RAIRO-Oper. Res. 54 (2020) 1593–1612
https://doi.org/10.1051/ro/2019114 **RAIRO Operations Research**
www.rairo-ro.org

OPTIMAL STRATEGIES AND PRICING ANALYSIS IN M/M/1 QUEUES WITH A SINGLE WORKING VACATION AND MULTIPLE VACATIONS

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Abstract. This paper considers the customers' equilibrium and socially optimal joining-balking behavior in single-server Markovian queues with a single working vacation and multiple vacations. Arriving customers decide whether to join the system or balk based on the system states and a linear reward-cost structure, which incorporates the desire of customers for service and their dislike to wait. We consider that the system states are almost unobservable and fully unobservable, respectively. For these two cases, we first analyze the stationary behavior of the system, and get the equilibrium strategies of the customers and compare them to socially optimal balking strategies using numerical examples. We also study the pricing problem that maximizes the server's profit and derive the optimal pricing strategy. Finally, the social benefits of the almost and fully unobservable queues are compared by numerical examples.

Mathematics Subject Classification. 60K25, 90B22.

Received January 16, 2019. Accepted December 6, 2019.

1. Introduction

In the past decades, queueing models with vacations have been developed as useful performance analysis tools for computer systems, communication networks and flexible manufacturing systems. Recently, more and more researchers begin to study strategic behavior in the vacation queues from an economic point of view. As to classical vacation queueing models, Burnetas and Economou [2] first studied a Markovian single-server queueing system with setup times. They derived equilibrium strategies of the customers under various levels of information and analyzed the stationary behavior of the system under these strategies. Economou *et al.* [4] extended this model to M/G/1 queue under unobservable case. Then, Economou and Kanta [3] considered the Markovian single-server queue with service breakdowns and repairs. They derived equilibrium thresholds in fully observable and almost observable queues. Li *et al.* [10] considered the almost and fully unobservable cases for the same model in [3] and obtained mixed balking strategies. Sun *et al.* [15, 16] studied customers' equilibrium and socially optimal balking strategies in the observable and unobservable queues with several types of setup/closedown policies, respectively. Guo and Hassin [5] discussed equilibrium and socially optimal strategies in observable and unobservable queues with N policy. This work was extended by Guo and Hassin [6] to heterogenous customers, and they studied both unobservable and observable queues and considered two

Keywords. Queueing system, working vacation, equilibrium balking strategy, social benefit, multiple vacations.

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Article published by EDP Sciences

situations regarding customers' delay sensitivity. Recently, Guo and Li [7] studied strategic behavior and social optimization in partially-observable Markovian vacation queues. Tian *et al.* [20] discussed optimal balking strategies in M/G/1 queue with N policy.

Liu and Wang [11] investigated the strategic joining behavior of customers in a single-server Markovian queueing system with Bernoulli vacation. Ma *et al.* [12] studied the equilibrium behaviors in a vacation queue model with two complementary services, in which the main and secondary services were complementary and a customer gained no benefit from obtaining only one of them. Zhang and Wang [23] analyzed equilibrium pricing strategy in an M/G/1 retrial queueing system with setup times in which the server kept idle for a reserved idle time after completion of a service. About comprehensive and excellent study on strategic queueing systems, readers are referred to the book of Hassin [8].

Recently, queueing systems with working vacations have been studied extensively where the server takes vacations once the system becomes empty and can still serve customers at a lower rate than regular one during the vacations. Research results of the steady-state performance measures on various working vacation queues can be consulted in the survey given by Tian *et al.* [19]. As for the work studying customers' behavior in queueing systems with working vacations, Zhang *et al.* [24] and Sun and Li [14] studied almost simultaneously the equilibrium balking strategies in M/M/1 queues with multiple working vacations for different cases with respect to the system information. Subsequently, Sun *et al.* [17, 18] considered the customers' optimal balking behavior in some single-server Markovian queues with two-stage working vacations and double adaptive working vacations, respectively. Wang and Zhang [22] considered the equilibrium strategies in the Markovian queues with a single working vacation. Then, Tian *et al.* [21] studied equilibrium and optimal strategies in M/M/1 queues with multiple working vacations and vacation interruptions under three different levels of system information. Lee [9] observed customer's equilibrium joining/balking behaviors in M/M/1 queues with a single working vacation and vacation interruptions.

The above-mentioned vacation policy is either a classic vacation policy or working vacation policy. However, to the best of the authors knowledge, researches for the equilibrium strategies in the Markovian queueing systems with a single working vacation and multiple vacations haven't been given. The proposed model has potential applications in the design of queueing system. In order to save operation costs, the server waits for the new customers and works with a lower service rate when the system becomes empty. After this period, the server may start a regular busy period if there are still customers in the system. Otherwise, the server may start vacations when there are still no customers in the system. This process includes working vacation and classic vacation, which we call a single working vacation and multiple vacations. For example, in order to keep the service system running efficiently for a long time, it is sometimes necessary to perform some routine maintenance on the service facility. If the system load becomes smaller, the server can choose to run at low speed (working vacation). The maintenance procedure is carried out once the system is vacant, and the time for the implementation of the maintenance is considered as "a server's vacation". Customers arriving during this period need to wait until the routine maintenance is completed before they can be received. This type of vacation is also applicable in banking systems and machine failure systems.

Motivated by the aforementioned studies and the possible application, we consider customers' optimal strategies and sever's profit optimization in an M/M/1 queueing model with a single working vacation and multiple vacations. We distinguish two cases: the almost unobservable queue and the fully unobservable queue, according to the information levels regarding system states. As to the almost unobservable case, a customer can only observe the state of the server and cannot know the number of the customers in the system. However, for the fully unobservable case, a customer have no information on the server's current status and the system occupancy. Our contributions are as follows. First, we not only obtain customer's equilibrium strategy but also compare it with socially optimal joining probability which maximizes social benefit in two cases. Second, we derive the optimal prices charged by the server in equilibrium and server's optimal profit under our model. Third, we demonstrate numerical results based on the impact of the system parameters as well as information level on the equilibrium behavior and social benefit to explore qualitative insights. The results of this paper have not been reported in the literature on M/M/1 queues with a single working vacation and multiple vacations.

This paper is organized as follows: descriptions of the model are given in Section 2. Sections 3, and 4 are devoted to the almost unobservable queue and the fully unobservable queue, respectively. For each type of queues, we derive and compare the customers' equilibrium and socially optimal joining-balking strategies. In Section 5, we present the relation of social benefit in two different cases and demonstrate the effect of system parameters by several numerical experiments. Finally, conclusions are given in Section 6.

2. Model description

In this paper, we consider an M/M/1 queue with a single working vacation and multiple vacations, where potential customers arrive according to a Poisson process with rate λ. The service times of the customers are assumed to be exponentially distributed with rate μ. Upon the completion of service, if there is no customer in the system, the server begins a working vacation and the vacation time is assumed to be exponentially distributed with the parameter θ_1. During the working vacation period, arriving customers can be served at a mean rate of μ_1 ($\mu_1 < \mu$). Upon the end of the working vacation, if there are also customers in the queue, the new regular busy period will start and the server will change the service rate from μ_1 to μ. Otherwise, the server begins a classic vacation and the vacation time is exponentially distributed with rate θ_0. Meanwhile, when a vacation ends, a regular busy period starts if there are customers in the queue. Otherwise, the server continues another vacation, and a new regular busy period starts when a customer arrival occurs. This policy is called single working vacation and multiple vacations (SWV+MV).

We assume that the inter-arrival times, the service times, the working vacation times and vacation times are mutually independent. In addition, the service discipline is first in first out (FIFO).

Denote by $N(t)$ the number of customers in the system at time t, and let $J(t)$ be the server state at time t:

$$J(t) = \begin{cases} 0, & \text{the server is in the vacation period at time t,} \\ 1, & \text{the server is in working vacation period at time t,} \\ 2, & \text{the server is busy at time t.} \end{cases}$$

The process $\{(N(t), J(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\Omega = \{0,0\} \cup \{0,1\} \cup \{(n,i)|n \geq 1, i = 0,1,2\}$.

Denote the stationary distribution as

$$\pi_{ni} = \lim_{t\to\infty} P\{N(t) = n, J(t) = i\}, \qquad (n,i) \in \Omega.$$

In this paper, we assume that arriving customers are identical and we are interested in the customers' strategic response as they can decide whether to join or balk upon arrival. If a tagged customer enter the system, he receives a reward of R units for completing service. There is a waiting cost of C units per time unit that the customer remains in the system. Based a reward-cost structure for each customer, a customer's utility is defined as $R - CW$ after service completion, where W is his mean sojourn time in the system. All customers want to maximize their expected utility by making decisions only at their arrival instants. Specifically, customers are risk neutral and maximize their expected net benefit. Finally, we assume that there are no retrials of balking customers nor reneging of waiting customers.

3. Almost unobservable queues

In this section, we turn our attention to the almost unobservable case, where arriving customers only can observe the state of the server at their arrival instant and do not observe the number of customers present.

3.1. Equilibrium and social optimization

A mixed strategy for a customer is specified by a vector (q_0, q_1, q_2), where q_i is the probability of joining when the server is in state i ($i = 0, 1, 2$). If all customers follow the same mixed strategy (q_0, q_1, q_2), then the

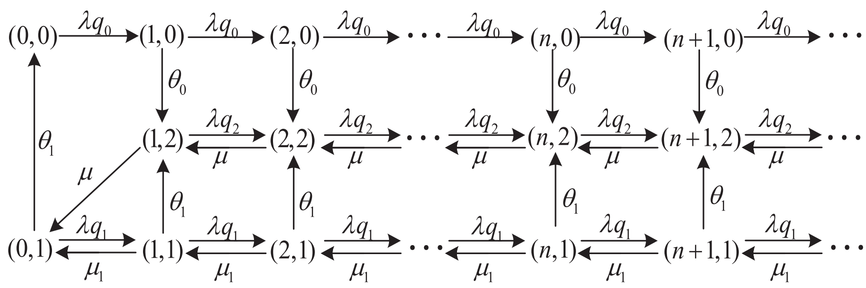

FIGURE 1. Transition rate diagram for the almost unobservable queues.

system follows a Markov chain where the arrival rate equals $\lambda_i = \lambda q_i$ when the server is in state i $(i = 0, 1, 2)$. The transition diagram is illustrated in Figure 1.

Using the lexicographical sequence for the states, the transition rate matrix $\boldsymbol{Q}$ can be written as the tridiagonal block matrix:

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{C}_0 & & & & \\ \boldsymbol{B}_1 & \boldsymbol{A} & \boldsymbol{C} & & & \\ & \boldsymbol{B} & \boldsymbol{A} & \boldsymbol{C} & & \\ & & \boldsymbol{B} & \boldsymbol{A} & \boldsymbol{C} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$\boldsymbol{A}_0 = \begin{pmatrix} -\lambda q_0 & 0 \\ \theta_1 & -(\lambda q_1 + \theta_1) \end{pmatrix}, \qquad \boldsymbol{C}_0 = \begin{pmatrix} \lambda q_0 & 0 & 0 \\ 0 & \lambda q_1 & 0 \end{pmatrix},$$

$$\boldsymbol{A} = \begin{pmatrix} -(\lambda q_0 + \theta_0) & 0 & \theta_0 \\ 0 & -(\lambda q_1 + \theta_1 + \mu_1) & \theta_1 \\ 0 & 0 & -(\lambda q_2 + \mu) \end{pmatrix}, \qquad \boldsymbol{B}_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mu_1 \\ 0 & \mu \end{pmatrix},$$

$$\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu \end{pmatrix}, \qquad \boldsymbol{C} = \begin{pmatrix} \lambda q_0 & 0 & 0 \\ 0 & \lambda q_1 & 0 \\ 0 & 0 & \lambda q_2 \end{pmatrix}.$$

The structure of $\boldsymbol{Q}$ indicates that $\{(N(t), J(t)), t \geq 0\}$ is a quasi birth and death (QBD) process, see Neuts [13]. To analyze this QBD process, it is necessary to solve for the minimal non-negative solution of the matrix quadratic equation

$$\boldsymbol{R}^2\boldsymbol{B} + \boldsymbol{R}\boldsymbol{A} + \boldsymbol{C} = 0, \tag{3.1}$$

and this solution is called the rate matrix and denoted by $\boldsymbol{R}$.

Because the coefficients of equation (3.1) are all upper triangular matrices, we can assume that $\boldsymbol{R}$ has the same structure as

$$\boldsymbol{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}.$$

Substituting $\boldsymbol{R}^2$ and $\boldsymbol{R}$ into (3.1), we obtain the matrix $\boldsymbol{R}$ whose expression is as follows,

$$\boldsymbol{R} = \begin{pmatrix} \sigma_0 & 0 & \rho_0 \\ 0 & r & \frac{\theta_1 r}{\mu(1-r)} \\ 0 & 0 & \rho_2 \end{pmatrix}, \tag{3.2}$$

where

$$\rho_0 = \frac{\lambda q_0}{\mu}, \quad \rho_2 = \frac{\lambda q_2}{\mu}, \quad \sigma_0 = \frac{\lambda q_0}{\lambda q_0 + \theta_0},$$
$$r = r(q_1) = \frac{\lambda q_1 + \mu_1 + \theta_1 - \sqrt{(\lambda q_1 + \mu_1 + \theta_1)^2 - 4\lambda q_1 \mu_1}}{2\mu_1} < 1, \tag{3.3}$$

and r satisfies

$$\lambda q_1 + \theta_1 + \mu_1(1-r) = \mu_1 + \frac{\theta_1}{1-r} = \frac{\lambda q_1}{r}. \tag{3.4}$$

Theorem 3.1. *Assumed that $\lambda < \mu$, the stationary probabilities $\{\pi_{ni} : (n,i) \in \Omega\}$ of the system are as follows:*

$$\begin{cases} \pi_{n0} = K \frac{\theta_1}{\lambda q_0} \sigma_0^n, & n \geq 0, \\ \pi_{n1} = K r^n, & n \geq 0, \\ \pi_{n2} = K \left(\frac{\theta_1 r}{\mu(1-r)} \sum_{j=0}^{n-1} r^j \rho_2^{n-1-j} + \frac{\theta_1}{\mu} \sum_{j=0}^{n-1} \sigma_0^j \rho_2^{n-1-j} \right), & n \geq 1, \end{cases} \tag{3.5}$$

where

$$K = \left(\frac{1}{1-r} + \frac{\theta_1}{\lambda q_0} \frac{1}{1-\sigma_0} + \frac{\theta_1 r}{\mu(1-r)^2(1-\rho_2)} + \frac{\theta_1}{\mu(1-\sigma_0)(1-\rho_2)} \right)^{-1}. \tag{3.6}$$

Proof. Using the matrix-geometric solution method, we have

$$(\pi_{n0}, \pi_{n1}, \pi_{n2}) = (\pi_{10}, \pi_{11}, \pi_{12}) \boldsymbol{R}^{n-1}, \quad n \geq 1, \tag{3.7}$$

and $(\pi_{00}, \pi_{01}, \pi_{10}, \pi_{11}, \pi_{12})$ satisfies the set of equations

$$(\pi_{00}, \pi_{01}, \pi_{10}, \pi_{11}, \pi_{12}) B[\boldsymbol{R}] = 0, \tag{3.8}$$

where

$$\boldsymbol{R}^{n-1} = \begin{pmatrix} \sigma_0^{n-1} & 0 & \rho_0 \sum_{j=0}^{n-2} \sigma_0^j \rho_2^{n-j-2} \\ 0 & r^{n-1} & \frac{\theta_1 r}{\mu(1-r)} \sum_{j=0}^{n-2} r^j \rho_2^{n-j-2} \\ 0 & 0 & \rho_2^{n-1} \end{pmatrix}, \tag{3.9}$$

and

$$B[\boldsymbol{R}] = \begin{pmatrix} \boldsymbol{A}_0 & \boldsymbol{C}_0 \\ \boldsymbol{B}_1 & \boldsymbol{A} + \boldsymbol{R}\boldsymbol{B} \end{pmatrix}.$$

Solving (3.8) and letting $\pi_{01} = K$, we get

$$(\pi_{00}, \pi_{01}) = K \left(\frac{\theta_1}{\lambda q_0}, 1 \right), \tag{3.10}$$
$$(\pi_{10}, \pi_{11}, \pi_{12}) = K \left(\frac{\theta_1}{\lambda q_0 + \theta_0}, r, \frac{\theta_1}{\mu(1-r)} \right). \tag{3.11}$$

Substituting the expression of (3.9) into equation (3.7), we obtain the probabilities π_{ni}. Finally, $\pi_{01} = K$ can be determined by the normalization condition $\sum_{n=0}^{\infty} \pi_{n0} + \sum_{n=0}^{\infty} \pi_{n1} + \sum_{n=1}^{\infty} \pi_{n2} = 1$. □

Then the probability that the system is in state i $(i = 0, 1, 2)$, denoted by π_i, can be derived as follows:

$$\pi_0 = \sum_{n=0}^{\infty} \pi_{n0} = \frac{K\theta_1}{\lambda q_0(1-\sigma_0)}, \tag{3.12}$$

$$\pi_1 = \sum_{n=0}^{\infty} \pi_{n1} = \frac{K}{1-r}, \tag{3.13}$$

$$\pi_2 = \sum_{n=1}^{\infty} \pi_{n2} = \frac{K}{1-\rho_2}\left(\frac{\theta_1 r}{\mu(1-r)^2} + \frac{\theta_1}{\mu(1-\sigma_0)}\right). \tag{3.14}$$

Let W_{ni} denotes his expected mean sojourn time given that he finds there are n customers and the server is at the state i $(i = 0, 1, 2)$ upon his arrival. We have the following results.

Lemma 3.2. *If the arriving customer finds the system at state* (n, i) *and decides to join the queue, his conditional mean sojourn time* W_{ni} *is given by*

$$W_{n0} = \frac{n+1}{\mu} + \frac{1}{\theta_0}, \qquad n \geq 0, \tag{3.15}$$

$$W_{n1} = \frac{n+1}{\mu} + \frac{\mu-\mu_1}{\mu\theta_1}\left(1 - \left(\frac{\mu_1}{\mu_1+\theta_1}\right)^{n+1}\right), \qquad n \geq 0, \tag{3.16}$$

$$W_{n2} = \frac{n+1}{\mu}, \qquad n \geq 1. \tag{3.17}$$

Proof. According to the previous assumptions, we have the following equations:

$$W_{00} = \frac{1}{\mu} + \frac{1}{\theta_0}, \quad W_{n0} = W_{n2} + \frac{1}{\theta_0}, \qquad n \geq 1, \tag{3.18}$$

$$W_{01} = \frac{\mu_1}{\mu_1+\theta_1}\frac{1}{\mu_1} + \frac{\theta_1}{\mu_1+\theta_1}\frac{1}{\mu}, \tag{3.19}$$

$$W_{n1} = \frac{\mu_1}{\mu_1+\theta_1}\left(\frac{1}{\mu_1} + W_{n-1,1}\right) + \frac{\theta_1}{\mu_1+\theta_1}W_{n2}, \qquad n \geq 1, \tag{3.20}$$

$$W_{n2} = \frac{n+1}{\mu}, \qquad n \geq 1. \tag{3.21}$$

From (3.19), we have

$$W_{01} = \frac{\mu+\theta_1}{\mu(\mu_1+\theta_1)} = \frac{1}{\mu} + \frac{\mu-\mu_1}{\mu\theta_1}\left(1 - \frac{\mu_1}{\mu_1+\theta_1}\right). \tag{3.22}$$

By iterating (3.20) and taking into account (3.21) and (3.22), we obtain

$$W_{n1} = \frac{n+1}{\mu} + \frac{\mu-\mu_1}{\mu\theta_1}\left(1 - \left(\frac{\mu_1}{\mu_1+\theta_1}\right)^{n+1}\right). \tag{3.23}$$

We assume throughout the paper that

$$R > \frac{C}{\mu} + \frac{C}{\theta_0},$$

which ensures that the reward for service exceeds the expected cost for a customer who finds the system empty. □

Consider an arriving customer who finds the server is on state i ($i = 0, 1, 2$). If the customer decides to enter the system in state i, his mean sojourn time is given by

$$W_0(q_0) = \frac{\sum_{n=0}^{\infty} W_{n0}\pi_{n0}}{\sum_{n=0}^{\infty} \pi_{n0}}, \quad W_1(q_1) = \frac{\sum_{n=0}^{\infty} W_{n1}\pi_{n1}}{\sum_{n=0}^{\infty} \pi_{n1}}, \quad W_2(q_0, q_1, q_2) = \frac{\sum_{n=1}^{\infty} W_{n2}\pi_{n2}}{\sum_{n=1}^{\infty} \pi_{n2}}. \tag{3.24}$$

By substituting (3.5), (3.15), (3.16) and (3.17) into (3.24), we have

$$W_0(q_0) = \frac{\lambda q_0 + \theta_0}{\mu\theta_0} + \frac{1}{\theta_0}, \tag{3.25}$$

$$W_1(q_1) = \frac{1}{\mu(1-r)} + \frac{\mu - \mu_1}{\mu(\theta_1 + \mu_1(1-r))}, \tag{3.26}$$

$$W_2(q_0, q_1, q_2) = \frac{1}{\mu - \lambda q_2} + \frac{r(1-\sigma_0)^2 + (1-r)^3}{\mu(r(1-r)(1-\sigma_0)^2 + (1-r)^3(1-\sigma_0))}. \tag{3.27}$$

Consider a tagged customer who finds the server at state i upon arrival. If he decides to enter the system, his expected net benefit is given by

$$U_0(q_0) = R - \frac{C(\lambda q_0 + \theta_0)}{\mu\theta_0} - \frac{C}{\theta_0}, \tag{3.28}$$

$$U_1(q_1) = R - \frac{C}{\mu(1-r)} - \frac{C(\mu - \mu_1)}{\mu(\theta_1 + \mu_1(1-r))}, \tag{3.29}$$

$$U_2(q_0, q_1, q_2) = R - \frac{C}{\mu - \lambda q_2} - \frac{C(r(1-\sigma_0)^2 + (1-r)^3)}{\mu(r(1-r)(1-\sigma_0)^2 + (1-r)^3(1-\sigma_0))}. \tag{3.30}$$

Obviously, $W_0(q_0)$ and $W_2(q_0, q_1, q_2)$ are monotonically increasing with respect to q_0 and q_2, respectively. From (3.26), we have

$$\begin{aligned} W_1'(q_1) &= \frac{\frac{\mathrm{d}r}{\mathrm{d}q_1}}{\mu(1-r)^2} + \frac{\mu_1(\mu - \mu_1)\frac{\mathrm{d}r}{\mathrm{d}q_1}}{\mu(\theta_1 + \mu_1(1-r))^2} \\ &= \left(\frac{1}{\mu(1-r)^2} + \frac{\mu_1(\mu - \mu_1)}{\mu(\theta_1 + \mu_1(1-r))^2}\right)\frac{\mathrm{d}r}{\mathrm{d}q_1}. \end{aligned} \tag{3.31}$$

Since $\lambda q_1 + \theta_1 + \mu_1(1-r) = \mu_1 + \frac{\theta_1}{1-r}$ and

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}q_1} &= \frac{\lambda}{2\mu_1}\left(1 - \frac{\lambda q_1 + \theta_1 - \mu_1}{\sqrt{(\lambda q_1 + \theta_1 - \mu_1)^2 + 4\mu_1\theta_1}}\right) \\ &= \frac{\lambda(1-r)}{\lambda q_1 + \theta_1 + \mu_1(1-r) - \mu_1 r} \\ &= \frac{\lambda(1-r)^2}{\mu_1(1-r)^2 + \theta_1} > 0, \end{aligned} \tag{3.32}$$

we can obtain $W_1'(q_1) > 0$ and $W_1(q_1)$ is increasing for $q_1 \in [0, 1]$.

Theorem 3.3. *In the almost unobservable M/M/1 queue, there exists a unique mixed equilibrium strategy* (q_0^e, q_1^e, q_2^e) *"observe* $J(t)$ *and enter with probability* $q_{J(t)}^e$*" where the vector* q_i^e *is given as follows.*

$$q_0^e = \begin{cases} \frac{1}{\lambda}\left(\frac{R\theta_0\mu}{C} - \theta_0 - \mu\right), & \frac{C}{\mu} + \frac{C}{\theta_0} \le R \le C\left(\frac{\lambda+\theta_0}{\mu\theta_0} + \frac{1}{\theta_0}\right), \\ 1, & R > C\left(\frac{\lambda+\theta_0}{\mu\theta_0} + \frac{1}{\theta_0}\right). \end{cases} \tag{3.33}$$

$$q_1^e = \begin{cases} 0, & R < \frac{C}{\mu} + \frac{C(\mu-\mu_1)}{\mu(\theta_1+\mu_1)}, \\ \frac{r_e(\mu_1(1-r_e)+\theta_1)}{\lambda(1-r_e)}, & \frac{C}{\mu} + \frac{C(\mu-\mu_1)}{\mu(\theta_1+\mu_1)} \le R \le \frac{C}{\mu(1-r(1))} + \frac{C(\mu-\mu_1)}{\mu(\theta_1+\mu_1(1-r(1)))}, \\ 1, & R > \frac{C}{\mu(1-r(1))} + \frac{C(\mu-\mu_1)}{\mu(\theta_1+\mu_1(1-r(1)))}. \end{cases} \tag{3.34}$$

$$q_2^e = \begin{cases} 0, & R < \frac{C}{\mu} + \frac{C}{\mu}\alpha(q_0^e, q_1^e), \\ \frac{1}{\lambda}\left(\mu - \frac{C\mu}{R\mu - C\alpha(q_0^e,q_1^e)}\right), & \frac{C}{\mu} + \frac{C}{\mu}\alpha(q_0^e, q_1^e) \le R \le \frac{C}{\mu-\lambda} + \frac{C}{\mu}\alpha(q_0^e, q_1^e), \\ 1, & R > \frac{C}{\mu-\lambda} + \frac{C}{\mu}\alpha(q_0^e, q_1^e). \end{cases} \tag{3.35}$$

Where

$$r_e = 1 + \frac{\mu(R\theta_1 - C) - \sqrt{\mu^2(R\theta_1 - C)^2 + 4R\mu\mu_1 C\theta_1}}{2R\mu\mu_1},$$

$$\alpha(q_0^e, q_1^e) = \frac{(\lambda q_0^e + \theta_0)^2(1 - r(q_1^e))^3 + r(q_1^e)\theta_0^2}{\theta_0(\lambda q_0^e + \theta_0)(1 - r(q_1^e))^3 + +r(q_1^e)(1 - r(q_1^e))\theta_0^2}.$$

Proof. From the previous analysis, U_i is monotonically decreasing with respect to q_i $(i = 0, 1, 2)$.

Consider a tagged customer who finds the server at state 0 upon arrival. Since condition $R > \frac{C}{\mu} + \frac{C}{\theta_0}$ ensures that q_0^e is positive, therefore we have two cases:

(1) $CW_0(0) \le R \le CW_0(1)$. In this case if all customers who find the system empty enter with probability $q_0^e = 1$, then the tagged customer suffers a negative expected benefit if he decides to enter. Hence, $q_0^e = 1$ does not lead to an equilibrium. Therefore, there exists a unique q_0^e, satisfying $U_0(q_0^e) = 0$, for which customers are indifferent between entering and balking.
(2) $R > CW_0(1)$. In this case, for every strategy of the other customers, the tagged customer has a positive expected net benefit if he decides to enter. Hence, $q_0^e = 1$.

Similarly, we consider the equilibrium strategy q_1^e in state 2. We know that the function $U_1(q_1)$ is strictly decreasing for q_1. We now consider the equation $U_1(q_1) = 0$ and solve for $r(q_1)$. We let r_e be the unique solution of equation for $r_e < 1$ and r_e is given by

$$r_e = 1 + \frac{\mu(R\theta_1 - C) - \sqrt{\mu^2(R\theta_1 - C)^2 + 4R\mu\mu_1 C\theta_1}}{2R\mu\mu_1}.$$

The corresponding unique $\overline{q}_1$ is found by considering (3.4) for $r = r_e$. Solving this equation with respect to q_1, we obtain

$$\overline{q}_1 = \frac{r_e(\mu_1(1 - r_e) + \theta_1)}{\lambda(1 - r_e)}.$$

Using the standard methodology of equilibrium analysis in unobservable queueing models, we derive the following equilibria:

(1) If $R < CW_1(0)$, then the equilibrium strategy is $q_1^e = 0$.
(2) If $CW_1(0) \le R \le CW_1(1)$, there exists a unique q_1^e, satisfying $U_1(q_1^e) = 0$, then the equilibrium strategy is denoted by $q_1^e = \overline{q}_1$.
(3) If $R > CW_1(1)$, then the equilibrium strategy is $q_1^e = 1$.

When q_0^e and q_1^e are determined, the expected net benefit of the arriving customer in state 2 is given by

$$U_2(q_0^e, q_1^e, q_2) = R - \frac{C}{\mu - \lambda q_2} - C\alpha(q_0^e, q_1^e). \tag{3.36}$$

To find q_2^e in equilibrium, we also consider the following subcases:

(1) If $U_2(q_0^e, q_1^e, 0) < 0$, then the equilibrium strategy is $q_2^e = 0$.
(2) If $U_2(q_0^e, q_1^e, 0) \geq 0$ and $U_2(q_0^e, q_1^e, 1) \leq 0$, there exists a unique q_2^e, satisfying $U_2(q_0^e, q_1^e, q_2^e) = 0$, then the equilibrium strategy is $q_1^e = \frac{1}{\lambda}\left(\mu - \frac{C\mu}{R\mu - C\alpha(q_0^e, q_1^e)}\right)$.
(3) If $U_2(q_0^e, q_1^e, 1) > 0$, then the equilibrium strategy is $q_2^e = 1$.

By rearranging the results, we can obtain the results of Theorem 3.3. This completes the proof. □

Next, we continue to consider the socially optimal behavior of customers. From (3.12) to (3.14), the effective arrival rate $\overline{\lambda}$ is equal to

$$\begin{aligned}\overline{\lambda} &= \lambda(q_0\pi_0 + q_1\pi_1 + q_2\pi_2)\\ &= K\left(\frac{\theta_1}{1-\sigma_0} + \frac{\lambda q_1}{1-r} + \frac{\lambda q_2}{1-\rho_2}\left(\frac{\theta_1}{\mu(1-r)^2} + \frac{\theta_1}{1-\sigma_0}\right)\right).\end{aligned} \tag{3.37}$$

And from (3.5), we get the mean number of the customers in the system, denoted by $E[L]$,

$$\begin{aligned}E[L] &= \sum_{n=0}^{\infty} n(\pi_{n0} + \pi_{n1}) + \sum_{n=1}^{\infty} n\pi_{n2}\\ &= K\left(\frac{r}{(1-r)^2} + \frac{\theta_1(\lambda q_0 + \theta_0)}{\theta_0^2} + \frac{\theta_1(1-\sigma_0\rho_2)}{\mu(1-\sigma_0)^2(1-\rho_2)^2} + \frac{\theta_1 r(1-r\rho_2)}{\mu(1-r)^3(1-\rho_2)^2}\right).\end{aligned} \tag{3.38}$$

The social benefit per time unit can now be easily computed as

$$\begin{aligned}S_{au}(q_0, q_1, q_2) &= \overline{\lambda}R - CE[L]\\ &= \overline{\lambda}R - CK\left(\frac{r}{(1-r)^2} + \frac{\theta_1(\lambda q_0 + \theta_0)}{\theta_0^2} + \frac{\theta_1(1-\sigma_0\rho_2)}{\mu(1-\sigma_0)^2(1-\rho_2)^2} + \frac{\theta_1 r(1-r\rho_2)}{\mu(1-r)^3(1-\rho_2)^2}\right).\end{aligned} \tag{3.39}$$

where $\overline{\lambda} = \lambda(q_0\pi_0 + q_1\pi_1 + q_2\pi_2)$.

The social planner expects to impose a socially optimal mixed strategy (q_0^*, q_1^*, q_2^*) to maximize the social benefit, where (q_0^*, q_1^*, q_2^*) can be obtained by $\max S_{au}(q_0, q_1, q_2)$ under conditions $0 \leq q_i \leq 1 (i = 0, 1, 2)$. It is easy to see that this is a nonlinear programming problem. Gradient projection methods or barrier and interior point methods can be used to find the optimal solution (see D.P. Bertsekas [1]). Let $Q = (q_0, q_1, q_2)$, we change the optimization problem to minimize the problem.

$$\begin{aligned}&\text{minimize} \quad f(Q) = -S_{au}(q_0, q_1, q_2) = -\overline{\lambda}R + CE[L]\\ &\text{subject to} \begin{cases} g_i(Q) = q_i \geq 0, & i = 0, 1, 2,\\ h_i(Q) = 1 - q_i \geq 0, & i = 0, 1, 2.\end{cases}\end{aligned} \tag{3.40}$$

Taking the barrier function methods as an example, the main steps of the proposed algorithm are as follows:

Step 1. $k = 1$. Let Q_1 is a inner point of the feasible region $\{Q | g_i(Q) \geq 0, h_i(Q) \geq 0, i = 0, 1, 2\}$. And ε is a sufficiently small positive number. Define a parameter sequence $\{\gamma_k\}$ with $0 < \gamma_{k+1} < \gamma_k$, $k = 0, 1, 2, \ldots$, and $\gamma_k \to 0$.

Step 2. Constructing an augmented objective function $F(Q,\gamma_k) = f(Q) + \gamma_k B(Q)$, where $B(Q)$ is a barrier function and its expression is defined as follows

$$B(Q) = -\sum_{i=0}^{2} \ln\{g_i(Q)\} - \sum_{i=0}^{2} \ln\{h_i(Q)\}.$$

Step 3. Solve the unconstrained optimization problem

$$\min F(Q,\gamma_k)$$

with Q_k as the starting point. We can use unconstrained methods such as Newton's method to obtain minimum point Q_{k+1}.

Step 4. If $0 < \gamma_k B(Q_{k+1}) \leq \varepsilon$, Q_{k+1} is the optimal solution. We stop the calculation. Otherwise, we change γ_k toγ_{k+1} and $k = k+1$, and repeat Step 2 to Step 4.

We need to note that Q_k is the inner point of the feasible region $\{Q|g_i(Q) \geq 0, h_i(Q) \geq 0, i = 0,1,2\}$.

Of course, we can simplify our calculation process by MATLAB software. In addition, we give a numerical results of optimal strategies.

Example: When we assume that $R = 7$, $C = 1$, $\lambda = 0.6$, $\mu = 0.9$, $\mu_1 = 0.5$, $\theta_0 = 0.2$ and $\theta_1 = 0.05$, we obtain socially optimal strategies $(q_0^*, q_1^*, q_2^*) = (0.3296, 0.5299, 0.8185)$, while equilibrium joining probabilities are $(q_0^e, q_1^e, q_2^e) = (0.2667, 0.8444, 0.9192)$. We observe that the individual optimal joining strategy is inconsistent with the socially optimal joining strategy.

3.2. Pricing analysis

From the above numerical example, we know that the individual optimal joining probability is different from the socially optimal joining probability, which shows that individual optimization deviates from the goals desired by the society. In equilibrium, the social welfare cannot attain the maximization. To eliminate the gap between the equilibrium and socially optimal strategies, the social planner can select a price p $(0 \leq p < R)$ and impose this charge on the customers that enter the queue. Then the reward of the customers that enter the system becomes $R - p$ in noncooperative case.

Let $W(q_0, q_1, q_2)$ be the mean sojourn time of a customer that enter the system. Based on the above, we obtain $W(q_0, q_1, q_2) = \pi_0 W_0(q_0) + \pi_1 W_1(q_1) + \pi_2 W_2(q_0, q_1, q_2)$. Then, the expected benefit of an arriving customer who decides to enter the system is

$$\begin{aligned} U &= \pi_0 U_0(q_0) + \pi_1 U_1(q_1) + \pi_2 U_2(q_0, q_1, q_2) \\ &= \pi_0(R - p - CW_0(q_0)) + \pi_1(R - p - CW_1(q_1)) + \pi_2(R - p - CW_2(q_0, q_1, q_2)) \\ &= R - p - CW(q_0, q_1, q_2). \end{aligned} \tag{3.41}$$

The monopoly does not give up the residual benefits of customers, so we have $p = R - CW(q_0, q_1, q_2)$. Let Π_s be the expected server profit per time unit. For the system, the monopoly's problem is a nonlinear programming problem to maximize Π_s with respect to q_0, q_1 and q_2:

$$\Pi_s = \overline{\lambda} p = \overline{\lambda}(R - CW(q_0, q_1, q_2)). \tag{3.42}$$

Then the objectives of a monopolistic server and the society are consistent with each other, so we can induce the socially optimal joining probabilities (q_0^*, q_1^*, q_2^*) by an appropriate price, which also maximizes profit of the server.

Moreover, as mentioned above, if the server chooses the optimal pricing strategy, the expected benefit of the customer equals

$$\begin{aligned} U &= R - (R - CW(q_0^*, q_1^*, q_2^*)) - CW(q_0, q_1, q_2) \\ &= C(W(q_0^*, q_1^*, q_2^*) - W(q_0, q_1, q_2)). \end{aligned} \tag{3.43}$$

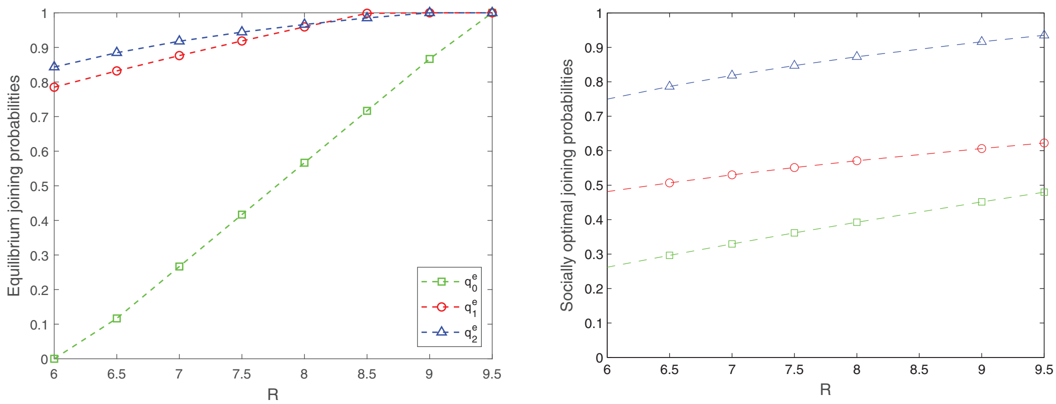

FIGURE 2. Social benefit of unobservable case when $C = 1$, $\theta_0 = 0.2$, $\theta_1 = 0.05$, $\mu_1 = 0.5$, $\mu = 0.9$, $\lambda = 0.6$.

It can be seen that (q_0^*, q_1^*, q_2^*) is the zero point of the function U. In other words, they are also customers equilibrium strategy.

Theorem 3.4. *In the almost unobservable queue, there exist unique equilibrium strategies where customers join the queue with probabilities* (q_0^*, q_1^*, q_2^*) *and the optimal price* $p^* = R - CW(q_0^*, q_1^*, q_2^*)$*, and the server's maximal profit is* $\Pi_s^* = S_{au}(q_0^*, q_1^*, q_2^*)$*.*

3.3. Numerical experiments

We investigate the effect of several system parameters on the behavior of customers in the almost unobservable queue and explore the sensitivity of equilibrium and socially optimal balking probabilities. It should be noted that the stable condition must be satisfied. The results are shown in Figures 2–5. It is observed that $q_1^* \le q_1^e$, $q_2^* \le q_2^e$, while the other two probabilities q_0^e and q_0^* are not able to determine the relationship. Figures 2 and 3 study the effect of equilibrium and socially optimal joining probabilities under service reward R and normal service rate μ, respectively. We observe that the joining probabilities increase with R and μ. It is very intuitive that customers can get higher utility from the large service reward and service rate. On the other hand, we observe that all probabilities decrease as λ increases in Figure 4. Figure 5 describes that the monotonicity of all probabilities is diverse. Among them, the probability q_1^e is monotonously increased, which indicates that more customers are willing to enter the queueing system during working vacation. This is because the shorter working vacation period allows the server to enter the regular busy period more quickly.

4. FULLY UNOBSERVABLE CASE

In this section, we focus on the fully unobservable case, where arriving customers cannot observe the state of the system at their arrival instant.

4.1. Equilibrium and social optimization

There are two pure strategies available for a customer, to join the queue or not to join the queue. A pure or mixed strategy can be described by a fraction q $(0 \le q \le 1)$, and arriving customers join the queue with probability q or leave the system with complementary probability $1 - q$. Then, the probability of joining is λq. The transition diagram is illustrated in Figure 6.

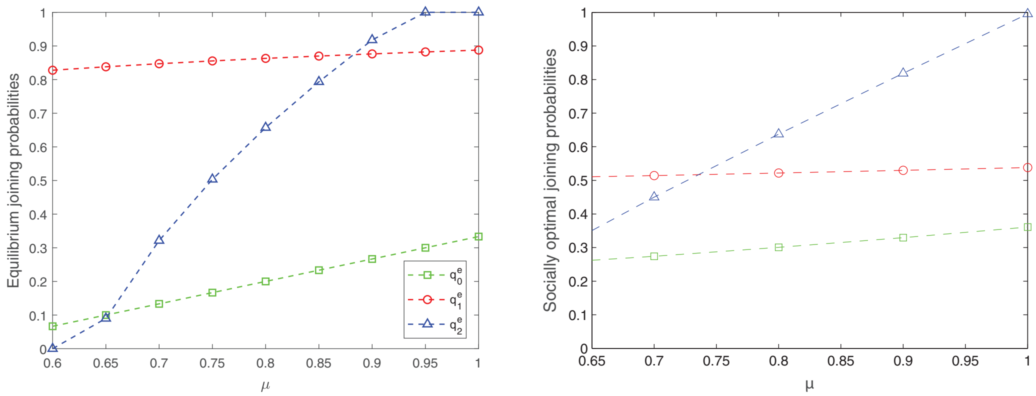

FIGURE 3. Social benefit of unobservable case when $R = 7$, $C = 1$, $\theta_0 = 0.2$, $\theta_1 = 0.05$, $\mu_1 = 0.5$, $\lambda = 0.6$.

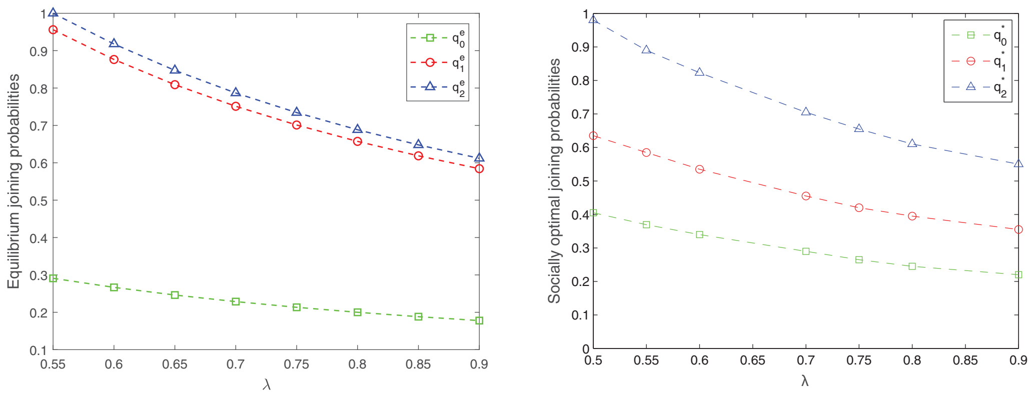

FIGURE 4. Social benefit of unobservable case when $R = 7$, $C = 1$, $\theta_0 = 0.2$, $\theta_1 = 0.05$, $\mu_1 = 0.5$, $\mu = 0.9$.

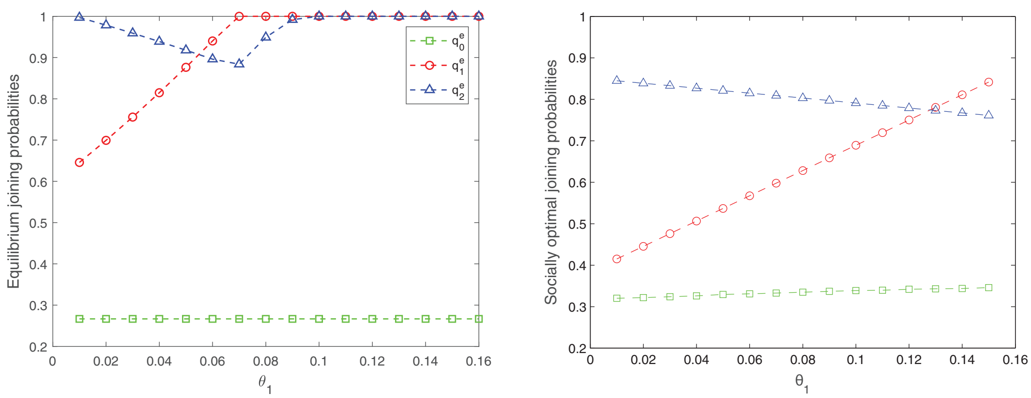

FIGURE 5. Social benefit of unobservable case when $R = 7$, $C = 1$, $\theta_0 = 0.2$, $\lambda = 0.6$, $\mu_1 = 0.5$, $\mu = 0.9$.

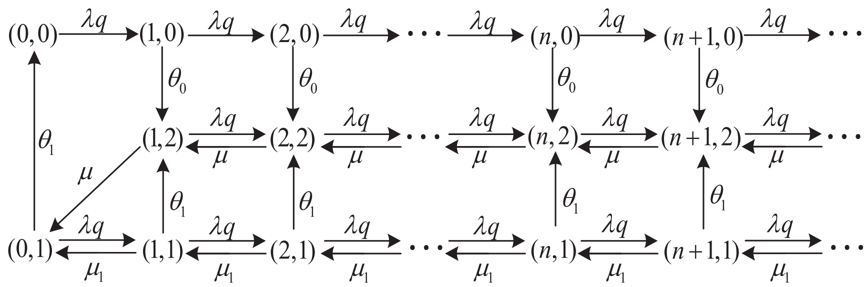

FIGURE 6. Transition rate diagram for the fully unobservable queues.

In the fully unobservable case, all customers have the same joining probability of q. Therefore, by considering $q_0 = q_1 = q_2 = q$ in (3.5), we can get the mean sojourn time of a joining customer (Fig. 6),

$$\begin{aligned} W(q) &= \frac{1}{\lambda q}\left(\sum_{n=0}^{\infty} n(\pi_{n0}+\pi_{n1}) + \sum_{n=1}^{\infty} n\pi_{n2}\right) \\ &= \frac{1}{\mu-\lambda q} + K_1\left((1-\beta)\left(1-\frac{\mu_1}{\mu}\right)\frac{r(q)}{\lambda q(1-r(q))} + \frac{\theta_1(1-r(q))}{\lambda q\theta_0}\right), \end{aligned} \tag{4.1}$$

where

$$\beta = \frac{\lambda q}{\lambda q+\theta_0}, \qquad r(q) = \frac{\lambda q+\mu_1+\theta_1-\sqrt{(\lambda q+\mu_1+\theta_1)^2-4\lambda q\mu_1}}{2\mu_1}, \tag{4.2}$$

$$K_1 = \left((1-\beta)(1-r(q))\frac{\mu_1}{\mu} + (1-\beta)\left(1-\frac{\mu_1}{\mu}\right) + \frac{\theta_1}{\lambda q}(1-r(q))\right)^{-1}. \tag{4.3}$$

We consider a tagged customer. If he decides to enter the system, his expected net benefit is

$$\begin{aligned} U(q) &= R - CW(q) \\ &= R - \frac{C}{\mu-\lambda q} - CK_1\left((1-\beta)\left(1-\frac{\mu_1}{\mu}\right)\frac{r(q)}{\lambda q(1-r(q))} + \frac{\theta_1(1-r(q))}{\lambda q\theta_0}\right). \end{aligned} \tag{4.4}$$

According to the previous method, we can get the solution $\bar{q}$ by solving the equation $U(q) = 0$. Due to the complex expression of $W(q)$, We can't get the concrete expression of $\bar{q}$. However, by using sufficient numerical experiments, we can get that $W(q)$ either monotonously increases with respect to q or first decreases then increases with q, and $W(q)$ is strictly convex with q and it has a minimum. This is due to the fact that an increase in q is equivalent to an increase in arrival rate. When customers join the system in vacation time, a shorter vacation time will trigger the server to switch to a regular busy period. The waiting time will decreases with the increase of the service rate. However, If the system is always in state 2, the waiting time of customers will increases as the arrival rate increases. Figure 7 shows that $W(q)$ is increasing with repent to q, so there exists a unique equilibrium strategy. However, in Figure 8 there may be two equilibrium strategies. When $R = 1.2, C = 1$, we have $q_e = (q_1, q_2)$. However, it is obvious that only the larger one is stable. So we only consider the customers' stable equilibrium strategy, which is unique. Therefore, there exists a unique mixed equilibrium strategy q_e and $q_e = \min\{\bar{q}, 1\}$.

The social benefit per time unit can now be easily computed as

$$\begin{aligned} S_{fu}(q) &= \lambda q(R - CW(q)) \\ &= \lambda q\left(R - \frac{C}{\mu-\lambda q} - CK_1\left((1-\beta)\left(1-\frac{\mu_1}{\mu}\right)\frac{r(q)}{\lambda q(1-r(q))} + \frac{\theta_1(1-r(q))}{\lambda q\theta_0}\right)\right). \end{aligned} \tag{4.5}$$

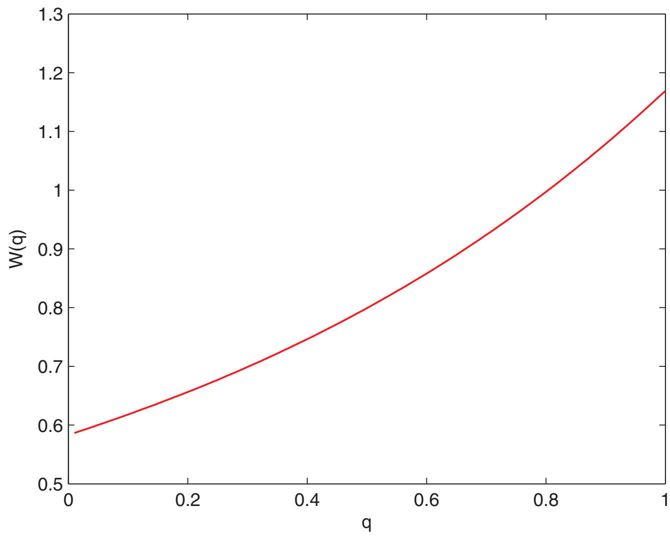

FIGURE 7. Mean sojourn time of the fully unobservable queue for $\lambda = 1, \mu = 3, \mu_1 = 1.5, \theta_0 = 4, \theta_1 = 0.2$.

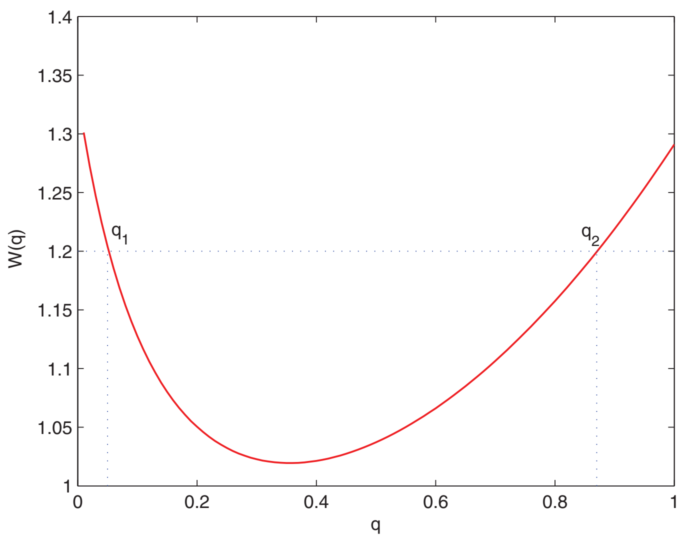

FIGURE 8. Mean sojourn time of the fully unobservable queue for $\lambda = 1, \mu = 3, \mu_1 = 1.5, \theta_0 = 1, \theta_1 = 0.2$.

From the perspective of social optimization, denote the customers' socially optimal mixed strategy as q^*, which can be obtained by max $S_{fu}(q)$ with respect to q and solving the equation.

Let x^* be the root of the first-order optimal condition $S'_{fu}(q) = 0$. They have the following relation.

Theorem 4.1. *If* $0 < x^* < 1$ *and* $S''_{fu}(q) \leq 0$, *then* $q^* = x^*$; *if* $0 < x^* < 1$ *and* $S''_{fu}(q) > 0$ *or* $x^* \geq 1$, *then* $q^* = 1$.

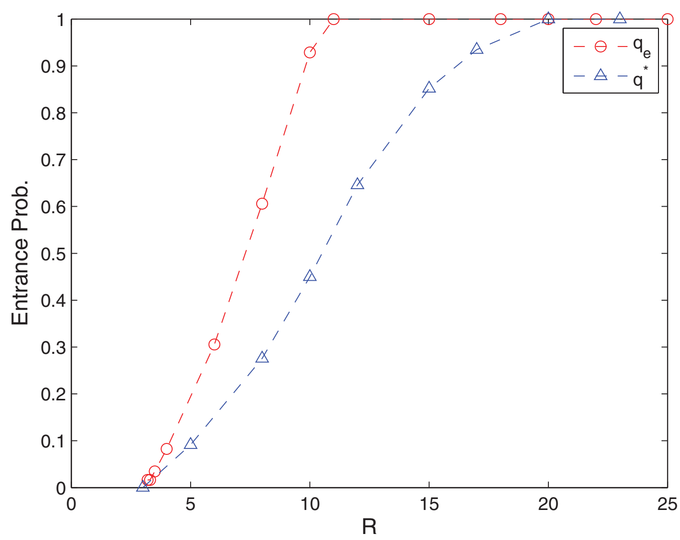

FIGURE 9. Equilibrium and socially optimal probabilities for fully unobservable queue for $C = 1$, $\theta_0 = 0.5$, $\theta_1 = 0.1$, $\mu_1 = 0.1$, $\mu = 0.9$, $\lambda = 0.6$.

4.2. Pricing analysis

For the fully unobservable queue, if the server imposes a price p on the customers that enter the queue, customer's equilibrium strategy satisfies $R - p - CW(q) = 0$. The monopoly's problem is to maximize $\Pi_s = \lambda qp$ and

$$\begin{aligned} p &= R - CW(q) \\ &= R - \frac{C}{\mu - \lambda q} - CK_1\left((1-\beta)\left(1 - \frac{\mu_1}{\mu}\right)\frac{r(q)}{\lambda q(1 - r(q))} + \frac{\theta_1(1 - r(q))}{\lambda q\theta_0}\right), \end{aligned} \tag{4.6}$$

so we have $\Pi_s = S_{fu}(q)$. Therefore, we can induce the socially optimal joining probabilities q^* by an appropriate price, which also maximizes profit of the server. At the same time, the optimal strategy q^* is also the customer's equilibrium strategy, and the optimal price $p^* = R - CW(q^*)$, and the server's maximal profit is $\Pi_s^* = S_{fu}(q^*)$.

4.3. Numerical experiments

We consider the fully unobservable system and explore the equilibrium and socially optimal probabilities with respect to some parameters of the system. The results are presented in Figures 9–12 and we find that equilibrium q_e is always greater than socially optimal probability q^*. This shows that individual optimization leads to queues that are longer than are socially desired, and always leads to system congestion. To eliminate this difference, social planners can set a price that allows customers to adopt socially optimal strategies and maximize monopoly profits, as discussed in the previous Section 4.2. Regarding the sensitivity of each parameter, an interesting conjecture arises from these figures. Equilibrium and socially optimal probabilities are increasing with respect R and μ, which is very intuitive. The probabilities are decreasing with respect to λ. When λ is higher, arriving customer are less willing to enter. The reason is that too many customers entering the queue will lead to an increase in waiting time. Moreover, all probabilities increase as θ_1 increases. The working vacation time becomes shorter, and the server can quickly switch to the regular busy period, which makes the waiting time of the customer shorter and the customer is more willing to enter the system.

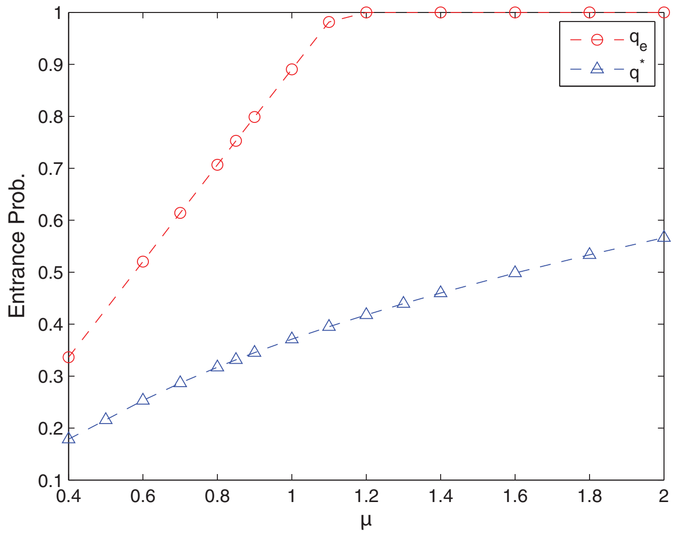

FIGURE 10. Equilibrium and socially optimal probabilities for fully unobservable queue for $R = 10$, $C = 1$, $\theta_0 = 0.5$, $\theta_1 = 0.1$, $\mu_1 = 0.1$, $\lambda = 0.6$.

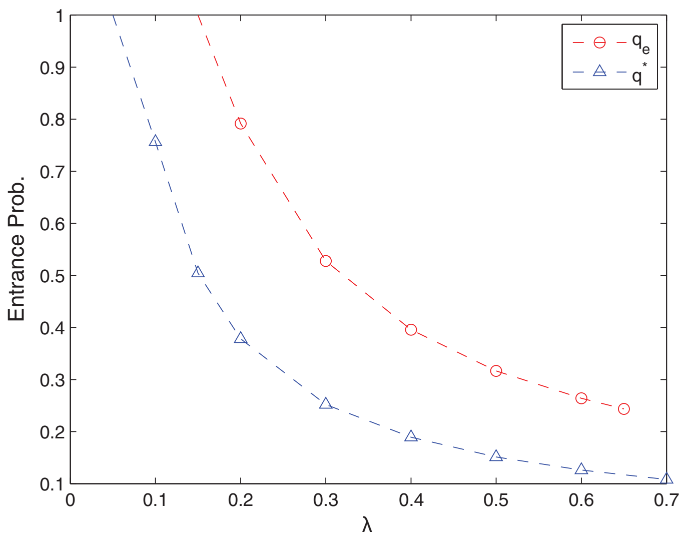

FIGURE 11. Equilibrium and socially optimal probabilities for fully unobservable system for $R = 10$, $C = 1$, $\theta_0 = 0.5$, $\theta_1 = 0.05$, $\mu_1 = 0.1$, $\mu = 0.9$.

5. Comparison of social benefit

In this section we consider the equilibrium and optimal social benefits for different system parameters in the almost and fully unobservable queues by numerical experiment. For the almost unobservable queues, when all customers follow the equilibrium strategy (q_0^e, q_1^e, q_2^e), the social welfare per time unit in equilibrium can be denoted by $S_{au}(q_0^e, q_1^e, q_2^e)$. On the other hand, the social planner wants to maximize his social welfare by an

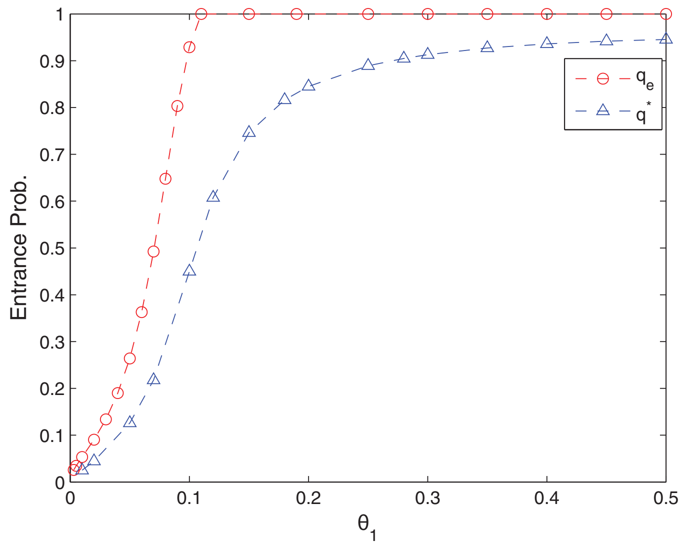

FIGURE 12. Equilibrium and socially optimal probabilities for fully unobservable system for $R = 10$, $C = 1$, $\theta_0 = 0.5$, $\mu_1 = 0.1$, $\mu = 0.9$, $\lambda = 0.6$.

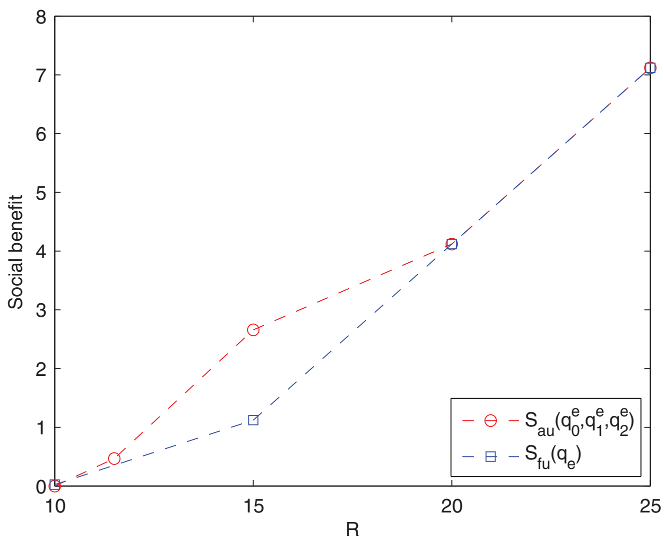

FIGURE 13. Equilibrium social benefit of two cases for $C = 1$, $\theta_0 = 0.1$, $\theta_1 = 0.05$, $\mu_1 = 0.3$, $\mu = 1$, $\lambda = 0.6$.

optimal strategy (q_0^*, q_1^*, q_2^*), which is expressed by $S_{au}(q_0^*, q_1^*, q_2^*)$. Similarly, for fully unobservable case, the equilibrium social benefit and optimal social benefit can be expressed by $S_{fu}(q_e)$ and $S_{fu}(q^*)$, respectively.

The numerical experiment is concerned with the social benefit under equilibrium strategy for the different information levels. Figure 13 shows that $S_{au}(q_0^e, q_1^e, q_2^e)$ (or $S_{fu}(q_e)$) is increasing with respect to the service reward R. Because The increase in service reward does not change the customer's waiting cost but will attract more customers into the system. Figure 14 shows that equilibrium social benefit first increases then decreases

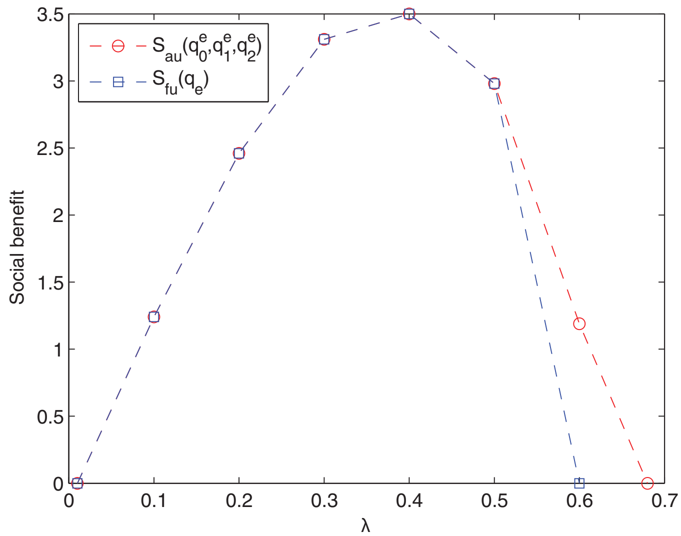

FIGURE 14. Equilibrium social benefit of two cases for $R = 20$, $C = 1$, $\theta_0 = 0.3$, $\theta_1 = 0.05$, $\mu_1 = 0.3$, $\mu = 0.7$.

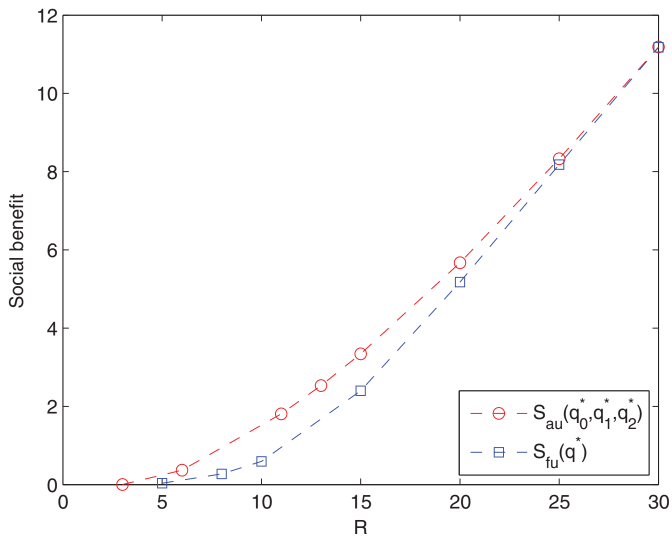

FIGURE 15. Socially optimal benefit of two cases for $C = 1$, $\theta_0 = 0.4$, $\theta_1 = 0.3$, $\mu_1 = 0.5$, $\mu = 0.9$, $\lambda = 0.6$.

with λ and it has a maximum. This is because λ is relatively small, there are fewer customers in the system and the customer has less waiting time. And as it continues to increase, the waiting time for customers increases, which leads to a reduction in social benefits. As to the optimal social benefit for the different information levels, the social benefit $S_{au}(q_0^*, q_1^*, q_2^*)$ (or $S_{fu}(q^*)$) is increasing with respect to the reward R in Figure 15. Figure 16 shows that optimal social benefit first monotonously increases, then achieves maximum and keeps invariable when λ exceeds the constant value.

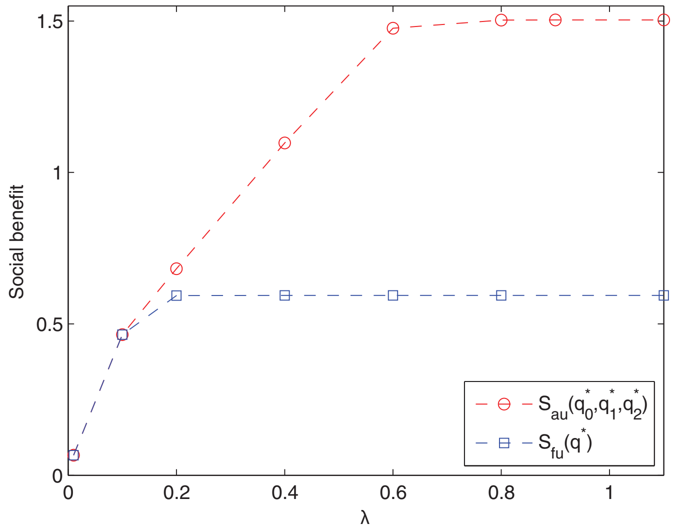

FIGURE 16. Socially optimal benefit of two cases for $R = 10$, $C = 1$, $\theta_0 = 0.5$, $\theta_1 = 0.1$, $\mu_1 = 0.1$, $\mu = 0.9$.

In addition, Figures 13 and 14 show that $S_{au}(q_0^e, q_1^e, q_2^e) \geq S_{fu}(q_e)$. This indicates that the system organizer should properly disclose the status of the server before the arriving customer makes a decision, when the arrival rate λ is relatively large or the reward R is relatively small. Similarly, we also find that $S_{au}(q_0^*, q_1^*, q_2^*) \geq S_{fu}(q^*)$. This indicates that the social planners and system organizers should choose similar decisions to maximize social benefits. Of course, through the previous discussion, we also know that the equilibrium strategy and the social optimal strategy are generally not uniform. There are two ways to improve this problem, either by revealing the server state information or by setting a toll to let the customer choose the optimal probability to enter the queueing system.

6. Conclusion

In this paper, we analyze the M/M/1 queueing system with a single working vacation and multiple vacations. Two different cases with respect to the levels of information provided to arriving customers have been investigated extensively. For these models, we present an extensive analysis of equilibrium joining strategies and socially optimal joining probability of the customers, and discuss the pricing strategy. And we compare the social benefit of two different cases numerically. In contrast with other related queueing models, this work combines working vacation and classic vacation queue. Furthermore, the direct generalization is the study of the corresponding observable model. One can also study the equilibrium strategies and pricing problem in the M/G/1 queue with working vacation and classic vacations.

Acknowledgements. The authors thank the anonymous referees for their detailed comments and helpful suggestions. This research was supported by the National Natural Science Foundation of China (No. 11601469), the Hebei Province Natural Science Foundation of China (No. A2018203088), the Science Research Project of Education Department of Hebei Province (No. ZD2018042) and the Science and Technology Research and Development Project of Qinhuangdao (No. 201502A221). Ruiling Tian would like to thank China Scholarship Council for supporting her visit to Simon Fraser University.

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