RAIRO-Oper. Res. 54 (2020) 1077–1086
https://doi.org/10.1051/ro/2019109

RAIRO Operations Research
www.rairo-ro.org

SIGNED DOMINATION AND MYCIELSKI'S STRUCTURE IN GRAPHS

Arezoo N. Ghameshlou[1,*], Athena Shaminezhad[2], Ebrahim Vatandoost[2] and Abdollah Khodkar[3]

Abstract. Let $G = (V, E)$ be a graph. The function $f : V(G) \to \{-1, 1\}$ is a signed dominating function if for every vertex $v \in V(G)$, $\sum_{x\in N_G[v]} f(x) \geq 1$. The value of $\omega(f) = \sum_{x\in V(G)} f(x)$ is called the weight of f. The signed domination number of G is the minimum weight of a signed dominating function of G. In this paper, we initiate the study of the signed domination numbers of Mycielski graphs and find some upper bounds for this parameter. We also calculate the signed domination number of the Mycielski graph when the underlying graph is a star, a wheel, a fan, a Dutch windmill, a cycle, a path or a complete bipartite graph.

Mathematics Subject Classification. 05C69, 05C75.

Received July 9, 2019. Accepted November 13, 2019.

1. Introduction

All graphs considered throughout this paper are simple, finite, undirected and connected. For the terminology and notations not defined here, we refer the reader to [11]. Let G be a graph with *vertex set* $V(G)$ and *edge set* $E(G)$. The *open neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. The *closed neighborhood* of a vertex v in graph G is $N_G[v] = N_G(v) \cup \{v\}$. Moreover, the *open* and *closed neighborhoods* of a subset $S \subseteq V(G)$ are $N_G(S) = \cup_{v\in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$, respectively. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) =| N_G(v) |$. A vertex $v \in V(G)$ is called an *odd* (*even*) vertex if $\deg_G(v)$ is odd (even). For a graph $G = (V, E)$, let V_o and V_e be the set of odd and even vertices, respectively. We denote the *maximum* degree of G with $\Delta(G)$ and its *minimum degree* with $\delta(G)$.

A vertex with degree of $| V(G) | -1$ is called a *universal* vertex. In a complete graph, K_n, all vertices are universal. Additionally, in *Stars*, $K_{1,n}$, and *Wheels*, W_n, the central vertex is universal. In this paper, we use two other special graphs having a universal vertex. A *Dutch windmill* graph D_3^n is the graph of order $2n + 1$ with vertex set $\{v, v_1, v_2, \ldots, v_{2n}\}$ and edge set $\{vv_i : 1 \leq i \leq 2n\} \cup \{v_iv_{i+1} : i = 1, 3, \ldots, 2n-1\}$. The *Fan* graph, F_n, is the graph of order $n+1$ with vertex set $\{v, v_1, v_2, \ldots, v_n\}$ and edge set $\{vv_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\}$.

For a function $f : V(G) \longrightarrow \{-1, 1\}$ and a subset S of $V(G)$, we define $f(S) = \sum_{x\in S} f(x)$. If $S = N_G[v]$ for some $v \in V(G)$, then we denote $f(S)$ by $f[v]$. Let $C_f = \{v \in V(G) : f[v] \geq 1\}$. A *signed dominating function*

Keywords. Signed domination number, Mycielski construction.

[1] Department of Irrigation and Reclamation Engineering, University of Tehran, Tehran, I.R. Iran.
[2] Department of Basic Science, Imam Khomeini International University, Qazvin, I.R. Iran.
[3] Department of Mathematics, University of West Georgia, Carrollton, GA 30118, USA.
*Corresponding author: a.ghameshlou@ut.ac.ir

of G is a function $f : V(G) \longrightarrow \{-1, 1\}$ such that for all vertices v of G, $v \in C_f$. The *weight* of a signed dominating function f is $\omega(f) = \sum_{v\in V(G)} f(v) = f(V(G))$. The *signed domination number* (SDN), $\gamma_s(G)$, is the minimum weight of a signed dominating function of G. A signed dominating function of weight $\gamma_s(G)$ is called a $\gamma_s(G)$-*function*. For a signed dominating function f of G we define $P_f = \{v \in V(G) \colon f(v) = 1\}$ and $M_f = \{v \in V(G) \colon f(v) = -1\}$.

The concept of the signed domination number of a graph was proposed in Dunbar *et al.* [1] and shown that the problem of determining the signed domination numbers for general graphs is NP-hard. Moreover, the authors of Dunbar *et al.* [1] proved that, there exist chordal and k-partite graphs with negative signed domination numbers. The signed domination number has been extensively studied in the literature; see *e.g.* Shon *et al.* [10] and references therein. The exact values of the signed domination numbers have been determined for some classes of graphs, including the complete graphs, path and cycles [4], complete bipartite graphs [12], Dutch windmill graphs, wheels, ladders and prisms [9] and grid graphs [3].

For a graph G with $V(G) = \{v_1, v_2, \dots, v_n\}$, let $U = \{u_1, u_2, \dots, u_n\}$ be a disjoint copy of $V(G)$ and let w be a new vertex. The *Mycielski graph* $\mu(G)$ of G is defined as follows:

$$V(\mu(G)) = V(G) \cup U \cup \{w\},$$
$$E(\mu(G)) = E(G) \cup \{v_i u_j \colon v_i v_j \in E(G)\} \cup \{w u_i \colon 1 \le i \le n\}.$$

The vertex w is called the *root* of $\mu(G)$ and the vertex $u_i = c(v_i)$ is called the *twin* of the vertex v_i, $i = 1, 2, \dots, n$. The Mycielski graph of a graph G was introduced by Mycielski in order to construct triangle-free graphs with an arbitrary large chromatic number [8]. In recent years, there have been results reported on Mycielski graphs related to various domination parameters [2, 5–7]. In [2], it was proved that $\gamma(\mu(G)) = \gamma(G) + 1$. This shows that the domination number of a Mycielski graph exceeds the domination number of its underlying graph G. As we will see such a result is not true for the signed domination number of Mycielski graphs.

In this paper, we initiate the study of the signed domination numbers of Mycielski graphs. In Section 2, we present some preliminary results on Mycielski graphs and their signed domination numbers. In Section 3, we show that the signed domination number of a Mycielski graph, whose underlying graph has at least one universal vertex, is at least 3. Then we calculate the exact values of $\gamma_s(\mu(G))$ when G is a star, a wheel, a fan, a Dutch windmill or a complete graph. In Section 4, we prove that if $\gamma_s(G) \ge 0$, then $\gamma_s(\mu(G)) \le 2\gamma_s(G)$, otherwise $\gamma_s(\mu(G)) \le \gamma_s(G) + 2$. Finally, in Section 5, we calculate $\gamma_s(\mu(G))$ when G is a cycle, a path or a complete bipartite graph. It is worth to note that there are graphs G, such as $K_{m,n}$, when $m = 1$ or m and n are both odd, with $\gamma_s(\mu(G)) < \gamma_s(G)$.

2. Preliminary results

For investigating the signed domination numbers of Mycielski graphs the following basic properties are useful.

Observation 2.1. Let G be a graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and let $\mu(G)$ be the graph obtained from G by Mycielski's construction.

(1) If $u_i = c(v_i)$, $i = 1, \dots, n$, and w is the root of $\mu(G)$, then $\deg_{\mu(G)}(v_i) = 2\deg_G(v_i)$, $\deg_{\mu(G)}(u_i) = \deg_G(v_i) + 1$ and $\deg_{\mu(G)}(w) = n$.

(2) $\Delta(\mu(G)) = \begin{cases} 2\Delta(G) & \text{if} \quad \Delta(G) \ge \lfloor \frac{n}{2} \rfloor \\ n & \text{otherwise.} \end{cases}$

(3) $\mid V_e(\mu(G)) \mid = \begin{cases} n + n_o + 1 & \text{if} \quad n \text{ is even,} \\ n + n_o & \text{if} \quad n \text{ is odd,} \end{cases}$
where n_o is the number of odd vertices in G.

(4) $\mid V_o(\mu(G)) \mid = \begin{cases} n - n_o & \text{if} \quad n \text{ is even,} \\ n - n_o + 1 & \text{if} \quad n \text{ is odd.} \end{cases}$

(5) The Mycielski graph $\mu(G)$ is r-regular if and only if $G = K_2$.

(6) The Mycielski graph $\mu(G)$ is connected if and only if the underlying graph G does not have any isolated vertices.

Observation 2.2. Let f be a signed dominating function of G.

(1) If $v \in V(G)$ is an even vertex, then $f[v] \geq 1$ while if v is an odd vertex, then $f[v] \geq 2$.
(2) $\omega(f) \equiv 1 \pmod 2$.
(3) If G has an isolated vertex, then $w \in P_f$, where w is the root vertex of $\mu(G)$.
(4) If G is a path or a cycle of order $n > 2$ and $w \in M_f$, then $\omega(f) = 2n - 1$.

3. Graphs with universal vertices

In this section, we show that the signed domination number of a Mycielski graph, whose underlying graph has at least one universal vertex, is at least 3.

Theorem 3.1. *Let G be a graph of order n. If G has a universal vertex, then $\gamma_s(\mu(G)) \geq 3$.*

Proof. Let v be a universal vertex of graph G, $u = c(v)$, and w be the root vertex. Suppose that f is a $\gamma_s(\mu(G))$-function. We consider two cases.

Case 1: n is odd. By using Observation 2.2, part 1, we have $f[u] = f(u) + f(V(G) \setminus \{v\}) + f(w) \geq 2$ and $f[w] = f(w) + f(U) \geq 2$. This implies that $f(V(G) \setminus \{v\}) \geq 0$ and $f(U \setminus \{u\}) \geq 0$. If $f(w) = -1$, then $f(V(G) \setminus \{v\}) \geq 2$ and $f(U) \geq 3$. Hence, $\gamma_s(\mu(G)) = f(v) + f(V(G) \setminus \{v\}) + f(U) + f(w) \geq 3$. Now assume $f(w) = 1$ and $f(u) = -1$. Then $f(V(G) \setminus \{v\}) \geq 2$ and $f(U \setminus \{u\}) \geq 2$. Hence, $\gamma_s(\mu(G)) = f(v) + f(V(G) \setminus \{v\}) + f(U) + f(w) \geq 3$. Finally, if $f(u) = f(w) = 1$, then $f(V(G) \setminus \{v\}) \geq 0$ and $f(U \setminus \{u\}) \geq 0$. Suppose that $f(V(G) \setminus \{v\}) = f(U \setminus \{u\}) = 0$. If $f(v) = -1$, then $f[v] < 1$. That is a contradiction. Hence, $\gamma_s(\mu(G)) = f(v) + f(V(G) \setminus \{v\}) + f(U \setminus \{u\}) + f(u) + f(w) \geq 1 + 2 = 3$. Now assume that $f(V(G) \setminus \{v\}) = 0$ and $f(U \setminus \{u\}) \geq 2$. Therefore, $\gamma_s(\mu(G)) = f(v) + f(V(G) \setminus \{v\}) + f(U) + f(w) \geq -1 + 2 + 2 = 3$. Similarly, if $f(V(G) \setminus \{v\}) \geq 2$ and $f(U \setminus \{u\}) = 0$, the $\gamma_s(\mu(G)) = f(v) + f(V(G) \setminus \{v\}) + f(U) + f(w) \geq -1 + 2 + 2 = 3$. Finally, if $f(V(G) \setminus \{v\}) \geq 2$ and $f(U \setminus \{u\}) \geq 2$, then we dedicate that $\gamma_s(\mu(G)) = f(v) + f(V(G) \setminus \{v\}) + f(U) + f(w) \geq -1 + 2 + 2 + 2 = 5$.

Case 2: n is even. By using Observation 2.2, part 1, we have $f[w] = f(w) + f(u) + \sum_{x \in U \setminus \{u\}} f(x) \geq 1$. Since $f(u) + f(w) \geq -2$, we have $\sum_{x \in U \setminus \{u\}} f(x) \geq -1$. Similarly, $f[u] = f(u) + f(w) + \sum_{x \in V(G) \setminus \{v\}} f(x) \geq 1$ and $\sum_{x \in V(G) \setminus \{v\}} f(x) \geq -1$.

If $\sum_{x \in U \setminus \{u\}} f(x) = \sum_{x \in V(G) \setminus \{v\}} f(x) = -1$, then $f[v] < 1$; that is a contradiction. Now, we consider the following subcases:

Subcase 2.1. $\sum_{x \in V(G) \setminus \{v\}} f(x) = 1$ and $\sum_{x \in U \setminus \{u\}} f(x) = -1$.

Then $f[v] = f(v) + \sum_{x \in V(G) \setminus \{v\}} f(x) + \sum_{x \in U \setminus \{u\}} f(x) \geq 1$ if and only if $f(v) = 1$. Furthermore, $f[w] = f(w) + f(u) + \sum_{x \in U \setminus \{u\}} f(x) \geq 1$ if and only if $f(u) = f(w) = 1$. Thus, $\gamma_s(\mu(G)) = f[v] + f(u) + f(w) \geq 1 + 1 + 1 = 3$.

Subcase 2.2. $\sum_{x \in V(G) \setminus \{v\}} f(x) = -1$ and $\sum_{x \in U \setminus \{u\}} f(x) = 1$.

Then $f[v] = f(v) + \sum_{x \in V(G) \setminus \{v\}} f(x) + \sum_{x \in U \setminus \{u\}} f(x) \geq 1$ if and only if $f(v) = 1$. Furthermore, $f[u] = f(u) + f(w) + \sum_{x \in V(G) \setminus \{v\}} f(x) \geq 1$ if and only if $f(u) = f(w) = 1$. Thus, $\gamma_s(\mu(G)) = f[v] + f(u) + f(w) \geq 1 + 1 + 1 = 3$.

Subcase 2.3. $\sum_{x \in V(G) \setminus \{v\}} f(x) = \sum_{x \in U \setminus \{u\}} f(x) = 1$.

Then $f[w] = f(w) + f(u) + \sum_{x \in U \setminus \{u\}} f(x) \geq 1$ if and only if $f(u) + f(w) \geq 0$. First let $f(u) = f(w) = 1$. Then $f[v] = f(v) + \sum_{x \in V(G) \setminus \{v\}} f(x) + \sum_{x \in U \setminus \{u\}} f(x) = f(v) + 1 + 1 = f(v) + 2 \geq 1$, and $\gamma_s(\mu(G)) = f[v] + f(u) + f(w) \geq 1 + 1 + 1 = 3$. Now let $f(u) + f(w) = 0$. We claim that $f(v) = 1$. Assume $f(v) = -1$.

Define $N=\sum_{y\in V(\mu(G))}\sum_{x\in N_{\mu(G)}[y]}f(x)$. We have

$$\begin{aligned} N &= \sum_{y\in V(\mu(G))}\sum_{x\in N_{\mu(G)}[y]} f(x) = \sum_{x\in V(\mu(G))} f[x] \\ &\geq \sum_{x\in V(\mu(G))} 1 = \mid V(\mu(G)) \mid = 2n+1. \end{aligned} \tag{3.1}$$

On the other hand, it is easy to verify that N counts the value $f(x)$ exactly $\mid N_{\mu(G)}[v]\mid = 1+\deg_{\mu(G)}(x)$ times for each $x\in V(\mu(G))$. So, $N=\sum_{x\in V(\mu(G))}(1+\deg_{\mu(G)}(x))f(x)$. Therefore,

$$\begin{aligned} N = &\sum_{x\in V(G)\setminus\{v\}}(1+\deg_{\mu(G)}(x))f(x)+\sum_{x\in U\setminus\{u\}}(1+\deg_{\mu(G)}(x))f(x) \\ &+(1+\deg_{\mu(G)}(w))f(w)+(1+\deg_{\mu(G)}(u))f(u)+(1+\deg_{\mu(G)}(v))f(v). \end{aligned}$$

By Observation 2.1, part 1, and the assumptions of Subcase 2.3 We have

$$\begin{aligned} N &= (n+1)(f(w)+f(u))+(2n-1)(-1) \\ &\quad+\sum_{x\in V(G)\setminus\{v\}}(1+\deg_{\mu(G)}(x))f(x)+\sum_{x\in U\setminus\{u\}}(1+\deg_{\mu(G)}(x))f(x) \\ &\leq 1-2n+(2n-1)\sum_{x\in V(G)\setminus\{v\}}f(x)+(n+1)\sum_{x\in U\setminus\{u\}}f(x)=n+1. \end{aligned} \tag{3.2}$$

Comparing Inequalities (3.1) and (3.2), we deduce that $2n+1\leq n+1$.

This contradicts $n\geq 1$. Therefore, $f(v)=1$, and $f[v]=f(v)+\sum_{x\in V(G)\setminus\{v\}}f(x)+\sum_{x\in U\setminus\{u\}}f(x)=3$. Thus, $\gamma_s(\mu(G))=f[v]+f(u)+f(w)=3+0=3$.

Subcase 2.4. $\sum_{x\in V(G)\setminus\{v\}}f(x)=1$ and $\sum_{x\in U\setminus\{u\}}f(x)\geq 3$.

In this case we have, $\gamma_s(\mu(G))=\sum_{x\in U\setminus\{u\}}f(x)+f[u]+f(v)\geq 3+1+(-1)=3$.

Subcase 2.5. $\sum_{x\in V(G)\setminus\{v\}}f(x)\geq 3$ and $\sum_{x\in U\setminus\{u\}}f(x)=1$.

It is easy to verify that $\gamma_s(\mu(G))=f[w]+f(v)+\sum_{x\in V(G)\setminus\{v\}}f(x)\geq 1+(-1)+3=3$.

Subcase 2.6. $\sum_{x\in V(G)\setminus\{v\}}f(x)\geq 3$ and $\sum_{x\in U\setminus\{u\}}f(x)\geq 3$.

Then

$$\begin{aligned} \gamma_s(\mu(G)) &= f(u)+f(w)+f(v)+\sum_{x\in V(G)\setminus\{v\}}f(x)+\sum_{x\in U\setminus\{u\}}f(x) \\ &\geq -1-1-1+3+3=3. \end{aligned}$$

□

We now show that the lower bound given in Theorem 3.1 can be achieved for some families of graphs. Recall that the Dutch windmill graph D_3^n is the graph of order $2n+1$ with vertex set $\{v,v_1,v_2,\dots,v_{2n}\}$ and edge set $\{vv_i\colon 1\leq i\leq 2n\}\cup\{v_iv_{i+1}\colon i=1,3,\dots,2n-1\}$.

Corollary 3.2. *For every graph $G\in\{K_n,K_{1,n},D_3^n\}$, $\gamma_s(\mu(G))=3$.*

Proof. First suppose that $G=K_n$. Let $V(G)=\{v_1,v_2,\dots,v_n\}$. Suppose that w is the root vertex of $\mu(G)$ and $u_i=c(v_i)$ for $i=1,2,\dots,n$. Let $f:V(\mu(G))\longrightarrow\{-1,1\}$ be the function which assigns 1 and -1 to the vertices $v_2,v_3,\dots,v_n,u_2,u_3,\dots,u_n$, respectively, and $f(v_1)=f(u_1)=f(w)=1$. Now let $G=K_{1,n}$. Suppose that $V(G)=\{v,v_1,v_2,\dots,v_n\}$, where v is the universal vertex in $K_{1,n}$. For $i=1,2,\dots,n$, set $u_i=c(v_i)$, $u=c(v)$ and assume w is the root vertex. Define $f:V(\mu(G))\longrightarrow\{-1,1\}$ with $f(v)=f(u)=f(w)=1$. For $i=1,\dots,n$, this function assigns 1 and -1 to the vertices $v_1,v_2,\dots,v_n,u_1,u_2,\dots,u_n$, respectively. Finally, let

$G = D_3^n$. Let $V(G) = \{v, v_1, v_2, \ldots, v_{2n}\}$ and v is the universal vertex in D_3^n. Suppose that w is the root vertex of $\mu(G)$, $u = c(v)$ and $u_i = c(v_i)$ for $i = 1, 2, \ldots, 2n$. Let $f : V(\mu(G)) \longrightarrow \{-1, 1\}$ be the function which assigns 1 and -1 to the vertices $v_1, v_2, \ldots, v_{2n}, u_1, u_2, \ldots, u_{2n}$, respectively, and $f(v) = f(u) = f(w) = 1$.

In each case, it is clear that f is an SDF. So $\gamma_s(\mu(G)) \leq \omega(f) = 3$. By Theorem 3.1, the proof is complete. □

Corollary 3.3. *Let W_n be the wheel graph of order $n+1$. Then $\gamma_s(\mu(W_n)) = 3$.*

Proof. Let v be the universal vertex of graph W_n and $(v_1, v_2, \ldots, v_n)$ be the n-cycle in W_n. Suppose that w is the root vertex of $\mu(W_n)$, $u = c(v)$ and $u_i = c(v_i)$ for $i = 1, 2, \ldots, n$. We define the function $f : V(\mu(W_n)) \to \{-1, 1\}$ with $f(u) = f(v) = f(w) = 1$.

If $n \equiv 0, 1 \pmod 4$, then for $i \equiv 1, 2 \pmod 4$ define $f(u_i) = 1$, otherwise $f(u_i) = -1$. Let $f(v_i) = -f(u_i)$ for $i \in \{1, 2, \ldots, n\}$.

If $n \equiv 2 \pmod 4$, then let $f(u_{n-1}) = -1$ and $f(u_n) = 1$. For $i = 1, 2 \ldots, n-2$, if $i \equiv 1, 2 \pmod 4$, define $f(u_i) = 1$, otherwise $f(u_i) = -1$. Let $f(v_i) = -f(u_i)$ for $i \in \{1, 2, \ldots, n\}$.

If $n \equiv 3 \pmod 4$, then let $f(u_n) = 1$. For $i = 1, 2 \ldots, n-1$, if $i \equiv 1, 2 \pmod 4$, define $f(u_i) = -1$, otherwise $f(u_i) = 1$. Let $f(v_i) = -f(u_i)$ for $i \in \{1, 2, \ldots, n\}$.

In each case, f is an SDF of G and $w(f) = 3$, so the result follows by Theorem 3.1. □

Recall that the Fan graph F_n is the graph of order $n+1$ with vertex set $\{v, v_1, v_2, \ldots, v_n\}$ and edge set $\{vv_i : 1 \leq i \leq n\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\}$.

Corollary 3.4. *Let F_n be the fan graph of order $n+1$. Then $\gamma_s(\mu(F_n)) = 3$.*

Proof. Suppose that w is the root vertex of $\mu(F_n)$, $u = c(v)$ and $u_i = c(v_i)$ for $i = 1, 2, \ldots, n$. We define the function $f : V(\mu(F_n)) \to \{-1, 1\}$ with $f(u) = f(v) = f(w) = 1$.

If $n \equiv 0, 1 \pmod 4$, then for $i \equiv 1, 2 \pmod 4$ define $f(u_i) = 1$, otherwise define $f(u_i) = -1$.

If $n \equiv 2 \pmod 4$, then let $f(u_{n-1}) = -1$ and $f(u_n) = 1$. For $i = 1, 2 \ldots, n-2$, if $i \equiv 1, 2 \pmod 4$ define $f(u_i) = 1$, otherwise $f(u_i) = -1$.

If $n \equiv 3 \pmod 4$, then for $i = 1, 2 \ldots, n$, if $i \equiv 1, 2 \pmod 4$ define $f(u_i) = 1$, otherwise $f(u_i) = -1$.

Finally, set $f(v_i) = -f(u_i)$ for $i \in \{1, 2, \ldots, n\}$.

In each case f is an SDF of G and $w(f) = 3$, so the result follows by Theorem 3.1. □

4. A relation between $\gamma_s(G)$ and $\gamma_s(\mu(G))$

In this section, we present an upper bound for the signed domination number of $\mu(G)$ in terms of the signed domination number of G.

Theorem 4.1. *For any graph G of order n,*

$$\gamma_s(\mu(G)) \leq \begin{cases} 2\gamma_s(G) + 1 & \text{if} \quad \gamma_s(G) \geq 0, \\ \gamma_s(G) + 2 & \text{if} \quad \gamma_s(G) \leq -1. \end{cases}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose that f is a $\gamma_s(G)$-function of graph G. Assume that w is the root vertex of $\mu(G)$. Define, $u_i = c(v_i)$ for $i = 1, 2, \ldots, n$. We consider two cases:

Case 1. $\gamma_s(G) \geq 0$.

Define a new function $g : V(\mu(G)) \to \{-1, 1\}$ by

$$g(x) = \begin{cases} 1 & \text{if} \quad x = w, \\ f(x) & \text{if} \quad x \in V(G), \\ f(v_i) & \text{if} \quad x = u_i. \end{cases}$$

It is easy to verify that g is a signed dominating function of graph $\mu(G)$. Therefore,

$$\gamma_s(\mu(G)) \leq \omega(g) = 2\gamma_s(G) + 1.$$

Case 2. $\gamma_s(G) \leq -1$.

Then $| M_f | \geq \lceil \frac{n}{2} \rceil$. Without loss of generality, let

$$f(v_1) = f(v_2) = \ldots = f(v_{|M_f|}) = -1.$$

Now define the function $g : V(\mu(G)) \to \{-1, 1\}$ by

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in V(G), \\ -1 & \text{if} \quad x \in \{u_1, u_2, \ldots, u_{\lfloor \frac{n}{2} \rfloor}\} \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, g is a signed dominating function of $\mu(G)$. Thus,

$$\gamma_s(\mu(G)) \leq \omega(g) = \gamma_s(G) + \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \gamma_s(G) + 2.$$

□

The bound presented in Theorem 4.1 for a graph G with $\gamma_s(G) \geq 0$ is sharp if $G = \overline{K_n}$. Additionally, Corollary 3.2 shows that this bound is sharp for $\mu(K_n)$ when n is odd.

5. Cycles, paths and complete bipartite graphs

In this section we find the signed domination number of $\mu(G)$ when G is a cycle, a path, or a complete bipartite graph.

Theorem 5.1. *For every cycle C_n of order n,*

$$\gamma_s(\mu(C_n)) = \begin{cases} \frac{n}{2} + 1 & \text{if} \quad n \equiv 0 \pmod 8, \\ \frac{n+5}{2} & \text{if} \quad n \equiv 1, 5 \pmod 8, \\ \frac{n}{2} + 2 & \text{if} \quad n \equiv 2, 6 \pmod 8, \\ \frac{n+7}{2} & \text{if} \quad n \equiv 3 \pmod 8, \\ \frac{n}{2} + 3 & \text{if} \quad n \equiv 4 \pmod 8, \\ \frac{n+3}{2} & \text{if} \quad n \equiv 7 \pmod 8. \end{cases}$$

Proof. Let C_n be a cycle with vertices $v_1, v_2, \ldots, v_n$. Let w be the root vertex of the graph $\mu(C_n)$ and for every $i = 1, 2, \ldots, n$, $u_i = c(v_i)$. Let f be an SDF of $\mu(C_n)$. If $f(w) = -1$, then by Observation 2.2, Part 4, $\omega(f) = 2n - 1$. Now let $f(w) = 1$ and

$$f(u_i) = \begin{cases} -1 & \text{if} \quad i = 1, \ldots, \lfloor \frac{n}{2} \rfloor, \\ 1 & \text{if} \quad \text{otherwise.} \end{cases} \tag{5.1}$$

So, $| M_f \cap U | = \lfloor \frac{n}{2} \rfloor$. Since $\deg_{\mu(C_n)}(u) = 3$ for $u \in U$, in order to satisfy the condition $f[u] \geq 1$, there can be at most one negative vertex in $N_{\mu(C_n)}[u]$. Hence, if $u_i \in M_f \cap U$, then $f[u_i] \geq 1$ if and only if $f(v_{i-1}) = f(v_{i+1}) = 1$. Therefore by equality (5.1), for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, $f[u_i] \geq 1$ if and only if

$$f(v_n) = f(v_1) = f(v_2) = \ldots = f(v_{\lfloor \frac{n}{2} \rfloor + 1}) = 1.$$

For $i = \lfloor \frac{n}{2} \rfloor + 2, \ldots, n-1$, assign $-1, -1, 1, 1, -1, -1, \ldots$ to $v_i \in \{v_{\lfloor \frac{n}{2} \rfloor + 2}, \ldots, v_{n-1}\}$, respectively. It is easy to verify that f is an SDF for a graph $\mu(C_n)$. Moreover, there is precisely one negative vertex in $N_{\mu(C_n)}[u]$. for every $u \in P_f$ except for u_n if $n \equiv 1, 2 \pmod 8$, u_{n-1} and u_n if $n \equiv 3, 4 \pmod 8$ and u_{n-1} if $n \equiv 5, 6 \pmod 8$, which have no negative vertices in their closed neighborhoods. Therefore f is a $\gamma_s(\mu(C_n))$-function.

A simple calculation shows that:

$$| M_f | = \begin{cases} \frac{3n}{4} & \text{if} \quad n \equiv 0 \pmod 8, \\ \frac{3n-3}{4} & \text{if} \quad n \equiv 1, 5 \pmod 8. \\ \frac{3n-2}{4} & \text{if} \quad n \equiv 2, 6 \pmod 8, \\ \frac{3n-5}{4} & \text{if} \quad n \equiv 3 \pmod 8, \\ \frac{3n-4}{4} & \text{if} \quad n \equiv 4 \pmod 8, \\ \frac{3n-1}{4} & \text{if} \quad n \equiv 7 \pmod 8). \end{cases}$$

Hence, $\gamma_s(\mu(C_n)) = \omega(f) = 2n + 1 - 2 \mid M_f \mid$. This completes the proof. □

The proof of the following result is straightforward.

Lemma 5.2. *Let P_n be a path of order n. Then*

$$\gamma_s(\mu(P_n)) = \begin{cases} 3 & \text{if} \quad n = 2, 3, 5, 7 \\ 1 & \text{if} \quad n = 4 \\ 5 & \text{if} \quad n = 6. \end{cases}$$

Theorem 5.3. *For a path P_n, $n \geq 8$,*

$$\gamma_s(\mu(P_n)) = \begin{cases} \frac{n+5}{2} & \text{if} \quad n \equiv 1 \pmod 8, \\ \frac{n+4}{2} & \text{if} \quad n \equiv 2 \pmod 8, \\ \frac{n+3}{2} & \text{if} \quad n \equiv 3, 7 \pmod 8, \\ \frac{n+2}{2} & \text{if} \quad n \equiv 0, 4 \pmod 8, \\ \frac{n+1}{2} & \text{if} \quad n \equiv 5 \pmod 8, \\ \frac{n}{2} & \text{if} \quad n \equiv 6 \pmod 8). \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n. Let w be the root vertex of graph $\mu(P_n)$ and for every $i = 1, 2, \ldots, n$, $u_i = c(v_i)$. If f is an SDF of P_n with $f(w) = -1$, then by Observation 2.2, Part 4, $\omega(f) = 2n - 1$. We now define a function $f : V(\mu(P_n)) \to \{-1, 1\}$ as follows: $f(w) = 1$ and $f(u_i) = f(v_i) = -1$ for $i \in \{1, n\}$. Then $f[u_i], f[v_i] \geq 1$ if and only if for every $i \in \{2, 3, n-1, n-2\}$, $f(u_i) = f(v_i) = 1$. Now for each $i = 4, \ldots, n-3$, define

$$f(u_i) = \begin{cases} -1 & \text{if} \quad i = 4, \ldots, \lfloor \frac{n+2}{2} \rfloor, \\ 1 & \text{if} \quad \text{otherwise}. \end{cases} \tag{5.2}$$

Clearly,

$$| M_f \cap U | = \left\lfloor \frac{n}{2} \right\rfloor.$$

According to the Mycielski's construction, for every $i = 2, \ldots, n-1$, $\deg_{\mu(P_n)}(u_i) = 3$. So if $u_i \in M_f \cap U$, then $f[u_i] \geq 1$ if and only if $f(v_i) = 1$, for $i = 4, 5, \ldots, \lfloor \frac{n+2}{2} \rfloor + 1$.

Assign $-1, -1, 1, 1, -1, -1, 1, 1, \ldots$ to the remaining unlabeled $v_i \in V(P_n)$ from the lowest index to the highest index, respectively. It is easy to verify that f is an SDF. Moreover, there is precisely one negative vertex in $N_{\mu(P_n)}[u]$ for every $u \in (P_f)$ except for u_3 if $n \equiv 5, 6 \pmod 8$, u_3, u_{n-2} if $n \equiv 0, 7 \pmod 8$, u_3, u_{n-3} if $n \equiv 3, 4$

(mod 8) and u_3, u_{n-2}, u_{n-3} if $n \equiv 1,2 \pmod 8$, which have no negative vertices in their closed neighborhoods. Hence, f has the minimum weight subject to the conditions $f(w) = 1$ and $f(u_i) = f(v_i) = -1$ for $i \in \{1, n\}$.

A simple calculation shows that:

$$| M_f | = \begin{cases} \frac{3n-3}{4} & \text{if } \quad n \equiv 1 \pmod 8, \\ \frac{3n-2}{4} & \text{if } \quad n \equiv 2 \pmod 8, \\ \frac{3n-1}{4} & \text{if } \quad n \equiv 3,7 \pmod 8, \\ \frac{3n}{4} & \text{if } n \equiv 0,4 \pmod 8, \\ \frac{3n+1}{4} & \text{if } \quad n \equiv 5 \pmod 8, \\ \frac{3n+2}{4} & \text{if } \quad n \equiv 6 \pmod 8. \end{cases}$$

Now let $g : V(\mu(P_n)) \to \{-1, 1\}$ be an SDF of P_n with $g(w) = 1$, $g(u_1) = -1$ and $g(u_n) = 1$. In order to minimize the weight of g we must have $g(u_2) = \ldots = g(u_{\lfloor \frac{n}{2} \rfloor}) = -1$. Then $g(v_1) = g(v_2) = \ldots = g(v_{\lfloor \frac{n}{2} \rfloor + 1}) = 1$ and $g(u_{\lfloor \frac{n}{2} \rfloor + 1}) = 1$. Let $x = \lceil \frac{n}{2} \rceil - 1$. If $x \equiv 0,1,3 \pmod 4$, then we assign $-1, -1, 1, 1, -1, -1, 1, 1, \ldots$ to $v_{\lfloor \frac{n}{2} \rfloor + 2}, \ldots, v_{n-2}$, respectively. If $x \equiv 2 \pmod 4$, then assign $-1, -1, 1, 1$ to $v_{\lfloor \frac{n}{2} \rfloor + 2}, \ldots, v_{n-2}$, respectively.

Finally, define $g(v_{n-1}) = -1$ and $g(v_n) = 1$. Notice that if $g(v_n) = g(v_{n-1}) = -1$, then $g[v_{n-1}] < 1$ when $x \equiv 2 \pmod 8$ or $n \equiv 5,6 \pmod 8$. Therefore, $| M_g \cap V(P_n) | = 2\lfloor \frac{x}{4} \rfloor + i$, where

$$i = \begin{cases} 0 & \text{if } \quad x \equiv 0 \pmod 4, \\ 1 & \text{if } \quad x \equiv 1,2 \pmod 4, \\ 2 & \text{if } \quad x \equiv 3 \pmod 4). \end{cases}$$

This shows that if $n \equiv 5,6 \pmod 8$, then $\omega(g) = \omega(f) + 2$, otherwise $\omega(g) = \omega(f)$. Finally, Let $g(v_1) = g(v_2) = 1$. If we define $g(v_2) = g(v_3) = -1$, then $g(v_4) = g(v_5) = g(u_2) = g(u_3) = g(u_4) = 1$. Now define $g(u_5) = g(u_6) = \ldots = g(u_{\lfloor \frac{n}{2} \rfloor + 4}) = -1$. Therefore we must have $g(u_{\lfloor \frac{n}{2} \rfloor + 5}) = \ldots = g(u_n) = 1$ and $g(v_6) = \ldots = g(v_{\lfloor \frac{n}{2} \rfloor + 5}) = 1$. Let $x = \lceil \frac{n}{2} \rceil - 5$. If $x \equiv 0,1,3 \pmod 4$, then assign $-1, -1, 1, -1, -, 1, \ldots$ to the remaining vertices in $V(P_n)$, respectively. If $x \equiv 2 \pmod 4$, then assign $-1, -1, 1, 1, -1, -1, \ldots$ to $v_{\lfloor \frac{n}{2} \rfloor + 6}, \ldots, v_{n-2}$, respectively. Finally, define $g(v_{n-1}) = -1$ and $g(v_n) = 1$. Notice that if $g(v_n) = g(v_{n-1}) = -1$, then $g[v_{n-1}] < 1$ if $x \equiv 2 \pmod 8$, or $n \equiv 5,6 \pmod 8$. Therefore, $| M_g \cap V(P_n) | = 2\lfloor \frac{x}{4} \rfloor + i$, where

$$i = \begin{cases} 0 & \text{if } \quad x \equiv 0 \pmod 4, \\ 1 & \text{if } \quad x \equiv 1,2 \pmod 4, \\ 2 & \text{if } \quad x \equiv 3 \pmod 4). \end{cases}$$

This shows that if $n \equiv 5,6 \pmod 8$, then $\omega(g) = \omega(f) + 2$, otherwise $\omega(g) = \omega(f)$.

Finally, if $g : V(\mu(P_n)) \to \{-1, 1\}$ is an SDF of P_n with $g(w) = 1$ and $g(u_1) = g(u_n) = 1$, in a similar fashion, it can be shown that $\omega(f) \le \omega(g)$. Hence, $\gamma_s(\mu(P_n)) = \omega(f) = 2n + 1 - 2 \mid M_f \mid$. □

Theorem 5.4. *For a complete bipartite graph $K_{m,n}$ with $m \ge n \ge 2$, $\gamma_s(\mu(K_{m,n})) = 5$.*

Proof. Suppose that $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ are the partite sets of $K_{m,n}$. In Mycielski's construction, define $c(x_i) = x'_i$ for $1 \le i \le m$, $X' = \{x'_1, \ldots, x'_m\}$, $c(y_j) = y'_j$ for $1 \le j \le n$ and $Y' = \{y'_1, \ldots, y'_n\}$. Let w be the root vertex of $\mu(K_{m,n})$.

Let f be an SDF of $\mu(K_{m,n})$ with $f(w) = -1$. For $2 \le n \le 4$ and $2 \le m \le 3$, one can see that $\omega(f) \ge 7$. Now let $m \ge 4$ and $n \ge 5$ and both m and n are odd. Since $f[w] \ge 1$, it follows that $\sum_{i=1}^{m} f(x'_i) + \sum_{i=1}^{n} f(y'_i) \ge 2$. For $x' \in X'$, the inequality $f[x'] \ge 1$ implies that

$$\sum_{i=1}^{n} f(y_i) \ge 1. \tag{5.3}$$

Similarly, if $y' \in Y'$, the inequality $f[y'] \geq 1$ implies that

$$\sum_{i=1}^{m} f(x_i) \geq 1. \tag{5.4}$$

On the other hand, for $x \in X$, the inequalities $f[x] \geq 1$ and (5.3) show that $\sum_{i=1}^{n} f(y'_i) \geq -1$. Similarly, if $y \in Y$, then by the inequalities $f[y] \geq 1$ and (5.4) we have $\sum_{i=1}^{m} f(x'_i) \geq -1$. However if $\sum_{i=1}^{m} f(x'_i) = \sum_{i=1}^{n} f(y'_i) = -1$, $\sum_{i=1}^{m} f(x'_i) = 1, \sum_{i=1}^{n} f(y'_i) = -1$ or $\sum_{i=1}^{m} f(x'_i) = -1, \sum_{i=1}^{n} f(y'_i) = 1$, then $f[w] < 1$, a contradiction. Therefore, $\sum_{i=1}^{m} f(x'_i) \geq 1$ and $\sum_{i=1}^{n} f(y'_i) \geq 1$. If $M_f \cap X' = \varnothing$, then $\gamma_s(K_{m,n}) \geq m + 2 \geq 7$. Similarly, if $M_f \cap Y' = \varnothing$, then $\gamma_s(\mu(K_{m,n})) \geq n + 2 \geq 7$. Now assume $M_f \cap X'$ and $M_f \cap Y'$ are not empty.

If $x' \in M_f \cap X'$, then $\sum_{i=1}^{n} f(y_i) \geq 3$. Similarly, for $y' \in M_f \cap Y'$, we have $\sum_{i=1}^{n} f(x_i) \geq 3$. Thus, $\omega(f) \geq 3 + 3 + 1 + 1 - 1 \geq 7$. For the cases m, n are even or m, n have different parity, in a similar fashion, we can deduce that $\omega(f) \geq 7$.

Now let f be an SDF of $\mu(K_{m,n})$ with $f(w) = 1$. Since $f[w] \geq 1$, it follows that $f(X') + f(Y') \geq 0$. If there is no negative vertex in X or in Y, then $f(X) \geq 2$ or $f(Y) \geq 2$, respectively, because $m \geq n \geq 2$. If there is a negative vertex in X or in Y, then $f(Y) + f(Y') \geq 2$ or $f(X) + f(X') \geq 2$, respectively. Hence,

$$\omega(f) = f(X) + f(Y) + f(X') + f(Y') + f(w) \geq 5.$$

Define $f : V(\mu(K_{m,n})) \longrightarrow \{-1, 1\}$ as follows: First set $f(w) = 1$. Then if m or n is even, assign $1, 1, -1, 1, -1, \ldots, -1, 1$ to all vertices of X from $i = 1$ to $i = m$ and all vertices of Y from $j = 1$ to $j = n$, respectively. Then assign $1, -1, 1, -1, \ldots$ to all vertices in X' and all vertices in Y', respectively. Similarly, if m and n are odd, assign $1, 1, -1, 1, -1, 1, \ldots, 1, -1$ to all vertices of X and all vertices of Y. Then assign $1, -1, 1, -1, \ldots$ to all vertices in X' and all vertices in Y', respectively. It is easy to verify that f is an SDF of $\mu(K_{m,n})$ and

$$\mid M_f \cap U \mid = \begin{cases} \frac{n+m}{2} & m \text{ and } n \text{ are even,} \\ \frac{n+m-1}{2} & m \text{ and } n \text{ have diffrent parity,} \\ \frac{n+m-2}{2} & m \text{ and } n \text{ are odd,} \end{cases}$$

and

$$\mid M_f \cap V(G) \mid = \begin{cases} \frac{n+m-4}{2} & m \text{ and } n \text{ are even,} \\ \frac{n+m-3}{2} & m \text{ and } n \text{ have diffrent parity,} \\ \frac{n+m-2}{2} & m \text{ and } n \text{ are odd.} \end{cases}$$

Thus, $\mid M_f \mid = n + m - 2$. So, $\gamma_s(\mu(K_{m,n})) \leq \omega(f) = 2n + 2m + 1 - 2n - 2m + 4 = 5$. This completes the proof. □

We conclude this paper with the following problem.

Problem 5.5. Classify the graph G for which there exists a signed dominating function f with $f(w) = -1$ and $\omega(f) = \gamma_s(\mu(G))$, where w is the root of $\mu(G)$.

Acknowledgements. The authors would like to thank the referees for their useful suggestions and comments. Research supported by the research council of Faculty of Agriculture Engineering and Technology, University of Tehran, through grant No. 322870/1/02.

References

[1] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning and P.J. Slater, Signed domination in graphs. In: Vol. 1 of *Graph Theory, Combinatorics and Applications*. John Wiley & Sons, Inc. (1995) 311–322.

[2] D.C. Fisher, P.A. McKenna and E.D. Boyer, Hamiltonicity, diameter, domination, packing and biclique partitions of Mycielski's graphs. *Discrete Appl. Math.* **84** (1998) 93–105.

[3] R. Hass and T.B. Wexler, Bounds on the signed domination numbers of a graph. *Discrete Math.* **283** (2004) 87–92.

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Advanced Topics. Marcel Deker Inc. (1998).

[5] F.T. Hu, M.Y. Sohn and J. Lee, Bondage numbers of Mycielski graphs. *Bull. Malays. Math. Soc.* **2** (2016) 229–245.

[6] A.P. Kazemi, Roman domination and Mycielski's structure in graphs. *Ars Combin.* **106** (2012) 277–287.

[7] D.A. Mojdeh and N. Jafari Rad, On domination and its forcing in Mycielski's graphs. *Sci. Iran.* **15** (2008) 218–222.

[8] J. Mycielski, Sur le coloriage des graphes. *Colloq. Math.* **3** (1955) 161–162.

[9] S.M. Sheikholeslami, Forcing signed domination numbers in graphs. *Oplooska* **59** (2007) 171–179.

[10] M.Y. Shon, J. Lee and Y.S. Kwon, Lower bounds of signed domination number of a graph. *Bull. Korean Math. Soc.* **41** (2004) 181–188.

[11] D.B. West, Introduction to Graph Theory. Prentice-Hall (2000).

[12] B. Zelinka, Signed and minus domination in bipartite graphs. *Czeckoslovak Math. J.* **56** (2006) 587–590.