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A SLACKS-BASED MEASURE APPROACH FOR EFFICIENCY DECOMPOSITION IN MULTI-PERIOD TWO-STAGE SYSTEMS

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Abstract. Two-stage production systems are often encountered in many real applications where the production process is divided into two processes. In contrast to the conventional data envelopment analysis (DEA) models, two-stage DEA models take the operations of the internal processes into account. A number of studies have used two-stage DEA models in order to evaluate the performance of decision making units (DMUs) having a network structure. In this paper, we use a non-radial DEA model called the network slacks-based measure (NSBM) model to measure the efficiency of a system with a multi-period two-stage structure. Then we describe the properties of the proposed model in details. Moreover, we shall decompose the overall efficiency of the system over a number of time periods as a weighted average of the efficiency in each period. The efficiency of the stages, in respect to the entire periods shall be decomposed in terms of the weighted average efficiency of the stages in each period. Finally, the real data of Mellat bank branches in Tehran extracted from extant literature is used to illustrate the proposed approach.

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1. Introduction

Data envelopment analysis is a non-parametric linear programming in order to estimate the efficiency of decision making units (DMUs) having multiple inputs and multiple outputs. Farrell [15] used the non-parametric methods to evaluate the efficiency of DMUs with multiple inputs and one output. Then, Charnes *et al.* [6] presented a method that was called data envelopment analysis (DEA), by extending Farrell's methods.

The conventional DEA measures the efficiency of DMUs, without taking the operations of the internal processes into consideration. But, in the present world, many of the DMUs have a complex internal structure such as network systems. Hence, a group of DEA models was presented in order to assess the efficiency of these systems. These models were called Network DEA (NDEA) models. For example Fukuyama and Mirdehghan [16] proposed a network approach that identifies the efficiency situations of systems and their divisions by providing

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a practical computational procedure. Lozano *et al.* [37] applied a directional distance approach to network DEA models in presence of undesirable outputs. Kao *et al.* [30] presented a multi-objective programming method presented to solve network DEA models. Also, Kao [25] reviewed research papers related to network systems. Boloori *et al.* [5] surveyed dual multiplier and envelopment DEA models to calculate the efficiency of systems with internal structure. The special case of these systems, are the multi-stage systems and in particular two-stage systems. Seiford and Zhu [41] presented models that measure the performance of the two-stage systems and their stages, independently. Based on their proposed models, the whole system may be efficient, but its stages are not efficient, which is a weakness of the model. Thus, Kao and Hwang [27] introduced a model for calculating the efficiency of two-stage systems under constant return to scale (CRS) and decomposing the overall efficiency of the system as the product of stages efficiencies. This model cannot apply under the variable return to scale (VRS) assumption. Therefore, Chen *et al.* [7] presented new models for measuring the overall efficiency of a two-stage system and efficiency of stages under both CRS and VRS assumptions. Based on these models, the overall efficiency of the system is equal to the weighted average of the stages efficiencies. According to the multiplicative and additive forms of DEA, many of the radial and non-radial models were introduced to calculate the performance of the developed two-stage systems. For instance, Amirteimoori [2] introduced a model for evaluating the efficiency of two-stage system with shared resources. Jianfeng [22] presented additive model by considering simultaneously the two-stage structure with shared inputs and free intermediate measures. Also, Aviles-Sacoto *et al.* [4] proposed a two-stage DEA approach when intermediate measures can be treated as outputs from the second stage. Wu *et al.* [45] evaluated the efficiency of two-stage network systems in presence of shared resources and resources recovered from undesirable outputs by using heuristic method. Guo and Zhu [17] indicated that non cooperative models can be transformed into a linear program by using Charnes–Cooper transformation. Also, Izadikhah *et al.* [21] suggested a novel two-stage model with freely distributed initial inputs and shared intermediate outputs. Kao [24] proposed a relational model for measuring the efficiency of multi-stage systems and stages, simultaneously. Based on this model, the efficiency of the whole system is expressed as a geometric average of the stages efficiencies. Despotis *et al.* [9] evaluated the efficiency of series multi-stage systems by using a network DEA model. Cook *et al.* [8] examined the structure of multi-stage systems and decomposed the overall efficiency as additive weighted average of the stages efficiencies. Ebrahimnejad *et al.* [11] proposed a three-stage DEA model with two independent parallel stages linking to a third stage and applied their model in the banking industry. Despotis *et al.* [10] presented a novel procedure to estimate unique and unbiased efficiency scores for two-stage systems by revisiting the additive and multiplicative efficiency decompositions methods and discussing their shortcomings. Finally, they obtained the efficiency of the overall system by using the aggregation method. Hinojosa *et al.* [19] estimated the efficiency of multi-stage production systems by using Nash bargaining. Using slacks-based models, Tone [43] introduced a non-radial slacks-based measure (SBM) model that measures the efficiency of black box DMUs. In order to evaluate the efficiency of systems with internal processes (Network), the network slack-based measure (NSBM) was introduced by Tone and Tsutsui [44]. In this model, the overall efficiency of system is decomposed into the weighted average of the processes that the weight of each process is defined by the decision maker (DM). Ashrafi *et al.* [3] presented a SBM model to measure the efficiency of two-stage systems. Their model is able to determine efficient projections for inefficient systems. Akther *et al.* [1] used a network slacks based inefficiency measure for two-stage systems and assessed the efficiency of Bangladesh bank. Lozano [36] evaluated the efficiency of general networks of processes in presence of desirable and undesirable outputs and also, determined the target value of inputs, outputs and intermediate measures. Kao [26] classified a slacks-based efficiency measures in network DEA with an analysis of the properties possessed. Kao [23], by extending that model, suggested a NSBM model that measures the efficiency of the system and internal processes. Whereas, the overall efficiency of system is also equal to the weighted average of the efficiency of processes (in output oriented) in which the weight of a typical process, is recognized as a ratio of the efficiency of that process to the sum of the efficiency of the total processes. The proposed NSBM models measure the efficiency of DMUs at the certain time. The mentioned models are focused on measuring the efficiency of systems at a specified time. In many applications, evaluating the efficiency of organizations is required over several time periods. In recently years, many of studies are proposed to measure the efficiency of black box systems over multi-period time. For example, Kao and Liu [29] proposed a model to measure the efficiency of systems over multi-period times. Park [39] introduced a new method to measure the efficiency of multi-period aggregative efficiency by developing the concept of Debreu-Farrell technical efficiency. A super-efficiency model for measuring aggregative efficiency of multi-period production systems was proposed by Esmaeilzadeh and Hadi-Vencheh [13]. Razavi Hajiagha *et al.* [18] proposed a two-stage approach based on the Chebyshev inequality bounds for finding interval efficiency of systems. Also, Kordrostami and Noveiri [33, 34] measured the efficiency of firms with negative data and fuzzy data in multi-period systems. Then, Noveiri and Kordrostami [38] evaluated the multi-period performance and efficiency changes of black box systems with undesirable outputs. Also, many researchers have presented models to measure the efficiency of multi-period multi-stage systems (and in particular multi-period two-stage systems). Kou *et al.* [32] used dynamic network DEA method to evaluate the efficiency of multi-period multi-division systems. Kao and Hwang [28] proposed a new approach for measuring the efficiency of multi-period two-stage systems. Moreover, they introduced a Malmquist productivity index to identify the situations of the efficiency over these time periods. Esfidani *et al.* [12] proposed an additive model to measure the multi-period efficiency of two-stage systems. Then, they introduced the new efficiency changes indexes related to the whole system and the first and second stages between two periods that identify the situations of the positive changes and the negative changes in the efficiency from a period to another period. Esmaeilzadeh and Kazemi matin [14] presented a novel multi-period DEA model to measure the efficiencies of parallel and series systems.

In the field of application, many studies have been presented to evaluate the efficiency of black box systems (and network systems). For example, Lozano *et al.* [37] used network DEA approach for evaluating the efficiency of airports by considering undesirable outputs. Kao and Liu [29] measured multi-period efficiency of Taiwanese commercial banks. The multi-period efficiency of East Asia airport companies was measured by Liu [35]. Tavassoli *et al.* [42] used a SBM-network DEA model to measure the efficiency of 11 domestic Iranian airlines in presence of shared inputs. Kao *et al.* [31] proposed alternative network DEA model to analyze cloud service businesses. Zhou *et al.* [46] proposed a multi-period three-stage model to measure the efficiency of banking systems under uncertainty. Parte and Alberca [40] proposed a multi-stage model for measuring the efficiency of bar industry.

So far, a model for evaluating the efficiency of multi-period two-stage systems based on non-radial slacks-based measure (SBM) model has not been presented. Hence, in this paper, we extend the model proposed by Kao [23] and propose a model for measuring the efficiency of the multi-period two-stage systems with the slacks-based measure so that the relative important of data is considered differently in time periods. In our proposed model, the overall efficiency of a system over time periods will be defined as the weighted average of the overall efficiency of each period. Our proposed non-radial method measures overall efficiency, stages efficiencies over whole time periods and each time period, simultaneously. The radial DEA models stand on the assumption that inputs or outputs change proportionality. However, when employ labor and paid profit as inputs or facilities and earned profit as outputs, for example, some of them are substitution and do not change proportionality. Our proposed linear model is unit invariant and identifies the sources of the inefficiency for both the whole system and stages in each period and whole time periods. Moreover, based on the obtained optimal solution, we will demonstrate that a multi-period two-stage system is efficient if and only if all its stages are efficient in the all periods.

The rest of this paper is organized as follows: we review the SBM model proposed by Tone [43] in Section 2. In Section 3, we first show the structure of a multi-period two-stage system and introduce a model to measure the efficiency of the two-stage system over whole time periods and each period, simultaneously, in a way the importance of data is considered differently in time periods. The properties of the proposed model are also discussed in this section. Moreover, the relationships between the overall efficiency over time periods and the overall efficiency in each period, and the relationships between efficiencies of the stages over time periods and the efficiencies of stages in each period are explored. In Section 4, we illustrate the proposed model by using the data of Mellat bank branches in Tehran. Section 5 presents our conclusions and future research directions.

2. Preliminaries

This section describes the non-radial slacks-based measure (SBM) model to calculate the efficiency score of black-box system that was proposed by Tone [43].

Suppose there are n DMUs that each DMU_j $(j = 1, \ldots, n)$ by consuming the input vector x_j produces output vector y_j. In order to measure the relative efficiency of DMU_o (DMU under evaluation), Tone [43] proposed the following non-radial model:

$$E_o^s = \min \frac{1 - \frac{1}{m} \sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s} \sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}}$$

s.t.

$$\begin{aligned} &\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} && i = 1, \ldots, m \\ &\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} && r = 1, \ldots, s \\ &\lambda_j, s_i^-, s_r^+ \geq 0, && j = 1, \ldots, n \quad i = 1, \ldots, m \quad r = 1, \ldots, s \end{aligned}$$

wherein, s_i^-, s_r^+, λ_j are input slack, output slack and intensity vector of DMU_j $(j = 1, \ldots, n)$, respectively.

In this model, $x_o > 0$ and $y_o > 0$. If $x_{io} = 0$, term $\frac{s_i^-}{x_{io}^-}$ will be removed from the objective function. Also, if $y_{ro} = 0$, term $y_{ro} = \varepsilon$ is replaced where ε is a very small positive number. Moreover, $E_o^s \leq \theta_{\text{CCR}}$ that θ_{CCR} is the efficiency score of CCR model. By using the Charnes–Cooper transformations, this model can be transformed to a linear form.

Definition 2.1. For each DMU_o, we have $E_o^s = 1$ if and only if $s_i^- = s_r^+ = 0 \quad \forall i, r$.

3. Multi-period two-stage system

In this section, we first introduce the structure of a multi-period two-stage system. Then we shall propose a model to calculate the efficiency of such systems with the slacks-based measure so that the relative importance of data is considered differently in time periods.

Consider we have n DMUs with a two-stage structure, so that their information is available in q time periods. In time period p, each DMU_j $(j = 1, \ldots, n)$, in stage 1, consumes inputs x_{ij}^p $(i = 1, \ldots, m)$ to produce intermediate products z_{dj}^p $(d = 1, \ldots, D)$ and then in stage 2, consumes the intermediate products to produces outputs y_{rj}^p $(r = 1, \ldots, s)$. In Figure 1 the structure of a multi-period two-stage system is shown.

By considering the input slack vector s^{-p}, the output slack vector s^{+p} and the intermediate products slacks vectors s^p (as output for stage 1) and t^p (as input for stage 2) and the intensity vectors λ^p (for stage 1) and λ'^p (for stage 2) in each time period p, the suggested non-radial model is given as follows:

$$E_o^s = \min \frac{\sum_{p=1}^{q} \left(1 - \frac{1}{m+D} \left(\sum_{i=1}^{m} \frac{s_i^{-p}}{x_{io}^p} + \sum_{d=1}^{D} \frac{t_d^p}{z_{do}^p} \right)\right)}{\sum_{p=1}^{q} \left(1 + \frac{1}{s+D} \left(\sum_{r=1}^{s} \frac{s_r^{+p}}{y_{ro}^p} + \sum_{d=1}^{D} \frac{s_d^p}{z_{do}^p} \right)\right)}$$

s.t.

$$\begin{array}{llllll}
\sum_{j=1}^{n} \lambda_j^p x_{ij}^p + s_i^{-p} = x_{io}^p & i=1,\ldots,m & p=1,\ldots,q & & & \\
\sum_{j=1}^{n} \lambda_j^p z_{dj} - s_d^p = z_{do}^p & d=1,\ldots,D & p=1,\ldots,q & & & \\
\sum_{j=1}^{n} \lambda_j'^p z_{dj}^p + t_d^p = z_{do}^p & d=1,\ldots,D & p=1,\ldots,q & & & \\
\sum_{j=1}^{n} \lambda_j'^p y_{rj}^p - s_r^{+p} = y_{ro}^p & r=1,\ldots,s & p=1,\ldots,q & & & \\
\lambda_j^p, \lambda_j'^p, s_i^{-p}, s_r^{+p}, t_d^p, s_d^p \geq 0 & j=1,\ldots,n & i=1,\ldots,m & r=1,\ldots,s & d=1,\ldots,D & p=1,\ldots,q.
\end{array} \tag{3.1}$$

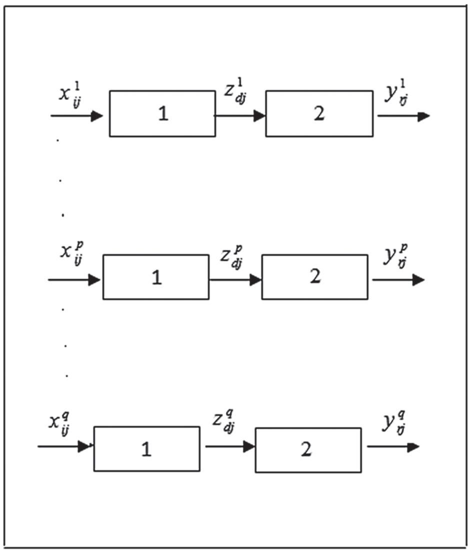

FIGURE 1. Multi-period two-stage system

This model measures the overall and period efficiencies, simultaneously.
Let

$$t = \frac{1}{\sum_{p=1}^{q} \left(1 + \frac{1}{s+D} \left(\sum_{r=1}^{s} \frac{s_r^{+p}}{y_{ro}^p} + \sum_{d=1}^{D} \frac{s_d^p}{z_{do}^p} \right)\right)}$$

and

$$ts_r^{+p} = \delta_r^{+p},\ ts_d^p = \delta_d^p, ts_i^{-p} = \delta_i^{-p},\ tt_d^p = \gamma_d^p, t\lambda_j^p = n_j^p, t\lambda_j'^p = n_j'^p.$$

In this case, the proposed model (3.1) is converted to the following linear model:

$$E_o^s = \min \sum_{p=1}^{q} \left(t - \frac{1}{m+D} \left(\sum_{i=1}^{m} \frac{\delta_i^{-p}}{x_{io}^p} + \sum_{d=1}^{D} \frac{\gamma_d^p}{z_{do}^p} \right)\right)$$

s.t.

$$\sum_{p=1}^{q} \left(t + \frac{1}{s+D} \left(\sum_{r=1}^{s} \frac{\delta_r^{+p}}{y_{ro}^p} + \sum_{d=1}^{D} \frac{\delta_d^p}{z_{do}^p} \right)\right) = 1$$

$$\begin{aligned}
&\sum_{j=1}^{n} n_j^p x_{ij}^p + \delta_i^{-p} = t x_{io}^p && i=1,\ldots,m, \quad p=1,\ldots,q\\
&\sum_{j=1}^{n} n_j^p z_{dj} - \delta_d^p = t z_{do}^p && d=1,\ldots,D, \quad p=1,\ldots,q\\
&\sum_{j=1}^{n} n_j'^p z_{dj}^p + \gamma_d^p = t z_{do}^p && d=1,\ldots,D, \quad p=1,\ldots,q\\
&\sum_{j=1}^{n} n_j'^p y_{rj}^p - \delta_r^{+p} = t y_{ro}^p && r=1,\ldots,s, \quad p=1,\ldots,q\\
&t > 0,\\
&n_j^p, n_j'^p, \delta_i^{-p}, \delta_r^{+p}, \gamma_d^p, \delta_d^p \geq 0, && j=1,\ldots,n \quad i=1,\ldots,m \quad r=1,\ldots,s \quad d=1,\ldots,D \quad p=1,\ldots,q.
\end{aligned} \tag{3.2}$$

If $\left(\delta^{-p^*}, \gamma^{p^*}, \delta^{p^*}, \delta^{+p^*}, \eta^{p^*}, \eta'^{p^*}, t^*\right)$ is an optimal solution of model (3.2), then the optimal solution of model (3.1) is given as follows:

$$\left(s^{-p^*} = \frac{\delta^{-p^*}}{t^*}, t^{p^*} = \frac{\gamma^{p^*}}{t^*}, s^{p^*} = \frac{\delta^{p^*}}{t^*}, s^{+p^*} = \frac{\delta^{+p^*}}{t^*}, \lambda^{p^*} = \frac{\eta^{p^*}}{t^*}, \lambda'^{p^*} = \frac{\eta'^{p^*}}{t^*}\right).$$

In this case, the efficiencies of the system are obtained as follows:

$$\begin{aligned}
E_o^s &= \frac{\sum_{p=1}^{q}\left(1 - \frac{1}{m+D}\left(\sum_{i=1}^{m}\frac{s_i^{-p*}}{x_{io}^p} + \sum_{d=1}^{D}\frac{t_d^{p*}}{z_{do}^p}\right)\right)}{\sum_{p=1}^{q}\left(1 + \frac{1}{s+D}\left(\sum_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p} + \sum_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}\right)\right)}, &
E_o^{s(p)} &= \frac{1 - \frac{1}{m+D}\left(\sum_{i=1}^{m}\frac{s_i^{-p*}}{x_{io}^p} + \sum_{d=1}^{D}\frac{t_d^{p*}}{z_{do}^p}\right)}{1 + \frac{1}{s+D}\left(\sum_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p} + \sum_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}\right)}\\
E_o^{\mathrm{I}} &= \frac{\sum_{p=1}^{q}\left(1 - \frac{1}{m}\sum_{i=1}^{m}\frac{s_i^{-p*}}{x_{io}^p}\right)}{\sum_{p=1}^{q}\left(1 + \frac{1}{D}\sum_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}\right)}, &
E_o^{\mathrm{II}} &= \frac{\sum_{p=1}^{q}\left(1 - \frac{1}{D}\sum_{d=1}^{D}\frac{t_d^{p*}}{z_{do}^p}\right)}{\sum_{p=1}^{q}\left(1 + \frac{1}{s}\sum_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p}\right)}\\
E_o^{\mathrm{I}(p)} &= \frac{\left(1 - \frac{1}{m}\sum_{i=1}^{m}\frac{s_i^{-p*}}{x_{io}^p}\right)}{\left(1 + \frac{1}{D}\sum_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}\right)}, &
E_o^{\mathrm{II}(p)} &= \frac{1 - \frac{1}{D}\sum_{d=1}^{D}\frac{t_d^{p*}}{z_{do}^p}}{1 + \frac{1}{s}\sum_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p}}
\end{aligned} \tag{3.3}$$

wherein, E_o^s, E_o^{I} and E_o^{II} demonstrate the overall efficiency of the system and efficiency of stages 1 and 2, over the whole q time periods, respectively. Similarly, $E_o^{s(p)}$, $E_o^{I(p)}$ and $E_o^{\mathrm{II}(p)}$ indicate the overall efficiency and efficiencies of stage 1 and 2 in period p, respectively.

According to proposed model (1), we define the following definitions:

Definition 3.1. DMU_o is overall efficient over q time periods if and only if $E_o^s = 1$.

Definition 3.2. DMU_o is efficient in stage 1 of whole q time periods if and only if $E_o^{\mathrm{I}} = 1$.

Definition 3.3. DMU_o is efficient in stage 2 of whole q time periods if and only if $E_o^{\mathrm{II}} = 1$.

3.1. Properties of the proposed model

In this sub-section, we shall explain some of the main properties of the proposed model (3.1).

(1) Model (3.1) is unit invariant. *i.e.* if we displace $x_{ij}^p, z_{dj}^p, y_{rj}^p$ to $\alpha_i x_{ij}^p, \theta_d z_{dj}^p, \beta_r y_{rj}^p$ the efficiency of system and stages do not change.

(2) The overall efficiency and efficiency of stages in each period p and over q time periods are in the range (0, 1], *i.e.* we have:

$$0 < E_o^s \leq 1, 0 < E_o^{s(p)} \leq 1, 0 < E_o^{\mathrm{I}} \leq 1, 0 < E_o^{\mathrm{I}(p)} \leq 1, 0 < E_o^{\mathrm{II}} \leq 1, 0 < E_o^{\mathrm{II}(p)} \leq 1.$$

(3) $E_o^s = 1$ if and only if $s_i^{-p} = t_d^p = s_d^p = s_r^{+p} = 0, i = 1, \ldots, m,\ d = 1, \ldots, D,\ r = 1, \ldots, s,\ p = 1, \ldots, q.$

Proof 1. By considering the constraints of model (3.1), we have:

$$s_i^{-p} = x_{io}^p - \sum_{j=1}^n \lambda_j^p x_{ij}^p, \qquad t_d^p = z_{do}^p - \sum_{j=1}^n \lambda_j'^p z_{dj}^p$$
$$s_d^p = \sum_{j=1}^n \lambda_j^p z_{dj}^p - z_{do}^p, \qquad s_r^{+p} = \sum_{j=1}^n \lambda_j'^p y_{rj}^p - y_{ro}^p.$$

Now, for each DMU$_j$, if $x_{ij}^p, z_{dj}^p, y_{rj}^p$ are changed to $\alpha_i x_{ij}^p, \theta_d z_{dj}^p, \beta_r y_{rj}^p (\forall i, \forall d, \forall r)$, the constraints of model (3.1) are converted to the following forms:

$$\alpha_i x_{io}^p - \sum_{j=1}^n \lambda_j^p \alpha_i x_{ij}^p = \alpha_i \left(x_{ij}^p - \sum_{j=1}^n \lambda_j^p x_{ij}^p \right) = \alpha_i s_i^{-p}$$

$$\theta_d z_{do}^p - \sum_{j=1}^n \lambda_j'^p \theta_d z_{dj}^p = \theta_d \left(z_{do}^p - \sum_{j=1}^n \lambda_j'^p z_{dj}^p \right) = \theta_d t_d^p$$

$$\sum_{j=1}^n \lambda_j^p \theta_d z_{dj}^p - \theta_d z_{do}^p = \theta_d \left(\sum_{j=1}^n \lambda_j^p z_{dj}^p - z_{do}^p \right) = \theta_d s_d^p$$

$$\sum_{j=1}^n \lambda_j'^p \beta_r y_{rj}^p - \beta_r y_{ro}^p = \beta_r \left(\sum_{j=1}^n \lambda_j'^p y_{rj}^p - y_{ro}^p \right) = \beta_r s_r^{+p}.$$

Thus, we have:

$$\frac{s_i^{-p}}{x_{io}^p} = \frac{\alpha_i s_i^{-p}}{\alpha_i x_{io}^p}, \quad \frac{t_d^p}{z_{do}^p} = \frac{\theta_d t_d^p}{\theta_d z_{do}^p}, \quad \frac{s_d^p}{z_{do}^p} = \frac{\theta_d s_d^p}{\theta_d z_{do}^p}, \quad \frac{s_r^{+p}}{y_{ro}^p} = \frac{\beta_r s_r^{+p}}{\beta_r y_{ro}^p}.$$

Therefore, regarding the equation (3.3), the efficiency of system and stages will not change and the proof is complete. □

Proof 2. According to the constraints of model (3.1), we have:

$$\frac{s_i^{-p}}{x_{io}^p} \leq 1, \quad \frac{t_d^p}{z_{do}^p} \leq 1.$$

This means that:

$$\sum_{i=1}^m \frac{s_i^{-p}}{x_{io}^p} \leq m, \quad \sum_{d=1}^D \frac{t_d^p}{z_{do}^p} \leq D.$$

Therefore,

$$\sum_{i=1}^m \frac{s_i^{-p}}{x_{io}^p} + \sum_{d=1}^D \frac{t_d^p}{z_{do}^p} \leq m + D.$$

Now, since $0 \le \frac{1}{m+D}\left(\sum_{i=1}^{m}\frac{s_i^{-p}}{x_{io}^p}+\sum_{d=1}^{D}\frac{t_d^p}{z_{do}^p}\right) \le 1$ and $1+\frac{1}{s+D}\left(\sum_{r=1}^{s}\frac{s_i^{+p}}{y_{ro}^p}+\sum_{d=1}^{D}\frac{s_d^p}{z_{do}^p}\right) \ge 1$, we conclude that:

$$0 \le \sum_{p=1}^{q}\frac{1}{m+D}\left(\sum_{i=1}^{m}\frac{s_i^{-p}}{x_{io}^p}+\sum_{d=1}^{D}\frac{t_d^p}{z_{do}^p}\right) \le q$$
$$\frac{1}{\sum_{p=1}^{q} 1+\frac{1}{s+D}\left(\sum_{r=1}^{s}\frac{s_r^{+p}}{y_{ro}^p}+\sum_{d=1}^{D}\frac{s_d^p}{z_{do}^p}\right)} \le \frac{1}{q}.$$

Therefore,

$$0 < E_o^s = \frac{\sum_{p=1}^{q}\frac{1}{m+D}\left(\sum_{i=1}^{m}\frac{s_i^{-p}}{x_{io}^p}+\sum_{d=1}^{D}\frac{t_d^p}{z_{do}^p}\right)}{\sum_{p=1}^{q} 1+\frac{1}{s+D}\left(\sum_{r=1}^{s}\frac{s_r^{+p}}{y_{ro}^p}+\sum_{d=1}^{D}\frac{s_d^p}{z_{do}^p}\right)} \le 1.$$

The other efficiencies are proved similarly. □

Proof 3. Suppose $E_o^s = 1$. In this case:

$$\sum_{p=1}^{q}\left(1-\frac{1}{m+D}\left(\sum_{i=1}^{m}\frac{s_i^{-p*}}{x_{io}^p}+\sum_{d=1}^{D}\frac{t_d^{p*}}{z_{do}^p}\right)\right) = \sum_{p=1}^{q}\left(1+\frac{1}{s+D}\left(\sum_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p}+\sum_{d=1}^{D}\frac{s_d^{p^*}}{z_{do}^p}\right)\right)$$

Therefore, we have $s_i^{-p} = t_d^p = s_d^p = s_r^{+p} = 0, \quad p = 1,\dots,q.$

Conversely, if, $s_i^{-p^*} = t_d^{p*} = s_d^{p*} = s_r^{+p*} = 0,\ i = 1,\dots,m,\ d = 1,\dots,D, r = 1,\dots,s$, according to the objective function of model (3.1), we have $E_o^s = 1$ and the proof is complete. □

Lemma 3.4. *For each* DMU_o*,* $E_o^s = 1$ *if and only if* $E_o^{\text{I}} = E_o^{\text{II}} = 1$*.*

Proof. Assume that

$$E_o^s = \frac{\sum_{p=1}^{q}\left(1-\frac{1}{m+D}\left(\sum_{i=1}^{m}\frac{s_i^{-p*}}{x_{io}^p}+\sum_{d=1}^{D}\frac{t_d^{p*}}{z_{do}^p}\right)\right)}{\sum_{p=1}^{q}\left(1+\frac{1}{s+D}\left(\sum_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p}+\sum_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}\right)\right)} = 1.$$

Thus with regard to the Property (3), we have:

$$s_i^{-p^*} = t_d^{p^*} = s_r^{+p^*} = s_d^{p^*} = 0, p = 1,\dots,q.$$

This means that $E_o^{\text{I}} = E_o^{\text{II}} = 1$.

Conversely, if $E_o^{\text{I}} = E_o^{\text{II}} = 1$, then we have $s_i^{-p^*} = t_d^{p^*} = s_r^{+p^*} = s_d^{p^*} = 0, p = 1,\dots,q$. Therefore, based on Property (3), we conclude that $E_o^s = 1$ and the proof is complete. □

Lemma 3.5. *For each* DMU_o*,* $E_o^{s(p)} = 1$ *if and only if* $E_o^{I(p)} = E_o^{\text{II}(p)} = 1$*.*

Proof. Let

$$E_o^{s(p)} = \frac{1-\frac{1}{m+D}\left(\sum_{i=1}^{m}\frac{s_i^{-p^*}}{x_{io}^p}+\sum_{d=1}^{D}\frac{t_d^{p^*}}{z_{do}^p}\right)}{1+\frac{1}{s+D}\left(\sum_{r=1}^{s}\frac{s_r^{+p^*}}{y_{ro}^p}+\sum_{d=1}^{D}\frac{s_d^{p^*}}{z_{do}^p}\right)} = 1.$$

Thus, with regard to Property (3), for each period p, we have

$$s_i^{-p^*} = t_d^{p^*} = s_r^{+p^*} = s_d^{p^*} = 0.$$

Therefore, according to the definitions of the stages efficiency, we have $E_o^{\mathrm{I}(p)} = E_o^{\mathrm{II}(p)} = 1$.

Conversely, if $E_o^{\mathrm{I}} = E_o^{\mathrm{II}} = 1$, we have $s_i^{-p^*} = s_d^{p^*} = t_d^{p^*} = s_r^{+p^*} = 0$. Hence, according to Property (3), $E_o^{s(p)} = 1$ and proof is complete. □

3.2. Decomposition of efficiencies

In this sub-section, we shall present the relationships between the overall efficiency and efficiencies of stages in each period and over whole time periods.

In order to communicate between the overall efficiency of system over time periods and overall efficiency of each period, we define weight $\omega^{(p)}$ as ratio output overall efficiency score of period p to the output overall efficiency score of q time periods:

$$\omega^{(p)} = \frac{1 + \frac{1}{s+D}\left(\sum\limits_{r=1}^{s} \frac{s_r^{+p*}}{y_{ro}^{p}} + \sum\limits_{d=1}^{D} \frac{s_d^{p^*}}{z_{do}^{p}}\right)}{\sum\limits_{p=1}^{q}\left(1 + \frac{1}{s+D}\left(\sum\limits_{r=1}^{s} \frac{s_r^{+p*}}{y_{ro}^{p}} + \sum\limits_{d=1}^{D} \frac{s_d^{p^*}}{z_{do}^{p}}\right)\right)}.$$

These weights are functions of the optimization variables. Note that $\omega^{(p)}$ is intended to represent the relative importance or contribution of the performance of each period to the over time periods. It must be noted that $\sum_{p=1}^{q} \omega^{(p)} = 1$.

Therefore, the overall efficiency of system over q time periods is defined as the weighted average of the efficiency of each period p as follows:

$$E_o^{s} = \sum_{p=1}^{q} \omega^{(p)} E_o^{s(p)}.$$

Now, we have

$$\begin{aligned}
\sum_{p=1}^{q} \omega^{(p)} E_o^{s(p)} &= \sum_{p=1}^{q} \left(\frac{1 + \frac{1}{s+D}\left(\sum\limits_{r=1}^{s} \frac{s_r^{+p*}}{y_{ro}^{p}} + \sum\limits_{d=1}^{D} \frac{s_d^{p*}}{z_{do}^{p}}\right)}{\sum\limits_{p=1}^{q}\left(1 + \frac{1}{s+D}\left(\sum\limits_{r=1}^{s} \frac{s_r^{+p*}}{y_{ro}^{p}} + \sum\limits_{d=1}^{D} \frac{s_d^{p*}}{z_{do}^{p}}\right)\right)} \right) \times \left(\frac{1 - \frac{1}{m+D}\left(\sum\limits_{i=1}^{m} \frac{s_i^{-p^*}}{x_{io}^{p}} + \sum\limits_{d=1}^{D} \frac{t_d^{p^*}}{z_{do}^{p}}\right)}{1 + \frac{1}{s+D}\left(\sum\limits_{r=1}^{s} \frac{s_r^{+p*}}{y_{ro}^{p}} + \sum\limits_{d=1}^{D} \frac{s_d^{p*}}{z_{do}^{p}}\right)} \right) \\
&= \left(\frac{1}{\sum\limits_{p=1}^{q}\left(1 + \frac{1}{s+D}\left(\sum\limits_{r=1}^{s} \frac{s_r^{+p*}}{y_{ro}^{p}} + \sum\limits_{d=1}^{D} \frac{s_d^{p*}}{z_{do}^{p}}\right)\right)} \right) \times \sum_{p=1}^{q} \left(1 - \frac{1}{m+D}\left(\sum_{i=1}^{m} \frac{s_i^{-p*}}{x_{io}^{p}} + \sum_{d=1}^{D} \frac{t_d^{p*}}{z_{do}^{p}}\right)\right) \\
&= \frac{\sum\limits_{p=1}^{q}\left(1 - \frac{1}{m+D}\left(\sum\limits_{i=1}^{m} \frac{s_i^{-p*}}{x_{io}^{p}} + \sum\limits_{d=1}^{D} \frac{t_d^{p*}}{z_{do}^{p}}\right)\right)}{\sum\limits_{p=1}^{q}\left(1 - \frac{1}{s+D}\left(\sum\limits_{r=1}^{s} \frac{s_r^{+p*}}{y_{ro}^{p}} + \sum\limits_{d=1}^{D} \frac{s_d^{p*}}{z_{do}^{p}}\right)\right)} = E_o^{s}.
\end{aligned}$$

Similarly, the relationships between the efficiencies of stages over q time periods and each period p can be concluded easily as follows:

$$E_o^{\mathrm{I}} = \sum_{p=1}^{q} \omega^{\mathrm{I}(p)} E_o^{\mathrm{I}(p)}, \quad \text{where} \quad \omega^{\mathrm{I}(p)} = \frac{1+\frac{1}{D}\sum\limits_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}}{\sum\limits_{p=1}^{q}\left(1+\frac{1}{D}\sum\limits_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}\right)}$$

$$E_o^{\mathrm{II}} = \sum_{p=1}^{q} \omega^{\mathrm{II}(p)} E_o^{\mathrm{II}(p)}, \quad \text{where} \quad \omega^{\mathrm{II}(p)} = \frac{1+\frac{1}{s}\sum\limits_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p}}{\sum\limits_{p=1}^{q}\left(1+\frac{1}{s}\sum\limits_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p}\right)}.$$

Here, $\omega^{\mathrm{I}(p)}$, $\omega^{\mathrm{II}(p)}$ are defined as ratio output efficiency score of period p to output efficiency score of q time periods in stages 1 and 2, respectively. It is clear that $\sum_{p=1}^{q}\omega^{\mathrm{I}(p)}=1, \sum_{p=1}^{q}\omega^{\mathrm{II}(p)}=1$.

Our argument is that the importance of a stage as measured by its weight. Note that in formula $\omega^{\mathrm{I}(p)}$, the value of $1+\frac{1}{D}\sum_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p} = \frac{1}{D}\sum_{d=1}^{D}\left(\frac{z_{do}^p+s_d^{p*}}{z_{do}^p}\right)$ is the mean proportional rate of output expansion of period p and the value of $\sum_{p=1}^{q}\left(1+\frac{1}{D}\sum_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}\right)$ represent the total mean proportional rate of outputs expansion in q time periods in the first stage. The same interpretation for $\omega^{\mathrm{II}(p)}$ can be expressed.

These relations are obtained as follows:

$$\sum_{p=1}^{q}\omega^{\mathrm{I}(p)}E_o^{\mathrm{I}(p)} = \sum_{p=1}^{q}\left(\frac{1+\frac{1}{D}\sum\limits_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}}{\sum\limits_{p=1}^{q}\left(1+\frac{1}{D}\sum\limits_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}\right)}\right)\times\left(\frac{1-\frac{1}{m}\sum\limits_{i=1}^{m}\frac{s_i^{-p*}}{x_{io}^p}}{1+\frac{1}{D}\sum\limits_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}}\right) = \frac{1}{\sum\limits_{p=1}^{q}\left(1+\frac{1}{D}\sum\limits_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}\right)}$$

$$\times\left(\sum_{p=1}^{q}\left(1-\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^{-p*}}{x_{io}^p}\right)\right) = \frac{\sum\limits_{p=1}^{q}\left(1-\frac{1}{m}\sum\limits_{i=1}^{m}\frac{s_i^{-p*}}{x_{io}^p}\right)}{\sum\limits_{p=1}^{q}\left(1+\frac{1}{D}\sum\limits_{d=1}^{D}\frac{s_d^{p*}}{z_{do}^p}\right)} = E_o^{\mathrm{I}}$$

$$\sum_{p=1}^{q}\omega^{\mathrm{II}(p)}E_o^{\mathrm{II}(p)} = \sum_{p=1}^{q}\left(\frac{1+\frac{1}{s}\sum\limits_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p}}{\sum\limits_{p=1}^{q}\left(1+\frac{1}{s}\sum\limits_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p}\right)}\right)\times\left(\frac{1-\frac{1}{D}\sum\limits_{d=1}^{D}\frac{t_d^{p*}}{z_{do}^p}}{1+\frac{1}{s}\sum\limits_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p}}\right)$$

$$= \frac{1}{\sum\limits_{p=1}^{q}\left(1+\frac{1}{s}\sum\limits_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p}\right)}\times\left(\sum_{p=1}^{q}\left(1-\frac{1}{D}\sum_{d=1}^{D}\frac{t_d^{p*}}{z_{do}^p}\right)\right) = \frac{\sum\limits_{p=1}^{q}\left(1-\frac{1}{D}\sum\limits_{d=1}^{D}\frac{t_d^{p*}}{z_{do}^p}\right)}{\sum\limits_{p=1}^{q}\left(1+\frac{1}{s}\sum\limits_{r=1}^{s}\frac{s_r^{+p*}}{y_{ro}^p}\right)} = E_o^{\mathrm{II}}.$$

Therefore, by using these decompositions, we conclude that a multi-period two-stage system is efficient if and only if it is efficient in all the periods. Also, a multi-period two-stage system is efficient in the stage under consideration if and only if it is efficient in the same stage for all periods. Such decompositions of efficiencies are useful for identifying the inefficiencies of the sources in the system and internal structures. Moreover, they help us to compare the efficiency of a period of a DMU, with other DMUs. Finally, by using the obtained decompositions and according to Property (3), Lemmas 3.4 and 3.5, we get the following theorem:

Theorem 3.6. *A multi-period two-stage system is efficient if and only if all its stages are efficient in the all periods.*

4. Case study

One of the most common applications of NDEA is the assessment of the efficiency of bank branches. In recent years, many scholars have focused on this topic. Hence, in this section, we will use the data of 20 Mellat bank branches in Tehran for three consecutive years (2011–2013) [20] to illustrate the proposed model (3.1). Model (3.1) calculates the stages efficiency and overall efficiency of multi-period two-stage systems over whole time periods and each period which have been introduced in Section 3. In this evaluation, we use two inputs "Personal score", "Paid profit", two intermediate measures "Total of four deposits", "Other resources" and three good outputs "Facilities", "Received handling fee", "Earned profit", and one bad output "Deferred claims". Note that in this application, each bank branch viewed as a DMU with multi-period two-stage structure. The obtained results are reported in Table 1.

Remark 4.1. It should be noted that, Iranian banks have very low capital, when they are established. Therefore, they practically cannot do business with their own capital. That's why the bank's activities are divided into two stages. In the first stage, the bank is trying to attract resources (different deposits), and in the second stage, these deposits are used for investment and earning money. The "Personnel score" includes the number of personnel, level of training, work experience, executive and job positions which consist of the weighted and normalized sum of these factors. Moreover, weights are determined by the bank experts. The Bank's staff strives more to attract resources and provide facilities. The higher the personnel score, the more resources are expected to be attracted. The higher personnel score indicates more cost of the branches. Note that instead of salary and wage, personnel score is applied in this case study.

Remark 4.2. The "Resources" (or Deposits) in the bank have many different types. Debt saving account, current debt saving account, short and long term accounts are defined as the four main deposits in the bank and other deposits such as housing deposits, youth deposits and etc. are named as "Other resources". All resources and deposits are the result of the Bank's first stage effort to get budget as an investment tool in order to get income. That is, the bank branches should use intermediate products to invest in an effective manner.

Remark 4.3. One of the main activities of the banks is providing loans and granting facilities to customers. Some customers do not repay received loans or pay late. The sum of all loans that customers do not pay on time to the bank is called "Deferred claims", which is a bad output.

The first and the second columns in Table 1, show the number of each DMU and the number of the period, respectively. And also, the columns 3, 4, 5 in Table 1, indicate the efficiency of stages 1, 2 and the overall efficiency in each period and whole time periods, respectively. Corresponding to each period the sixth, seventh and eighth columns demonstrate the weights of the two stages and whole system which are introduced in Section 3.2.

According to the results of Table 1, in whole time periods, there is no efficient DMU in stage 1 and stage 2. Hence, as it is clear, all of DMUs are inefficient in whole system. DMU_{18} and DMU_{16} have the highest and the lowest efficiency in stage 2 and whole system, respectively. The best and the worst efficiency in stage 1 belongs to DMU_{16} and DMU_1 with scores 0.8896, 0.0288, respectively.

In period 1, DMUs 5, 12, 15 and 18 in stage 1 and DMUs 1, 5, 10, 13, 18 and 19 in stage 2 are efficient. Hence, DMU_5 and DMU_{18} are overall efficient. Among the inefficient DMUs, the highest efficiency in stage 1 and in stage 2 belongs to DMU_4 and DMU_2 with scores 0.8798, 0.4798, respectively. In stage 1, the lowest efficiency belongs toDMU_{10}. And also, DMU_{16} has the lowest efficiency in stage 2 and whole system with scores 0.0098 and 0.0129, respectively. Moreover, DMU_9 with score 0.8087 has the best efficiency in whole system.

DMUs 8, 15 and 16 in stage 1 and DMUs 1, 6, 7, 8, 9, 14, 15, 17 and 18 in stage 2, are efficient in period 2. Other DMUs are inefficient in these stages of period 2. Note that DMU_8 and DMU_{15} are efficient in stage 1 and stage 2. Hence, these DMUs are overall efficient in period 2. Also, DMU_{11} has the highest efficiency in stage 1 and stage 2 with scores 0.9782, 0.5410, respectively. Moreover, DMU_{17} and DMU_{16} have the lowest efficiency

TABLE 1. The stages and overall efficiencies.

DMU	Period	Eff. of Stage 1	Eff. of Stage 2	Overall Eff.	W. of Stage 1	W. of Stage 2	W. of Whole system
1	1	0.0162	1	0.0472	0.9352	0.076	0.6898
	2	0.2332	1	0.6142	0.0156	0.076	0.0328
	3	0.2032	0.0185	0.0836	0.0492	0.848	0.2773
	Whole period	0.0288	0.2102	0.0759	–	–	–
2	1	0.8236	0.4798	0.6418	0.0447	0.0255	0.0296
	2	0.5891	0.0242	0.029	0.0447	0.9504	0.7574
	3	0.0449	1	0.1284	0.9106	0.0242	0.213
	Whole period	0.104	0.0594	0.0683	–	–	–
3	1	0.1934	0.4035	0.2946	0.7211	0.1136	0.2449
	2	0.5445	0.087	0.0993	0.1395	0.8095	0.6647
	3	0.6812	1	0.8406	0.1395	0.0769	0.0904
	Whole period	0.3104	0.1931	0.2142	–	–	–
4	1	0.8798	0.1889	0.2674	0.1455	0.4741	0.3821
	2	0.8613	0.2389	0.3135	0.1455	0.4126	0.3378
	3	0.1741	1	0.4034	0.7089	0.1133	0.2801
	Whole period	0.3768	0.3014	0.321	–	–	–
5	1	1	1	1	0.0499	0.1234	0.0827
	2	0.0544	0.1392	0.0902	0.8813	0.6611	0.7828
	3	0.5409	0.3325	0.4085	0.0689	0.2155	0.1344
	Whole period	0.1351	0.287	0.2083	–	–	–
6	1	0.4086	0.0786	0.1254	0.4686	0.6621	0.6309
	2	0.8132	1	0.9066	0.1915	0.0737	0.0927
	3	0.3261	0.0662	0.2016	0.34	0.2642	0.2765
	Whole period	0.458	0.1717	0.2188	–	–	–
7	1	0.7467	0.0498	0.0637	0.1862	0.8906	0.8156
	2	0.4735	1	0.7307	0.3501	0.0444	0.0769
	3	0.1935	0.2581	0.3087	0.4637	0.0651	0.1075
	Whole period	0.3945	0.1168	0.1413	–	–	–
8	1	0.4658	0.0829	0.1207	0.1836	0.8113	0.6171
	2	1	1	1	0.1053	0.0944	0.0977
	3	0.1422	1	0.3359	0.7111	0.0944	0.2851
	Whole period	0.2919	0.2559	0.268	–	–	–
9	1	0.6145	1	0.8087	0.1931	0.2541	0.2293
	2	0.2817	1	0.5518	0.5689	0.2541	0.382
	3	0.5289	0.4136	0.4304	0.2381	0.4918	0.3887
	Whole period	0.4048	0.7116	0.5635	–	–	–
10	1	0.0134	1	0.0405	0.9721	0.0782	0.7368
	2	0.7824	0.105	0.1352	0.014	0.7206	0.1999
	3	1	0.0714	0.3674	0.014	0.2012	0.0632
	Whole period	0.0379	0.1934	0.0801	–	–	–
11	1	0.2398	0.0241	0.0359	0.6071	0.8214	0.8066
	2	0.9782	0.541	0.6447	0.1965	0.0505	0.0606
	3	1	0.0447	0.2519	0.1964	0.1281	0.1328
	Whole period	0.5342	0.0657	0.1015	–	–	–
12	1	1	0.3949	0.5408	0.0595	0.391	0.1833
	2	0.0664	0.4777	0.1558	0.8811	0.4096	0.705
	3	0.6594	1	0.8297	0.0595	0.1994	0.1117
	Whole period	0.1572	0.5495	0.3016	–	–	–
13	1	0.077	1	0.2002	0.6433	0.1601	0.4586
	2	0.2796	0.2786	0.2981	0.1769	0.3891	0.258
	3	0.2033	0.1081	0.2431	0.1799	0.4507	0.2834
	Whole period	0.1355	0.3907	0.2376	–	–	–
14	1	0.6025	0.0562	0.0683	0.0588	0.8884	0.6214
	2	0.0666	1	0.1763	0.8825	0.0558	0.8825
	3	1	1	1	0.0588	0.0558	0.0567
	Whole period	0.1529	0.1615	0.156	–	–	–
15	1	1	0.1641	0.273	0.0318	0.6717	0.2104
	2	1	1	1	0.0318	0.1642	0.0687
	3	0.0339	1	0.0954	0.9364	0.1642	0.7208

Table 1. Continued.

DMU	Period	Eff. of Stage 1	Eff. of Stage 2	Overall Eff.	W. of Stage 1	W. of Stage 2	W. of Whole system
	Whole period	0.0954	0.4385	0.1949	–	–	–
16	1	0.6689	0.0098	0.0129	0.3333	0.6331	0.6299
	2	1	0.0174	0.0356	0.3333	0.2322	0.2333
	3	1	0.0089	0.0636	0.3333	0.1347	0.1368
	Whole period	0.8896	0.0148	0.0251	–	–	–
17	1	0.7574	0.0126	0.0175	0.0503	0.9721	0.8598
	2	0.0507	1	0.1439	0.8993	0.014	0.1218
	3	0.5699	1	0.785	0.0503	0.014	0.0184
	Whole period	0.1124	0.0402	0.047	–	–	–
18	1	1	1	1	0.159	0.2373	0.2038
	2	0.4233	1	0.6944	0.298	0.2373	0.2632
	3	0.2927	0.1717	0.3825	0.5431	0.5253	0.5329
	Whole period	0.4441	0.712	0.5905	–	–	–
19	1	0.1627	1	0.3683	0.6782	0.1701	0.3912
	2	0.6354	0.2577	0.28	0.1104	0.6599	0.4207
	3	0.3118	1	0.6119	0.2114	0.1701	0.1881
	Whole period	0.2464	0.5102	0.377	–	–	–
20	1	0.6689	0.0391	0.048	0.3333	0.8398	0.816
	2	0.6354	0.2577	0.28	0.3333	0.1274	0.1371
	3	0.5699	1	0.785	0.3333	0.0328	0.0469
	Whole period	0.6248	0.0985	0.1144	–	–	–

with scores 0.0507 and 0.0174 in stage 1 and stage 2, respectively. The best efficiency and the worst efficiency in whole system belongs to DMU_6 and DMU_2 with scores 0.9066 and 0.0290, respectively.

In period 3, DMUs 10, 11, 14 and 16 in stage 1 and DMUs 2, 3, 4, 8, 12, 14, 15, 17, 19 and 20 in stage 2, are efficient. Hence, DMU_{14} is overall efficient and all of other DMUs are overall inefficient. Among the inefficient DMUs, DMU_3 has the best efficiency in stage 1 and whole system with scores 0.6812 and 0.8406, respectively. DMU_{15} has the lowest efficiency in stage 1. Also, the worst efficiency in stage 2 and whole system belongs toDMU_{16}. In stage 2, the highest efficiency belongs to DMU_9 with score 0.4136. Finally, for more illustration of the presented decompositions in Section 3.2, we investigate these decompositions for DMU_8 as follows:

$$E_o^s = \omega^{(1)}E_o^{s(1)} + \omega^{(2)}E_o^{s(2)} + \omega^{(3)}E_o^{s(3)} = ((0.6171 \times 0.1207) + (0.0977 \times 1) + (0.2851 \times 0.3359)) = 0.2680$$
$$E_o^{\mathrm{I}} = \omega^{\mathrm{I}(1)}E_o^{\mathrm{I}(1)} + \omega^{\mathrm{I}(1)}E_o^{\mathrm{I}(2)} + \omega^{\mathrm{I}(1)}E_o^{\mathrm{I}(3)} = ((0.1836 \times 0.4658) + (0.1053 \times 1) + (0.7111 \times 0.1422)) = 0.2919$$
$$E_o^{\mathrm{II}} = \omega^{\mathrm{II}(1)}E_o^{\mathrm{II}(1)} + \omega^{\mathrm{II}(1)}E_o^{\mathrm{II}(2)} + \omega^{\mathrm{II}(1)}E_o^{\mathrm{II}(3)} = ((0.8113 \times 0.0829) + (0.0944 \times 1) + (0.0944 \times 1)) = 0.2559.$$

Note that the utilized weights are extracted from columns 6, 7 and 8 of Table 1. Also, DMU_{14} is efficient in stage 2 of periods 2 and 3. Therefore, as it is clear from Table 1 the weights of these periods are identical. Also, DMU_9 is efficient in stage 2 of periods 1 and 2. Hence, the corresponding weights of this DMU in these periods are 0.2541.

Kao and Hwang [28], proposed a radial model to measure the efficiency of multi-period two-stage systems with the multiplier form of DEA. Hence, in order to compare the results of our non-radial model with their model, we applied the data of bank to their model. For example, in the proposed method in period 1, we see that DMU_5 and DMU_{18} are efficient. But using the radial method, all DMUs are overall inefficient. In period 2 of both methods, DMU_{16} is efficient in stage 1. Also, in period 3, DMU_3 has the best overall efficiency in our model but in their model the best efficiency belongs to DMU_6. Actually, these two models cannot be fairly comparable. But as we see in the example, we can find gaps among two models, which must be caused by the difference between the radial and non-radial assumption.

Since in the most branches of bank the second stage performance is better than the first stage, this means that in part of resource absorption, it performs fewer attempts than resource allocation. In other words, in the field of collecting deposits, due to more interest paid (profit paid) to the depositors, the performance declines in the first stage. On the other hand, due to the high demand of customers for high profitable facilities (bank loans), the bank is relatively successful in the second stage and has better performance.

Based on these results, it is suggested in order to increase the efficiency of the first stage and the overall efficiency, it is necessary for the bank to strive to attract low-cost (low-paid) resources. The bank's policies should be to increase shareholders' equity, attract cheap resources, and raise fees by providing more customer service.

5. Conclusions

In the real world, many of the systems have two-stage structure. The traditional two-stage DEA models measure the efficiency of two-stage systems at the certain time, while the calculation of the efficiency of these systems is especially important during the multi-period time. In this paper, we proposed a model to measure the overall efficiency and efficiency of the stages of multi-period two-stage systems with slacks-based measure, so that the relative importance of data is taken into account in different time periods. Then some of the properties of this model was described .The proposed model decomposed the overall efficiency of the system, over whole time periods, into a weighted average of the overall efficiencies of each period. Furthermore, the efficiencies of stages over total periods were decomposed into the weighted average of the stages efficiencies in each period. Finally, we used a case of 20 Mellat bank branches in Tehran to illustrate the suggested model.

This work can be extended along at least three directions. First, one can generalize the proposed model to determine the type of return to scale of each stage in period p and over whole time period q. Second, further research on extending the proposed method to measure the efficiency of the same systems in the presence of undesirable outputs is an interesting stream of future research. Finlay, the proposed model does not consider the situation when input and output data are fuzzy, which may often be seen in real-world applications. Further research may consider this problem, and extend the fuzzy DEA models to the proposed multi-period two-stage slacks-based DEA model.

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