% Negative distribution and 80% positive distribution.

Figure 8 summarizes the results for 50% from Negative Linear and 50% from Positive Linear distributions, the results for 80% from Negative Linear and 20% from Positive Linear distributions are summarized Figure 9 while the results for 20% from Negative Linear and 80% from Positive Linear distributions are summarized in Figure 10. As can be seen, the results summarized in Figures 8–10 are very similar to the results summarized in Figure 7 for non-mixed distributions. The results in general were similar and hence, mixing the distributions do not seem to change the results. We guess this is due to the fact that we conducted 1000 replications which is very large, and hence, the results do not change.

5. Concluding remarks

The four-machine flowshop scheduling problem is investigated in this paper with the objective of minimizing makespan. The processing times are uncertain and only the lower and upper bounds of these times are known. Some dominance relations are established, and twelve algorithms are proposed. The algorithms are based on the well-known Johnson's algorithm, which is known to yield the optimal solution for the two-machine case. They are also based on the weights given to the lower and upper bounds of processing times on the four machines.

The proposed twelve algorithms are evaluated through computational experiments. The latter reveal that one of the algorithms, *Algorithm A*7, is performing significantly better than the rest for different distributions considered to generate the processing times of the jobs between the given lower and upper bounds. Specifically, the error percentage of *Algorithm A*7 is from 3.6% to 43% of the error percentages of the other eleven considered algorithms. Therefore, the error percentage of *Algorithm A*7 is about 43% of the error percentage of the next best performing algorithm. These results are confirmed by using confidence intervals and test of hypothesis. Therefore, *Algorithm A*7 is recommended for the considered problem.

It is assumed in this paper that setup times are included in processing times. This assumption is valid for most of manufacturing environments. However, this assumption may not be valid for some other manufacturing environments, Therefore, an extension of the research conducted in this paper is to address the four-machine flowshop scheduling problem with uncertain processing times (within some intervals) to minimize makespan for the environments where setup times are treated as separate from processing times.

Appendix A.

Proof of Theorem 2.1. Let $\phi_{[\alpha,2]}(\pi_r)$ denote $\phi_{[\alpha,2]}$ for the sequence π_r, $r = 1, 2$. It follows from the definition of $\phi_{[j,2]}$ that for $j = \alpha$ and $\alpha + 1$,

$$\phi_{[\alpha,2]}(\pi_1) = \mathrm{TP}_{[\alpha-1,1]}(\pi_1) + t_{h,1} - \mathrm{TP}_{[\alpha-1,2]}(\pi_1). \quad (A.1)$$

$$\phi_{[\alpha,2]}(\pi_2) = \mathrm{TP}_{[\alpha-1,1]}(\pi_2) + t_{g,1} - \mathrm{TP}_{[\alpha-1,2]}(\pi_2). \quad (A.2)$$

$$\phi_{[\alpha+1,2]}(\pi_1) = \mathrm{TP}_{[\alpha-1,1]}(\pi_1) + t_{h,1} + t_{g,1} - \mathrm{TP}_{[\alpha-1,2]}(\pi_1) - t_{h,2}. \quad (A.3)$$

$$\phi_{[\alpha+1,2]}(\pi_2) = \mathrm{TP}_{[\alpha-1,1]}(\pi_2) + t_{g,1} + t_{h,1} - \mathrm{TP}_{[\alpha-1,2]}(\pi_2) - t_{g,2}. \quad (A.4)$$

It should be noted that $t_{h,1}$, $t_{h,2}$, $t_{g,1}$, and $t_{g,2}$ satisfy the following inequalities,

$$Lt_{h,1} \le t_{h,1} \le Ut_{h,1} \quad (A.5)$$

$$Lt_{g,1} \le t_{g,1} \le Ut_{g,1} \quad (A.6)$$

$$Lt_{h,2} \le t_{h,2} \le Ut_{h,2} \quad (A.7)$$

$$Lt_{g,2} \le t_{g,2} \le Ut_{g,2} \quad (A.8)$$

It should be also noted that

$$\mathrm{TP}_{[\alpha-1,1]}(\pi_1) = \mathrm{TP}_{[\alpha-1,1]}(\pi_2) \quad \text{and} \quad \mathrm{TP}_{[\alpha-1,2]}(\pi_1) = \mathrm{TP}_{[\alpha-1,2]}(\pi_2) \quad (A.9)$$

since both sequences π_1 and π_2 have the same jobs in positions $1, \ldots, \alpha - 1$.

It follows from equations (A.1) to (A.9) and C1-1 of the theorem that

$$\phi_{[\alpha,2]}(\pi_2) \le \phi_{[\alpha,2]}(\pi_1) \quad \text{and} \quad \phi_{[\alpha+1,2]}(\pi_2) \le \phi_{[\alpha+1,2]}(\pi_1). \quad (A.10)$$

By C1-2 of theorem and equations (A.2)–(A.9), we have

$$\phi_{[\alpha,2]}(\pi_2) \le \phi_{[\alpha+1,2]}(\pi_2) \quad \text{and} \quad \phi_{[\alpha+1,2]}(\pi_2) \le \phi_{[\alpha+1,2]}(\pi_1). \quad (A.11)$$

By C1-3 of the theorem and equations (A.1), (A.2), (A.4)–(A.6), (A.8) and (A.9) we obtain

$$\phi_{[\alpha,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_1) \quad \text{and} \quad \phi_{[\alpha+1,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_1). \tag{A.12}$$

By C1-4 of the theorem and equations (A.2)–(A.5) and (A.7)–(A.9),

$$\phi_{[\alpha,2]}(\pi_2) \leq \phi_{[\alpha+1,2]}(\pi_1) \quad \text{and} \quad \phi_{[\alpha+1,2]}(\pi_2) \leq \phi_{[\alpha+1,2]}(\pi_1). \tag{A.13}$$

By equations (A.1), (A.2), (A.4)–(A.6), (A.8), (A.9) and C1-5 of the theorem,

$$\phi_{[\alpha,2]}(\pi_2) \leq \phi_{[\alpha+1,2]}(\pi_2) \quad \text{and} \quad \phi_{[\alpha+1,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_1). \tag{A.14}$$

By equations (A.1), (A.2), (A.4)–(A.6), (A.8), (A.9) and C1-6 of the theorem,

$$\phi_{[\alpha+1,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_2) \quad \text{and} \quad \phi_{[\alpha,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_1). \tag{A.15}$$

Finally, by equations (A.2)–(A.5), (A.7)–(A.9) and C1-7 of the theorem, we have

$$\phi_{[\alpha+1,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_2) \quad \text{and} \quad \phi_{[\alpha,2]}(\pi_2) \leq \phi_{[\alpha+1,2]}(\pi_1). \tag{A.16}$$

If any of the Inequalities of (A.10)–(A.16) hold, then,

$$\max\{\phi_{[\alpha,2]}(\pi_2), \phi_{[\alpha+1,2]}(\pi_2)\} \leq \max\{\phi_{[\alpha,2]}(\pi_1), \phi_{[\alpha+1,2]}(\pi_1)\}.$$

□

Proof of Theorem 2.2. Let $\Delta_{[\alpha,2]}(\pi_r)$ denote $\Delta_{[\alpha,2]}$ for the sequence π_r, $r = 1, 2$. It follows from the definition of $\phi_{[j,3]}$ that for $j = \alpha$ and $\alpha + 1$,

$$\phi_{[\alpha,3]}(\pi_1) = \max\left\{\Delta_{[\alpha-1,2]}(\pi_1), \phi_{[\alpha,2]}(\pi_1)\right\} + \mathrm{TP}_{[\alpha-1,2]}(\pi_1) + t_{h,2} - \mathrm{TP}_{[\alpha-1,3]}(\pi_1). \tag{A.17}$$

$$\phi_{[\alpha,3]}(\pi_2) = \max\left\{\Delta_{[\alpha-1,2]}(\pi_2), \phi_{[\alpha,2]}(\pi_2)\right\} + \mathrm{TP}_{[\alpha-1,2]}(\pi_2) + t_{g,2} - \mathrm{TP}_{[\alpha-1,3]}(\pi_2). \tag{A.18}$$

$$\begin{aligned}\phi_{[\alpha+1,3]}(\pi_1) = {} & \max\left\{\Delta_{[\alpha-1,2]}(\pi_1),\ \phi_{[\alpha,2]}(\pi_1), \phi_{[\alpha+1,2]}(\pi_1)\right\} + \mathrm{TP}_{[\alpha-1,2]}(\pi_1) + t_{h,2} + t_{g,2} \\ & - \mathrm{TP}_{[\alpha-1,3]}(\pi_1) - t_{h,3}.\end{aligned} \tag{A.19}$$

$$\begin{aligned}\phi_{[\alpha+1,3]}(\pi_2) = {} & \max\left\{\Delta_{[\alpha-1,2]}(\pi_2),\ \phi_{[\alpha,2]}(\pi_2), \phi_{[\alpha+1,2]}(\pi_2)\ \right\} + \mathrm{TP}_{[\alpha-1,2]}(\pi_2) + t_{g,2} + t_{h,2} \\ & - \mathrm{TP}_{[\alpha-1,3]}(\pi_2) - t_{g,3}.\end{aligned} \tag{A.20}$$

It should be noted that

$$\mathrm{TP}_{[\alpha-1,3]}(\pi_1) = \mathrm{TP}_{[\alpha-1,3]}(\pi_2) \quad \text{and} \quad \Delta_{[\alpha-1,2]}(\pi_1) = \Delta_{[\alpha-1,2]}(\pi_2) \tag{A.21}$$

since both sequences π_1 and π_2 have the same jobs in positions $1, \ldots, \alpha - 1$. It should be also noted that $t_{g,3}$, and $t_{h,3}$ satisfy the following inequalities,

$$Lt_{h,3} \leq t_{h,3} \leq Ut_{h,3} \tag{A.22}$$

$$Lt_{g,3} \leq t_{g,3} \leq Ut_{g,3} \tag{A.23}$$

It follows by C2-1 of the theorem and equations (A.7), (A.9), (A.10) and (A.18)–(A.23) that

$$\phi_{[\alpha,3]}(\pi_2) \leq \phi_{[\alpha+1,3]}(\pi_2) \quad \text{and} \quad \phi_{[\alpha+1,3]}(\pi_2) \leq \phi_{[\alpha+1,3]}(\pi_1). \tag{A.24}$$

By C2-2 of the theorem and equations (A.7)–(A.9), (A.12), (A.17), (A.18), (A.20), (A.21) and (A.23),

$$\phi_{[\alpha,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_1) \quad \text{and} \quad \phi_{[\alpha+1,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_1). \tag{A.25}$$

By C2-3 of the theorem and equations (A.7), (A.9), (A.10) and (A.18)–(A.23),

$$\phi_{[\alpha,3]}\left(\pi_2\right) \leq \phi_{[\alpha+1,3]}\left(\pi_1\right) \quad \text{and} \quad \phi_{[\alpha+1,3]}\left(\pi_2\right) \leq \phi_{[\alpha+1,3]}\left(\pi_1\right). \tag{A.26}$$

By equations (A.7)–(A.9), (A.14), (A.17), (A.18), (A.20), (A.21), (A.23) and C2-4 of the theorem,

$$\phi_{[\alpha,3]}\left(\pi_2\right) \leq \phi_{[\alpha+1,3]}\left(\pi_2\right) \quad \text{and} \quad \phi_{[\alpha+1,3]}\left(\pi_2\right) \leq \phi_{[\alpha,3]}\left(\pi_1\right). \tag{A.27}$$

By equations (A.7)–(A.9), (A.15), (A.17), (A.18), (A.20), (A.21), (A.23) and C2-5 of the theorem,

$$\phi_{[\alpha+1,3]}\left(\pi_2\right) \leq \phi_{[\alpha,3]}\left(\pi_2\right) \quad \text{and} \quad \phi_{[\alpha,3]}\left(\pi_2\right) \leq \phi_{[\alpha,3]}\left(\pi_1\right). \tag{A.28}$$

Lastly, from equations (A.1), (A.2), (A.4), (A.7), (A.9), (A.18)–(A.23) and C2-6 of the theorem, it follows that

$$\phi_{[\alpha+1,3]}\left(\pi_2\right) \leq \phi_{[\alpha,3]}\left(\pi_2\right) \quad \text{and} \quad \phi_{[\alpha,3]}\left(\pi_2\right) \leq \phi_{[\alpha+1,3]}\left(\pi_1\right). \tag{A.29}$$

If any of the Inequalities (A.24)–(A.29) hold, then,

$$\max\{\phi_{[\alpha,3]}\left(\pi_2\right), \phi_{[\alpha+1,3]}\left(\pi_2\right)\} \leq \max\{\phi_{[\alpha,3]}\left(\pi_1\right), \phi_{[\alpha+1,3]}\left(\pi_1\right)\}.$$

□

Proof of Theorem 2.3. It follows from the definition of $\phi_{[j,4]}$ that for $j = \alpha$ and $\alpha + 1$

$$\phi_{[\alpha,4]}\left(\pi_1\right) = \max\left\{\Delta_{[\alpha-1,3]}\left(\pi_1\right), \phi_{[\alpha,3]}\left(\pi_1\right)\right\} + \mathrm{TP}_{[\alpha-1,3]}\left(\pi_1\right) + t_{h,3} - \mathrm{TP}_{[\alpha-1,4]}\left(\pi_1\right). \tag{A.30}$$

$$\phi_{[\alpha,4]}\left(\pi_2\right) = \max\left\{\Delta_{[\alpha-1,3]}\left(\pi_2\right), \phi_{[\alpha,3]}\left(\pi_2\right)\right\} + \mathrm{TP}_{[\alpha-1,3]}\left(\pi_2\right) + t_{g,3} - \mathrm{TP}_{[\alpha-1,4]}\left(\pi_2\right). \tag{A.31}$$

$$\begin{aligned}\phi_{[\alpha+1,4]}\left(\pi_1\right) = \max&\left\{\Delta_{[\alpha-1,3]}\left(\pi_1\right), \phi_{[\alpha,3]}\left(\pi_1\right), \phi_{[\alpha+1,3]}\left(\pi_1\right)\right\} + \mathrm{TP}_{[\alpha-1,3]}\left(\pi_1\right) + t_{h,3} + t_{g,3}\\ &- \mathrm{TP}_{[\alpha-1,4]}\left(\pi_1\right) - t_{h,4}.\end{aligned} \tag{A.32}$$

$$\begin{aligned}\phi_{[\alpha+1,4]}\left(\pi_2\right) = \max&\left\{\Delta_{[\alpha-1,3]}\left(\pi_2\right),\ \phi_{[\alpha,3]}\left(\pi_2\right), \phi_{[\alpha+1,3]}\left(\pi_2\right)\ \right\} + \mathrm{TP}_{[\alpha-1,3]}\left(\pi_2\right) + t_{g,3} + t_{h,3}\\ &- \mathrm{TP}_{[\alpha-1,4]}\left(\pi_2\right) - t_{g,4}.\end{aligned} \tag{A.33}$$

It should be noted that

$$\mathrm{TP}_{[\alpha-1,4]}\left(\pi_1\right) = \mathrm{TP}_{[\alpha-1,4]}\left(\pi_2\right) \quad \text{and} \quad \Phi_{[\alpha-1,3]}\left(\pi_1\right) = \Phi_{[\alpha-1,3]}\left(\pi_2\right) \tag{A.34}$$

since both sequences π_1 and π_2 have the same jobs in positions $1, \ldots, \alpha - 1$. It should be also noted that $t_{g,3}$, and $t_{h,3}$ satisfy the following inequalities,

$$Lt_{h,4} \leq t_{h,4} \leq Ut_{h,4} \tag{A.35}$$

$$Lt_{g,4} \leq t_{g,4} \leq Ut_{g,4} \tag{A.36}$$

Notice that if C3-1 of the Theorem holds, then, $\phi_{[\alpha,3]}\left(\pi_2\right) \leq \phi_{[\alpha,3]}\left(\pi_1\right)$ and $\phi_{[\alpha,2]}\left(\pi_2\right) \leq \phi_{[\alpha,2]}\left(\pi_1\right)$ and $\phi_{[\alpha+1,3]}\left(\pi_2\right) \leq \phi_{[\alpha,3]}\left(\pi_2\right)$ and $\phi_{[\alpha+1,2]}\left(\pi_2\right) \leq \phi_{[\alpha,2]}\left(\pi_2\right)$. Therefore, it follows from equations (A.5)–(A.8), (A.21)–(A.23), (A.30)–(A.36) and C3-1 of the theorem that

$$\phi_{[\alpha,4]}\left(\pi_2\right) \leq \phi_{[\alpha,4]}\left(\pi_1\right) \quad \text{and} \quad \phi_{[\alpha+1,4]}\left(\pi_2\right) \leq \phi_{[\alpha+1,4]}\left(\pi_1\right). \tag{A.37}$$

If C3-2 of the theorem holds, then, $\phi_{[\alpha+1,3]}\left(\pi_2\right) \leq \phi_{[\alpha,3]}\left(\pi_2\right)$ and $\phi_{[\alpha+1,2]}\left(\pi_2\right) \leq \phi_{[\alpha,2]}\left(\pi_2\right)$. Thus, we have from C3-2 of the theorem and equations (A.5)–(A.8), (A.21)–(A.23) and (A.31)–(A.36) that

$$\phi_{[\alpha,4]}\left(\pi_2\right) \leq \phi_{[\alpha+1,4]}\left(\pi_2\right) \quad \text{and} \phi_{[\alpha+1,4]}\left(\pi_2\right) \leq \phi_{[\alpha+1,4]}\left(\pi_1\right). \tag{A.38}$$

When C3-3 of the Theorem holds, $\phi_{[\alpha,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_1)$ and $\phi_{[\alpha,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_1)$, and furthermore, $\phi_{[\alpha+1,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_1)$ and $\phi_{[\alpha+1,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_1)$. By, from C3-3 of the theorem and equations (A.5)–(A.8), (A.21)–(A.23), (A.30), (A.31) and (A.33)–(A.36),

$$\phi_{[\alpha,4]}(\pi_2) \leq \phi_{[\alpha,4]}(\pi_1) \quad \text{and} \quad \phi_{[\alpha+1,4]}(\pi_2) \leq \phi_{[\alpha,4]}(\pi_1). \tag{A.39}$$

On the other hand, given that C3-4 of the Theorem holds, $\phi_{[\alpha,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_1)$ and $\phi_{[\alpha,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_1)$, and $\phi_{[\alpha+1,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_2)$ and $\phi_{[\alpha+1,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_2)$. Hence, by follows by C3-4 of the theorem and equations (A.5)–(A.8), (A.21)–(A.23) and (A.31)–(A.36),

$$\phi_{[\alpha,4]}(\pi_2) \leq \phi_{[\alpha+1,4]}(\pi_1) \quad \text{and} \quad \phi_{[\alpha+1,4]}(\pi_2) \leq \phi_{[\alpha+1,4]}(\pi_1). \tag{A.40}$$

By C3-5 of the Theorem, $\phi_{[\alpha,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_1)$ and $\phi_{[\alpha,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_1)$, and $\phi_{[\alpha+1,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_2)$ and $\phi_{[\alpha+1,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_2)$. Thus, it follows from equations (A.5)–(A.8), (A.21)–(A.23), (A.30), (A.31), (A.33)–(A.36) and C3-5 of the theorem that

$$\phi_{[\alpha,4]}(\pi_2) \leq \phi_{[\alpha+1,4]}(\pi_2) \quad \text{and} \quad \phi_{[\alpha+1,4]}(\pi_2) \leq \phi_{[\alpha,4]}(\pi_1). \tag{A.41}$$

By C3-6 of the theorem, $\phi_{[\alpha,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_1)$ and $\phi_{[\alpha,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_1)$, and $\phi_{[\alpha+1,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_2)$ and $\phi_{[\alpha+1,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_2)$. Thus, it follows from equations (A.5)–(A.8), (A.21)–(A.23), (A.30), (A.31), (A.33)–(A.36) and C3-6 of the theorem that

$$\phi_{[\alpha+1,4]}(\pi_2) \leq \phi_{[\alpha,4]}(\pi_2) \quad \text{and} \quad \phi_{[\alpha,4]}(\pi_2) \leq \phi_{[\alpha,4]}(\pi_1). \tag{A.42}$$

Finally, by C3-7 of the theorem $\phi_{[\alpha,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_1)$ and $\phi_{[\alpha,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_1)$, and $\phi_{[\alpha+1,3]}(\pi_2) \leq \phi_{[\alpha,3]}(\pi_2)$ and $\phi_{[\alpha+1,2]}(\pi_2) \leq \phi_{[\alpha,2]}(\pi_2)$. Hence, by equations (A.5)–(A.8), (A.21)–(A.23), (A.31)–(A.36) and C3-7 of the theorem,

$$\phi_{[\alpha+1,4]}(\pi_2) \leq \phi_{[\alpha,4]}(\pi_2) \quad \text{and} \quad \phi_{[\alpha,4]}(\pi_2) \leq \phi_{[\alpha+1,4]}(\pi_1). \tag{A.43}$$

If any of the Inequalities of (A.37)–(A.43) hold, then,

$$\max\{\phi_{[\alpha,4]}(\pi_2), \phi_{[\alpha+1,4]}(\pi_2)\} \leq \max\{\phi_{[\alpha,4]}(\pi_1), \phi_{[\alpha+1,4]}(\pi_1)\}.$$

□

Proof of Lemma 2.5. From the definition, for $r = \alpha+2, \ldots, n$,

$$\begin{aligned} \phi_{[r,2]}(\pi_1) = \mathrm{TP}_{[\alpha-1,1]}(\pi_1) + t_{h,1} + t_{g,1} + \sum_{p=\alpha+2}^{r} t_{[r,1]}(\pi_1) \\ - \mathrm{TP}_{[\alpha-1,2]}(\pi_1) - t_{h,2} \quad - t_{g,2} - \sum_{p=\alpha+2}^{r-1} t_{[r,2]}(\pi_1), \end{aligned} \tag{A.44}$$

$$\begin{aligned} \phi_{[\alpha+1,2]}(\pi_2) = \mathrm{TP}_{[\alpha-1,1]}(\pi_2) + t_{g,1} + t_{h,1} + \sum_{p=\alpha+2}^{r} t_{[r,1]}(\pi_2) \\ - \mathrm{TP}_{[\alpha-1,2]}(\pi_2) - t_{g,2} - t_{h,2} - \sum_{p=\alpha+2}^{r-1} t_{[r,2]}(\pi_2), \end{aligned} \tag{A.45}$$

where $\sum_{p=\alpha+2}^{\alpha+1} t_{[r,2]}(\pi_1) = \sum_{p=\alpha+2}^{\alpha+1} t_{[r,2]}(\pi_2) = 0$. It should be noted that both sequences π_1 and π_2 have the same jobs in positions $r = \alpha+2, \ldots, n$, therefore,

$$\sum_{p=\alpha+2}^{r} t_{[r,1]}(\pi_1) = \sum_{p=\alpha+2}^{r} t_{[r,1]}(\pi_2) \quad \text{and} \quad \sum_{p=\alpha+2}^{r-1} t_{[r,2]}(\pi_1) = \sum_{p=\alpha+2}^{r-1} t_{[r,2]}(\pi_2). \tag{A.46}$$

Hence, it follows from equations (A.9), (A.44) and (A.45) that

$$\phi_{[r,2]}(\pi_2) = \phi_{[r,2]}(\pi_1) \quad \text{for } r = \alpha - 1,\ \alpha + 2, \ldots, n. \tag{A.47}$$

□

Proof of Lemma 2.6. Note that for $r = \alpha + 2, \ldots, n$,

$$\begin{aligned} \phi_{[r,3]}(\pi_1) = \max\left\{\Delta_{[\alpha-1,2]}(\pi_1),\ \phi_{[\alpha,2]}(\pi_1), \phi_{[\alpha+1,2]}(\pi_1), \ldots, \phi_{[r,2]}(\pi_1)\right\} + \mathrm{TP}_{[\alpha-1,2]}(\pi_1) \\ + t_{h,2} + t_{g,2} + \sum_{p=\alpha+2}^{r} t_{[r,2]}(\pi_1) - \mathrm{TP}_{[\alpha-1,3]}(\pi_1) \\ - t_{h,3} - t_{g,3} - \sum_{p=\alpha+2}^{r-1} t_{[r,3]}(\pi_1), \end{aligned} \tag{A.48}$$

$$\begin{aligned} \phi_{[r,3]}(\pi_2) = \max\left\{\Delta_{[\alpha-1,2]}(\pi_2),\ \phi_{[\alpha,2]}(\pi_2),\ \phi_{[\alpha+1,2]}(\pi_2), \ldots, \phi_{[r,2]}(\pi_2)\right\} + \mathrm{TP}_{[\alpha-1,2]}(\pi_2) \\ + t_{g,2} + t_{h,2} + \sum_{p=\alpha+2}^{r} t_{[r,2]}(\pi_2) - \mathrm{TP}_{[\alpha-1,3]}(\pi_2) \\ - t_{g,3} - t_{h,3} - \sum_{p=\alpha+2}^{r-1} t_{[r,3]}(\pi_2), \end{aligned} \tag{A.49}$$

where $\sum_{p=\alpha+2}^{\alpha+1} t_{[r,3]}(\pi_1) = \sum_{p=\alpha+2}^{\alpha+1} t_{[r,3]}(\pi_2) = 0$. It should be noted that both sequences π_1 and π_2 have the same jobs in positions $r = \alpha + 2, \ldots, n$, therefore,

$$\sum_{p=\alpha+2}^{r} t_{[r,2]}(\pi_1) = \sum_{p=\alpha+2}^{r} t_{[r,2]}(\pi_2) \quad \text{and} \quad \sum_{p=\alpha+2}^{r-1} t_{[r,3]}(\pi_1) = \sum_{p=\alpha+2}^{r-1} t_{[r,3]}(\pi_2). \tag{A.50}$$

Then, it follows from equations (A.9), (A.21) and (A.48)–(A.50) that when

$$\max\{\phi_{[\alpha,2]}(\pi_2), \phi_{[\alpha+1,2]}(\pi_2)\} \leq \max\{\phi_{[\alpha,2]}(\pi_1), \phi_{[\alpha+1,2]}(\pi_1)\}$$

we have

$$\phi_{[r,3]}(\pi_2) \leq \phi_{[r,3]}(\pi_1) \quad \text{for } r = \alpha + 2, \ldots, n.$$

□

Proof of Lemma 2.7. From the definition of $\phi_{[r,4]}$ for $r = \alpha + 2, \ldots, n$,

$$\begin{aligned} \phi_{[r,4]}(\pi_1) = \max\left\{\Delta_{[\alpha-1,3]}(\pi_1),\ \phi_{[\alpha,3]}(\pi_1),\ \phi_{[\alpha+1,3]}(\pi_1), \ldots, \phi_{[r,3]}(\pi_1)\right\} + \mathrm{TP}_{[\alpha-1,3]}(\pi_1) \\ + t_{h,3} + t_{g,3} + \sum_{p=\alpha+2}^{r} t_{[r,3]}(\pi_1) - \mathrm{TP}_{[\alpha-1,4]}(\pi_1) \\ - t_{h,4} - t_{g,4} - \sum_{p=\alpha+2}^{r-1} t_{[r,4]}(\pi_1), \end{aligned} \tag{A.51}$$

$$\begin{aligned} \phi_{[r,4]}(\pi_2) = \max\left\{\Delta_{[\alpha-1,3]}(\pi_2),\ \phi_{[\alpha,3]}(\pi_2),\ \phi_{[\alpha+1,3]}(\pi_2), \ldots, \phi_{[r,3]}(\pi_2)\right\} + \mathrm{TP}_{[\alpha-1,3]}(\pi_2) \\ + t_{g,3} + t_{h,3} + \sum_{p=\alpha+2}^{r} t_{[r,3]}(\pi_2) - \mathrm{TP}_{[\alpha-1,4]}(\pi_2) \end{aligned}$$

$$- t_{g,4} - t_{h,4} - \sum_{p=\alpha+2}^{r-1} t_{[r,4]}\left(\pi_2\right), \tag{A.52}$$

where $\sum_{p=\alpha+2}^{\alpha+1} t_{[r,4]}\left(\pi_1\right) = \sum_{p=\alpha+2}^{\alpha+1} t_{[r,4]}\left(\pi_2\right) = 0$. Since both sequences π_1 and π_2 have the same jobs in positions $r = \alpha+2, \ldots, n$,

$$\sum_{p=\alpha+2}^{r} t_{[r,3]}\left(\pi_1\right) = \sum_{p=\alpha+2}^{r} t_{[r,3]}\left(\pi_2\right) \quad \text{and} \quad \sum_{p=\alpha+2}^{r-1} t_{[r,4]}\left(\pi_1\right) = \sum_{p=\alpha+2}^{r-1} t_{[r,4]}\left(\pi_2\right). \tag{A.53}$$

Then, it follows from equations (A.21), (A.34) and (A.51)–(A.53) that

$$\max\{\phi_{[\alpha,3]}\left(\pi_2\right),\ \phi_{[\alpha+1,3]}\left(\pi_2\right)\} \le \max\{\phi_{[\alpha,3]}\left(\pi_1\right),\ \phi_{[\alpha+1,3]}\left(\pi_1\right)\}$$

implies

$$\phi_{[r,4]}\left(\pi_2\right) \le \phi_{[r,4]}\left(\pi_1\right) \quad \text{for} \quad r = \alpha+2, \ldots, n.$$

□

Proof of Theorem 2.8. If follows from Lemma 2.4 that

$$\phi_{[r,4]}\left(\pi_2\right) = \phi_{[r,4]}\left(\pi_1\right) \quad \text{for} \quad r = 1, 2, \ldots, \alpha-1. \tag{A.54}$$

Moreover, if any of the conditions C4-1 to C4-5 of the theorem holds, it follows from Theorem 2.3 that

$$\max\{\phi_{[\alpha,4]}\left(\pi_2\right), \phi_{[\alpha+1,4]}\left(\pi_2\right)\} \le \max\{\phi_{[\alpha,4]}\left(\pi_1\right), \phi_{[\alpha+1,4]}\left(\pi_1\right)\}. \tag{A.55}$$

Furthermore, Lemma 2.7 implies

$$\phi_{[r,4]}\left(\pi_2\right) \le \phi_{[r,4]}\left(\pi_1\right) \quad \text{for} \quad r = \alpha+2, \ldots, n \tag{A.56}$$

since $\max\{\phi_{[\alpha,3]}\left(\pi_2\right), \phi_{[\alpha+1,3]}\left(\pi_2\right)\} \le \max\{\ \phi_{[\alpha,3]}\left(\pi_1\right), \phi_{[\alpha+1,3]}\left(\pi_1\right)\}$ if any of the conditions C4-1 to C4-5 of the theorem holds.

Hence, by equations (A.54)–(A.56)

$$\Phi_{[r,4]}\left(\pi_2\right) \le \Phi_{[r,4]}\left(\pi_1\right). \tag{A.57}$$

Finally, we obtain from equations (2.4) and (A.57) that

$$C_{\max}(\pi_2) \le C_{\max}(\pi_1).$$

□

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