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REMARKS ON PATH FACTORS IN GRAPHS

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Abstract. A spanning subgraph of a graph is defined as a path factor of the graph if its component are paths. A $P_{\geq n}$-factor means a path factor with each component having at least n vertices. A graph G is defined as a $(P_{\geq n}, m)$-factor deleted graph if $G - E'$ has a $P_{\geq n}$-factor for every $E' \subseteq E(G)$ with $|E'| = m$. A graph G is defined as a $(P_{\geq n}, k)$-factor critical graph if after deleting any k vertices of G the remaining graph of G admits a $P_{\geq n}$-factor. In this paper, we demonstrate that (i) a graph G is $(P_{\geq 3}, m)$-factor deleted if $\kappa(G) \geq 2m + 1$ and $bind(G) \geq \frac{3}{2} - \frac{1}{4m+4}$; (ii) a graph G is $(P_{\geq 3}, k)$-factor critical if $\kappa(G) \geq k + 2$ and $bind(G) \geq \frac{5+k}{4}$.

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1. Introduction

All graphs considered in this paper are finite simple graphs. Let G be a graph. We denote by $V(G)$ and $E(G)$ its vertex set and edge set, respectively. For $x \in V(G)$, we use $N_G(x)$ to denote the set of vertices adjacent to x in G, and $d_G(x) = |N_G(x)|$ is the degree of x in G. For a vertex subset X of G, we denote by $N_G(X)$ the union of $N_G(x)$ for each $x \in X$ and by $G[X]$ the subgraph of G induced by X. Let $G - X = G[V(G) \setminus X]$. For $E' \subseteq E(G)$, the graph obtained from G by deleting edges of E' is denoted by $G - E'$. The number of connected components and isolated vertices in G are denoted by $\omega(G)$ and $i(G)$, respectively. We use $\lambda(G)$ and $\kappa(G)$ to denote the edge connectivity and the vertex connectivity of G, respectively. The binding number $bind(G)$ of G is defined as

$$bind(G) = \min \left\{ \frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G) \right\}.$$

Let n be an integer with $n \geq 2$. We use P_n to denote the path with n vertices and $n - 1$ edges. A spanning subgraph F of G is defined as a path factor of G if each component of F is a path. A $P_{\geq n}$-factor means a path factor with each component having at least n vertices. A graph G is defined as a $(P_{\geq n}, m)$-factor deleted graph if $G - E'$ has a $P_{\geq n}$-factor for every $E' \subseteq E(G)$ with $|E'| = m$. It is easy to see that a $(P_{\geq n}, 0)$-factor deleted graph admits a $P_{\geq n}$-factor. If $m = 1$, then a $(P_{\geq n}, m)$-factor deleted graph is simply called a $P_{\geq n}$-factor deleted graph. A graph G is defined as a $(P_{\geq n}, k)$-factor critical graph if after deleting any k vertices of G the remaining graph of G admits a $P_{\geq n}$-factor. Obviously, a $(P_{\geq n}, 0)$-factor critical graph has a $P_{\geq n}$-factor.

Keywords. Graph, path factor, binding number, connectivity.

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A spanning subgraph F of a graph G is said to be a 1-factor of G if $d_F(x) = 1$ for every $x \in V(G)$. A graph R is defined as a factor-critical graph if $R - \{x\}$ has a 1-factor for every $x \in V(R)$. A graph H is defined as a sun if $H = K_1$, $H = K_2$ or H is the corona of a factor-critical graph R with order at least 3, i.e., H is obtained from R by adding a new vertex $w = w(v)$ together with a new edge vw for any $v \in V(R)$. A sun with at least six vertices is called a big sun. A component of G is said to be a sun component if it is isomorphic to a sun. We denote by $sun(G)$ the number of sun components of G.

Kaneko [7] presented a necessary and sufficient condition for a graph to admit a $P_{\geq 3}$-factor. Kano *et al.* [8] gave a simpler proof.

Theorem 1.1 (Kaneko [7] and Kano *et al.* [8])**.** *A graph G has a $P_{\geq 3}$-factor if and only if $sun(G - S) \leq 2|S|$ for any $S \subseteq V(G)$.*

A claw is a graph isomorphic to $K_{1,3}$. A graph is called a claw-free graph if it does not contain induced claw. Kelmans [10] showed the following results on $\{P_3\}$-factors in claw-free graphs.

Theorem 1.2 (Kelmans [10])**.** *Let G be a 2-connected claw-free graph of order n. If $n \equiv 1$ (mod 3), then $G - x$ has a $\{P_3\}$-factor for any $x \in V(G)$.*

Theorem 1.3 (Kelmans [10])**.** *Let G be a 2-connected claw-free graph of order n. If $n \equiv 0$ (mod 3), then $G - e$ has a $\{P_3\}$-factor for any $e \in E(G)$.*

Note that $\{x\}$ is a subset of $V(G)$ for any $x \in V(G)$, and $\{e\}$ is an subset of $E(G)$ for any $e \in E(G)$. Naturally, motivated by the above theorems, we consider the more general problem.

Problem. Find sufficient conditions for a graph to be a $(P_{\geq 3}, m)$-factor critical/deleted graph.

Many results on the binding number conditions for the existence of graph factors were acquired by Nam [12], Plummer and Saito [13], Zhou [17,18], Robertshaw and Woodall [14]. Some results on factor deleted graphs see [4,5,19], and some results on factor critical graphs see [2,3,15,16,21]. Many authors [1,6,9,11,20,22,23] studied the existence of path factors in graphs. In this paper, we study the existence of $(P_{\geq 3}, m)$-factor deleted graphs and $(P_{\geq 3}, k)$-factor critical graphs and obtain some sufficient conditions for graphs to be $(P_{\geq 3}, m)$-factor deleted graphs and $(P_{\geq 3}, k)$-factor critical graphs depending on binding numbers, which are shown in Sections 2 and 3. Furthermore, our main results in this paper give solutions for the problem above.

2. $(P_{\geq 3}, m)$-FACTOR DELETED GRAPHS

Theorem 2.1. *Let m be a nonnegative integer, and let G be a graph. If $\kappa(G) \geq 2m+1$ and $bind(G) \geq \frac{3}{2} - \frac{1}{4m+4}$, then G is a $(P_{\geq 3}, m)$-factor deleted graph.*

If $m = 0$ in Theorem 2.1, then we get the following corollary.

Corollary 2.2. *Let G be a graph with $\kappa(G) \geq 1$. If $bind(G) \geq \frac{5}{4}$, then G admits a $P_{\geq 3}$-factor.*

If $m = 1$ in Theorem 2.1, then we get the following corollary.

Corollary 2.3. *Let G be a graph with $\kappa(G) \geq 3$. If $bind(G) \geq \frac{11}{8}$, then G is a $P_{\geq 3}$-factor deleted graph.*

Remark 2.4. In the following, we show that the conditions $\kappa(G) \geq 2m + 1$ and $bind(G) \geq \frac{3}{2} - \frac{1}{4m+4} = \frac{6m+5}{4m+4}$ in Theorem 2.1 can not be replaced by $\kappa(G) \geq 2m$ and $bind(G) \geq \frac{6m+5}{4m+5}$.

Let m be an nonnegative integer with $m \leq 2$. Set $G = K_{2m} \vee (2m+3)K_2$, (see Fig. 1), where $\vee$ denotes "join". For any $x \in V((2m+3)K_2)$, we write $X = V((2m+3)K_2) \setminus \{x\}$. It is obvious that $\kappa(G) = 2m$ and $bind(G) = \frac{|N_G(X)|}{|X|} = \frac{6m+5}{4m+5}$. For any $E' \subseteq E((2m+3)K_2)$ with $|E'| = m$, we write $G' = G - E'$. We choose $S = V(K_{2m}) \subseteq V(G')$. Thus, we have

$$sun(G' - S) = 2m + (m + 3) = 3m + 3 > 4m = 2|S|.$$

In terms of Theorem 1.1, G' has no $P_{\geq 3}$-factor, that is, G is not a $(P_{\geq 3}, m)$-factor deleted graph.

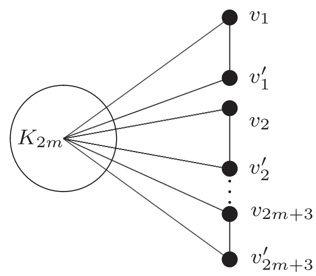

FIGURE 1. $K_{2m} \vee (2m+3)K_2$.

Proof of Theorem 2.1. Let $G' = G - E'$ for any $E' \subseteq E(G)$ with $|E'| = m$. Obviously, $V(G') = V(G)$ and $E(G') = E(G) \setminus E'$. In order to verify Theorem 2.1, we only need to prove that G' has a $P_{\geq 3}$-factor. By contradiction, suppose that G' has no $P_{\geq 3}$-factor. Then by Theorem 1.1, there exists some subset S of $V(G')$ such that

$$sun(G' - S) > 2|S|. \tag{2.1}$$

If $S = \emptyset$, then from (2.1) we have

$$sun(G') \geq 1. \tag{2.2}$$

On the other hand, note that $\lambda(G) \geq \kappa(G) \geq 2m+1$, $|E'| = m$ and $G' = G - E'$. Thus, we obtain

$$sun(G') \leq \omega(G') = 1. \tag{2.3}$$

In terms of (2.2) and (2.3), we have

$$sun(G') = \omega(G') = 1. \tag{2.4}$$

Since $\lambda(G) \geq \kappa(G) \geq 2m+1$, $|E'| = m$ and $G' = G - E'$, we have $\lambda(G') \geq m+1$ and $|V(G')| = |V(G)| \geq 2m+2$.

If $m \geq 1$, then $\lambda(G') \geq 2$, and so $sun(G') = 0$, which contradicts (2.4).

If $m = 0$, then $G = G'$. Combining this with $\lambda(G') \geq 1$ and (2.4), we obtain that $G = K_2$ or G is a big sun. If $G = K_2$, then it easy to see that $bind(G) = 1$, which contradicts $bind(G) \geq \frac{3}{2} - \frac{1}{4m+4}$. If G is a big sun, then we write R for the factor-critical subgraph of G. Thus, we obtain

$$bind(G) \leq \frac{|N_G(V(G) \setminus V(R))|}{|V(G) \setminus V(R)|} = 1,$$

which contradicts $bind(G) \geq \frac{3}{2} - \frac{1}{4m+4}$. Hence, we may assume that $S \neq \emptyset$. In the following, we consider two cases.

Case 1. S is not a vertex cut set of G.

In this case, $\omega(G - S) = \omega(G) = 1$. If $|S| \geq \frac{m+1}{2}$, then we have

$$sun(G' - S) = sun(G - S - E') \leq \omega(G - S - E') \leq \omega(G - S) + m = m + 1,$$

and so

$$sun(G' - S) \leq m + 1 \leq 2|S|,$$

which contradicts (2.1).

If $|S| < \frac{m+1}{2}$, then we obtain

$$\lambda(G-S) \geq \kappa(G-S) \geq \kappa(G) - |S| > 2m+1-\frac{m+1}{2} > m.$$

According to the integrity of $\lambda(G-S)$,

$$\lambda(G-S) \geq m+1.$$

Thus, we have $\lambda(G'-S) = \lambda(G-S-E') \geq \lambda(G-S) - m \geq 1$, and so $\omega(G'-S) = 1$. It follows from $S \neq \emptyset$, (2.1) and $\omega(G'-S) = 1$ that

$$2 \leq 2|S| < sun(G'-S) \leq \omega(G'-S) = 1,$$

which is a contradiction.

Case 2. S is a vertex cut set of G.

In this case, $\omega(G-S) \geq 2$ and $|S| \geq 2m+1$. Note that $sun(G'-S) = sun(G-S-E') \leq sun(G-S)+2m$, that is,

$$sun(G-S) \geq sun(G'-S) - 2m. \tag{2.5}$$

It follows from (2.1), (2.5) and $|S| \geq 2m+1$ that

$$sun(G-S) \geq sun(G'-S) - 2m > 2|S| - 2m \geq 2(2m+1) - 2m = 2m+2. \tag{2.6}$$

Suppose that there exist a isolated vertices, bK_2's and c big sun components $H_1, H_2, \cdots, H_c$, where $|V(H_i)| \geq 6$ for $1 \leq i \leq c$, in $G-S$. Obviously, $sun(G-S) = a+b+c$. For each H_i, R_i denotes the factor-critical subgraph of H_i. We write

$$Y = V(aK_1) \cup V(bK_2) \cup V(\cup_{i=1}^{c} H_i).$$

Clearly, we have

$$|Y| = a + 2b + \sum_{i=1}^{c} |V(H_i)|. \tag{2.7}$$

Claim 1. $b \geq 1$.

Proof. Assume that $b = 0$. If $c = 0$, then $sun(G-S) = a > 2|S| - 2m \geq 2$ by (2.6) and $sun(G-S) = a+b+c$. Combining this with $|S| \geq 2m+1$, the definition of $bind(G)$ and the hypothesis of Theorem 2.1, we obtain

$$\begin{aligned}\frac{3}{2} - \frac{1}{4m+4} \leq bind(G) \leq \frac{|N_G(V(aK_1))|}{|V(aK_1)|} \leq \frac{|S|}{a} \\ < \frac{|S|}{2|S|-2m} = \frac{1}{2} + \frac{m}{2|S|-2m} \leq \frac{1}{2} + \frac{m}{2(2m+1)-2m} = \frac{3}{2} - \frac{m+2}{2m+2},\end{aligned}$$

which is a contradiction.

If $c \geq 1$, then $sun(G-S) = a+c > 2|S| - 2m \geq 2$ by (2.6) and $sun(G-S) = a+b+c$. For any $x \in V(\cup_{i=1}^{c} H_i) \setminus V(\cup_{i=1}^{c} R_i)$, there exists $y \in V(\cup_{i=1}^{c} R_i)$ such that $xy \in E(\cup_{i=1}^{c} H_i)$. Note that $d_{G-S}(x) = 1$. Thus, we obtain

$$|N_G(Y \setminus \{y\})| \leq |S| + \sum_{i=1}^{c} |V(H_i)| - 1. \tag{2.8}$$

It follows from (2.7), (2.8), $b = 0$, the definition of $bind(G)$ and the hypothesis of Theorem 2.1 that

$$\frac{3}{2} - \frac{1}{4m+4} \leq bind(G) \leq \frac{|N_G(Y \setminus \{y\})|}{|Y \setminus \{y\}|} \leq \frac{|S| + \sum_{i=1}^{c} |V(H_i)| - 1}{a + \sum_{i=1}^{c} |V(H_i)| - 1},$$

that is,

$$(4m+4)|S| \geq (6m+5)a + (2m+1)\sum_{i=1}^{c}|V(H_i)| - (2m+1).$$

Combining this with $a + c > 2|S| - 2m$ and $|V(H_i)| \geq 6$ for $1 \leq i \leq c$, we have

$$\begin{aligned}(4m+4)|S| &\geq (6m+5)a + (2m+1)\sum_{i=1}^{c}|V(H_i)| - (2m+1)\\ &\geq (4m+2)a + 6(2m+1)c - (2m+1)\\ &= (4m+2)(a+c) + 4(2m+1)c - (2m+1)\\ &> (4m+2)(a+c) > (4m+2)(2|S|-2m),\end{aligned}$$

which implies

$$4m|S| < 4m(2m+1). \tag{2.9}$$

It follows from (2.9) and $|S| \geq 2m+1$ that

$$4m(2m+1) \leq 4m|S| < 4m(2m+1),$$

which is a contradiction. Claim 1 is proved. □

In terms of Claim 1, we obtain $|V(bK_2)| = 2b \geq 2$. For any $z \in V(bK_2) \subseteq Y$, we have

$$|N_G(Y \setminus \{z\})| \leq |S| + (2b-1) + \sum_{i=1}^{c}|V(H_i)|. \tag{2.10}$$

It follows from (2.7), the definition of $bind(G)$ and the hypothesis of Theorem 2.1 that

$$\frac{3}{2} - \frac{1}{4m+4} \leq bind(G) \leq \frac{|N_G(Y \setminus \{z\})|}{|Y \setminus \{z\}|} \leq \frac{|S| + (2b-1) + \sum_{i=1}^{c}|V(H_i)|}{a + (2b-1) + \sum_{i=1}^{c}|V(H_i)|}. \tag{2.11}$$

In terms of (2.6), (2.11), $|V(H_i)| \geq 6$ and $sun(G-S) = a+b+c$, we obtain

$$\begin{aligned}|S| &\geq \left(\frac{3}{2} - \frac{1}{4m+4}\right)\left(a + 2b + \sum_{i=1}^{c}|V(H_i)| - 1\right) - \left(2b + \sum_{i=1}^{c}|V(H_i)| - 1\right)\\ &\geq \left(\frac{1}{2} - \frac{1}{4m+4}\right)\left(2a + 2b + \sum_{i=1}^{c}|V(H_i)| - 1\right)\\ &\geq \left(\frac{1}{2} - \frac{1}{4m+4}\right)(2a+2b+2c-1)\\ &= \left(1 - \frac{1}{2m+2}\right)(a+b+c) - \left(\frac{1}{2} - \frac{1}{4m+4}\right)\\ &= \left(1 - \frac{1}{2m+2}\right)\cdot sun(G-S) - \left(\frac{1}{2} - \frac{1}{4m+4}\right)\\ &\geq \left(1 - \frac{1}{2m+2}\right)(2|S|-2m+1) - \left(\frac{1}{2} - \frac{1}{4m+4}\right)\\ &> \left(1 - \frac{1}{2m+2}\right)(2|S|-2m),\end{aligned}$$

which implies

$$2m|S| < 2m(2m+1).$$

If $m = 0$, then we have $0 < 0$, which is contradiction. If $m \geq 1$, then we obtain $|S| < 2m+1$, which contradicts that $|S| \geq 2m+1$. This completes the proof of Theorem 2.1. □

3. $(P_{\geq 3}, k)$-FACTOR CRITICAL GRAPHS

Theorem 3.1. *Let k be a nonnegative integer, and let G be a graph with $\kappa(G) \geq k+2$. If $bind(G) \geq \frac{5+k}{4}$, then G is a $(P_{\geq 3}, k)$-factor critical graph.*

Proof. Theorem 3.1 holds for $k = 0$ by Corollary 2.2. In the following, we assume that $k \geq 1$.

Let $G' = G - U$ for any $U \subseteq V(G)$ with $|U| = k$. To prove Theorem 3.1, we only need to verify that G' admits a $P_{\geq 3}$-factor. By contradiction, we assume that G' has no $P_{\geq 3}$-factor. Then from Theorem 1.1, there exists some subset S of $V(G')$ satisfying

$$sun(G' - S) \geq 2|S| + 1. \tag{3.1}$$

Claim 2. $|S| \geq 2$.

Proof. Let $S = \emptyset$. Then it follows from (3.1) that

$$sun(G') \geq 1. \tag{3.2}$$

By $\kappa(G) \geq k+2$, $|U| = k$ and $G' = G - U$, we have

$$\lambda(G') \geq \kappa(G') = \kappa(G-U) \geq \kappa(G) - |U| = \kappa(G) - k \geq 2. \tag{3.3}$$

According to (3.3) and the definition of a sun, we obtain

$$sun(G') = 0,$$

which contradicts (3.2).

Let $|S| = 1$. By (3.1), we have

$$sun(G' - S) \geq 3. \tag{3.4}$$

Note that $\lambda(G'-S) \geq \kappa(G'-S) = \kappa(G-U-S) \geq \kappa(G) - |U| - |S| = \kappa(G) - k - 1 \geq 1$, and so, $\omega(G'-S) = 1$. Thus, we obtain

$$sun(G' - S) \leq \omega(G' - S) = 1,$$

which contradicts (3.4). This completes the proof of Claim 2. □

Suppose that there exist a isolated vertices, bK_2's and c big sun components $H_1, H_2, \cdots, H_c$, where $|V(H_i)| \geq 6$ for $1 \leq i \leq c$, in $G' - S$. Combining this with (3.1) and Claim 2, we have

$$sun(G' - S) = a + b + c \geq 2|S| + 1 \geq 5. \tag{3.5}$$

We now consider the following two cases by the value of a.

Case 1. $a = 0$.

In this case, $b + c \geq 5$ by (3.5). We write $Q = (bK_2) \cup (\cup_{i=1}^{c} H_i)$. Clearly, there exist $x, y \in V(Q)$ such that $d_Q(x) = 1$ and $xy \in E(Q)$. Thus, we have

$$|N_G(V(Q) \setminus \{y\})| \leq |U| + |S| + 2b + \sum_{i=1}^{c} |V(H_i)| - 1 = |S| + k + 2b + \sum_{i=1}^{c} |V(H_i)| - 1.$$

Combining this with the definition of $bind(G)$ and the hypothesis of Theorem 3.1, we obtain

$$\frac{5+k}{4} \le bind(G) \le \frac{|N_G(V(Q) \setminus \{y\})|}{|V(Q) \setminus \{y\}|} \le \frac{|S| + k + 2b + \sum_{i=1}^{c} |V(H_i)| - 1}{2b + \sum_{i=1}^{c} |V(H_i)| - 1}. \tag{3.6}$$

According to (3.5), (3.6), $|V(H_i)| \ge 6$, $k \ge 1$ and $b + c \ge 5$, we have

$$\begin{aligned}
4|S| &\ge (5+k)\left(2b + \sum_{i=1}^{c} |V(H_i)| - 1\right) - 4\left(2b + \sum_{i=1}^{c} |V(H_i)| - 1\right) - 4k \\
&= (1+k)\left(2b + \sum_{i=1}^{c} |V(H_i)| - 1\right) - 4k \\
&\ge (1+k)(2b + 6c - 1) - 4k \\
&\ge (1+k)(b + c + 4) - 4k \\
&\ge (1+k)(2|S| + 1 + 4) - 4k \\
&= 2(1+k)|S| + 5 + k \ge 4|S| + 6,
\end{aligned}$$

which is a contradiction.

Case 2. $a \ge 1$.

Let $Y = V(aK_1) \cup V(bK_2) \cup V(\cup_{i=1}^{c} H_i)$. It is obvious that $|N_G(Y)| \le |U| + |S| + 2b + \sum_{i=1}^{c} |V(H_i)| = |S| + k + 2b + \sum_{i=1}^{c} |V(H_i)|$. In view of $bind(G) \ge \frac{5+k}{4}$ and the definition of $bind(G)$, we obtain

$$\frac{5+k}{4} \le bind(G) \le \frac{|N_G(Y)|}{|Y|} \le \frac{|S| + k + 2b + \sum_{i=1}^{c} |V(H_i)|}{a + 2b + \sum_{i=1}^{c} |V(H_i)|}. \tag{3.7}$$

By (3.5), (3.7), $|V(H_i)| \ge 6$, $a \ge 1$ and $k \ge 1$, we have

$$\begin{aligned}
4|S| &\ge (5+k)\left(a + 2b + \sum_{i=1}^{c} |V(H_i)|\right) - 4\left(2b + \sum_{i=1}^{c} |V(H_i)|\right) - 4k \\
&= (1+k)\left(a + 2b + \sum_{i=1}^{c} |V(H_i)|\right) + 4a - 4k \\
&\ge (1+k)(a + 2b + 6c) + 4a - 4k \\
&\ge (1+k)(a + b + c) + 4a - 4k \\
&\ge (1+k)(2|S| + 1) + 4a - 4k \\
&\ge (2k+2)|S| + 2 + 4a - 4k,
\end{aligned}$$

that is,

$$4|S| \ge (2k+2)|S| + 2 + 4a - 4k. \tag{3.8}$$

If $k = 1$, then by (3.8) and $a \ge 1$, we have $4|S| \ge 4|S| + 2 + 4a - 4 \ge 4|S| + 2$, which is a contradiction. If $k \ge 2$, then it follows from (3.8) and $a \ge 1$ that

$$|S| \le \frac{4k - 2 - 4a}{2k - 2} \le \frac{4k - 4 - 2a}{2k - 2} < 2,$$

which contradicts Claim 2. Theorem 3.1 is proved. □

Remark 3.2. In the following, we show that the conditions $\kappa(G) \ge k + 2$ and $bind(G) \ge \frac{5+k}{4}$ in Theorem 3.1 can not be replaced by $\kappa(G) \ge k + 1$ and $bind(G) \ge \frac{5+k}{5}$.

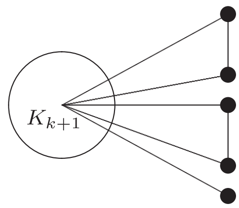

FIGURE 2. $K_{k+1} \vee (2K_2 \cup K_1)$.

Let k be an nonnegative integer. Let $G = K_{k+1} \vee (2K_2 \cup K_1)$, (see Fig. 2), where $\vee$ denotes "join". We write $X = V(2K_2 \cup K_1)$. It is easy to see that $\kappa(G) = k+1$ and $bind(G) = \frac{|N_G(X)|}{|X|} = \frac{5+k}{5}$. For any $U \subseteq V(K_{k+1})$ with $|U| = k$, we write $G' = G - U$. We choose $S = V(K_{k+1}) \setminus U \subseteq V(G')$, and so $|S| = 1$. Thus, we obtain

$$sun(G' - S) = 3 > 2 = 2|S|.$$

According to Theorem 1.1, G' has no $P_{\geq 3}$-factor, that is, G is not a $(P_{\geq 3}, k)$-factor critical graph.

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