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A DESCENT DERIVATIVE-FREE ALGORITHM FOR NONLINEAR MONOTONE EQUATIONS WITH CONVEX CONSTRAINTS

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Abstract. In this paper, we present a derivative-free algorithm for nonlinear monotone equations with convex constraints. The search direction is a product of a positive parameter and the negation of a residual vector. At each iteration step, the algorithm generates a descent direction independent from the line search used. Under appropriate assumptions, the global convergence of the algorithm is given. Numerical experiments show the algorithm has advantages over the recently proposed algorithms by Gao and He (*Calcolo* **55** (2018) 53) and Liu and Li (*Comput. Math. App.* **70** (2015) 2442–2453).

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1. Introduction

In this paper, we address the solution of systems of nonlinear monotone equations with convex constraints. This kind of problem has various applications, such as the ℓ_1-norm problem arising from compressing sensing [21, 39], and optimal power flow equations [13, 38], etc.

Newton's method, quasi-Newton methods, and inexact-Newton methods are well-known methods for the unconstrained systems of nonlinear equations [29, 33]. Moreover, several Jacobian-free methods for solving unconstrained systems of nonlinear equations are proposed over the last decade (see *e.g.* [3, 9, 17, 25, 27, 36, 37]). Specifically, La Cruz *et al.* [15, 16] extended the spectral gradient method [8, 30, 31] to solve large-scale systems of nonlinear equations.

By combining Newton's method and projection strategy, Solodov and Svaiter [32] proposed an inexact Newton method for systems of monotone equations which is globally convergent without the regularity assumptions. Following the work of Solodov and Svaiter, several algorithms have been proposed for solving nonlinear monotone equations, see for example [1, 2, 7, 10, 18, 19, 22, 26, 40, 42, 43]. Wang *et al.* [34] extended the work by Solodov and Svaiter to solve convex constrained monotone equations. Ma and Wang [24] proposed a modified projection method for solving systems of monotone equations with convex constraints. Though the projection methods for convex constraints monotone equations proposed in [24, 34] have a very good numerical performance, they are not suitable for solving large-scale monotone equations because they require matrix storage.

By extending the idea of Zhang and Zhou [42], Yu *et al.* [41] proposed a constrained version of the spectral gradient projection algorithm for solving nonlinear monotone equations in which computing the sequence of steps

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does not need matrix storage as well as the solution of linear systems of equations. Interested readers may refer to the references ([4–6,12,14,21,35] among others), for alternative proposals. Recently, Liu and Feng [20] proposed a derivative-free spectral conjugate gradient-type projection algorithm for nonlinear monotone equations with convex constraints, which can generate a sufficient descent direction at each iteration.

Motivated by the development of the algorithms that can generate descent directions for nonlinear monotone equations with convex constraints, in this paper, we propose an alternative derivative-free algorithm that can be used to generate a descent direction at each iteration. The algorithm is capable of improving the numerical performance for some nonlinear monotone equations. Furthermore, the proposed direction requires less number of backtracks in search of the step length compared with some recent existing directions. Under the monotonicity and Lipschitz continuity assumptions, we show that the proposed algorithm is globally convergent.

The remaining part of this paper is organized as follows. In Section 2, we present the motivation and general algorithm of the proposed method. In Section 3, we prove the global convergence. In Section 4, we present the numerical experiments, as well as some conclusions. Unless otherwise stated, throughout this paper $\|\cdot\|$ stands for the Euclidean norm of vectors, $\langle\cdot\,,\ \cdot\rangle$ is the inner product of vectors in $\mathbb{R}^n$.

2. Motivation and algorithm

Consider the problem of finding the solution of the nonlinear systems of equations of the form

$$F(x) = 0, \tag{2.1}$$

where $F:\mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous and monotone. The Lipschitz continuity of F means there exists a constant $L > 0$ such that $\|F(x) - F(y)\| \le L\|x-y\|, \quad \forall x, y \in \mathbb{R}^n$, while the monotonicity of F means $\langle F(x) - F(y), x-y\rangle \ge 0, \quad \forall x, y \in \mathbb{R}^n$. Most of the methods used for solving (2.1) are iterative in which given an initial iterate x_0, the next iterate is obtained via $x_{k+1} = x_k + s_k, \quad k = 0, 1, 2, \ldots$, where $s_k = \alpha_k d_k$, α_k is the step length obtained by some line search and d_k is the search direction usually satisfying

$$\langle F(x_k),\ d_k\rangle \le -c\|F(x_k)\|^2, \tag{2.2}$$

where c is a positive constant.

To solve convex constrained nonlinear monotone equations, Yu *et al.* [41] popularized the work of Zhang and Zhou [42] in which the search direction is given as

$$d_k = -\theta_k F(x_k), \tag{2.3}$$

where θ_k is a parameter similar to the well-known long Barzilai–Borwein spectral coefficient [8] given as

$$\theta_k = \frac{\langle s_{k-1},\ s_{k-1}\rangle}{\langle \bar{y}_{k-1},\ s_{k-1}\rangle}, \tag{2.4}$$

with $s_{k-1} = x_k - x_{k-1},\ \bar{y}_{k-1} = F(x_k) - F(x_{k-1}) + rs_{k-1},\ r > 0$.

It has been shown in [42] that

(i) the parameter θ_k is positive for all k provided the solution of (2.1) is not attained and
(ii) the sequence of the direction d_k is bounded for all k.

Our aim in this paper is to use different positive parameter that is similar to the well-known short Barzilai–Borwein spectral coefficient [8] given as

$$\bar{\theta}_k = \frac{\langle \gamma_{k-1},\ s_{k-1}\rangle}{\langle \gamma_{k-1},\ \gamma_{k-1}\rangle}, \tag{2.5}$$

where

$$\gamma_{k-1} = y_{k-1} + r_{k-1}d_{k-1}, \quad y_{k-1} = F(x_k) - F(x_{k-1}) \text{ and } r_{k-1} = 1 + \max\left\{0, -\frac{\langle y_{k-1},\ d_{k-1}\rangle}{\|F(x_{k-1})\|^2}\right\}.$$

From the definition of γ_{k-1} and the monotonicity of F, we have

$$\begin{aligned}\langle \gamma_{k-1},\ s_{k-1}\rangle &= \langle y_{k-1},\ s_{k-1}\rangle + \frac{1}{\alpha_{k-1}} r_{k-1}\|s_{k-1}\|^2 \\ &\geq \frac{1}{\alpha_{k-1}} r_{k-1}\|s_{k-1}\|^2 > 0.\end{aligned}$$

This shows that $\langle \gamma_{k-1},\ s_{k-1}\rangle$ is always positive provided the solution of (2.1) is not attained.

The choice of $\bar{\theta}_k$ in (2.5) is always positive provided that the function F is monotone, and it is shown to be more significant and more effective when used as a scalar multiple of the residual vector to compute the search direction [23, 28]. It is shown that this approach is superior (numerically) to some modified conjugate residual approaches for solving nonlinear monotone equations with convex constraints (see Sect. 4).

To describe our algorithm for solving (2.1), we start with the definition of a projection operator and its remarkable property.

Definition 2.1. Let Ω be a nonempty closed convex subset of $\mathbb{R}^n$. A map P from $\mathbb{R}^n$ to Ω defined as

$$P_\Omega[x] = arg\min\{\|x - y\| : y \in \Omega\}, \tag{2.6}$$

is called the projection of $x \in \mathbb{R}^n$ onto the closed convex set Ω.

One of the interesting property of the projection map is that it is nonexpansive, *i.e.*

$$\|P_\Omega[x] - P_\Omega[y]\| \leq \|x - y\| \quad \forall x, y \in \mathbb{R}^n. \tag{2.7}$$

Algorithm 1: Descent derivative-free projection method (DDPM).

input : Given $x_0 \in \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$, β, ρ, $\sigma \in (0,1)$, $0 < \ell \leq u$ and $tol > 0$.

Set $k = 0$;
Compute $\|F(x_k)\|$;
while $\|F(x_k)\| > tol$ **do**
  if $k = 0$, **then**
    set $d_k = -F(x_k)$;
  else
    Compute $d_k = -\bar{\theta}_k F(x_k)$, where,

$$\bar{\theta}_k = \min\left\{\max\left\{\frac{\langle \gamma_{k-1},\ s_{k-1}\rangle}{\langle \gamma_{k-1},\ \gamma_{k-1}\rangle},\ \ell\right\},\ u\right\}$$

    with $\gamma_{k-1} = y_{k-1} + r_{k-1}d_{k-1}$, $r_{k-1} = 1 + \max\left\{0, -\frac{\langle y_{k-1},\ d_{k-1}\rangle}{\|F(x_{k-1})\|^2}\right\}$ and $y_{k-1} = F(x_k) - F(x_{k-1})$, $s_{k-1} = x_k - x_{k-1}$;
  end
  Initialize $m = 0$;
  while $\langle F(x_k + \beta\rho^m d_k),\ d_k\rangle > -\sigma\beta\rho^m\|F(x_k + \beta\rho^m d_k)\|\|d_k\|^2$ **do**
    $m = m + 1$;
    $\alpha = \beta\rho^m$;
  end
  Set $\alpha_k = \alpha$;
  Compute $z_k = x_k + \alpha_k d_k$;
  if $z_k \in \Omega$ *and* $F(z_k) \leq tol$ **then**
    $x_{k+1} = z_k$;
  else
    $x_{k+1} = P_\Omega[x_k - \xi_k F(z_k)]$, where $\xi_k = \frac{\langle x_k - z_k, F(z_k)\rangle}{\|F(z_k)\|^2}$;
  end
  Set $k = k + 1$.
end

Remark 2.2. The parameter $\bar{\theta}_k$ positive and uniformly bounded for each k.

The proof of the following lemma is omitted because it follows directly from the definition of the search direction d_k and Remark 2.2.

Lemma 2.3. *Let the sequence $\{x_k\}$ be generated by Algorithm 1, then the search direction satisfies the following conditions:*

(a) $\langle F(x_k),\ d_k\rangle \leq -c\|F(x_k)\|^2$, *and*
(b) $m_1\|F(x_k)\| \leq \|d_k\| \leq m_2\|F(x_k)\|\quad \forall k\geq 0$, *where c, m_1 and m_2 are positive constants.*

3. Convergence results

We now turn to analyze the global convergence of Algorithm 1 (DDPM). In what follows, we assume that $F(z_k)\neq 0,\ F(x_k)\neq 0$, otherwise the solution of (2.1) has been obtained.

Lemma 3.1. *If the sequences $\{x_k\}$ and $\{d_k\}$ are generated by Algorithm 1, then there exist a step-size α_k satisfying*

$$-\langle F(x_k+\alpha_k d_k),\ d_k\rangle \geq \sigma\alpha_k\|F(x_k+\alpha_k d_k)\|\|d_k\|^2. \tag{3.1}$$

Proof. Suppose that there exists $k_0\geq 0$, such that

$$\langle F(x_{k_0}+\beta\rho^m d_{k_0}),\ d_{k_0}\rangle > -\sigma\beta\rho^m\|F(x_{k_0}+\beta\rho^m d_{k_0})\|\|d_{k_0}\|^2,\quad \forall m\geq 0.$$

By continuity of F, allowing $m\to+\infty$, we have

$$\langle F(x_{k_0}),\ d_{k_0}\rangle \geq 0,$$

which contradicts the conclusion of Lemma 2.3 (a). □

The proof of the following lemma is similar to Lemma 3.2 in [21].

Lemma 3.2. *If $\{x_k\}$ and $\{z_k\}$ are sequences generated by Algorithm 1, then $\{x_k\}$ and $\{z_k\}$ are bounded. In addition,*

$$\lim_{k\to+\infty}\|x_k-z_k\| = 0, \tag{3.2}$$

and

$$\lim_{k\to+\infty}\|x_{k+1}-x_k\| = 0. \tag{3.3}$$

Proof. From Line 9 of Algorithm 1

$$\begin{aligned}\langle F(z_k),\ x_k-z_k\rangle &= -\alpha_k\langle F(z_k),\ d_k\rangle\\ &\geq \sigma\alpha_k^2\|F(z_k)\|\|d_k\|^2\\ &= \sigma\|F(z_k)\|\|x_k-z_k\|^2\\ &> 0.\end{aligned} \tag{3.4}$$

Let $\tilde{x}\in\Omega$ such the $F(\tilde{x})=0$, then by monotonicity of F, it holds that

$$\begin{aligned}\langle F(z_k),\ x_k-\tilde{x}\rangle &= \langle F(z_k),\ x_k-z_k\rangle + \langle F(z_k),\ z_k-\tilde{x}\rangle\\ &\geq \langle F(z_k),\ x_k-z_k\rangle + \langle F(\tilde{x}),\ z_k-\tilde{x}\rangle\\ &= \langle F(z_k),\ x_k-z_k\rangle.\end{aligned} \tag{3.5}$$

Also, using (2.7), (3.4), (3.5) and the projection of $x_k - \xi_k F(z_k)$ on Ω we have

$$\begin{aligned}\|x_{k+1} - \tilde{x}\|^2 &= \|P_\Omega[x_k - \zeta_k F(z_k)] - P_\Omega(\tilde{x})\|^2 \\ &\le \|x_k - \xi_k F(z_k) - \tilde{x}\|^2 \\ &= \|x_k - \tilde{x}\|^2 - 2\xi_k\langle F(z_k),\ x_k - \tilde{x}\rangle + \|\xi_k F(z_k)\|^2 \\ &= \|x_k - \tilde{x}\|^2 - \frac{\langle F(z_k),\ x_k - z_k\rangle^2}{\|F(z_k)\|^2} \\ &\le \|x_k - \tilde{x}\|^2 - \frac{\sigma^2\|F(z_k)\|^2\|x_k - z_k\|^4}{\|F(z_k)\|^2} \\ &= \|x_k - \tilde{x}\|^2 - \sigma^2\|x_k - z_k\|^4. \end{aligned} \tag{3.6}$$

Thus, the sequence $\{\|x_k - \tilde{x}\|\}$ is nonincreasing and convergent, and hence $\{x_k\}$ is bounded. Inequality (3.4), monotonicity of F and Cauchy–Schwarz inequality yield

$$0 < \sigma\|F(z_k)\|\|x_k - z_k\|^2 \le \langle F(z_k),\ x_k - z_k\rangle \le \|F(z_k)\|\|x_k - z_k\|,$$

which implies that

$$\sigma\|x_k - z_k\| \le 1. \tag{3.7}$$

Now, from (3.7) we have that the sequence $\{z_k\}$ is bounded.

It follows from (3.6) that

$$\sigma^2\sum_{k=0}^{\infty}\|x_k - z_k\|^4 \le \sum_{k=0}^{\infty}(\|x_k - \tilde{x}\|^2 - \|x_{k+1} - \tilde{x}\|^2) < \infty, \tag{3.8}$$

which implies

$$\lim_{k\to+\infty}\|x_k - z_k\| = 0. \tag{3.9}$$

As $x_k \in \Omega$ and $\xi_k = \frac{\langle x_k - z_k,\ F(z_k)\rangle}{\|F(z_k)\|^2}$, by (2.6) and Cauchy–Schwarz inequality, it holds that

$$\begin{aligned}\|x_{k+1} - x_k\| &= \|P_\Omega[x_k - \xi_k F(z_k)] - P_\Omega(x_k)\| \\ &\le \|x_k - \xi_k F(z_k) - x_k\| \\ &= \|\xi_k F(z_k)\| \\ &\le \|x_k - z_k\|, \end{aligned}$$

which yields

$$\lim_{k\to+\infty}\|x_{k+1} - x_k\| = 0.$$

□

Remark 3.3. By (3.9) and definition of z_k, we have

$$\lim_{k\to+\infty}\alpha_k\|d_k\| = 0. \tag{3.10}$$

The following lemma is similar to Lemma 2.4 of Li and Li [19].

Lemma 3.4. *If $\{x_k\}$ and $\{z_k\}$ are sequences generated by Algorithm 1, then we have*

$$\alpha_k \geq \min\left\{\beta, \frac{c\rho\|F(x_k)\|^2}{(L+\sigma\|F(z_k')\|\|d_k\|^2}\right\}, \tag{3.11}$$

where $z_k' = x_k + \alpha_k' d_k$ and $\alpha_k' = \alpha_k \rho$.

Proof. From the line search procedure, when $\alpha_k \neq \beta$, then $\alpha_k' = \alpha_k \rho^{-1}$ does not satisfy Line 9 of Algorithm 1. This implies that

$$\langle F(z_k'),\ d_k\rangle > -\sigma\alpha_k'\|F(z_k')\|\|d_k\|^2.$$

Applying the Lipschitz continuity of F and Lemma 2.3 (a), we have

$$\begin{aligned} c\|F(x_k)\|^2 &\leq -\langle F(x_k),\ d_k\rangle \\ &\leq \langle F(z_k') - F(x_k),\ d_k\rangle + \sigma\alpha_k'\|F(z_k')\|\|d_k\|^2 \\ &\leq \alpha_k'(L+\sigma\|F(z_k')\|)\|d_k\|^2. \end{aligned}$$

This yields the desired result (3.11). □

The proof of the following theorem follows by contradiction and the results from the previous lemmas.

Theorem 3.5. *If $\{x_k\}$ and $\{z_k\}$ are sequences generated by Algorithm 1, then*

$$\liminf_{k\to+\infty} \|F(x_k)\| = 0. \tag{3.12}$$

Furthermore, the sequence $\{x_k\}$ converges to $\tilde{x} \in \Omega$ such that $F(\tilde{x}) = 0$.

4. Numerical experiments

This section presents some numerical experiment to evaluate the performance of the proposed algorithm DDPM compared with the efficient three-term conjugate gradient method proposed by Gao and He [12] denoted by ETTC and another method proposed by Liu and Li [21] denoted by PCGM for solving nonlinear monotone equations with convex constraints. We chose the following parameters for the implementation of DDPM algorithm $\rho = 0.5$, $\sigma = 0.01$, $\beta = 1$, $\ell = 10^{-30}$, and $u = 10^{30}$. The parameters in ETTC and PCGM algorithms are chosen as in [12,21].

The list of the test problems used for the numerical experiments is given below, where the function F is taken as $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^T$.

Problem 1 [16]**.** Exponential problem

$$\begin{aligned} &F_1(x) = e^{x_1} - 1, \\ &F_i(x) = e^{x_i} + x_i - 1,\ \text{for } i = 2,3,\ldots,n, \\ &\text{and } \Omega = \mathbb{R}^n_+. \end{aligned}$$

Problem 2 [16]**.** Modified logarithmic problem

$$\begin{aligned} &F_i(x) = \ln(x_i+1) - \frac{x_i}{n},\ \text{for } i = 2,3,\ldots,n, \\ &\text{and } \Omega = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i \leq n, x_i > -1, i = 1,2,\ldots,n\}. \end{aligned}$$

Problem 3 [43]**.** Nonsmooth problem I

$$F_i(x) = 2x_i - \sin|x_i|,\ i = 1, 2, 3, \dots, n,$$
$$\text{and } \Omega = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \le n, x_i \ge 0, i = 1, 2, \dots, n\}.$$

It is clear that Problem 3 is nonsmooth at $x = 0$.

Problem 4 [16]**.** Strictly convex problem I

$$F_i(x) = e^{x_i} - 1,\ \text{for } i = 1, 2, \dots, n,$$
$$\text{and } \Omega = \mathbb{R}^n_+.$$

Problem 5 [16]**.** Strictly convex problem II

$$F_i(x) = \frac{i}{n} e^{x_i} - 1,\ \text{for } i = 1, 2, \dots, n,$$
$$\text{and } \Omega = \mathbb{R}^n_+.$$

Problem 6 [41]**.** Nonsmooth problem II

$$F_i(x) = x_i - \sin|x_i - 1|,\ i = 1, 2, 3, \dots, n,$$
$$\text{and } \Omega = \left\{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \le n, x_i \ge -1, i = 1, 2, \dots, n\right\}.$$

It is clear that Problem 6 is nonsmooth at $x = 1$.

Problem 7 [16]**.** Discrete boundary value problem

$$\begin{aligned} F_1(x) &= 2x_1 + 0.5h^2(x_1 + h)^3 - x_2, \\ F_i(x) &= 2x_i - x_{i-1} + x_{i+1} + 0.5h^2(x_i + ih)^3,\ \text{for } i = 2, \dots, n-1, \\ F_n(x) &= 2x_n - x_{n-1} + 0.5h^2(x_n + nh)^3, \\ h &= \frac{1}{n+1} \text{ and } \Omega = \mathbb{R}^n_+. \end{aligned}$$

We used nine different initial points for all the test problems, the initial points are $x^1 = (1, \dots, 1)^T$, $x^2 = (0.1, \dots, 0.1)^T$, $x^3 = \left(\frac{1}{2}, \dots, \frac{1}{2^n}\right)^T$, $x^4 = \left(1 - \frac{1}{n}, \dots, n-1\right)^T$, $x^5 = \left(0, \frac{1}{n}, \dots, \frac{n-1}{n}\right)^T$, $x^6 = \left(1, \frac{1}{2}, \dots, \frac{1}{n}\right)^T$, $x^7 = \left(\frac{n-1}{n}, \frac{n-2}{n}, \dots, 0\right)^T$, $x^8 = \left(\frac{1}{n}, \frac{2}{n}, \dots, 1\right)^T$, and $x^9 = rand(n, 1)$, which means the n randomly generated numbers from $(0, 1)$. Furthermore, we used five different dimensions namely; 1000, 5000, 10 000, 50 000, 100 000.

All methods were coded in MATLAB R2018a and run on a PC with Intel CORE i5 processor, 4GB of RAM and CPU speed of 2.3GHZ. The iteration process is terminated when $\|F(x_k)\| \le 10^{-5}$. We use "–" to declare a failure of the algorithm when at least 1000 iterations are completed without achieving the convergence. Tables 1–7 report the numerical results obtained from the experiments, in which ITER, FVAL, TIME and NORM represent the number of iterations, number of function evaluations, CPU time (in seconds) and the norm of the residual at the approximate solution respectively. Also, we used the performance profiles introduced by Dolan and Moré in [11] to plot Figures 1–3. As described in [11], the performance profiles are defined in terms of a performance measure $t_{\rho,\ s} > 0$ obtained for each problem $\rho \in \mathbb{P}$ and solver $s \in \mathbb{S}$. For any pair (ρ, s) of problem ρ and solver s, the performance ratio is defined as

$$r_{\rho,\ s} = \frac{t_{\rho,\ s}}{\min\{t_{\rho,\ s} \mid s \in \mathbb{S}\}}.$$

TABLE 1. Numerical Results for DDPM, ETTC and PCGM for Problem 1 with given initial points and dimensions.

		DDPM				ETTC				PCGM			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x^1	12	**17**	1.7133	3.33E-06	**11**	32	**0.68876**	2.7E-07	75	389	0.82854	9.23E-06
	x^2	11	**16**	0.053233	5.06E-06	**9**	23	**0.052446**	7.44E-06	61	316	0.17662	9.77E-06
	x^3	12	**14**	**0.095149**	4.47E-06	**9**	18	0.42197	6.33E-06	50	252	0.12742	9.84E-06
	x^4	14	**19**	**0.046254**	1.07E-06	**11**	23	0.046805	3.05E-06	74	384	0.19509	9.30E-06
	x^5	14	**19**	**0.023248**	1.07E-06	**11**	23	0.038575	2.97E-06	75	382	0.20009	9.14E-06
	x^6	15	20	**0.017463**	7.95E-06	**9**	**18**	0.030294	4.23E-06	55	276	0.13541	8.95E-06
	x^7	14	**19**	**0.012663**	1.07E-06	**11**	23	0.035613	3.05E-06	74	384	0.18071	9.30E-06
	x^8	14	**19**	0.05582	1.07E-06	**11**	23	**0.046895**	2.98E-06	74	377	0.18191	9.09E-06
	x^9	14	**19**	**0.13585**	1.03E-06	**12**	25	0.13591	3.36E-06	87	451	0.26761	9.28E-06
5000	x^1	12	**17**	**0.20135**	6.66E-06	**11**	23	0.27868	2.56E-06	76	394	0.69443	7.62E-06
	x^2	11	16	0.075023	8.56E-06	**10**	**29**	**0.029806**	2.53E-07	62	320	0.53869	9.71E-06
	x^3	12	**14**	0.028435	4.47E-06	**9**	18	**0.020158**	6.33E-06	49	247	0.4153	8.19E-06
	x^4	14	**19**	0.042681	2.4E-06	**11**	23	**0.025818**	6.69E-06	74	385	0.70698	7.15E-06
	x^5	14	**19**	**0.043942**	2.4E-06	**11**	23	0.092539	6.66E-06	80	407	0.66247	9.21E-06
	x^6	15	20	0.040354	7.89E-06	**9**	**18**	**0.02302**	4.23E-06	53	266	0.43926	9.74E-06
	x^7	14	**19**	0.036299	2.4E-06	**11**	23	**0.024423**	6.69E-06	74	385	0.65025	7.15E-06
	x^8	14	**19**	0.069729	2.4E-06	**11**	23	**0.027958**	6.66E-06	80	407	0.70279	9.52E-06
	x^9	14	**19**	**0.032056**	2.4E-06	**11**	23	0.044427	6.68E-06	90	468	0.81457	7.05E-06
10 000	x^1	12	**17**	0.11658	9.27E-06	**11**	32	**0.086133**	7.01E-07	78	404	1.3073	9.27E-06
	x^2	12	30	0.06031	3.81E-06	**10**	**29**	**0.052551**	3.58E-07	65	336	1.075	8.55E-06
	x^3	12	**14**	0.049315	4.47E-06	**9**	18	**0.037459**	6.33E-06	49	247	0.81323	8.17E-06
	x^4	14	**19**	0.065195	3.39E-06	**11**	23	**0.04548**	9.44E-06	72	374	1.2394	9.28E-06
	x^5	14	**19**	0.058889	3.39E-06	**11**	23	**0.056483**	9.42E-06	84	425	1.3862	9.76E-06
	x^6	15	20	0.059327	7.9E-06	**9**	**18**	**0.036668**	4.23E-06	53	266	0.89633	9.72E-06
	x^7	14	**19**	0.060889	3.39E-06	**11**	23	**0.052131**	9.44E-06	72	374	1.2779	9.28E-06
	x^8	14	**19**	0.061588	3.39E-06	**11**	23	**0.046482**	9.42E-06	84	425	1.3909	9.07E-06
	x^9	14	**19**	0.059187	3.38E-06	**11**	23	**0.055107**	9.37E-06	88	458	1.5289	9.11E-06
50 000	x^1	16	37	0.31866	3.82E-06	**11**	**32**	**0.24079**	1.54E-06	88	455	6.0355	9.63E-06
	x^2	12	30	**0.19633**	3.88E-06	**10**	**29**	0.20208	7.99E-07	74	382	5.098	9.62E-06
	x^3	12	**14**	0.17456	4.47E-06	**9**	18	**0.17387**	6.33E-06	49	247	3.34	8.16E-06
	x^4	15	34	**0.24803**	4.15E-06	**12**	**25**	0.26737	4.21E-06	79	409	5.5001	8.28E-06
	x^5	14	**20**	**0.1789**	9.77E-06	**12**	25	0.23548	4.21E-06	77	386	5.1792	9.66E-06
	x^6	15	20	0.19008	7.92E-06	**9**	**18**	**0.14401**	4.23E-06	53	266	3.5394	9.72E-06
	x^7	15	34	**0.22943**	4.15E-06	**12**	**25**	0.24174	4.21E-06	79	409	5.4154	8.28E-06
	x^8	14	**20**	**0.16939**	9.77E-06	**12**	25	0.27987	4.21E-06	82	414	5.4923	9.54E-06
	x^9	15	34	**0.23178**	4.01E-06	**12**	**25**	0.26896	4.21E-06	93	484	6.4091	9.07E-06
100 000	x^1	17	40	0.77196	3.3E-06	**12**	**25**	**0.47779**	2.19E-06	88	456	12.8932	7.95E-06
	x^2	12	30	**0.35675**	3.95E-06	**10**	**29**	0.41328	1.13E-06	75	387	10.6182	9.98E-06
	x^3	12	**14**	**0.25509**	4.47E-06	**9**	18	0.28336	6.33E-06	49	247	6.8603	8.16E-06
	x^4	15	35	0.4507	7.31E-06	**12**	**25**	**0.43937**	5.96E-06	84	435	12.32	8.81E-06
	x^5	15	35	0.50674	7.28E-07	**12**	**25**	**0.44449**	5.96E-06	77	394	12.5185	9.52E-06
	x^6	15	20	**0.36666**	7.92E-06	**9**	**18**	0.41761	4.23E-06	53	266	8.5851	9.72E-06
	x^7	15	35	0.43206	7.31E-06	**12**	**25**	**0.41439**	5.96E-06	84	435	14.5536	8.81E-06
	x^8	15	35	0.43595	7.28E-07	**12**	**25**	**0.42832**	5.96E-06	77	395	12.7393	8.03E-06
	x^9	15	35	**0.44319**	3.91E-06	**12**	**25**	0.44804	5.97E-06	94	489	16.1043	9.81E-06

TABLE 2. Numerical Results for DDPM, ETTC and PCGM for Problem 2 with given initial points and dimensions.

		DDPM				ETTC				PCGM			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x^1	17	**17**	**0.17746**	2.57E-07	**6**	22	1.1958	2.46E-06	7	15	1.0195	4.31E-06
	x^2	11	11	0.011757	2.87E-07	**3**	**3**	**0.004513**	5.17E-07	5	13	0.010311	8.84E-06
	x^3	15	15	0.01247	9.48E-06	**8**	**12**	**0.007347**	1.71E-06	–	–	–	–
	x^4	17	17	0.02182	1.47E-07	**9**	**13**	**0.011744**	7.45E-06	–	–	–	–
	x^5	17	17	0.015295	1.47E-07	**9**	**13**	**0.011267**	7.45E-06	–	–	–	–
	x^6	17	17	0.013645	1.34E-08	**10**	**16**	**0.012793**	6.22E-06	10	16	0.035876	6.58E-06
	x^7	17	17	0.015947	1.47E-07	**9**	**13**	**0.008994**	7.45E-06	–	–	–	–
	x^8	17	17	0.01666	1.48E-07	**9**	**13**	**0.010563**	7.47E-06	–	–	–	–
	x^9	17	17	0.0199	1.49E-07	**9**	**13**	**0.009598**	7.7E-06	9	37	0.054132	5.41E-06
5000	x^1	17	**17**	0.1339	1.1E-07	**6**	24	**0.075544**	3.06E-06	7	15	0.2951	7.99E-06
	x^2	11	11	0.036028	1.21E-07	**3**	**3**	**0.008229**	1.75E-07	6	35	0.032728	1.79E-06
	x^3	15	**15**	0.068326	9.45E-06	**11**	18	**0.032255**	8.73E-08	–	–	–	–
	x^4	17	**17**	0.063459	6.42E-08	**12**	19	**0.038022**	9.5E-07	–	–	–	–
	x^5	17	**17**	0.053111	6.42E-08	**12**	19	**0.040947**	9.5E-07	–	–	–	–
	x^6	17	**17**	0.05955	2.64E-09	**13**	22	0.052009	3.16E-07	19	34	**0.050809**	3.14E-06
	x^7	17	**17**	0.056279	6.42E-08	**12**	19	**0.036755**	9.5E-07	–	–	–	–
	x^8	17	**17**	0.057113	6.42E-08	**12**	19	**0.048019**	9.51E-07	–	–	–	–
	x^9	17	**17**	0.054061	6.43E-08	**12**	19	**0.040853**	9.74E-07	–	–	–	–
10 000	x^1	17	**17**	0.087662	7.45E-08	**6**	25	**0.063698**	8.39E-06	8	37	0.15876	9.48E-07
	x^2	11	11	0.05386	8.16E-08	**3**	**3**	**0.014232**	1.21E-07	6	35	0.11835	2.17E-06
	x^3	15	**15**	0.071802	9.44E-06	**12**	20	**0.057353**	3.07E-08	–	–	–	–
	x^4	17	**17**	0.09035	4.42E-08	**13**	21	**0.074668**	4.6E-07	–	–	–	–
	x^5	17	**17**	**0.090102**	4.42E-08	**13**	21	0.10199	4.6E-07	–	–	–	–
	x^6	17	**17**	0.081409	1.3E-09	**14**	24	**0.075687**	1.12E-07	17	45	0.098276	4.14E-06
	x^7	17	**17**	**0.090834**	4.42E-08	**13**	21	0.12438	4.6E-07	–	–	–	–
	x^8	17	**17**	0.10319	4.42E-08	**13**	21	**0.075875**	4.61E-07	–	–	–	–
	x^9	17	**17**	0.14902	4.43E-08	**13**	21	**0.085206**	4.65E-07	–	–	–	–
50 000	x^1	19	**21**	0.41724	1.97E-08	**6**	25	0.2125	4.79E-06	8	37	0.67891	1.81E-06
	x^2	11	11	0.21509	2.21E-08	**3**	**3**	**0.050057**	6.32E-08	6	35	0.22498	4.20E-06
	x^3	**15**	**15**	**0.24756**	9.44E-06	16	28	0.33261	6.34E-10	–	–	–	–
	x^4	17	**17**	**0.32833**	1.57E-08	**16**	27	0.35616	4.95E-08	–	–	–	–
	x^5	17	**17**	**0.38994**	1.57E-08	**16**	27	0.42528	4.95E-08	–	–	–	–
	x^6	**16**	**16**	0.41098	4.1E-10	17	30	**0.3665**	5.27E-09	–	–	–	–
	x^7	17	**17**	0.56091	1.57E-08	**16**	27	**0.36136**	4.95E-08	–	–	–	–
	x^8	17	**17**	**0.26715**	1.57E-08	**16**	27	0.36915	4.95E-08	–	–	–	–
	x^9	17	**17**	**0.36713**	1.57E-08	**16**	27	0.39799	5.04E-08	–	–	–	–
100 000	x^1	19	**22**	0.63202	4.12E-09	**6**	25	**0.41514**	4.33E-06	8	37	0.86145	2.50E-06
	x^2	11	11	0.31372	2.99E-09	**3**	**3**	**0.096282**	5.4E-08	6	35	0.34583	5.83E-06
	x^3	**15**	**15**	**0.42237**	9.44E-06	17	31	0.58174	9.18E-06	–	–	–	–
	x^4	**18**	**19**	**0.57584**	7.46E-09	21	51	1.0181	3E-07	–	–	–	–
	x^5	**18**	**19**	**0.59921**	7.46E-09	21	51	1.0077	3E-07	–	–	–	–
	x^6	**16**	**16**	**0.49971**	1.98E-10	19	36	0.77306	7.34E-06	17	49	0.93197	5.66E-06
	x^7	**18**	**19**	**0.59938**	7.46E-09	21	51	1.0118	3E-07	–	–	–	–
	x^8	**18**	**19**	**0.60921**	7.46E-09	21	51	1.0164	3E-07	–	–	–	–
	x^9	**18**	**19**	**0.7224**	7.46E-09	21	51	1.1021	2.99E-07	–	–	–	–

TABLE 3. Numerical Results for DDPM, ETTC and PCGM for Problem 3 with given initial points and dimensions.

		DDPM				ETTC				**PCGM**			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x^1	**6**	**18**	0.1734	3.86E-14	26	56	0.40232	5.11E-06	**6**	23	**0.012229**	6.22E-06
	x^2	**3**	**15**	0.011854	6.68E-14	22	48	0.02489	4.62E-06	5	21	**0.007028**	8.55E-06
	x^3	**8**	**17**	**0.012421**	5.86E-06	22	44	0.023161	7.34E-06	–	–	–	–
	x^4	**9**	**19**	**0.015498**	8.46E-15	25	51	0.027446	8.7E-06	–	–	–	–
	x^5	**9**	**19**	**0.013427**	8.46E-15	25	51	0.024021	8.7E-06	–	–	–	–
	x^6	**10**	**18**	**0.0165**	7.28E-06	23	46	0.024082	9.1E-06	–	–	–	–
	x^7	**9**	**19**	**0.016207**	8.46E-15	25	51	0.022866	8.7E-06	–	–	–	–
	x^8	**9**	**19**	**0.012868**	8.51E-15	25	51	0.026401	8.71E-06	195	1047	0.19152	3.31E-07
	x^9	**9**	**19**	**0.014744**	8.62E-15	25	51	0.026035	8.8E-06	615	3497	0.7834	2.69E-06
5000	x^1	**6**	**18**	0.04067	8.62E-14	27	62	0.10155	3.3E-06	7	50	**0.033582**	1.22E-06
	x^2	**3**	**15**	0.033554	1.49E-13	23	54	0.063346	2.98E-06	6	48	**0.033127**	1.68E-06
	x^3	**11**	**17**	**0.037618**	5.86E-06	22	44	0.0551	7.34E-06	–	–	–	–
	x^4	**12**	**19**	**0.043923**	1.9E-14	26	56	0.066669	7.01E-06	–	–	–	–
	x^5	**12**	**19**	**0.048658**	1.9E-14	26	56	0.082992	7.01E-06	627	3246	2.159	4.21E-06
	x^6	**13**	**18**	**0.044785**	7.28E-06	23	46	0.088818	9.1E-06	–	–	–	–
	x^7	**12**	**19**	**0.039034**	1.9E-14	26	56	0.069477	7.01E-06	–	–	–	–
	x^8	**12**	**19**	**0.0457**	1.9E-14	26	56	0.085932	7.01E-06	–	–	–	–
	x^9	**12**	**19**	**0.065429**	1.91E-14	26	56	0.12767	7.05E-06	–	–	–	–
10 000	x^1	**6**	**18**	0.072498	1.22E-13	27	62	0.13866	4.66E-06	7	50	**0.051828**	1.73E-06
	x^2	**3**	**15**	0.061342	2.11E-13	23	54	0.11598	4.22E-06	6	48	**0.054244**	2.37E-06
	x^3	**12**	**17**	**0.074055**	5.86E-06	22	44	0.10335	7.34E-06	–	–	–	–
	x^4	**13**	**19**	**0.099482**	2.68E-14	26	56	0.12341	9.91E-06	–	–	–	–
	x^5	**13**	**19**	**0.095645**	2.68E-14	26	56	0.13518	9.91E-06	–	–	–	–
	x^6	**14**	**18**	**0.091556**	7.28E-06	23	46	0.10327	9.1E-06	–	–	–	–
	x^7	**13**	**19**	**0.07945**	2.68E-14	26	56	0.11718	9.91E-06	–	–	–	–
	x^8	**13**	**19**	**0.084169**	2.68E-14	26	56	0.12519	9.91E-06	–	–	–	–
	x^9	**13**	**19**	**0.10344**	2.68E-14	26	56	0.15012	9.9E-06	–	–	–	–
50 000	x^1	**6**	**22**	0.31782	2.91E-13	28	69	0.5686	1.6E-07	7	50	**0.19323**	3.86E-06
	x^2	**3**	**15**	**0.2125**	4.73E-13	23	54	0.44224	9.43E-06	6	48	0.25449	5.31E-06
	x^3	**16**	**17**	**0.23887**	5.86E-06	22	44	0.40029	7.34E-06	–	–	–	–
	x^4	**16**	**21**	**0.4047**	3.43E-14	27	62	0.51986	6.4E-06	–	–	–	–
	x^5	**16**	**21**	**0.28357**	3.43E-14	27	62	0.55361	6.4E-06	–	–	–	–
	x^6	**17**	**18**	**0.25083**	7.28E-06	23	46	0.4458	9.1E-06	–	–	–	–
	x^7	**16**	**21**	**0.2934**	3.43E-14	27	62	0.56596	6.4E-06	–	–	–	–
	x^8	**16**	**21**	**0.28616**	3.43E-14	27	62	0.59423	6.4E-06	–	–	–	–
	x^9	**16**	**21**	**0.37522**	3.42E-14	27	62	0.65299	6.4E-06	–	–	–	–
100 000	x^1	**6**	**23**	0.70371	9.48E-13	28	69	0.99836	2.27E-07	7	50	**0.34656**	5.46E-06
	x^2	**3**	**15**	0.3937	6.68E-13	24	61	0.92002	2.05E-07	6	48	**0.33406**	7.50E-06
	x^3	**17**	**17**	**0.45311**	5.86E-06	22	44	0.7802	7.34E-06	–	–	–	–
	x^4	**21**	**21**	**0.53907**	4.84E-14	27	62	0.95975	9.06E-06	–	–	–	–
	x^5	**21**	**21**	**0.6219**	4.84E-14	27	62	0.97533	9.06E-06	–	–	–	–
	x^6	**19**	**18**	**0.56973**	7.28E-06	23	46	0.80074	9.1E-06	–	–	–	–
	x^7	**21**	**21**	**0.53069**	4.84E-14	27	62	0.92895	9.06E-06	–	–	–	–
	x^8	**21**	**21**	**0.63599**	4.84E-14	27	62	0.96025	9.06E-06	–	–	–	–
	x^9	**21**	**21**	**0.87478**	4.84E-14	27	62	1.163	9.06E-06	–	–	–	–

TABLE 4. Numerical Results forDDPM, ETTC and PCGM for Problem 4 with given initial points and dimensions.

		DDPM				ETTC				PCGM			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x^1	17	**19**	0.050024	1.66E-09	24	52	0.10585	6.68E-06	**6**	25	**0.006814**	1.68E-06
	x^2	14	**15**	0.009432	1.87E-09	21	46	0.015648	7.09E-06	**5**	21	**0.006892**	2.31E-06
	x^3	16	17	0.012548	8.21E-06	21	42	0.011128	8.78E-06	**6**	**11**	**0.005285**	8.27E-07
	x^4	19	**21**	0.015882	4.63E-10	24	49	0.020959	8.25E-06	**14**	43	**0.010178**	8.03E-07
	x^5	19	**21**	0.013926	4.63E-10	24	49	0.019983	8.25E-06	**14**	43	**0.011895**	8.03E-07
	x^6	18	20	**0.012267**	7.03E-06	22	44	0.018524	8.68E-06	**8**	**18**	0.013573	6.89E-07
	x^7	19	**21**	0.014931	4.63E-10	24	49	0.019183	8.25E-06	**14**	43	**0.012638**	8.03E-07
	x^8	19	**21**	**0.013733**	4.65E-10	24	49	0.022347	8.26E-06	**13**	38	0.014792	1.27E-06
	x^9	19	**21**	0.021861	4.58E-10	24	49	0.0188	8.18E-06	**14**	46	**0.011258**	6.34E-07
5000	x^1	17	**19**	0.054725	3.72E-09	25	58	0.064323	4.31E-06	**6**	25	**0.016971**	3.76E-06
	x^2	14	**15**	0.029887	4.19E-09	22	52	0.058551	4.57E-06	**5**	21	**0.018714**	5.16E-06
	x^3	16	17	0.042822	8.21E-06	21	42	0.038254	8.78E-06	**6**	**11**	**0.012135**	8.27E-07
	x^4	19	**21**	0.04778	1.04E-09	25	54	0.051762	6.65E-06	**14**	42	**0.030856**	9.90E-06
	x^5	19	**21**	0.050052	1.04E-09	25	54	0.061453	6.65E-06	**14**	42	**0.031997**	9.90E-06
	x^6	18	20	0.049672	7.03E-06	22	44	0.078129	8.69E-06	**8**	**18**	**0.016464**	6.89E-07
	x^7	19	**21**	0.031958	1.04E-09	25	54	0.052884	6.65E-06	**14**	42	**0.029429**	9.90E-06
	x^8	19	**21**	0.038877	1.04E-09	25	54	0.054468	6.65E-06	**14**	42	**0.03024**	7.71E-07
	x^9	19	**21**	0.045111	1.04E-09	25	54	0.054447	6.63E-06	**14**	48	**0.034582**	5.37E-07
10 000	x^1	17	**19**	0.054876	5.26E-09	25	58	0.090132	6.09E-06	**6**	25	**0.025914**	5.31E-06
	x^2	14	**15**	0.045278	5.93E-09	22	52	0.10134	6.47E-06	**5**	21	**0.021886**	7.29E-06
	x^3	16	17	0.053802	8.21E-06	21	42	0.078872	8.78E-06	**6**	**11**	**0.017628**	8.27E-07
	x^4	19	**21**	0.10496	1.47E-09	25	54	0.11913	9.4E-06	**16**	52	**0.062862**	6.94E-07
	x^5	19	**21**	**0.0598**	1.47E-09	25	54	0.11708	9.4E-06	**16**	52	0.0684	6.94E-07
	x^6	18	20	0.070989	7.03E-06	22	44	0.089728	8.69E-06	**8**	**18**	**0.028402**	6.89E-07
	x^7	19	**21**	0.1029	1.47E-09	25	54	0.098592	9.4E-06	**16**	52	**0.051348**	6.94E-07
	x^8	19	**21**	0.068188	1.47E-09	25	54	0.10645	9.4E-06	**17**	49	**0.054121**	3.54E-06
	x^9	19	**21**	0.086594	1.48E-09	25	54	0.087947	9.42E-06	**16**	60	**0.070782**	6.59E-07
50 000	x^1	19	**23**	0.29752	1.61E-08	26	65	0.46689	1.86E-07	**7**	52	**0.1526**	9.39E-07
	x^2	14	**15**	0.21191	1.33E-08	23	59	0.38495	1.96E-07	**6**	48	**0.12668**	1.23E-06
	x^3	16	17	0.21798	8.21E-06	21	42	0.26447	8.78E-06	**6**	**11**	**0.058938**	8.27E-07
	x^4	19	**22**	0.26653	4.89E-09	26	60	0.37473	6.07E-06	**18**	64	**0.2276**	9.17E-07
	x^5	19	**22**	0.33455	4.89E-09	26	60	0.36743	6.07E-06	**18**	64	**0.22521**	9.17E-07
	x^6	18	20	0.28007	7.03E-06	22	44	0.27685	8.69E-06	**8**	**18**	**0.082535**	6.89E-07
	x^7	19	**22**	0.32143	4.89E-09	26	60	0.36431	6.07E-06	**18**	64	**0.22849**	9.17E-07
	x^8	19	**22**	0.29868	4.89E-09	26	60	0.36685	6.07E-06	**17**	51	**0.2088**	7.01E-07
	x^9	19	**22**	0.26604	4.89E-09	26	60	0.36817	6.07E-06	**18**	54	**0.24536**	7.73E-07
100 000	x^1	20	**25**	0.45445	1.01E-08	26	65	0.73565	2.63E-07	**7**	52	**0.29219**	1.33E-06
	x^2	14	**15**	0.29737	1.87E-08	23	59	0.7102	2.77E-07	**6**	48	**0.25593**	1.74E-06
	x^3	16	17	0.36506	8.21E-06	21	42	0.60532	8.78E-06	**6**	**11**	**0.11754**	8.27E-07
	x^4	**19**	**22**	0.4704	6.91E-09	26	60	0.82523	8.58E-06	**19**	60	**0.44032**	5.72E-06
	x^5	**19**	**22**	0.67228	6.91E-09	26	60	0.92822	8.58E-06	**19**	60	**0.45182**	5.72E-06
	x^6	18	20	0.50315	7.03E-06	22	44	0.56893	8.69E-06	**8**	**18**	**0.15106**	6.89E-07
	x^7	**19**	**22**	0.54649	6.91E-09	26	60	0.89053	8.58E-06	**19**	60	**0.43136**	5.72E-06
	x^8	**19**	**22**	0.49297	6.91E-09	26	60	0.93237	8.58E-06	21	60	**0.45684**	4.67E-06
	x^9	**19**	**22**	**0.60235**	6.92E-09	26	60	0.9517	8.59E-06	46	436	2.0631	3.54E-07

TABLE 5. Numerical Results for DDPM, ETTC and PCGM for Problem 5 with given initial points and dimensions.

		DDPM				ETTC				PCGM			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x^1	23	**25**	0.025441	7.12E-07	37	65	0.18635	7.62E-06	**6**	**25**	**0.006814**	1.68E-06
	x^2	20	33	0.021475	1.37E-06	30	55	0.052738	4.59E-06	**5**	**21**	0.006892	2.31E-06
	x^3	47	71	0.031597	5.17E-06	–	–	–	–	**6**	**11**	**0.005285**	8.27E-07
	x^4	34	54	0.039719	3.19E-06	–	–	–	–	**14**	**43**	**0.010178**	8.03E-07
	x^5	36	63	0.035652	8.4E-06	40	71	0.033502	3.16E-06	**14**	**43**	**0.011895**	8.03E-07
	x^6	86	112	0.037471	4.03E-06	28	52	0.024595	6.61E-06	**8**	**18**	**0.013573**	6.89E-07
	x^7	34	54	0.019909	3.19E-06	–	–	–	–	**14**	**43**	**0.012638**	8.03E-07
	x^8	36	62	0.041105	6.56E-06	40	71	0.033486	3.16E-06	**13**	**38**	**0.014792**	1.27E-06
	x^9	36	67	0.042494	7.53E-06	37	68	0.031532	5.9E-06	**14**	**46**	**0.011258**	6.34E-07
5000	x^1	23	38	0.086065	7.25E-07	37	69	0.11584	6.35E-06	**6**	**25**	**0.016971**	3.76E-06
	x^2	20	33	0.085614	3.9E-06	30	55	0.090644	4.57E-06	**5**	**21**	**0.018714**	5.16E-06
	x^3	31	49	0.073807	7.01E-06	–	–	–	–	**6**	**11**	**0.012135**	8.27E-07
	x^4	28	72	0.084676	9.51E-06	–	–	–	–	**14**	**42**	**0.030856**	9.90E-06
	x^5	28	**33**	0.094617	3.13E-06	48	79	0.1086	7.46E-06	**14**	42	**0.031997**	9.90E-06
	x^6	70	96	0.17331	2.47E-06	32	60	0.11158	7.47E-06	**8**	**18**	**0.016464**	6.89E-07
	x^7	28	72	0.12217	9.51E-06	–	–	–	–	**14**	**42**	**0.029429**	9.90E-06
	x^8	28	**33**	0.11425	3.5E-06	48	79	0.10041	7.46E-06	**14**	42	**0.03024**	7.71E-07
	x^9	44	106	0.18149	8.14E-06	41	82	0.098875	9.84E-06	**14**	**48**	**0.034582**	5.37E-07
10 000	x^1	23	38	0.14135	1.14E-06	37	69	0.20997	9.07E-06	**6**	**25**	**0.025914**	5.31E-06
	x^2	20	33	0.13331	6.12E-06	30	55	0.12079	9.42E-06	**5**	**21**	**0.021886**	7.29E-06
	x^3	–	–	–	–	–	–	–	–	**6**	**11**	**0.017628**	8.27E-07
	x^4	–	–	–	–	–	–	–	–	**16**	**52**	**0.062862**	6.94E-07
	x^5	28	**33**	0.15687	3.57E-06	47	82	0.14754	3.85E-06	**16**	52	**0.0684**	6.94E-07
	x^6	32	60	0.14277	7.37E-06	30	55	0.13992	9.46E-06	**8**	**18**	**0.028402**	6.89E-07
	x^7	–	–	–	–	–	–	–	–	**16**	**52**	**0.051348**	6.94E-07
	x^8	28	**33**	0.13123	4.1E-06	46	81	0.15826	3.85E-06	**17**	49	**0.054121**	3.54E-06
	x^9	46	113	0.27681	8.89E-06	45	88	0.17862	8.41E-06	**16**	**60**	**0.070782**	6.59E-07
50 000	x^1	24	**40**	0.42257	2.45E-06	39	76	0.76059	5.94E-06	**7**	52	**0.1526**	9.39E-07
	x^2	38	55	0.57096	4.32E-06	60	171	1.4206	6.46E-06	**6**	**48**	**0.12668**	1.23E-06
	x^3	–	–	–	–	–	–	–	–	**6**	**11**	**0.058938**	8.27E-07
	x^4	44	74	0.51361	4.67E-06	–	–	–	–	**18**	**64**	**0.2276**	9.17E-07
	x^5	26	**42**	0.40649	8.99E-07	46	81	0.62267	8.8E-06	**18**	64	**0.22521**	9.17E-07
	x^6	36	58	0.45165	5.12E-06	54	148	0.93692	6.2E-06	**8**	**18**	**0.082535**	6.89E-07
	x^7	44	74	0.53898	4.67E-06	–	–	–	–	**18**	**64**	**0.22849**	9.17E-07
	x^8	26	**42**	0.40099	8.99E-07	46	81	0.62293	8.8E-06	**17**	51	**0.2088**	7.01E-07
	x^9	28	**30**	0.38416	3.39E-06	49	99	0.71061	6.32E-06	**18**	54	**0.24536**	7.73E-07
100 000	x^1	24	**41**	0.65968	4.9E-06	39	76	1.0395	8.43E-06	**7**	52	**0.29219**	1.33E-06
	x^2	93	122	1.5632	8.69E-06	77	260	2.97	9.05E-06	**6**	**48**	**0.25593**	1.74E-06
	x^3	68	103	1.2605	4.5E-06	–	–	–	–	**6**	**11**	**0.11754**	8.27E-07
	x^4	–	–	–	–	–	–	–	–	**19**	**60**	**0.44032**	5.72E-06
	x^5	26	**43**	0.73223	3.13E-06	49	85	1.2349	7.52E-06	**19**	60	**0.45182**	5.72E-06
	x^6	45	81	0.97656	7.38E-06	95	335	3.5779	9.87E-06	**8**	**18**	**0.15106**	6.89E-07
	x^7	–	–	–	–	–	–	–	–	**19**	**60**	**0.43136**	5.72E-06
	x^8	26	**43**	0.72698	3.13E-06	49	85	1.1922	7.52E-06	**21**	60	**0.45684**	4.67E-06
	x^9	**26**	**42**	**0.68144**	1.09E-06	56	126	1.6091	7.67E-06	46	436	2.0631	3.54E-07

TABLE 6. Numerical Results for DDPM, ETTC and PCGM for Problem 6 with given initial points and dimensions.

		DDPM				ETTC				PCGM			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x^1	12	**25**	**0.026395**	1.01E-06	**11**	26	1.1229	4.48E-06	24	123	0.68669	6.71E-06
	x^2	**7**	131	0.023566	5.34E-06	14	154	0.040037	3.47E-06	16	**26**	**0.010789**	3.45E-07
	x^3	**11**	**24**	**0.010728**	9.75E-06	**11**	37	0.01444	7.12E-06	–	–	–	–
	x^4	13	**17**	0.012665	8.19E-06	**11**	26	**0.011223**	3.41E-06	14	44	0.012685	6.63E-06
	x^5	13	**17**	**0.011049**	8.19E-06	**11**	26	0.0117	3.41E-06	79	790	0.13784	1.32E-07
	x^6	14	**18**	**0.010742**	1.69E-06	**12**	28	0.012013	1.75E-06	9	40	0.034921	2.95E-06
	x^7	13	**17**	0.014376	8.19E-06	**11**	26	**0.010396**	3.41E-06	14	44	0.016852	6.63E-06
	x^8	13	**17**	0.013902	8.21E-06	**11**	26	**0.011983**	3.41E-06	66	547	0.081472	5.96E-06
	x^9	13	**17**	0.015753	8.19E-06	**11**	26	**0.013531**	3.45E-06	78	652	0.09855	5.30E-06
5000	x^1	**12**	**25**	0.057801	2.25E-06	**12**	33	**0.050372**	3.3E-06	72	689	0.36581	2.01E-07
	x^2	**8**	177	0.13841	2.26E-06	14	154	**0.13181**	7.76E-06	29	**75**	0.061658	5.30E-06
	x^3	**12**	**41**	**0.049439**	2.42E-06	**12**	47	0.050564	5.18E-06	–	–	–	–
	x^4	14	27	0.045481	1.56E-06	**11**	**26**	**0.036509**	7.62E-06	19	90	0.054744	9.56E-09
	x^5	14	27	0.056395	1.56E-06	**11**	**26**	**0.035917**	7.62E-06	147	1946	1.0496	1.92E-08
	x^6	14	**18**	0.043706	3.78E-06	**12**	28	**0.033969**	3.95E-06	12	70	0.04399	6.45E-06
	x^7	14	27	0.053423	1.56E-06	**11**	**26**	**0.031611**	7.62E-06	19	90	0.051965	9.56E-09
	x^8	14	27	0.056072	1.56E-06	**11**	**26**	**0.034529**	7.62E-06	144	1907	0.92334	1.84E-06
	x^9	14	27	0.056653	1.56E-06	**11**	**26**	**0.034497**	7.6E-06	150	1939	0.8903	5.51E-07
10 000	x^1	**12**	**25**	0.070712	3.18E-06	**12**	33	**0.069801**	4.67E-06	96	1205	1.1393	1.97E-06
	x^2	**8**	177	0.22782	3.2E-06	15	173	**0.22388**	2.19E-06	23	**44**	0.13182	4.05E-07
	x^3	**12**	**41**	0.1077	3.42E-06	**12**	47	**0.077682**	7.33E-06	19	75	0.088929	2.30E-06
	x^4	14	**27**	0.09561	2.21E-06	**12**	33	**0.061986**	3.55E-06	–	–	–	–
	x^5	14	**27**	0.093952	2.21E-06	**12**	33	**0.064858**	3.55E-06	171	2405	1.8565	2.86E-07
	x^6	14	**18**	0.099189	5.35E-06	**12**	28	**0.05831**	5.6E-06	17	61	0.069022	9.70E-07
	x^7	14	**27**	0.08379	2.21E-06	**12**	33	**0.062272**	3.55E-06	–	–	–	–
	x^8	14	**27**	0.091591	2.21E-06	**12**	33	**0.061694**	3.55E-06	179	2672	2.1318	5.16E-06
	x^9	14	**27**	0.085958	2.22E-06	**12**	33	**0.0688**	3.58E-06	186	2833	2.1238	1.66E-06
50 000	x^1	14	**28**	**0.23269**	8.16E-06	**13**	41	0.29341	1.95E-06	177	2658	8.3361	6.76E-06
	x^2	**8**	177	**0.75351**	7.16E-06	15	**173**	0.81147	4.89E-06	–	–	–	–
	x^3	**12**	**41**	**0.23691**	7.64E-06	13	58	0.36307	2.78E-06	23	94	0.46603	1.75E-08
	x^4	14	**27**	0.27151	4.94E-06	**12**	33	**0.24401**	7.94E-06	–	–	–	–
	x^5	14	**27**	0.27429	4.94E-06	**12**	33	**0.19597**	7.94E-06	284	5179	15.5753	8.87E-06
	x^6	15	**28**	0.33283	1.02E-06	**13**	35	**0.20457**	4.14E-06	23	66	0.3247	7.56E-06
	x^7	14	**27**	0.27502	4.94E-06	**12**	33	**0.19034**	7.94E-06	–	–	–	–
	x^8	14	**27**	0.28735	4.94E-06	**12**	33	**0.24482**	7.94E-06	245	4046	13.5162	1.53E-06
	x^9	14	**27**	0.30907	4.95E-06	**12**	33	**0.234**	7.94E-06	338	5801	19.0926	1.56E-06
100 000	x^1	15	46	0.62555	2.48E-06	**13**	**41**	**0.51972**	2.76E-06	221	3893	22.7321	9.25E-07
	x^2	**9**	241	1.9379	2.77E-06	15	**173**	**1.4647**	6.91E-06	–	–	–	–
	x^3	**13**	66	0.72285	7.74E-06	**13**	**58**	**0.68667**	3.94E-06	–	–	–	–
	x^4	14	**27**	**0.40432**	6.99E-06	**13**	41	0.53463	2.1E-06	22	61	0.58063	3.41E-06
	x^5	14	**27**	**0.46088**	6.99E-06	**13**	41	0.53607	2.1E-06	288	5244	30.6097	3.32E-06
	x^6	15	45	0.5932	1.43E-06	**13**	**35**	**0.51349**	5.85E-06	25	74	0.7058	1.02E-07
	x^7	14	**27**	**0.50743**	6.99E-06	**13**	41	0.52999	2.1E-06	22	61	0.60277	3.41E-06
	x^8	14	**27**	0.58727	6.99E-06	**13**	41	**0.53473**	2.1E-06	349	6852	39.4304	2.39E-08
	x^9	14	**27**	0.70336	6.98E-06	**13**	41	**0.62852**	2.1E-06	399	7671	44.277	6.03E-06

TABLE 7. Numerical Results for DDPM, ETTC and PCGM for Problem 7 with given initial points and dimensions.

		DDPM				ETTC				PCGM			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
	x^1	**34**	**58**	**0.16592**	8.76E-06	347	718	0.93945	9.95E-06	75	389	0.82854	9.23E-06
	x^2	**27**	**45**	**0.059422**	9.92E-06	336	691	0.51497	9.81E-06	61	316	0.17662	9.77E-06
	x^3	**25**	**41**	**0.051433**	9.05E-06	91	184	0.14943	8.99E-06	50	252	0.12742	9.84E-06
	x^4	**33**	**57**	**0.083585**	9.56E-06	381	786	0.61742	9.71E-06	74	384	0.19509	9.30E-06
	x^5	**28**	**35**	**0.064956**	7.42E-06	326	658	0.49821	9.88E-06	75	382	0.20009	9.14E-06
	x^6	**26**	**39**	**0.082907**	9.41E-06	71	142	0.11774	9.11E-06	55	276	0.13541	8.95E-06
	x^7	**33**	**57**	**0.096982**	9.56E-06	381	786	0.5754	9.71E-06	74	384	0.18071	9.30E-06
1000	x^8	**26**	**31**	**0.068784**	7.65E-06	314	635	0.48368	9.83E-06	74	377	0.18191	9.09E-06
	x^9	**32**	**48**	**0.086804**	7.68E-06	421	862	0.62142	9.99E-06	87	451	0.26761	9.28E-06
	x^1	**34**	**59**	**0.28084**	5.08E-06	340	704	1.9723	8.66E-06	76	394	0.69443	7.62E-06
	x^2	**31**	**55**	**0.22188**	9.77E-06	333	685	1.943	8.17E-06	62	320	0.53869	9.71E-06
	x^3	**25**	**41**	**0.20725**	6.15E-06	91	184	0.56391	9.8E-06	49	247	0.4153	8.19E-06
	x^4	**32**	**56**	**0.24403**	8.88E-06	335	694	1.9595	9.81E-06	74	385	0.70698	7.15E-06
	x^5	**22**	**24**	**0.13971**	7.09E-06	50	100	0.31336	9.76E-06	80	407	0.66247	9.21E-06
	x^6	**26**	**38**	**0.16512**	6.99E-06	71	142	0.44498	9.92E-06	53	266	0.43926	9.74E-06
	x^7	**32**	**56**	**0.23927**	8.88E-06	335	694	1.9428	9.81E-06	74	385	0.65025	7.15E-06
5000	x^8	**22**	**24**	**0.13776**	7.17E-06	58	116	0.37112	9.88E-06	80	407	0.70279	9.52E-06
	x^9	**31**	**43**	**0.19192**	9.88E-06	346	712	1.9934	9.31E-06	90	468	0.81457	7.05E-06
	x^1	**31**	**51**	**0.34783**	8.52E-06	–	–	–	–	78	404	1.3073	9.27E-06
	x^2	**30**	**54**	**0.36767**	8.94E-06	–	–	–	–	65	336	1.075	8.55E-06
	x^3	**25**	**41**	**0.29419**	6.13E-06	91	184	1.0184	9.81E-06	49	247	0.81323	8.17E-06
	x^4	**31**	**53**	**0.36595**	7.57E-06	339	702	3.8185	8.9E-06	72	374	1.2394	9.28E-06
	x^5	**21**	**23**	**0.20659**	6.79E-06	43	86	0.52649	8.61E-06	84	425	1.3862	9.76E-06
	x^6	**25**	**36**	**0.27122**	8.49E-06	71	142	0.82616	9.93E-06	53	266	0.89633	9.72E-06
	x^7	**31**	**53**	**0.37864**	7.57E-06	339	702	3.7655	8.9E-06	72	374	1.2779	9.28E-06
10 000	x^8	**21**	**23**	**0.19954**	6.85E-06	43	86	0.52115	8.92E-06	84	425	1.3909	9.07E-06
	x^9	**31**	**42**	**0.32204**	9.15E-06	352	723	3.8675	7.77E-06	88	458	1.5289	9.11E-06
	x^1	**30**	**50**	**1.3231**	9.58E-06	–	–	–	–	88	455	6.0355	9.63E-06
	x^2	**29**	**50**	**1.3277**	8.2E-06	298	612	14.1298	9.92E-06	74	382	5.098	9.62E-06
	x^3	**25**	**41**	**1.1395**	6.12E-06	91	184	4.3292	9.81E-06	49	247	3.34	8.16E-06
	x^4	**29**	**46**	**1.4016**	5.29E-06	340	704	16.1922	8.51E-06	79	409	5.5001	8.28E-06
	x^5	**22**	**25**	**0.87711**	8.23E-06	44	88	2.1325	7.8E-06	77	386	5.1792	9.66E-06
	x^6	**26**	**38**	**1.1053**	5.87E-06	71	142	3.3468	9.93E-06	53	266	3.5394	9.72E-06
	x^7	**29**	**46**	**1.2984**	5.29E-06	340	704	16.0892	8.51E-06	79	409	5.4154	8.28E-06
50 000	x^8	**22**	**25**	**0.79977**	8.23E-06	44	88	2.1115	7.81E-06	82	414	5.4923	9.54E-06
	x^9	**30**	**39**	**1.1767**	7.85E-06	427	1537	30.1988	9.99E-06	93	484	6.4091	9.07E-06
	x^1	**32**	**61**	**3.1517**	9.77E-06	–	–	–	–	88	456	12.8932	7.95E-06
	x^2	**28**	**48**	**2.6182**	5.65E-06	299	614	30.5511	9.71E-06	75	387	10.6182	9.98E-06
	x^3	**25**	**41**	**2.2649**	6.12E-06	91	184	9.3364	9.81E-06	49	247	6.8603	8.16E-06
	x^4	**27**	**42**	**2.4224**	5.95E-06	339	702	34.8864	8.23E-06	84	435	12.32	8.81E-06
	x^5	**22**	**26**	**1.7101**	6.48E-06	44	88	4.4276	8.76E-06	77	394	12.5185	9.52E-06
	x^6	**25**	**37**	**2.1496**	9.91E-06	71	142	7.0849	9.93E-06	53	266	8.5851	9.72E-06
	x^7	**27**	**42**	**2.4023**	5.95E-06	339	702	34.5878	8.23E-06	84	435	14.5536	8.81E-06
100 000	x^8	**22**	**26**	**1.723**	6.48E-06	44	88	4.4578	8.76E-06	77	395	12.7393	8.03E-06
	x^9	**31**	**42**	**2.4607**	8.14E-06	–	–	–	–	94	489	16.1043	9.81E-06

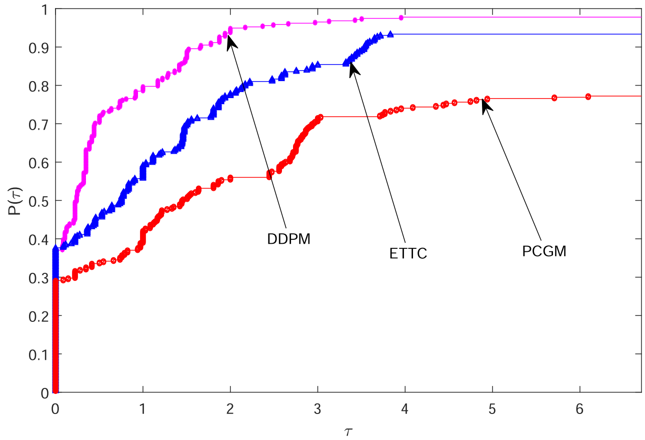

FIGURE 1. Performance profiles for the number of iterations.

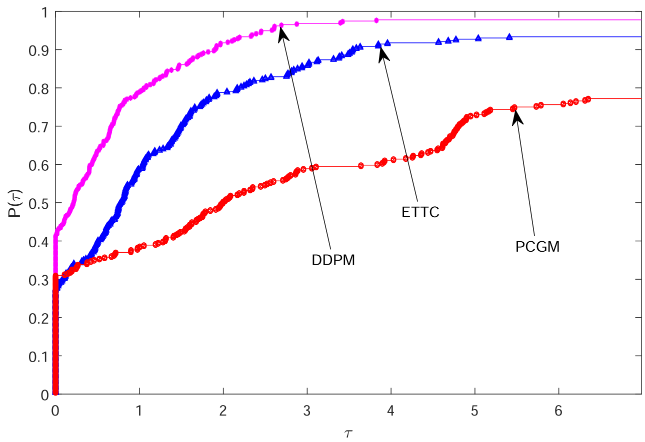

FIGURE 2. Performance profiles for the CPU time (in seconds).

The best solver for a particular problem attains the lower bound $r_{\rho,\ s} = 1$. If a solver s fails to satisfy the convergence test on problem p, then $r_{\rho,\ s}$ is set as ∞. The performance profile of a solver $s \in \mathbb{S}$ is defined as the fraction of problems where the performance ratio is at most τ, that is,

$$P(\tau) = \frac{1}{n_\rho}\text{size}\{\rho \in \mathbb{P} \mid r_{\rho,\ s} \leq \tau\},$$

where n_ρ is the number of problems. In each of Figures 1–3, the x-axis represent τ and the y-axis represent $P(\tau)$.

It can be observed from Tables 1 and 4, that starting from any of the nine initial points, all the three methods terminate successfully at an approximate solution of the problem. However, in most of the cases, the DDPM has the least number of function evaluations (see the bolded numbers in the tables), this shows that the DDPM direction has the least dependency on the line search compared to the existing methods. This superiority of the DDPM can be seen in Figure 3. In Tables 2, 3 and 6 the PCGM fails for some initial points which indicate that the PCGM is more sensitive to the initial points than DDPM and ETTC. The efficiency of the DDPM can also be seen in Table 7 where it outperforms the PCGM and ETTC in all the three metrics. This is an indication

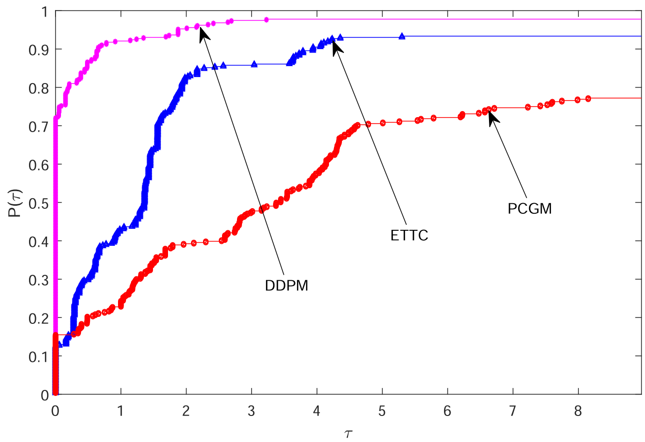

FIGURE 3. Performance profiles for the number of function evaluations.

that DDPM may perform well in solving some discretized problems. In general, based on the experiments, the DDPM performs well on all the nine different initial points except for some initial points in Table 5. Thus, it is more robust than ETTC and PCGM.

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