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INTEGRATED PRODUCTION INVENTORY MODEL WITH VARIABLE PRODUCTION RATE ON QUALITY OF PRODUCTS INVOLVING PROBABILISTIC DEFECTIVE UNDER VARIABLE SETUP COST

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Abstract. A predetermined production rate in a supply chain model with economic production lot size is quite appropriate for this type of situations as production rate can be changed in some cases to fulfill demand of customers. This paper investigates an integrated production inventory model with variable production rate on quality of products involving probabilistic defective under variable setup cost. As a rate of production has a direct impact on system performance, the production rate is considered as a variable along with the production cost. This production process gone through a long run system as a result after some specific time the production gone out-of-control state due to different issues and produced defective items. In addition, we consider that the defective follows three types of probability distribution function such as, (i) uniform, (ii) triangular and (iii) beta distributions. Two types of lead time crashed concept considering in this model and also we consider three types of continuous probabilistic defective function to find the associated cost of the system. The main objective is to find an optimal solution for an order quantity, safety factor, production cost, setup cost and to analyze how the flexibility of the production rate affects the process quality. An efficient iterative algorithm is designed to obtain the optimal solution of the model numerically and sensitivity analysis table formulate to show the impact of different parameter.

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1. Introduction

In aggressive business market of current conditions, numerous vendors and buyers might want to make long-term co-employable relationship as integrated of free market activity to limit costs and improve by and large quantity. Hence, lately, the two parties make a community oriented exertion between the associations for progress.

In the fundamental inventory model, the rate of production was expected as consistent. Be that as it may, as a rule. The machine production rate may effortlessly change. Machine apparatus cost. Likewise increments with increasing production rate. Additionally as the production rate expands, the probability of failure to the process may gradually increases which make the product to be deteriorated at some percentage. Existence of defective items is quite an obvious case in any long run manufacturing system. Moreover, the rate of defective production

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may increase due to inability of machines and high production rate. Due to the increasing production rate and inability of machines, the defective items may discover after the stage of inspection and also as high production, the emission are emitted from the industrial sectors.

Inspection in manufacturing is conducting inspection during the production process. This methodology of review controls the quality of products by helping to fix the sources of defects immediately after they are detected and it is helpful for any factory that needs to improve profitability, diminish imperfection, reduce the cost of rework and also reduce the defective rate. Those articles share one common characteristic about the defective rate which is known. But in real world applications, the defective rate be inexact and imprecise in nature probabilistic is the practical and suitable factor in this environment.

Inventory costs become more significant in the manufacturing organizations because of a changing environmental situations. Consequently, numerous organizations spend additional cash to preparing the laborers, procedural changes and concentrated hardware procurement so as to diminish the stock cost such as setup cost. For example, in India, little scale industries spend an additional expense by methods of investment and it is relied upon to have an outcome to decrease the total cost of the supply chain.

Due to the increasing production rate and inability of machines, the defective items may discover after the stage of inspection and also as high production, the emission are emitted from the industrial sectors. The vendor spend more money to reduce the setup cost to advertise the market competition.

2. Literature review

Traditional inventory models tend to obtain an economic (optimal) order quantity (EOQ) or economic production quantity (EPQ) based on the ordering/setup cost and the inventory carrying cost. They make many assumptions while coming up with a closed form solution for the most economical batch size in a stock or a production facility. One assumption is that the items produced by the facility are all of a perfect quality. Another is that the screening process that identifies the defective items in a lot is error-free. That is, the defective items from a lot can be screened out through 100% inspection. This is an idealistic approach. In practice the production lot may contain a substantial number of defective items, possibly because of long-run production process, deficient planned maintenance, inadequate work instructions and/or damage in transit. During the production, the system may transfer to out-of-control state from in-control state at any random time. Thus, producing items are not always perfect. Generally it depends on the condition of the production process. Also, the screening process, *e.g.*, at the end of an assembly line, is never perfect. Hence, there is a need to determine an optimal order quantity when the inspection process is prone to making errors while screening a defective lot.

Khan *et al.* [11] investigated an EOQ (economic order quantity) model with imperfect quality and inspection errors. Taheri-Tolgari *et al.* [24] proposed a discounted cash-flow approach to an imperfect production-inventory model under inflationary conditions and inspection errors. Hsu and Hsu [7] developed two EPQ (economic production quantity) models with imperfect production process, inspection errors, planned backorders and sales returns. Hsu and Hsu [8] developed an EOQ model with imperfect quality items, inspection errors, shortage backordering, and sales returns. Jauhari *et al.* [10] investigated an integrated inventory model for manufacturer–retailer system assuming inspection errors. Pal and Mahapatra [16] developed a manufacturing oriented supply chain with inspection errors. Dey and Giri [5] developed a learning in inspection in an integrated vendor–buyer model with imperfect production process [1] analyzed a supply chain model with vendor-managed inventory, consignment and quality inspection errors. Priyan and Uthayakumar [20] proposed an integrated production-distribution inventory system involving probabilistic defective and errors in quality inspection under variable setup cost. Dey and Giri [4] proposed a new approach to deal with learning in inspection in an integrated vendor–buyer model with imperfect production process. Jaber *et al.* [9] considered the supply chain coordination with emission reduction incentives. Palanivel and Uthayakumar [18] proposed a production-inventory model with promotional effort, variable production cost and probabilistic deterioration. Palanivel and Uthayakumar [17] an EPQ model with variable production, probabilistic deterioration and partial backlogging under inflation.

In the basic supply chain model, the rate of production was assumed as constant. However, in many cases, the machine production rate may easily change [12]. Machine tool cost also increases with increasing production rate. Moreover, as the production rate increases, the probability of failure to the process may gradually increase, which causes the product quality to be deteriorated at some percentage. As an illustration of this idea, assume a production process with robotic assembly system, robot repeatability decreases with increasing production rate [14]. Rosenblatt and Lee [21] considered the production process acts perfectly at the initial stage, but after an amount of time τ, it shifts from in control state to out of control state and starts producing imperfect products. The elapsed time until the production process reaches to the "out-of-control" state was considered to be a negative exponentially distributed random variable with mean $1/\mu$. Khouja and Mehrez [12] reconsidered the Rosenblatt and Lee [21] work and assumed the mean of the negative exponentially distributed random variable as a function of production rate, namely, quality function. They introduced a linear and quadratic from which are increasing function of production rate. This study is motivated from the idea of Khouja and Mehrez [12]. Sarkar *et al.* [23] developed an effects of variable production rate on quality of products in a single-vendor multi-buyer supply chain management. Zanoni *et al.* [25] proposed a vendor-managed inventory with consignment stock agreement for single-vendor single-buyer under the emission-trading scheme.

To improve customer service and to reduce stock out loss, it is important to reduce lead time. Liao and Shyu [13] first incorporated a probabilistic inventory model assuming lead time as a unique decision variable. Ben-Daya and Raouf [3] considered an inventory model as an extension of Liao and Shyu [13] model where lead time is one of the decision variables. Ben-Daya and Raouf [3] model dealt with no shortage and continuous lead time. Ouyang *et al.* [15] extended Ben-Daya and Raouf [3] model by assuming discrete lead time and shortages. Pan and Yang [19] analyzed an integrated inventory model with lead time in a controllable manner. Annadurai and Uthayakumar [2] developed a periodic review inventory model under controllable lead time and lost sales reduction.

Each and every production company wants to sell their product more, so they wants to produce more perfect and reliable product compare to others. Customers/retailers always wants more profitable products. Today's customers wants more perfect product, they don't bother about the cost. Customer always wants more quality product, so up gradation of quality of any product is one of the main target of the production industries. The quality of product can be improved by some investment discussed by Sarkar and Moon [22]. They also reduced setup cost for an imperfect production process in this model. Hemapriya and Uthayakumar [6] developed an inventory model with uncertain demand and lost sales reduction under service level constraint.

Different researcher developed different types of model under the consideration of imperfect production, multi-product production system with safety stock, and setup cost reduction, but no one developed any model for single-vendor single-buyer for defective products involving probabilistic defective with the role of human factors such as errors in quality inspection and along with the consideration of partial backorder, normally distributed lead time, shortage, variable production cost, quality function, greenhouse gas and and investment strategy to reduce/control setup cost. In addition, we consider that the defective follows three different types of probability distribution function such as (i) uniform, (ii) triangular, and (iii) beta distribution. Therefore, this research paper intends to fill this remarkable gap in the inventory literature. There is a big research gap in this direction, which is fulfilled by this research. The comparison of present stochastic model with some other existing literatures are tabulated in Table 1.

3. Notation and assumptions

3.1. Notation

The following notation is used in this model.

3.1.1. General parameters

D Average demand per year (units/year)

S_0 The initial setup cost of the vendor ($/setup)

Table 1. A comparison of the present model with related existing models.

Model	Imperfect production	Inspection errors	Rework	Greenhouse gases emission	$MTTF$	Investment
Alfares and Attia [1]	✓	✓				
Dey and Giri [4]	✓					
Hsu and Hsu [7]	✓	✓				
Annadurai and Uthayakumar [2]						✓
Khouja and Mehrez [12]	✓				✓	
Priyan and Uthayakumar [20]	✓	✓				✓
Jaber *et al.* [9]				✓		
Zanoni *et al.* [25]				✓		
Sarkar *et al.* [23]			✓		✓	
Present paper	✓	✓	✓	✓	✓	✓

A	The ordering cost of the buyer (\$/order)
h_v	The holding cost rate of the vendor per unit per unit time (\$/unit/year)
L	Length of lead time (days)
π	Backorder cost per unit time for vendor (\$/unit)
F	Buyer's transportation cost per unit of the product shipped (\$/shipment)
R	Rework cost (\$/unit)
s	Safety stock
$R(L)$	Crashing cost
T	Transportation time
σ	Standard deviation (unit/week)
ζ_1	Increasing machining cost due to single unit increase in production rate (\$/unit time)
ζ_2	Machine running cost per unit time (\$/unit time)
N	Number of defective goods in a production cycle (units)
t	Actual production run time (time/ units)
$\eta(P)$	Elapsed time that the process goes "out-of-control" (Exponential random variable).
$C(P)$	Production cost (\$/time)

3.1.2. Defective sector parameters

h_{b1}	The holding cost rate of the buyer for defective items per item per year (\$/unit/year)
h_{b2}	The holding cost rate of the buyer for non-defective items per item per year (\$/unit/year)
w	Buyer's unit screening cost
x	Buyer's screening rate (units/year)
y	Defective percentage among the lot-size Q
y_1	Probability of 1st type error (classifying a usable product as defective)
y_2	Probability of 2nd type error (classifying a defective product as usable)
γ	Percentage of defective products supplied by the vendor
Y_e	The percentage of defective products observed by the buyer through screening
$f(y_1)$	Probability density function of m_1
$f(y_2)$	Probability density function of m_2
$f(y)$	Probability density function of y
c_a	Cost of falsely accepting a defective product
c_r	Cost of falsely rejecting a non-defective product

3.1.3. Emission sector parameters

a	Emissions function parameter (ton.year^2/unit^3)
b	Emissions function parameter (ton.year/unit^2)
c	Emissions function parameter (ton/unit)
c_t	Emissions tax (\$/ton)
c_p	Emissions penalty (\$/year) for exceeding emissions limit
E	Greenhouse gas (CO_2) emissions (ton/unit)
$E_{\lim}$	Emissions limit (ton/year)
α	minimum production-demand ratio, where $\alpha \geq 1$
$P_{\max}$	Maximum attainable production rate (unit/year)
Y	Emissions limit variable or emissions limit, which is 1 if the emissions exceed the allowable limit and 0 otherwise

3.1.4. Decision variables

Q	Size of the shipment form the vendor to the buyer (units)
n	Number of lots in which the item are delivered from the vendor (integer)
S	Setup cost for the vendor (\$/setup)
P	Production rate (units/time)
k	Safety factor

3.2. Assumptions

(1) The integrated supply chain model consists of a single-vendor and a single-buyer for a single product.
(2) The vendor's production rate P is finite and it is restricted to upper and lower limits $P_{\min}$ and $P_{\max}$ with $P_{\min} = \alpha D$.
(3) Since $\alpha \geq 1$, the minimum production rate $P_{\min}$ will be greater than D, it avoids the shortages.
(4) Replenishments are made when the on hand inventory reaches the reorder point R (the inventory is reviewed continuously).
(5) If lead time is high then lost sale increases which causes a huge loss to the industry. The known mean and standard deviation of lead time demand are DL and $DL(T)$, respectively and the corresponding standard deviation as $\sigma\sqrt{L}$ and $\sigma\sqrt{T}$, respectively. Thus the safety stock for the first batch $s = k\sigma\sqrt{L}$ and the safety stock for the second batch to onwards is defined as $s = k\sigma\sqrt{T}$.
(6) Defective follows continuous probability distribution function as (i) uniform distribution, (ii) triangular distribution, and (iii) beta distribution.
(7) For all products, the lead time L consists of n mutually independent components. The ith component has a normal duration b_i, minimum duration a_i, and crashing cost per unit time c_i such that $c_1 \leq c_2 \leq \ldots \leq c_n$. The components of lead time are crashed one at a time starting from the first component because it has the minimum unit crashing cost, and then the second component, and so on. Let $L_0 = \sum_{i=1}^{m} b_i$, and L_i be the length of the lead time with components $1, 2, \ldots, i$ crashed to their minimum duration, then L_i can be expressed as $L_i = L_0 - \sum_{j=1}^{i}(b_j - a_j)$, $i = 1, 2, \ldots, m$; and for all products, the lead time crashing cost per cycle $C(L)$ is given by $R(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$, $L \in [L_i, L_{i-1}]$.
(8) The elapsed time after the production system goes out-of-control is an exponentially distributed random variable and the mean of the exponential distribution is a decreasing function of the production rate [12].

4. Mathematical model

In this section, we tend to develop the essential vendor–buyer integrated model, in which buyer place an order of total size nQ to the vendor. The vendor produced nQ (non-defective) items to the buyer with the finite production rate P. The produced lot is shipped to the buyer is $n-$ equal sized shipments. Therefore the vendor will make a delivery at regular intervals of $\frac{Q(1-y)}{D}$ units until the inventory level reach the reorder point where we have presumed that each lot contains y percentage of defective items. Therefore, the length of one complete production cycle is $\frac{nQ(1-y)}{D}$.

4.1. Buyer's perspective

The buyer places an associate order of size Q for non-defective items to the vendor. The vendor delivers these items once a relentless time interval L. On arrival of an order, the buyer inspects the items at a set screening rate x. Screening is ideal. (*i.e.*) all defective items are found. Therefore, the buyer kept these defective items in their warehouse and return these defective items to the vendor at the time of next delivery. Hence the buyer have two holding cost (i) defective holding cost (ii) non-defective holding cost. Therefore, the buyer's average inventory level of non-defective items (before the end of the screening time) in a cycle is similar [4] which is given by,

$$\frac{nQ(1-y)}{D}\left[k\sigma\sqrt{L}+\frac{Q(1-y)}{2}+\frac{DQy}{2x(1-y)}\right].$$

Similarly, the average inventory level for defective items after the screening time in a cycle is given by,

$$nQ^2y\left[\frac{1-y}{D}-\frac{1}{2x}\right].$$

As a production machine produced defective items in long run, shortages must occur, the backorder quantity for buyers are $E(x_1-r_1)^+$ and $E(x-r_2)^+$, then the shortage cost per item per unit time is given by,

$$\frac{D\pi}{Q}E(x_1-r_1)^+ + \frac{D\pi(n-1)}{Q}E(x_2-r_2)^+.$$

Therefore, the total cost for the buyer including the ordering cost, transportation cost, holding cost, shortage cost and lead time crashing cost is given by,

$$\begin{aligned}TC_b = {}& \frac{AD}{nQ(1-y)}+h_{b1}\left[Qy-\frac{DQy}{2x(1-y)}\right]+h_{b2}\left[k\sigma\sqrt{L}+\frac{Q(1-y)}{2}+\frac{DQy}{2x(1-y)}\right]\\ &+\frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-y)}+\frac{D\pi(n-1)\sigma\sqrt{T}\psi(k)}{Q(1-y)}+\frac{D(F+R(L))}{Q(1-y)}+\frac{wD}{1-y}\end{aligned} \tag{4.1}$$

where $\psi(k)=\int_k^\infty (z-k)\phi(z)dz$, $\phi(z)$ be the standard normal probability density function.

4.2. Vendor's perspective

During the production period, the vendor produces Q items in the first stage and delivers those to the buyer. After that the vendor delivers a amount Q to the buyer each T units of time where $T=\frac{Q(1-y)}{D}$. This continues until the vendor's production run is completed. By assumptions, vendor's production for non-defective items rate is bigger than the demand rate. Therefore the vendor's inventory level step by step will increase until the production is over. Then the vendor's average level inventory holding cost is similar to Dey and Giri [4] which is given by,

$$h_v\frac{Q}{2}\left[n\left(1-\frac{D}{P(1-y)}\right)-1+\frac{2D}{P(1-y)}\right].$$

To fulfill buyer's demand, vendor produced Q quantity at a production rate P, and the production cost is $C(P)$, as it is too much difficult to guess how much production is needed and in this model a variable production rate P with variable production cost is considered for the vendor which is given by, $C(P)=\zeta_1P+\frac{\zeta_2}{P}$ and $P_{\min}>D$. A continuous investment S is used to reduce the setup cost of vendor that is, $b\log\left(\frac{S_0}{S}\right)$.

Consequently, the total cost for the vendor incorporating the setup cost, holding cost and production cost is given by,

$$TC_v = \frac{SD}{nQ(1-y)} + h_v \frac{Q}{2}\left[n\left(1 - \frac{D}{P(1-y)}\right) - 1 + \frac{2D}{P(1-y)}\right] + \frac{D}{(1-y)}(\zeta_1 P + \frac{\zeta_2}{P}) + b\log\left(\frac{S_0}{S}\right). \tag{4.2}$$

The joint total cost for the both vendor and buyer is obtained as follows:

$$\begin{aligned}\mathrm{JTC} = {} & \frac{(A+S)D}{nQ(1-y)} + h_{b1}\left[Qy - \frac{DQy}{2x(1-y)}\right] + h_{b2}\left[k\sigma\sqrt{L} + \frac{Q(1-y)}{2} + \frac{DQy}{2x(1-y)}\right] \\ & + \frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-y)} + \frac{D\pi(n-1)\sigma\sqrt{T}\psi(k)}{Q(1-y)} + \frac{D(F+R(L))}{Q(1-y)} + \frac{D}{1-y}\left(w + \left(\zeta_1 P + \frac{\zeta_2}{P}\right)\right) \\ & + h_v\frac{Q}{2}\left[n\left(1 - \frac{D}{P(1-y)}\right) - 1 + \frac{2D}{P(1-y)}\right] + b\log\left(\frac{S_0}{S}\right).\end{aligned} \tag{4.3}$$

4.3. Integrated approach

The integrated system continually tries to maximize the average total cost of the system. The expected annual total cost of the integrated system is obtained as follows:

$$\begin{aligned}\mathrm{JETC}(Q,P,A,k,L,n) = {} & \frac{(A+S)D}{nQ(1-E[y])} + h_{b1}\left[QE[y] - \frac{DQE[y]}{2x(1-E[y])}\right] + h_{b2}\left[k\sigma\sqrt{L}\right. \\ & \left. + \frac{Q(1-E[y])}{2} + \frac{DQE[y]}{2x(1-E[y])}\right] + \frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y])} + \frac{D\pi(n-1)}{Q(1-E[y])} \\ & \times \sigma\sqrt{T}\psi(k) + \frac{D(F+R(L))}{Q(1-E[y])} + \frac{D}{1-E[y]}\left(w + \left(\zeta_1 P + \frac{\zeta_2}{P}\right)\right) \\ & + h_v\frac{Q}{2}\left[n\left(1 - \frac{D}{P(1-E[y])}\right) - 1 + \frac{2D}{P(1-E[y])}\right] \\ & + b\log\left(\frac{S_0}{S}\right)\end{aligned} \tag{4.4}$$

subject to $0 < S \leq S_0$ where $E[T] = (1 - E[y]Q/D)$.

4.4. Inspection errors

In this subsection, we consider a situation that the inspectors can commit two types of errors at the buyer's end while screening the items. Type I error in which some non-defective items are categorized as defective (*i.e.*, $(1-y)y_1$) and Type II error in which some defective items are categorized as non-defective (*i.e.*, (yy_2)). These errors might have associate adverse result on the flexibility of associate scrutiny attempt to guarantee product quality Thus the percentage of defective units would be:

$$y_e = (1-y)y_1 + y(1-y_2).$$

We assume that y, y_1 and y_2 are mutually independent variables. Thus,

$$[y_e] = (1-E[y])E[y_1] + E[y](1-E[y_2]).$$

So the time interval between two successive shipments would be,

$$E[T] = \frac{(1-E[y_e])Q}{D} = \frac{(1-(1-E[y])E[y_1] - E[y](1-E[y_2])Q)}{D}.$$

The errors in screening will cause the buyer's costs of misclassification. That is cost of false rejection of non-defective items (Type I error) and the cost of false acceptance of defective items (Type II error). We assume that the the cost of false acceptance is greater than the false rejection. Since in the case of critical components, such as in health care industries where safety is the highest priority. Consolidating these costs and rework cost, the joint expected total cost given in equation (4.4), becomes

$$\begin{aligned}\mathrm{JETC}(Q,P,A,k,L,n) = {}& \frac{(A+S)D}{nQ(1-E[y_e])} + h_{b1}\left[QE[y_e] - \frac{DQE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\left[k\sigma\sqrt{L}\right.\\ &\left. + \frac{Q(1-E[y_e])}{2} + \frac{DQE[y_e]}{2x(1-E[y_e])}\right] + \frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y_e])} + \frac{D\pi(n-1)\sigma\sqrt{T}}{Q(1-E[y_e])}\\ &\times \psi(k) + \frac{D(F+R(L))}{Q(1-E[y_e])} + \frac{D}{1-E[y_e]}\left(w + \left(\zeta_1 P + \frac{\zeta_2}{P}\right)\right) + b\log\left(\frac{S_0}{S}\right)\\ &+ h_v\frac{Q}{2}\left[n\left(1-\frac{D}{P(1-E[y_e])}\right) - 1 + \frac{2D}{P(1-E[y_e])}\right] + \frac{c_a E[y]E[y_2]D}{(1-E[y_e])}\\ &+ \frac{c_r(1-E[y])E[y_1]D}{(1-E[y_e])}. \end{aligned} \tag{4.5}$$

Subject to: $0 < S \le S_0$.

4.5. Greenhouse gases emissions

In order to reduce the emissions, from production and to shield the atmosphere, some legislative systems have taken several actions such as implementing taxes and penalties. The production rate is assumed to be finite and ranges over a finite interval $[P_{\min}, P_{\max}]$. The relationship between the production rate and GHG emissions are given below (as in [9, 25])

$$E = a^*P^2 - b^*P + c^*. \tag{4.6}$$

The case where emissions can be expressed as a convex function of production rate. Since the consumption of energy is usually associated with the generation of GHG emissions, it is therefore reasonable to infer a function of the form given in equation (4.6). The total cost term for emissions is the sum of cost of emission of GHG per year and the total penalty paid for exceeding the emission limit E_{lim}. *i.e.*, Total GHG emissions cost is given as

$$\mathrm{TEC} = E \times D \times C_t + Y \times C_p, \tag{4.7}$$

Therefore, the total cost per unit of time is the sum of the supply chain relevant cost and the greenhouse emission cost per unit of time including penalties are,

$$\begin{aligned}\mathrm{Min}\,\mathrm{JETC}(Q,P,A,k,L,n) = {}& \frac{(A+S)D}{nQ(1-E[y_e])} + h_{b1}\left[QE[y_e] - \frac{DQE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\left[k\sigma\sqrt{L}\right.\\ &\left. + \frac{Q(1-E[y_e])}{2} + \frac{DQE[y_e]}{2x(1-E[y_e])}\right] + \frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y_e])} + \frac{D\pi(n-1)\sigma}{Q(1-E[y_e])}\\ &\times \sqrt{T}\psi(k) + \frac{D(F+R(L))}{Q(1-E[y_e])} + \frac{D}{1-E[y_e]}\left(w + \left(\zeta_1 P + \frac{\zeta_2}{P}\right)\right) + h_v\frac{Q}{2}\left[n\right.\\ &\left.\times\left(1-\frac{D}{P(1-E[y_e])}\right) - 1 + \frac{2D}{P(1-E[y_e])}\right] + \frac{D}{(1-E[y_e])}\\ &+ b\log\left(\frac{S_0}{S}\right) + \frac{c_a E[y]E[y_2]D}{(1-E[y_e])} + \frac{c_r(1-E[y])E[y_1]D}{(1-E[y_e])}\\ &+ E\times D\times C_t + Y \times C_p, \end{aligned} \tag{4.8}$$

subject to:

$$Y = \begin{cases} 1 & \text{if} \quad E \times D > E_{\lim} \\ 0 & \text{else.} \end{cases} \tag{4.9}$$

$$0 < S \leq S_0$$

$$P_{\min} = \alpha D \leq P \leq P_{\max}. \tag{4.10}$$

Clearly the function (4.6) attains minimum when $P = P_0 = \frac{b}{2a}$ but P_0 is not an optimal production rate for the system. Here, the penalties depend on the worth of emissions and consequently on the worth of P. Condition (4.9) ensures that Y takes on the value 1 once the corresponding emissions limit $E_{\lim}$ is exceeded and condition (4.10) restricts the production rate to vary between a minimum and maximum which can result to technical reasons. Therefore, the best answer P is up to $P_{\min}$ or P_0 or it lies between these two values.

In order to establish the relationship between the production rate and the process quality, we assumed that $f(P)$ as an increasing function of P and it denotes the number of failure of production process with an increased production rate. Appropriately, $1/f(P)$ denotes the mean time to failure and it becomes a decreasing function of the production rate P [12]. Therefore from the above analysis, it shows that when the production rate is increased, the mean time to failure decreases.

The number of defective units in a production cycle is,

$$N = \begin{cases} 0 & \text{if} \quad \eta \geq t \\ \alpha P(t - \eta(P)) & \text{if} \quad \eta \leq t. \end{cases}$$

Therefore, the expected number of defective units in a production cycle is,

$$E(N) = \alpha P \left[\frac{Q}{P} + \frac{1}{f(P)} e^{-(\frac{Qf(P)}{P})} - \frac{1}{f(P)} \right].$$

For the small value of $f(P)$, using the Maclaurin series for, $e^{\left(-\frac{Qf(P)}{P}\right)}$ which yields

$$e^{\left(-\frac{Qf(P)}{P}\right)} = 1 - \frac{Qf(P)}{P} + \frac{(Qf(P))^2}{2P^2}.$$

From the above two equations, we obtain $E(N) = \alpha f(P) \frac{Q^2}{2P}$ and thus the expected rework cost is given by, $R \frac{D}{Q} E(N) = RD\alpha f(P) \frac{Q}{2P}$. When the machines are inoperative, *i.e.*, the production process ceases, there is no chance of any defective products to be created. As the machines change into operation mode, the chances of the arrival of defective goods appear. Thus, assuming the mean time to failure, independent of production rate is inappropriate in practical aspect. An experimental observation or relationship between production rate and mean time to failure is unavailable. Three different cases are introduced with three different functions to define the mean time to failure.

Case 1 $\dfrac{1}{f(P)} = \dfrac{1}{b_1 P} \times$ (The quality function $f(P)$ is linear in P)

Case 2 $\dfrac{1}{f(P)} = \dfrac{1}{b_2 P + c_2 P^2} \times$ (The quality function $f(P)$ is quadratic in P)

Case 3 $\dfrac{1}{f(P)} = \dfrac{1}{b_3 P + c_3 P^2 + d_3 P^3} \times$ (The quality function $f(P)$ is cubic in P)

where b_1, b_2, c_2, b_3, c_3 and d_3 are non-negative real numbers that provide the best fit for the function $f(P)$ as well as $\frac{1}{f(P)}$.

Now, we consider the buyer's defective rate y follows a three types of distributions (i) uniform distributions, (ii) triangular distribution and (iii) beta distribution with expected value $E[y]$ which is similar to Priyan and Uthayakumar [20]. The three cases of defective rate with three cases of $\frac{1}{f(P)}$, result in 9 cases of this model, we now solve for each of the nine cases.

4.6. Defective rate follows uniform distribution

In this section, we assume that the defective rate follows a uniform distribution with expected value $E[y] = \frac{a+b}{2}$, $a > 0, b > 0, a < b$.

Case 1: $f(P)$ is linear in P

Now, the joint expected total cost with linear function $f(P)$ is given by,

$$\begin{aligned}\text{Min JETC}_{11}(Q,P,A,k,L,n) = {} & \frac{(A+S)D}{nQ(1-E[y_e])} + h_{b1}\left[QE[y_e] - \frac{DQE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\Bigg[k\sigma\sqrt{L} \\ & + \frac{Q(1-E[y_e])}{2} + \frac{DQE[y_e]}{2x(1-E[y_e])}\Bigg] + \frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y_e])} + \frac{D\pi(n-1)\sigma}{Q(1-E[y_e])} \\ & \times \sqrt{T}\psi(k) + \frac{D(F+R(L))}{Q(1-E[y_e])} + \frac{D}{1-E[y_e]}\left(w + \left(\zeta_1 P + \frac{\zeta_2}{P}\right)\right) + h_v\frac{Q}{2}\Bigg[n \\ & \times\left(1 - \frac{D}{P(1-E[y_e])}\right) - 1 + \frac{2D}{P(1-E[y_e])}\Bigg] + \frac{D}{(1-E[y_e])} \\ & + b\log\left(\frac{S_0}{S}\right) + RD\alpha b_1 P\frac{Q}{(1-E[y_e])2P} + \frac{c_a E[y]E[y_2]D}{(1-E[y_e])} \\ & + \frac{c_r(1-E[y])E[y_1]D}{(1-E[y_e])} + E\times D\times C_t + Y\times C_p, \end{aligned} \tag{4.11}$$

subject to:

$$0 < S \le S_0,$$
$$P_{\min} = \alpha D \le P \le P_{\max},$$

where $E[y_e] = (1 - \frac{a+b}{2})E[y_1] + \frac{a+b}{2}(1 - E[y_2])$.

The problem expressed in the previous section occurs as constrained non-linear program. To solve this kind of nonlinear problem, we pursue the similar procedure of most of the literature dealing with nonlinear problem. That is, first we temporarily ignore the constraints then to determine the optimum solutions. For fixed Q, P, k, S and n, $\text{JETC}_{11}(Q,P,S,k,L,n)$ is a concave function of $L \in [L_i, L_{i-1}]$, since it is easy to see that

$$\frac{\partial^2 \text{JETC}_{11}}{\partial L^2} = -\frac{h_{b2}k\sigma}{4L\sqrt{L}} - \frac{D\pi\sigma\psi(k)}{4L\sqrt{L}Q(1-E[y_e])} < 0. \tag{4.12}$$

Therefore, for the fixed Q, P, S, k and $L \in [L_i, L_{i-1}]$, $\text{JETC}_{11}(Q,P,S,k,L,n)$ is a convex in n. Since,

$$\frac{\partial^2 \text{JETC}_{11}}{\partial n^2} = \frac{2(A+S)D}{n^3 Q(1-E[y_e])} > 0. \tag{4.13}$$

Now, for fixed n and $L \in [L_i, L_{i-1}]$, equating to zero the first derivatives of JETC_{11} with respect to Q, P, S and k, we have

$$\begin{aligned}\frac{\partial^2 \text{JETC}_{11}}{\partial Q} = {} & -\frac{1}{Q^2}\Bigg(\frac{DA}{n(1-E[y_e])} + \frac{D\pi\psi(k)\sigma\sqrt{L}}{1-E[y_e]} + \frac{D\pi(n-1)\sigma\sqrt{T}\psi(k)}{1-E[y_e]} + \frac{D(F+R(L))}{1-E[y_e]} \\ & + \frac{SD}{n(1-E[y_e])}\Bigg) + G(n,y) \end{aligned} \tag{4.14}$$

where

$$G(n,y) = \left(h_{b1}\left[E[y_e] - \frac{DE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\left[\frac{1-E[y_e]}{2} + \frac{DE[y_e]}{2x(1-E[y_e])}\right] + h_v\left[n\left(1 - \frac{D}{P(1-E[y_e])} - 1 + \frac{2D}{P(1-E[y_e])}\right)\right] + \frac{RD\alpha b_1 P}{(1-E[y_e])2P}\right)$$

$$\frac{\partial^2 \mathrm{JETC}_{11}}{\partial P} = \frac{1}{P^2}\left(\frac{h_v QD(n-2)}{2(1-E[y_e])} - \frac{D\zeta_2}{1-E[y_e]}\right) + (2aP-b)Dc_t + \frac{D\zeta_1}{1-E[y_e]} \tag{4.15}$$

$$\frac{\partial^2 \mathrm{JETC}_{11}}{\partial S} = \frac{D}{nQ(1-E[y_e])} - \frac{b}{S} \tag{4.16}$$

$$\frac{\partial^2 \mathrm{JETC}_{11}}{\partial k} = (\Phi(k)-1)\left[\frac{D\pi\sigma\sqrt{L}}{Q(1-E[y_e])} + \frac{D\pi(n-1)\sigma\sqrt{T}}{Q(1-E[y_e])}\right] + h_{b2}\sigma\sqrt{L}. \tag{4.17}$$

Therefore, we obtain

$$Q = \left[\frac{\left(\frac{DA}{n} + D\pi\psi(k)\sigma\sqrt{L} + D\pi(n-1)\sigma\sqrt{T}\psi(k) + D(F+R(L)) + \frac{SD}{n}\right)}{(1-E[y_e])G(n,y)}\right]^{\frac{1}{2}} \tag{4.18}$$

$$P = \left[\frac{\zeta_2 + h_v Q(2-n)}{2c_t(1-E[y_e])(2aP-b) + 2\zeta_1}\right]^{\frac{1}{2}} \tag{4.19}$$

$$S = \frac{bnQ(1-E[y_e])}{D} \tag{4.20}$$

$$\Phi(k) = 1 - \frac{h_{b2}Q\sigma\sqrt{L}(1-E[y_e])}{D\pi(\sigma\sqrt{L} + (n-1)\sigma\sqrt{T})}. \tag{4.21}$$

Theoretically, for fixed n, from equations (4.18) to (4.21), we can obtain the values of Q, P, S and k when $0 < S \leq S_0$ is inactive. For fixed n, the Hessian matrix for $\mathrm{JETC}_{11}(Q,P,S,k,L,n)$ is positive definite at point (Q,P,S,k,L,n). (see Appendix A). We cannot obtain the explicit general solution for Q, P, S and k by solving equations (4.18)–(4.21), because the evaluation of each of the expressions needs knowledge of the value of the other. However, we can obtain the optimal value of Q, P, S and k by using an iterative procedure. The following algorithm is developed to find the optimal values for the order quantity, production, setup cost, safety factor and lead time.

Algorithm

Step 1: Set $n = 1$.

Step 2: For each $L \in [L_i, L_{i-1}]$, determine the $\mathrm{JETC}_{11}(Q_{11}, P_{11}, S_{11}, k_{11}, L, n)$ using the sub-algorithm.

Step 3: Find $\mathrm{Min}\,\mathrm{JETC}_{11}(Q_{11}, P_{11}, S_{11}, k_{11}, L, n)$ for every $L \in [L_i, L_{i-1}]$.

Step 4: Let $\mathrm{JETC}^*_{11}(Q^*_{11}, P^*_{11}, S^*_{11}, k^*_{11}, L^*, n^*) = \mathrm{Min}\,\mathrm{JETC}_{11}(Q_{11}, P_{11}, S_{11}, k_{11}, L, n)$ for every $L \in [L_i, L_{i-1}]$ then $\mathrm{JETC}^*_{11}(Q^*_{11}, P^*_{11}, S^*_{11}, k^*_{11}, L^*, n^*)$ minimum joint expected total cost of the proposed model and $(Q^*_{11}, P^*_{11}, S^*_{11}, k^*_{11}, L^*, n^*)$ is the optimal solution.

Sub-Algorithm

Step 1: Repeat step (1.1)–(1.3) until no change occur in the values Q_{11}, P_{11}, S_{11} and k_{11}. Denote the solution by $(Q^*_{11}, P^*_{11}, S^*_{11}, k^*_{11}, L^*, n^*)$.

Step 1.1: Start with $P_1 = P_{\min}$, $S_1 = S_0$ and $k_1 = 0$

Step 1.2 Substituting P_1, S_1 and k_1 into equation (4.18) evaluates Q_1.

Step 1.3 Utilizing Q_1 to determine P_2, S_2 and k_2 from equations (4.19) to (4.21).

Step 2: Compare S_{11}^* with S_0.
Step 2.1: If $S_{11}^* < S_0$ then go to step 3.
Step 2.2: If $S_{11}^* > S_0$, then for this given $L \in [L_i, L_{i-1}]$, let If $S_{11}^* = S_0$ and utilize equation (4.18) to determine new Q_{11}^* and go to step 3.
Step 3: Compute the corresponding $\mathrm{JETC}_{11}(Q_{11}, P_{11}, S_{11}, k_{11}, L, n)$ by putting $Q = Q_{11}$, $P = P_{11}$, $S = S_{11}$ and $k = k_{11}$ in equation (4.11).

Case 2: $f(P)$ is quadratic in P

Now, the joint expected total cost with quadratic function $f(P)$ is given by,

$$\begin{aligned}\mathrm{Min\,JETC}_{12}(Q,P,A,k,L,n) &= \frac{(A+S)D}{nQ(1-E[y_e])} + h_{b1}\left[QE[y_e] - \frac{DQE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\Bigg[k\sigma\sqrt{L} \\ &+ \frac{Q(1-E[y_e])}{2} + \frac{DQE[y_e]}{2x(1-E[y_e])}\Bigg] + \frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y_e])} + \frac{D\pi(n-1)\sigma}{Q(1-E[y_e])} \\ &\times \sqrt{T}\psi(k) + \frac{D(F+R(L))}{Q(1-E[y_e])} + \frac{D}{1-E[y_e]}\left(w + \left(\zeta_1 P + \frac{\zeta_2}{P}\right)\right) + h_v\frac{Q}{2}\Bigg[n \\ &\times \left(1 - \frac{D}{P(1-E[y_e])}\right) - 1 + \frac{2D}{P(1-E[y_e])}\Bigg] + \frac{D}{(1-E[y_e])} \\ &+ b\log\left(\frac{S_0}{S}\right) + RD\alpha(b_2P + c_2P^2)\frac{Q}{(1-E[y_e])2P} + \frac{c_aE[y]E[y_2]D}{(1-E[y_e])} \\ &+ \frac{c_r(1-E[y])E[y_1]D}{(1-E[y_e])} + E\times D\times C_t + Y\times C_p, \end{aligned} \tag{4.22}$$

subject to:

$$0 < S \le S_0,$$
$$P_{\min} = \alpha D \le P \le P_{\max},$$

where $E[y_e] = (1 - \frac{a+b}{2})E[y_1] + \frac{a+b}{2}(1 - E[y_2])$.

Now, for fixed n and $L \in [L_i, L_{i-1}]$, equating to zero the first derivatives of JETC_{12} with respect to Q, P, S and k, we obtain,

$$Q = \left[\frac{\left(\frac{DA}{n} + D\pi\psi(k)\sigma\sqrt{L} + D\pi(n-1)\sigma\sqrt{T}\psi(k) + D(F+R(L)) + \frac{SD}{n}\right)}{(1-E[y_e])G(n,y)}\right]^{\frac{1}{2}} \tag{4.23}$$

where

$$\begin{aligned} G(n,y) = &\Bigg(h_{b1}\left[E[y_e] - \frac{DE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\left[\frac{1-E[y_e]}{2} + \frac{DE[y_e]}{2x(1-E[y_e])}\right] \\ &+ h_v\left[n\left(1 - \frac{D}{P(1-E[y_e])} - 1 + \frac{2D}{P(1-E[y_e])}\right)\right] + \frac{RD\alpha(b_2P + c_2P^2}{(1-E[y_e])2P}\Bigg) \end{aligned}$$

$$P = \left[\frac{\zeta_2 + h_vQ(2-n)}{2c_t(1-E[y_e])(2aP-b) + 2\zeta_1 + R\alpha c_2Q}\right]^{\frac{1}{2}} \tag{4.24}$$

$$S = \frac{bnQ(1-E[y_e])}{D} \tag{4.25}$$

$$\Phi(k) = 1 - \frac{h_{b2}Q\sigma\sqrt{L}(1-E[y_e])}{D\pi(\sigma\sqrt{L} + (n-1)\sigma\sqrt{T})}. \tag{4.26}$$

Case 3: $f(P)$ is cubic in P

Now, the joint expected total cost with cubic function $f(P)$ is given by,

$$\begin{aligned}\text{Min JETC}_{13}(Q,P,A,k,L,n) &= \frac{(A+S)D}{nQ(1-E[y_e])} + h_{b1}\left[QE[y_e] - \frac{DQE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\left[k\sigma\sqrt{L}\right.\\ &\quad \left. + \frac{Q(1-E[y_e])}{2} + \frac{DQE[y_e]}{2x(1-E[y_e])}\right] + \frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y_e])} + \frac{D\pi(n-1)\sigma}{Q(1-E[y_e])}\\ &\quad \times\sqrt{T}\psi(k) + \frac{D(F+R(L))}{Q(1-E[y_e])} + \frac{D}{1-E[y_e]}(w+(\zeta_1 P + \frac{\zeta_2}{P})) + h_v\frac{Q}{2}\left[n\right.\\ &\quad \left.\times\left(1-\frac{D}{P(1-E[y_e])}\right) - 1 + \frac{2D}{P(1-E[y_e])}\right] + \frac{D}{(1-E[y_e])}\\ &\quad + b\log\left(\frac{S_0}{S}\right) + RD\alpha(b_3P + c_3P^2 + d_3P^3)\frac{Q}{(1-E[y_e])2P}\\ &\quad + \frac{c_aE[y]E[y_2]D}{(1-E[y_e])} + \frac{c_r(1-E[y])E[y_1]D}{(1-E[y_e])} + E\times D\times C_t\\ &\quad + Y\times C_p, \end{aligned} \tag{4.27}$$

subject to:

$$0 < S \le S_0,$$
$$P_{\min} = \alpha D \le P \le P_{\max},$$

where $E[y_e] = (1-\frac{a+b}{2})E[y_1] + \frac{a+b}{2}(1-E[y_2])$.

Now, for fixed n and $L \in [L_i, L_{i-1}]$, equating to zero the first derivatives of JETC_{13} with respect to Q, P, S and k, we obtain,

$$Q = \left[\frac{\left(\frac{DA}{n} + D\pi\psi(k)\sigma\sqrt{L} + D\pi(n-1)\sigma\sqrt{T}\psi(k) + D(F+R(L)) + \frac{SD}{n}\right)}{(1-E[y_e])G(n,y)}\right]^{\frac{1}{2}} \tag{4.28}$$

where

$$\begin{aligned} G(n,y) = &\left(h_{b1}\left[E[y_e] - \frac{DE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\left[\frac{1-E[y_e]}{2} + \frac{DE[y_e]}{2x(1-E[y_e])}\right]\right.\\ &\left. + h_v\left[n\left(1-\frac{D}{P(1-E[y_e])} - 1 + \frac{2D}{P(1-E[y_e])}\right)\right] + \frac{RD\alpha(b_3P + c_3P^2 + d_3P^3}{(1-E[y_e])2P}\right) \end{aligned}$$

$$P = \left[\frac{\zeta_2 + h_vQ(2-n)}{c_t(1-E[y_e])(2aP-b) + \zeta_1 + R\alpha Q(\frac{c_3}{2} + Pd_3)}\right]^{\frac{1}{2}} \tag{4.29}$$

$$S = \frac{bnQ(1-E[y_e])}{D} \tag{4.30}$$

$$\Phi(k) = 1 - \frac{h_{b2}Q\sigma\sqrt{L}(1-E[y_e])}{D\pi(\sigma\sqrt{L} + (n-1)\sigma\sqrt{T})}. \tag{4.31}$$

4.7. Defective rate follows triangular distribution

In this section, we assume that the defective rate follows a triangular distribution with expected value $E[y] = \frac{a+b+c}{3}$, where a is lower limit, b is upper limit, and mode c as well as $a < b$ and $a \le c \le b$.

Case 1: $f(P)$ is linear in P

Now, the joint expected total cost with linear function $f(P)$ is given by,

$$\begin{aligned}\text{Min}\,\text{JETC}_{21}(Q,P,A,k,L,n) &= \frac{(A+S)D}{nQ(1-E[y_e])} + h_{b1}\left[QE[y_e] - \frac{DQE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\Big[k\sigma\sqrt{L} \\ &+ \frac{Q(1-E[y_e])}{2} + \frac{DQE[y_e]}{2x(1-E[y_e])}\Big] + \frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y_e])} + \frac{D\pi(n-1)\sigma}{Q(1-E[y_e])} \\ &\times \sqrt{T}\psi(k) + \frac{D(F+R(L))}{Q(1-E[y_e])} + \frac{D}{1-E[y_e]}\left(w + \left(\zeta_1 P + \frac{\zeta_2}{P}\right)\right) + h_v\frac{Q}{2}\Big[n \\ &\times \left(1 - \frac{D}{P(1-E[y_e])}\right) - 1 + \frac{2D}{P(1-E[y_e])}\Big] + \frac{D}{(1-E[y_e])} \\ &+ b\log\left(\frac{S_0}{S}\right) + RD\alpha b_1 P\frac{Q}{(1-E[y_e])2P} + \frac{c_a E[y]E[y_2]D}{(1-E[y_e])} \\ &+ \frac{c_r(1-E[y])E[y_1]D}{(1-E[y_e])} + E\times D\times C_t + Y\times C_p, \end{aligned} \tag{4.32}$$

subject to:

$$0 < S \le S_0,$$
$$P_{\min} = \alpha D \le P \le P_{\max},$$

where $E[y_e] = (1 - \frac{a+b+c}{3})E[y_1] + \frac{a+b+c}{3}(1-E[y_2])$.

Similarly, for fixed n and $L \in [L_i, L_{i-1}]$, equating to zero the first derivatives of JETC_{21} with respect to Q, P, S and k, we obtain the values of Q, P, S and k are,

$$Q = \left[\frac{\left(\frac{DA}{n} + D\pi\psi(k)\sigma\sqrt{L} + D\pi(n-1)\sigma\sqrt{T}\psi(k) + D(F+R(L)) + \frac{SD}{n}\right)}{(1-E[y_e])G(n,y)}\right]^{\frac{1}{2}} \tag{4.33}$$

$$P = \left[\frac{\zeta_2 + h_v Q(2-n)}{2c_t(1-E[y_e])(2aP-b) + 2\zeta_1}\right]^{\frac{1}{2}} \tag{4.34}$$

$$S = \frac{bnQ(1-E[y_e])}{D} \tag{4.35}$$

$$\Phi(k) = 1 - \frac{h_{b2}Q\sigma\sqrt{L}(1-E[y_e])}{D\pi(\sigma\sqrt{L} + (n-1)\sigma\sqrt{T})}. \tag{4.36}$$

Case 2: $f(P)$ is quadratic in P

Now, the joint expected total cost with quadratic function $f(P)$ is given by,

$$\begin{aligned}\text{Min}\,\text{JETC}_{22}(Q,P,A,k,L,n) &= \frac{(A+S)D}{nQ(1-E[y_e])} + h_{b1}\left[QE[y_e] - \frac{DQE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\Big[k\sigma\sqrt{L} \\ &+ \frac{Q(1-E[y_e])}{2} + \frac{DQE[y_e]}{2x(1-E[y_e])}\Big] + \frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y_e])} + \frac{D\pi(n-1)\sigma}{Q(1-E[y_e])} \\ &\times \sqrt{T}\psi(k) + \frac{D(F+R(L))}{Q(1-E[y_e])} + \frac{D}{1-E[y_e]}\left(w + \left(\zeta_1 P + \frac{\zeta_2}{P}\right)\right) + h_v\frac{Q}{2}\Big[n \end{aligned}$$

$$\times\left(1-\frac{D}{P(1-E[y_e])}\right)-1+\frac{2D}{P(1-E[y_e])}\Bigg]+\frac{D}{(1-E[y_e])}$$
$$+b\log\left(\frac{S_0}{S}\right)+RD\alpha(b_2P+c_2P^2)\frac{Q}{(1-E[y_e])2P}+\frac{c_aE[y]E[y_2]D}{(1-E[y_e])}$$
$$+\frac{c_r(1-E[y])E[y_1]D}{(1-E[y_e])}+E\times D\times C_t+Y\times C_p, \tag{4.37}$$

subject to:

$$0<S\le S_0,$$
$$P_{\min}=\alpha D\le P\le P_{\max},$$

where $E[y_e]=(1-\frac{a+b+c}{3})E[y_1]+\frac{a+b+c}{3}(1-E[y_2])$.

Similarly, for fixed n and $L\in[L_i,L_{i-1}]$, equating to zero the first derivatives of JETC_{22} with respect to Q, P, S and k, we obtain the values of Q, P, S and k are,

$$Q=\left[\frac{\left(\frac{DA}{n}+D\pi\psi(k)\sigma\sqrt{L}+D\pi(n-1)\sigma\sqrt{T}\psi(k)+D(F+R(L))+\frac{SD}{n}\right)}{(1-E[y_e])G(n,y)}\right]^{\frac{1}{2}} \tag{4.38}$$

where

$$G(n,y)=\Bigg(h_{b1}\left[E[y_e]-\frac{DE[y_e]}{2x(1-E[y_e])}\right]+h_{b2}\left[\frac{1-E[y_e]}{2}+\frac{DE[y_e]}{2x(1-E[y_e])}\right]$$
$$+h_v\left[n\left(1-\frac{D}{P(1-E[y_e])}-1+\frac{2D}{P(1-E[y_e])}\right)\right]+\frac{RD\alpha(b_2P+c_2P^2)}{(1-E[y_e])2P}\Bigg)$$
$$P=\left[\frac{\zeta_2+h_vQ(2-n)}{2c_t(1-E[y_e])(2aP-b)+2\zeta_1+R\alpha c_2Q}\right]^{\frac{1}{2}} \tag{4.39}$$
$$S=\frac{bnQ(1-E[y_e])}{D} \tag{4.40}$$
$$\Phi(k)=1-\frac{h_{b2}Q\sigma\sqrt{L}(1-E[y_e])}{D\pi(\sigma\sqrt{L}+(n-1)\sigma\sqrt{T})}. \tag{4.41}$$

Case 3: $f(P)$ is cubic in P

Now, the joint expected total cost with cubic function $f(P)$ is given by,

$$\text{Min JETC}_{23}(Q,P,A,k,L,n)=\frac{(A+S)D}{nQ(1-E[y_e])}+h_{b1}\left[QE[y_e]-\frac{DQE[y_e]}{2x(1-E[y_e])}\right]+h_{b2}\Bigg[k\sigma\sqrt{L}$$
$$+\frac{Q(1-E[y_e])}{2}+\frac{DQE[y_e]}{2x(1-E[y_e])}\Bigg]+\frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y_e])}+\frac{D\pi(n-1)\sigma}{Q(1-E[y_e])}$$
$$\times\sqrt{T}\psi(k)+\frac{D(F+R(L))}{Q(1-E[y_e])}+\frac{D}{1-E[y_e]}\left(w+\left(\zeta_1P+\frac{\zeta_2}{P}\right)\right)+h_v\frac{Q}{2}\Bigg[n$$
$$\times\left(1-\frac{D}{P(1-E[y_e])}\right)-1+\frac{2D}{P(1-E[y_e])}\Bigg]+\frac{D}{(1-E[y_e])}$$
$$+b\log\left(\frac{S_0}{S}\right)+RD\alpha(b_3P+c_3P^2+d_3P^3)\frac{Q}{(1-E[y_e])2P}$$

$$+\frac{c_a E[y]E[y_2]D}{(1-E[y_e])}+\frac{c_r(1-E[y])E[y_1]D}{(1-E[y_e])}+E\times D\times C_t$$
$$+Y\times C_p, \tag{4.42}$$

subject to:

$$0<S\leq S_0,$$
$$P_{\min}=\alpha D\leq P\leq P_{\max},$$

where $E[y_e]=(1-\frac{a+b+c}{3})E[y_1]+\frac{a+b+c}{3}(1-E[y_2])$.

Similarly, for fixed n and $L\in[L_i,L_{i-1}]$, equating to zero the first derivatives of JETC_{23} with respect to Q, P, S and k, we obtain the values of Q, P, S and k are,

$$Q=\left[\frac{\left(\frac{DA}{n}+D\pi\psi(k)\sigma\sqrt{L}+D\pi(n-1)\sigma\sqrt{T}\psi(k)+D(F+R(L))+\frac{SD}{n}\right)}{(1-E[y_e])G(n,y)}\right]^{\frac{1}{2}} \tag{4.43}$$

where

$$G(n,y)=\left(h_{b1}\left[E[y_e]-\frac{DE[y_e]}{2x(1-E[y_e])}\right]+h_{b2}\left[\frac{1-E[y_e]}{2}+\frac{DE[y_e]}{2x(1-E[y_e])}\right]\right.$$
$$\left.+\,h_v\left[n\left(1-\frac{D}{P(1-E[y_e])}-1+\frac{2D}{P(1-E[y_e])}\right)\right]+\frac{RD\alpha(b_3P+c_3P^2+d_3P^3)}{(1-E[y_e])2P}\right)$$

$$P=\left[\frac{\zeta_2+h_vQ(2-n)}{c_t(1-E[y_e])(2aP-b)+\zeta_1+R\alpha Q(\frac{c_3}{2}+Pd_3)}\right]^{\frac{1}{2}} \tag{4.44}$$

$$S=\frac{bnQ(1-E[y_e])}{D} \tag{4.45}$$

$$\Phi(k)=1-\frac{h_{b2}Q\sigma\sqrt{L}(1-E[y_e])}{D\pi(\sigma\sqrt{L}+(n-1)\sigma\sqrt{T})}. \tag{4.46}$$

4.8. Defective rate follows beta distribution

In this section, we assume that the defective rate follows a beta distribution with expected value $E[y]=\frac{\beta}{\beta+\gamma}$, Here, it is a continuous probability distributions defined on the interval $(0,1)$ parameterized by two positive parameters, denoted by β and γ.

Case 1: $f(P)$ is linear in P

Now, the joint expected total cost with linear function $f(P)$ is given by,

$$\mathrm{Min\,JETC}_{31}(Q,P,A,k,L,n)=\frac{(A+S)D}{nQ(1-E[y_e])}+h_{b1}\left[QE[y_e]-\frac{DQE[y_e]}{2x(1-E[y_e])}\right]+h_{b2}\left[k\sigma\sqrt{L}\right.$$
$$\left.+\frac{Q(1-E[y_e])}{2}+\frac{DQE[y_e]}{2x(1-E[y_e])}\right]+\frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y_e])}+\frac{D\pi(n-1)\sigma}{Q(1-E[y_e])}$$
$$\times\sqrt{T}\psi(k)+\frac{D(F+R(L))}{Q(1-E[y_e])}+\frac{D}{1-E[y_e]}\left(w+\left(\zeta_1P+\frac{\zeta_2}{P}\right)\right)+h_v\frac{Q}{2}\left[n\right.$$
$$\left.\times\left(1-\frac{D}{P(1-E[y_e])}\right)-1+\frac{2D}{P(1-E[y_e])}\right]+\frac{D}{(1-E[y_e])}$$
$$+b\log\left(\frac{S_0}{S}\right)+RD\alpha b_1P\frac{Q}{(1-E[y_e])2P}+\frac{c_aE[y]E[y_2]D}{(1-E[y_e])}$$

$$+\frac{c_r(1-E[y])E[y_1]D}{(1-E[y_e])}+E\times D\times C_t+Y\times C_p, \tag{4.47}$$

subject to:

$$0<S\leq S_0,$$
$$P_{\min}=\alpha D\leq P\leq P_{\max},$$

where $E[y_e]=(1-\frac{\beta}{\beta+\gamma})E[y_1]+\frac{\beta}{\beta+\gamma}(1-E[y_2])$.

Similarly, for fixed n and $L\in[L_i,L_{i-1}]$, equating to zero the first derivatives of JETC_{31} with respect to Q, P, S and k, we obtain the values of Q, P, S and k are

$$Q=\left[\frac{\left(\frac{DA}{n}+D\pi\psi(k)\sigma\sqrt{L}+D\pi(n-1)\sigma\sqrt{T}\psi(k)+D(F+R(L))+\frac{SD}{n}\right)}{(1-E[y_e])G(n,y)}\right]^{\frac{1}{2}} \tag{4.48}$$

$$P=\left[\frac{\zeta_2+h_vQ(2-n)}{2c_t(1-E[y_e])(2aP-b)+2\zeta_1}\right]^{\frac{1}{2}} \tag{4.49}$$

$$S=\frac{bnQ(1-E[y_e])}{D} \tag{4.50}$$

$$\Phi(k)=1-\frac{h_{b2}Q\sigma\sqrt{L}(1-E[y_e])}{D\pi(\sigma\sqrt{L}+(n-1)\sigma\sqrt{T})}\cdot \tag{4.51}$$

Case 2: $f(P)$ is quadratic in P

Now, the joint expected total cost with quadratic function $f(P)$ is given by,

$$\begin{aligned}\mathrm{Min}\,\mathrm{JETC}_{32}(Q,P,A,k,L,n)=&\frac{(A+S)D}{nQ(1-E[y_e])}+h_{b1}\left[QE[y_e]-\frac{DQE[y_e]}{2x(1-E[y_e])}\right]+h_{b2}\Bigg[k\sigma\sqrt{L}\\&+\frac{Q(1-E[y_e])}{2}+\frac{DQE[y_e]}{2x(1-E[y_e])}\Bigg]+\frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y_e])}+\frac{D\pi(n-1)\sigma}{Q(1-E[y_e])}\\&\times\sqrt{T}\psi(k)+\frac{D(F+R(L))}{Q(1-E[y_e])}+\frac{D}{1-E[y_e]}(w+(\zeta_1P+\frac{\zeta_2}{P}))+h_v\frac{Q}{2}\Bigg[n\\&\times\left(1-\frac{D}{P(1-E[y_e])}\right)-1+\frac{2D}{P(1-E[y_e])}\Bigg]+\frac{D}{(1-E[y_e])}\\&+b\log\left(\frac{S_0}{S}\right)+RD\alpha(b_2P+c_2P^2)\frac{Q}{(1-E[y_e])2P}+\frac{c_aE[y]E[y_2]D}{(1-E[y_e])}\\&+\frac{c_r(1-E[y])E[y_1]D}{(1-E[y_e])}+E\times D\times C_t+Y\times C_p,\end{aligned} \tag{4.52}$$

subject to:

$$0<S\leq S_0,$$
$$P_{\min}=\alpha D\leq P\leq P_{\max},$$

where $E[y_e]=(1-\frac{\beta}{\beta+\gamma})E[y_1]+\frac{\beta}{\beta+\gamma}(1-E[y_2])$.

Similarly, for fixed n and $L \in [L_i, L_{i-1}]$, equating to zero the first derivatives of JETC$_{32}$ with respect to Q, P, S and k, we obtain the values of Q, P, S and k are

$$Q = \left[\frac{\left(\frac{DA}{n} + D\pi\psi(k)\sigma\sqrt{L} + D\pi(n-1)\sigma\sqrt{T}\psi(k) + D(F+R(L)) + \frac{SD}{n}\right)}{(1-E[y_e])G(n,y)} \right]^{\frac{1}{2}} \tag{4.53}$$

where

$$\begin{aligned} G(n,y) = &\Bigg(h_{b1}\left[E[y_e] - \frac{DE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\left[\frac{1-E[y_e]}{2} + \frac{DE[y_e]}{2x(1-E[y_e])}\right] \\ &+ h_v\left[n\left(1 - \frac{D}{P(1-E[y_e])} - 1 + \frac{2D}{P(1-E[y_e])}\right)\right] + \frac{RD\alpha(b_2P + c_2P^2)}{(1-E[y_e])2P}\Bigg) \end{aligned}$$

$$P = \left[\frac{\zeta_2 + h_vQ(2-n)}{2c_t(1-E[y_e])(2aP-b) + 2\zeta_1 + R\alpha c_2 Q} \right]^{\frac{1}{2}} \tag{4.54}$$

$$S = \frac{bnQ(1-E[y_e])}{D} \tag{4.55}$$

$$\Phi(k) = 1 - \frac{h_{b2}Q\sigma\sqrt{L}(1-E[y_e])}{D\pi(\sigma\sqrt{L} + (n-1)\sigma\sqrt{T})}. \tag{4.56}$$

Case 3: $f(P)$ is cubic in P

Now, the joint expected total cost with cubic function $f(P)$ is given by,

$$\begin{aligned} \text{Min JETC}_{33}(Q,P,A,k,L,n) = &\frac{(A+S)D}{nQ(1-E[y_e])} + h_{b1}\left[QE[y_e] - \frac{DQE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\Bigg[k\sigma\sqrt{L} \\ &+ \frac{Q(1-E[y_e])}{2} + \frac{DQE[y_e]}{2x(1-E[y_e])}\Bigg] + \frac{D\pi\sigma\sqrt{L}\psi(k)}{Q(1-E[y_e])} + \frac{D\pi(n-1)\sigma}{Q(1-E[y_e])} \\ &\times \sqrt{T}\psi(k) + \frac{D(F+R(L))}{Q(1-E[y_e])} + \frac{D}{1-E[y_e]}\left(w + \left(\zeta_1 P + \frac{\zeta_2}{P}\right)\right) + h_v\frac{Q}{2}\Bigg[n \\ &\times \left(1 - \frac{D}{P(1-E[y_e])}\right) - 1 + \frac{2D}{P(1-E[y_e])}\Bigg] + \frac{D}{(1-E[y_e])} \\ &+ b\log\left(\frac{S_0}{S}\right) + RD\alpha(b_3P + c_3P^2 + d_3P^3)\frac{Q}{(1-E[y_e])2P} \\ &+ \frac{c_aE[y]E[y_2]D}{(1-E[y_e])} + \frac{c_r(1-E[y])E[y_1]D}{(1-E[y_e])} + E\times D\times C_t \\ &+ Y \times C_p, \end{aligned} \tag{4.57}$$

subject to:

$$0 < S \le S_0,$$
$$P_{\min} = \alpha D \le P \le P_{\max},$$

where $E[y_e] = (1 - \frac{\beta}{\beta+\gamma})E[y_1] + \frac{\beta}{\beta+\gamma}(1 - E[y_2])$.

TABLE 2. Lead time component with data.

Lead time component i	Normal duration b_i (days)	Minimum duration cost a_i (days)	Unit crashing c_i (\$/days)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Similarly, for fixed n and $L \in [L_i, L_{i-1}]$, equating to zero the first derivatives of JETC_{33} with respect to Q, P, S and k, we obtain the values of Q, P, S and k are

$$Q = \left[\frac{\left(\frac{DA}{n} + D\pi\psi(k)\sigma\sqrt{L} + D\pi(n-1)\sigma\sqrt{T}\psi(k) + D(F + R(L)) + \frac{SD}{n}\right)}{(1-E[y_e])G(n,y)}\right]^{\frac{1}{2}} \tag{4.58}$$

where

$$G(n,y) = \left(h_{b1}\left[E[y_e] - \frac{DE[y_e]}{2x(1-E[y_e])}\right] + h_{b2}\left[\frac{1-E[y_e]}{2} + \frac{DE[y_e]}{2x(1-E[y_e])}\right] + h_v\left[n\left(1 - \frac{D}{P(1-E[y_e])} - 1 + \frac{2D}{P(1-E[y_e])}\right)\right] + \frac{RD\alpha(b_3P + c_3P^2 + d_3P^3)}{(1-E[y_e])2P}\right)$$

$$P = \left[\frac{\zeta_2 + h_vQ(2-n)}{c_t(1-E[y_e])(2aP-b) + \zeta_1 + R\alpha Q(\frac{c_3}{2} + Pd_3)}\right]^{\frac{1}{2}} \tag{4.59}$$

$$S = \frac{bnQ(1-E[y_e])}{D} \tag{4.60}$$

$$\Phi(k) = 1 - \frac{h_{b2}Q\sigma\sqrt{L}(1-E[y_e])}{D\pi(\sigma\sqrt{L} + (n-1)\sigma\sqrt{T})}\cdot \tag{4.61}$$

5. NUMERICAL ANALYSIS

Example 5.1. The parametric values for finding the optimality are taken from Sarkar *et al.* [23] and Dey and Giri [4]. Then by using software Mat lab, one can obtain the optimum result for the above three cases which follows a uniform distribution is shown below.

$D = 600$, $A = 150$, $h_{b1} = 0.5$, $x = 175\,200$, $h_{b2} = 1.7$, $\sigma = 7$, $t_s = 0.17$, $\pi = 80$, $T = 1.9$, $F = 30$, $w = 0.25$, $h_v = 1.7$, $\zeta_1 = 0.04$, $\zeta_2 = 0.00333$, $b = 20$, $S_0 = 400$, $R = 100$, $\alpha = 0.3$, $b_2 = 0.0004$, $a^* = 3 \times 10^{-7}$, $b^* = 0.012$, $c^* = 1.4$, $c_t = 118$, $c_p = 4000$, $c_a = 200$, $c_r = 50$ $a = 0.15$, $b = 0.25$, $a_0 = 0.015$, $b_0 = 0.025$, $[P_{\min}, P_{\max}] = [200, 800]$. The lead time has three components with data shown in Table 2.

Now applying the algorithm for the three cases, the outcomes of the solution procedure are outlined in Tables 3–5. Here likewise, the consequences of the no investment approach in a Tables 6–8 are inclined to show the impacts of setup cost reduction. Also the corresponding curves of the joint expected total cost are plotted as in Figure 1.

Example 5.2. Most of the parameters are obtained from an Example 5.1. $D = 600$, $A = 150$, $h_{b1} = 0.5$, $x = 175\,200$, $h_{b2} = 1.7$, $\sigma = 7$, $t_s = 0.17$, $\pi = 80$, $T = 1.9$, $F = 30$, $w = 0.25$, $h_v = 1.7$, $\zeta_1 = 0.04$, $\zeta_2 = 0.00333$,

Table 3. Expected total cost for an optimal solution in uniform distribution case 1. (linear $f(P)$)

L	n	Q	P	k	S	JETC_{11}
8	1	274	265	2.42	22	5 705 530
6	1	251	290	2.46	20	5 945 000
4	1	282	268	2.42	22	6 341 530
3	1	295	274	2.40	23	7 479 245

Table 4. Expected total cost for an optimal solution in uniform distribution case 2. (quadratic $f(P)$)

L	n	Q	P	k	S	JETC_{12}
8	1	275	264	2.42	22	5 705 529
6	1	277	266	2.42	22	5 932 827
4	1	282	269	2.42	22	6 402 370
3	1	295	275	2.40	23	7 538 881

Table 5. Expected total cost for an optimal solution in uniform distribution case 3. (cubic $f(P)$)

L	n	Q	P	k	S	JETC_{13}
8	1	274	264	2.42	21	5 646 407
6	1	276	265	2.42	21	5 819 893
4	1	281	268	2.42	22	6 283 723
3	1	295	274	2.40	23	7 475 276

Table 6. Expected total cost for the fixed and reduced setup cost in uniform distribution case 1. (linear $f(P)$)

n	L	Q	JETC for reduced setup cost	Q	JETC for fixed setup cost
1	8	274	5705530	401	7 720 865

Table 7. Expected total cost for the fixed and reduced setup cost in uniform distribution case 2. (quadratic $f(P)$)

n	L	Q	JETC for reduced setup cost	Q	JETC for fixed setup cost
1	8	275	5 705 529	402	7 789 900

Table 8. Expected total cost for the fixed and reduced setup cost in uniform distribution case 3. (cubic $f(P)$)

n	L	Q	JETC for reduced setup cost	Q	JETC for fixed setup cost
1	8	274	5 646 407	404	7 927 970

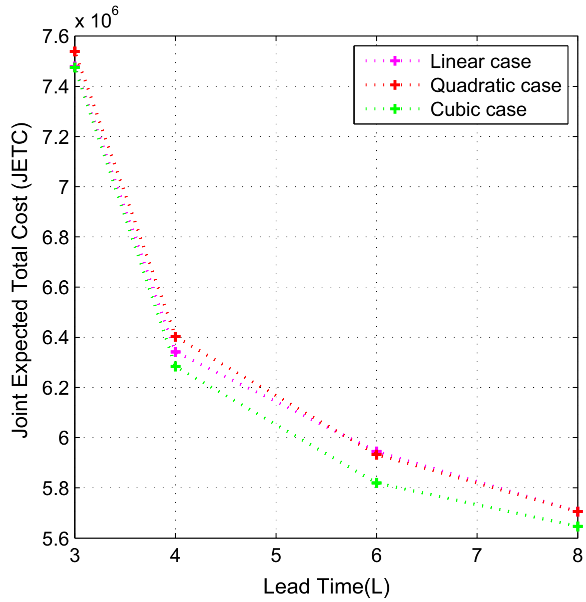

FIGURE 1. Graphical representation for Example 5.1.

$b = 20$, $S_0 = 400$, $R = 100$, $\alpha = 0.3$, $b_2 = 0.0004$, $a^* = 3 \times 10^{-7}$, $b^* = 0.012$, $c^* = 1.4$, $c_t = 118$, $c_p = 4000$, $c_a = 200$, $c_r = 50$ $a = 0.15$, $b = 0.25$, $c = 0.25$, $a_0 = 0.015$, $b_0 = 0.025$, $c_0 = 0.025$, $[P_{\min}, P_{\max}] = [200, 800]$. Here, the percentage of inspection errors (type I and type II) follows a triangular distribution with probability density function is,

$$f(y_1) = \begin{cases} \frac{2(y_1 - a_0)}{(b_0 - a_0)(c_0 - a_0)}, & a_0 \leq y_1 \leq c_0 \\ \frac{2(b_0 - y_1)}{(b_0 - a_0)(b_0 - c_0)}, & c_0 \leq y_1 \leq b_0 \\ 0 & \text{otherwise} \end{cases}$$

$$f(y_2) = \begin{cases} \frac{2(y_2 - a_0)}{(b_0 - a_0)(c_0 - a_0)}, & a_0 \leq y_2 \leq c_0 \\ \frac{2(b_0 - y_2)}{(b_0 - a_0)(b_0 - c_0)}, & c_0 \leq y_2 \leq b_0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus we have,

$$E(y_1) = \frac{a_0 + b_0 + c_0}{3} = 0.025$$
$$E(y_2) = \frac{a_0 + b_0 + c_0}{3} = 0.025.$$

Now applying the algorithm for the three cases, the outcomes of the solution procedure are outlined in Tables 9–11. Here likewise, the consequences of the no investment approach in a Tables 12–14 are inclined to show the impacts of setup cost reduction. Also the corresponding curves of the joint expected total cost are plotted as in Figure 2.

Example 5.3. Most of the parameters are obtained from an Example 5.1. $D = 600$, $A = 150$, $h_{b1} = 0.5$, $x = 175\,200$, $h_{b2} = 1.7$, $\sigma = 7$, $t_s = 0.17$, $\pi = 80$, $T = 1.9$, $F = 30$, $w = 0.25$, $h_v = 1.7$, $\zeta_1 = 0.04$, $\zeta_2 = 0.00333$,

Table 9. Expected total cost for an optimal solution in triangular distribution case 1. (linear $f(P)$)

L	n	Q	P	k	S	JETC_{21}
8	1	286	271	2.43	21	7 903 176
6	1	289	271	2.43	21	8 090 583
4	1	296	275	2.42	22	8 800 846
3	1	308	280	2.41	23	9 916 623

Table 10. Expected total cost for an optimal solution in triangular distribution case 2. (quadratic $f(P)$)

L	n	Q	P	k	S	JETC_{22}
8	1	286	270	2.43	21	7 841 239
6	1	288	271	2.43	21	8 028 134
4	1	295	274	2.42	22	8 669 520
3	1	307	281	2.41	23	9 923 059

Table 11. Expected total cost for an optimal solution in triangular distribution case 3. (cubic $f(P)$)

L	n	Q	P	k	S	JETC_{23}
8	1	286	270	2.43	21	7 841 240
6	1	288	271	2.43	21	8 028 139
4	1	296	274	2.42	22	8 800 940
3	1	308	281	2.41	23	9 987 889

Table 12. Expected total cost for the fixed and reduced setup cost in triangular distribution case 1. (linear $f(P)$)

n	L	Q	JETC for reduced setup cost	Q	JETC for fixed setup cost
1	8	286	7 903 176	423	31 922 276

Table 13. Expected total cost for the fixed and reduced setup cost in uniform distribution case 2. (quadratic $f(P)$)

n	L	Q	JETC for reduced setup cost	Q	JETC for fixed setup cost
1	8	286	7 841 239	421	31 868 580

TABLE 14. Expected total cost for the fixed and reduced setup cost in uniform distribution case 3. (cubic $f(P)$)

n	L	Q	JETC for reduced setup cost	Q	JETC for fixed setup cost
1	8	286	7 841 240	420	31 792 974

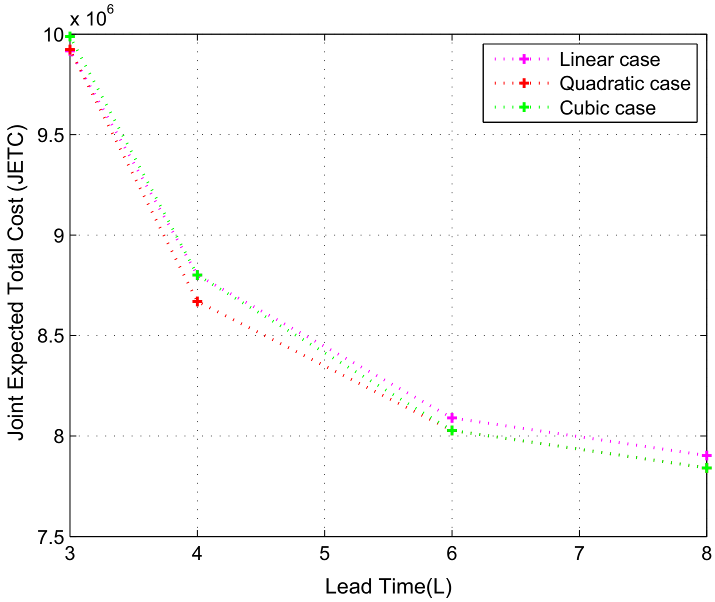

FIGURE 2. Graphical representation for Example 5.2.

$b = 20$, $S_0 = 400$, $R = 100$, $\alpha = 0.3$, $b_2 = 0.0004$, $a^* = 3 \times 10^{-7}$, $b^* = 0.012$, $c^* = 1.4$, $c_t = 118$, $c_p = 4000$, $c_a = 200$, $c_r = 50$, $\beta = 0.15$, $\gamma = 0.35$, $\beta_0 = 0.025$, $\gamma_0 = 0.75$, $[P_{\min}, P_{\max}] = [200, 800]$. Here, the percentage of inspection errors (type I and type II) follows a beta distribution with probability density function is,

$$f(y_1) = \begin{cases} \frac{y_1^{\beta_0 - 1}(1-y_1)^{\gamma_0 - 1}}{B(\beta_0, \gamma_0)}, & 0 < y_1 < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f(y_2) = \begin{cases} \frac{y_2^{\beta_0 - 1}(1-y_2)^{\gamma_0 - 1}}{B(\beta_0, \gamma_0)}, & 0 < y_2 < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Thus we have,

$$E(y_1) = \frac{\beta_0}{\beta_0 + \gamma_0} = 0.03$$

$$E(y_2) = \frac{\beta_0}{\beta_0 + \gamma_0} = 0.03.$$

Now applying the algorithm for the three cases, the outcomes of the solution procedure are outlined in Tables 15–17. Here likewise, the consequences of the no investment approach in a Tables 18–20 are inclined to show the impacts of setup cost reduction. Also the corresponding curves of the joint expected total cost are plotted as in Figure 3.

TABLE 15. Expected total cost for an optimal solution in beta distribution case 1. (linear $f(P)$)

L	n	Q	P	k	S	$JETC_{31}$
8	1	300	277	2.44	21	10 571 347
6	1	302	278	2.44	21	10 782 876
4	1	308	281	2.43	21	20 423 402
3	1	323	287	2.42	22	20 943 901

TABLE 16. Expected total cost for an optimal solution in beta distribution case 2. (quadratic $f(P)$)

L	n	Q	P	k	S	$JETC_{32}$
8	1	299	277	2.44	21	10 502 903
6	1	301	278	2.44	21	10 714 205
4	1	308	280	2.44	21	20 347 370
3	1	323	287	2.42	22	20 844 133

TABLE 17. Expected total cost for an optimal solution in beta distribution case 3. (cubic $f(P)$)

L	n	Q	P	k	S	$JETC_{33}$
8	1	299	276	2.44	21	10 429 010
6	1	302	277	2.44	21	10 708 268
4	1	308	280	2.43	21	20 347 394
3	1	322	287	2.42	22	30 873 283

TABLE 18. Expected total cost for the fixed and reduced setup cost in triangular distribution case 1. (linear $f(P)$)

n	L	Q	JETC for reduced setup cost	Q	JETC for fixed setup cost
1	8	300	10 571 347	441	60 651 981

TABLE 19. Expected total cost for the fixed and reduced setup cost in uniform distribution case 2. (quadratic $f(P)$)

n	L	Q	JETC for reduced setup cost	Q	JETC for fixed setup cost
1	8	299	10 502 903	440	60 460 225

TABLE 20. Expected total cost for the fixed and reduced setup cost in beta distribution case 3. (cubic $f(P)$)

n	L	Q	JETC for reduced setup cost	Q	JETC for fixed setup cost
1	8	299	10 429 010	440	60 568 586

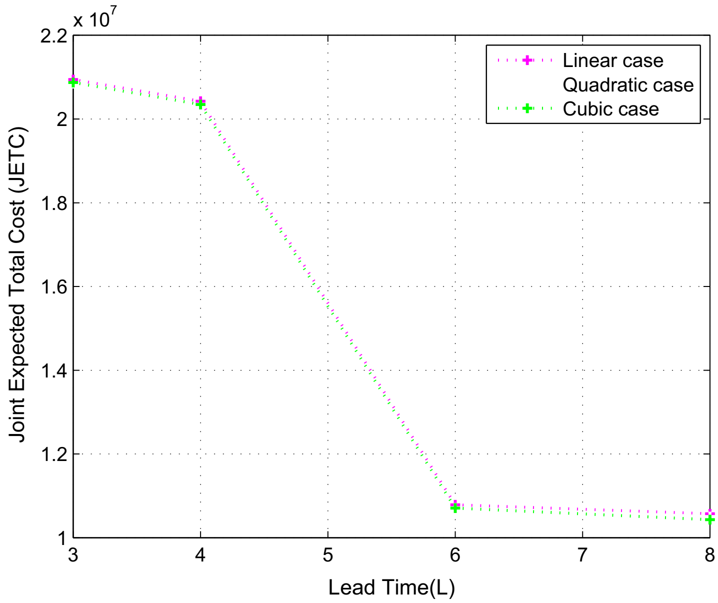

FIGURE 3. Graphical representation for Example 5.3.

From the above examples, we observe that the lead time, at which the total supply chain cost is at a minimum is 8 weeks for all the buyers for each case. The joint expected total cost in uniform distribution for cases 1, 2 and 3 are \$5 705 530, \$5 705 529 and \$5 646 407, the joint expected total cost in triangular distribution for cases 1, 2 and 3 are \$7 903 176, \$7 841 239 and \$7 841 240 and the joint expected total cost in beta distribution for cases 1, 2 and 3 are \$10 571 347, \$10 502 903 and \$10 429 010, respectively. Presently in the event that we look at the quality function $f(P)$ as showed in three cases under three distributions. We can see that little changes in the production rate may result in bigger deviations in the quality function of case 1 than those of cases 2 and 3. In addition, cases 2 and 3 can be diminished to case 1 by assigning c_2, c_3 and d_3 to be zero. Thus case 3 harmonizes with case 2 if $d_3 = 0$. Hence if similar estimations of the co-efficients of the direct and quadratic terms are utilized in the quality function for all cases. It is seem that the expected total cost from case 1 to case 3 decreases.

6. SENSITIVITY ANALYSIS

In this segment, the deviation of the joint expected total cost with the difference in a cost parameters are present in the supply chain model is contemplated. The cost parameters are increased and decreased +50%, +25%, −25% and −50% from their actual values as utilized in the above models, with a step length of +25% and −25%. The changes in the joint expected total cost with differing parameters are exhibited in Tables 21–29 and the effects of parameters A and π on the joint expected total cost for three cases under three distributions are depicted in Figures 4–9, respectively.

(1) From Tables 21, 22 and 23, we observed that the joint expected total cost using uniform distribution for three cases are −0.82% to 0.99%, −0.78% to 0.95% and −1.5% to 2% respectively, of savings can be achieved. As usual increasing values of parameters give the increasing total cost. Machinery cost ζ_1 and A are more sensitive rather than the other parameters to the total cost.

(2) In Tables 24, 25 and 26, it is interesting to note that the joint expected total cost using triangular distribution for linear, quadratic and cubic cases −0.86% to 0.43%, −0.06% to 0.08% and −0.87% to 0.43% respectively,

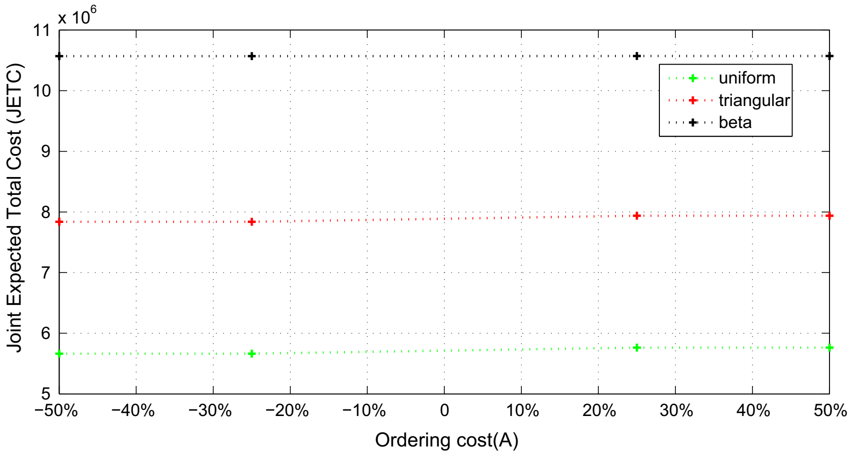

FIGURE 4. Effect of A on the JETC for linear case under three distributions.

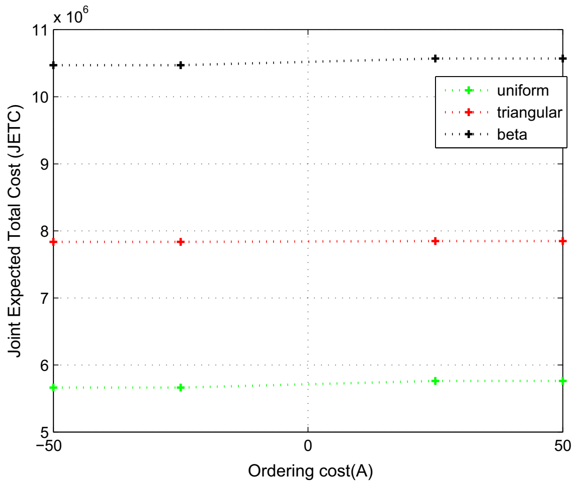

FIGURE 5. Effect of A on the JETC for quadratic case under three distributions.

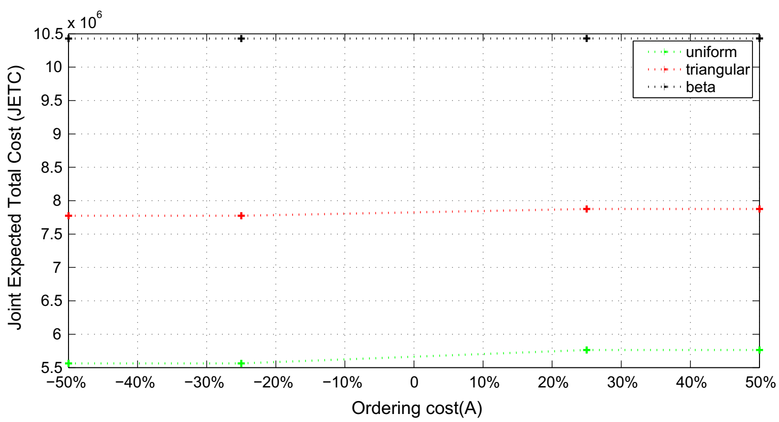

FIGURE 6. Effect of A on the JETC for cubic case under three distributions.

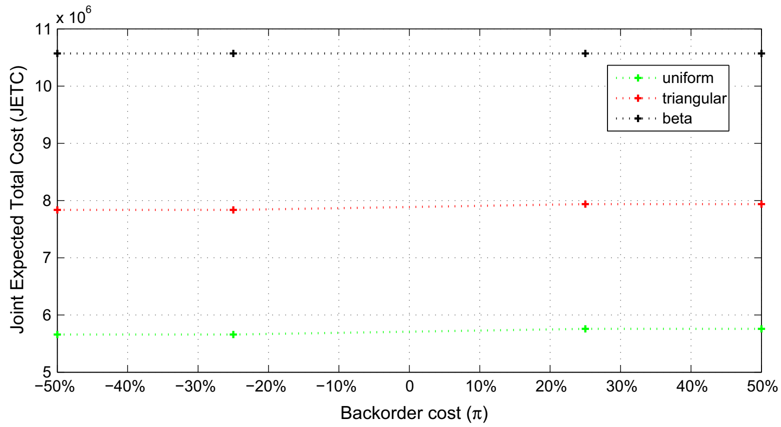

FIGURE 7. Effect of π on the JETC for linear case under three distributions.

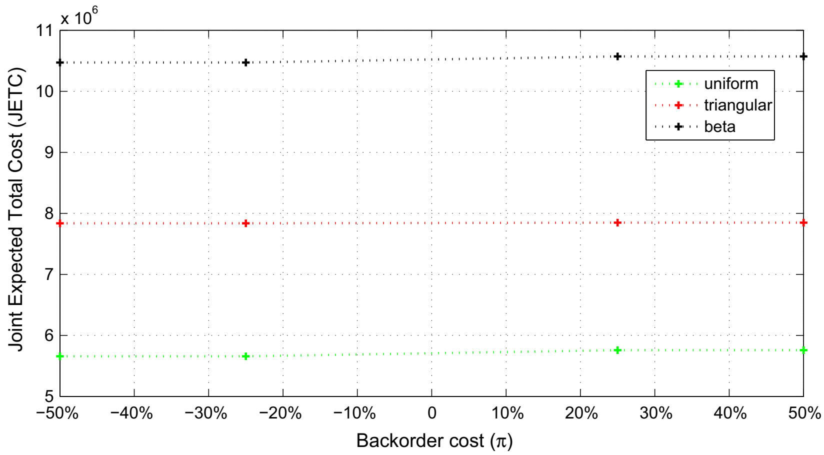

FIGURE 8. Effect of π on the JETC for quadratic case under three distributions.

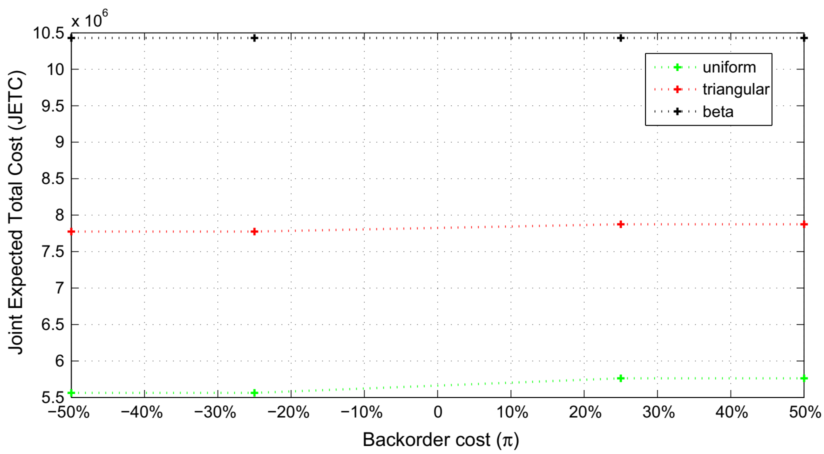

FIGURE 9. Effect of π on the JETC for cubic case under three distributions.

TABLE 21. Sensitivity analysis using uniform distribution for linear case.

Parameters	Changes in %	$JETC_{11}$	Parameters	Changes in %	$JETC_{11}$
	+50	5 762 883		+50	5 764 044
A	+25	5 762 790	ζ_1	+25	5 763 371
	−50	5 662 514		−50	5 661 353
	−25	5 662 606		−25	5 662 026
	+50	5 762 726		+50	5 762 699
F	+25	5 762 712	ζ_2	+25	5 762 699
	−50	5 662 685		−50	5 662 699
	−25	5 662 670		−25	5 662 699
	+50	5 762 718		+50	5 758 701
σ	+25	5 762 708	π	+25	5 758 699
	−50	5 662 679		−50	5 658 695
	−25	5 662 689		−25	5 658 697

TABLE 22. Sensitivity analysis using uniform distribution for quadratic case.

Parameters	Changes in %	$JETC_{12}$	Parameters	Changes in %	$JETC_{12}$
	+50	5 762 882		+50	5 764 042
A	+25	5 762 787	ζ_1	+25	5 763 365
	−50	5 662 512		−50	5 661 350
	−25	5 662 604		−25	5 662 022
	+50	5 762 723		+50	5 762 696
F	+25	5 762 709	ζ_2	+25	5 762 696
	−50	5 662 681		−50	5 662 696
	−25	5 662 665		−25	5 662 696
	+50	5 766 714		+50	5 758 698
σ	+25	5 762 703	π	+25	5 758 697
	−50	5 662 674		−50	5 658 693
	−25	5 662 683		−25	5 658 695

of savings can be achieved. When the parameters A, F, σ, ζ_1 and ζ_2 and π decreases, the expected total cost decreases. Machinery costs ζ_1 are more sensitive and A is next sensitive rather than the other parameters to the total cost.

(3) From Tables 27, 28 and 29, the joint expected total cost using beta distribution for 3 cases are −0.015% to 0.05%, 0.32% to 0.68$ and −0.017% to 0.03%, respectively, of savings can be achieved and also it is more sensitive than the above two models.

(4) The production rate are considered as variable, which is more realistic rather than constant.

(5) From the comparison of three models under three different cases, we can observe that the joint expected total cost of the uniform distribution for the cubic case is lower than others.

Table 23. Sensitivity analysis using uniform distribution for cubic case.

Parameters	Changes in %	$JETC_{13}$	Parameters	Changes in %	$JETC_{13}$
	+50	5 762 883		+50	5 764 044
A	+25	5 762 791	ζ_1	+25	5 763 371
	−50	5 562 513		−50	5 561 353
	−25	5 562 606		−25	5 562 026
	+50	5 762 726		+50	5 762 698
F	+25	5 762 712	ζ_2	+25	5 762 698
	−50	5 562 671		−50	5 562 698
	−25	5 562 685		−25	5 562 698
	+50	5 762 718		+50	5 762 700
σ	+25	5 762 708	π	+25	5 762 699
	−50	5 562 679		−50	5 562 697
	−25	5 562 689		−25	5 562 698

Table 24. Sensitivity analysis using triangular distribution for linear case.

Parameters	Changes in %	$JETC_{21}$	Parameters	Changes in %	$JETC_{21}$
	+50	7 937 420		+50	7 938 695
A	+25	7 937 325	ζ_1	+25	7 937 963
	−50	7 837 041		−50	7 835 766
	−25	7 837 136		−25	7 836 498
	+50	7 937 259		+50	7 937 230
F	+25	7 937 245	ζ_2	+25	7 937 230
	−50	7 837 202		−50	7 837 230
	−25	7 837 216		−25	7 837 230
	+50	7 937 260		+50	7 937 231
σ	+25	7 937 250	π	+25	7 937 232
	−50	7 837 221		−50	7 837 229
	−25	7 837 230		−25	7 837 230

7. Conclusion

Because of inaccessibility of any experimental data about the relation between quality and production rate, three cases containing three different types of functions under three distribution namely, (i) uniform (ii) triangular and (iii) beta were considered. This study minimized the joint expected total cost along with simultaneously optimizing the order quantity, safety factor, setup cost, production cost. One of the consequences of this work is that if the setup cost per setup could be diminished effectively the joint expected total cost could be automatically minimized. From this model, we conclude that the joint expected total cost of the uniform distribution for the cubic case is lower than the others. This model can be extended by considering automation policy for inspection along with the different types of warehouse. This paper can also be extended by multi-buyer, multi-vendor with multi-items etc.

TABLE 25. Sensitivity analysis using triangular distribution for quadratic case.

Parameters	Changes in %	$JETC_{22}$	Parameters	Changes in %	$JETC_{22}$
	+50	7 847 430		+50	7 848 704
A	+25	7 847 335	ζ_1	+25	7 847 972
	−50	7 837 050		−50	7 835 775
	−25	7 837 145		−25	7 836 508
	+50	7 847 268		+50	7 847 240
F	+25	7 847 254	ζ_2	+25	7 847 240
	−50	7 837 226		−50	7 837 240
	−25	7 837 211		−25	7 837 240
	+50	7 847 259		+50	7 847 246
σ	+25	7 847 250	π	+25	7 847 242
	−50	7 837 220		−50	7 837 238
	−25	7 837 230		−25	7 837 239

TABLE 26. Sensitivity analysis using triangular distribution for cubic case.

Parameters	Changes in %	$JETC_{23}$	Parameters	Changes in %	$JETC_{23}$
	+50	7 875 193		+50	7 876 468
A	+25	7 875 099	ζ_1	+25	7 875 736
	−50	7 774 813		−50	7 773 539
	−25	7 774 908		−25	7 774 271
	+50	7 875 032		+50	7 875 004
F	+25	7 875 018	ζ_2	+25	7 875 004
	−50	7 774 975		−50	7 775 004
	−25	7 774 989		−25	7 775 004
	+50	7 875 023		+50	7 875 005
σ	+25	7 875 013	π	+25	7 875 004
	−50	7 774 984		−50	7 775 001
	−25	7 774 994		−25	7 775 002

TABLE 27. Sensitivity analysis using beta distribution for linear case.

Parameters	Changes in %	$JETC_{31}$	Parameters	Changes in %	$JETC_{31}$
	+50	10 571 541		+50	10 572 957
A	+25	10 5714 44	ζ_1	+25	10 572 152
	−50	10 571 153		−50	10 569 736
	−25	10 571 250		−25	10 570 542
	+50	10 571 376		+50	10 571 347
F	+25	10 571 361	ζ_2	+25	10 571 347
	−50	10 571 318		−50	10 571 347
	−25	10 571 332		−25	10 571 347
	+50	10 571 366		+50	10 571 350
σ	+25	10 571 356	π	+25	10 571 348
	−50	10 571 328		−50	10 571 344
	−25	10 571 337		−25	10 571 346

TABLE 28. Sensitivity analysis using beta distribution for quadratic case.

Parameters	Changes in %	JETC_{32}	Parameters	Changes in %	JETC_{32}
	+50	10 571 541		+50	10 572 958
A	+25	10 571 443	ζ_1	+25	10 572 153
	−50	10 471 154		−50	10 469 737
	−25	10 471 251		−25	10 470 542
	+50	10 571 376		+50	10 571 345
F	+25	10 571 362	ζ_2	+25	10 571 345
	−50	10 471 333		−50	10 471 345
	−25	10 471 318		−25	10 471 345
	+50	10 571 367		+50	10 571 351
σ	+25	10 571 357	π	+25	10 571 349
	−50	10 471 338		−50	10 471 345
	−25	10 471 328		−25	10 471 347

TABLE 29. Sensitivity analysis using beta distribution for cubic case.

Parameters	Changes in %	JETC_{33}	Parameters	Changes in %	JETC_{33}
	+50	10 429 205		+50	10 430 615
A	+25	10 429 108	ζ_1	+25	10 429 813
	−50	10 428 816		−50	10 427 406
	−25	10 428 914		−25	10 428 208
	+50	10 429 040		+50	10 429 011
F	+25	10 429 025	ζ_2	+25	10 429 011
	−50	10 428 996		−50	10 429 011
	−25	10 428 982		−25	10 429 011
	+50	10 429 030		+50	10 429 012
σ	+25	10 429 020	π	+25	10 429 011
	−50	10 428 991		−50	10 429 009
	−25	10 429 001		−25	10 429 010

APPENDIX A.

For a given value of $L \in [L_i, L_{i-1}]$, we first obtain the Hessian Matrix $\mathbf{H}$ as follows:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial Q^2} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial Q \partial P} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial Q \partial S} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial Q \partial k} \\ \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial P \partial Q} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial P^2} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial P \partial S} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial P \partial k} \\ \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial S \partial Q} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial S \partial P} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial S^2} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial S \partial k} \\ \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial k \partial Q} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial k \partial P} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial k \partial S} & \frac{\partial^2 \text{JETC}(Q,P,S,k\theta,L,n)}{\partial k^2} \end{bmatrix}.$$

For the first minor, one can easily obtain as,

$$|H_{11}| = \det\left[\frac{\partial^2}{\partial Q^2}\text{JETC}(Q,P,S,k,L,n)\right] = \frac{2D}{Q^3(1-E[y_c])}\left[\frac{A+S}{n} + \pi\psi(k)\Big(\sigma\sqrt{L} + (n-1) \times \sigma\sqrt{T}\Big) + F + R(L)\right] > 0.$$

For the second minor, one can easily obtain easily as,

$$\begin{aligned}|H_{22}| &= \det\begin{bmatrix} \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial Q^2} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial Q \partial P} \\ \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P \partial Q} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P^2} \end{bmatrix} \\ &= \frac{2D}{Q^3(1-E[y_c])}\left[\frac{A+S}{n} + \pi\psi(k)\left(\sigma\sqrt{L} + (n-1)\sigma\sqrt{T}\right) + F + R(L)\right] \\ &\quad \times \frac{h_v Q}{P^3}\left(\frac{2D}{1-E[y_c]} - \frac{nD}{1-E[y_c]}\right) + 2aDc_t + \frac{2D\zeta_2}{P^3(1-E[y_c])} - h_v\left[\frac{2D}{1-E[y_c]}\right. \\ &\quad \left.-\frac{nD}{1-E[y_c]}\right] > 0\end{aligned}$$

since,

$$\begin{aligned}&\frac{2D}{Q^3(1-E[y_c])}\left[\frac{A+S}{n} + \pi\psi(k)\left(\sigma\sqrt{L} + (n-1)\sigma\sqrt{T}\right) + F + R(L)\right] \\ &\quad \times \frac{h_v Q}{P^3}\left(\frac{2D}{1-E[y_c]} - \frac{nD}{1-E[y_c]}\right) + 2aDc_t + \frac{2D\zeta_2}{P^3(1-E[y_c])} \\ &\quad > h_v\left[\frac{2D}{1-E[y_c]} - \frac{nD}{1-E[y_c]}\right].\end{aligned}$$

Therefore, $|H_{22}| > 0$.

$$\begin{aligned}|H_{33}| &= \det\begin{bmatrix} \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial Q^2} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial Q \partial P} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial Q \partial S} \\ \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P \partial Q} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P^2} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P \partial S} \\ \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial S \partial Q} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial S \partial P} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial S^2} \end{bmatrix} \\ &= \det\begin{bmatrix} A & B & -\frac{D}{nQ^2(1-E[y_c])} \\ B & C & 0 \\ -\frac{D}{nQ^2(1-E[y_c])} & 0 & \frac{b}{S^2} \end{bmatrix} \\ &= \frac{2D}{Q^3(1-E[y_c])}\left[\frac{A+S}{n} + \pi\psi(k)\left(\sigma\sqrt{L} + (n-1)\sigma\sqrt{T}\right) + F + R(L)\right] \times \frac{b}{S^2} \\ &\quad \times\left(\frac{h_v Q}{P^3}\left(\frac{2D}{1-E[y_c]} - \frac{nD}{1-E[y_c]}\right) + 2aDc_t + \frac{2D\zeta_2}{P^3(1-E[y_c])}\right) - {h_v}^2\left[\frac{2D}{1-E[y_c]} - \frac{nD}{1-E[y_c]}\right]^2 \\ &\quad \times \frac{b}{S^2} + \left(-\frac{D}{nQ^2(1-E[y_c])}\right)^2 \times \left(\frac{h_v Q}{P^3}\left(\frac{2D}{1-E[y_c]} - \frac{nD}{1-E[y_c]}\right) + 2aDc_t + \frac{2D\zeta_2}{P^3(1-E[y_c])}\right) \\ &= \frac{2D}{Q^3(1-E[y_c])}\left[\frac{A+S}{n} + \pi\psi(k)\left(\sigma\sqrt{L} + (n-1)\sigma\sqrt{T}\right) + F + R(L)\right] \times \frac{b}{S^2} \\ &\quad \times\left(\frac{h_v Q}{P^3}\left(\frac{2D}{1-E[y_c]} - \frac{nD}{1-E[y_c]}\right) + 2aDc_t + \frac{2D\zeta_2}{P^3(1-E[y_c])}\right) > {h_v}^2\left[\frac{2D}{1-E[y_c]} - \frac{nD}{1-E[y_c]}\right]^2 \\ &\quad \times \frac{b}{S^2} + \left(-\frac{D}{nQ^2(1-E[y_c])}\right)^2 \times \left(\frac{h_v Q}{P^3}\left(\frac{2D}{1-E[y_c]} - \frac{nD}{1-E[y_c]}\right) + 2aDc_t + \frac{2D\zeta_2}{P^3(1-E[y_c])}\right).\end{aligned}$$

Thus, $|H_{33}| > 0$.

Finally, for the 4th minor, the optimum value can be obtained as,

$$|H_{44}| = \begin{bmatrix} \frac{\partial^2 \mathrm{JETC}(Q,P,S,kL,n)}{\partial Q^2} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,kL,n)}{\partial Q \partial P} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial Q \partial S} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial Q \partial k} \\ \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P \partial Q} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P^2} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P \partial S} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P \partial k} \\ \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial S \partial Q} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial S \partial P} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial S^2} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial S \partial k} \\ \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial k \partial Q} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial k \partial P} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial k \partial S} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial k^2} \end{bmatrix}$$

$$= -\frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial Q \partial k} \begin{bmatrix} \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P \partial Q} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P^2} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial P \partial S} \\ \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial S \partial Q} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial S \partial P} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial S^2} \\ \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial k \partial Q} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial k \partial P} & \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial k \partial S} \end{bmatrix}$$

$$+ \frac{\partial^2 \mathrm{JETC}(Q,P,S,k,L,n)}{\partial k^2} |H_{33}|$$

$$= -\frac{D\pi(\Phi(k)-1)}{Q^2(1-E[y_c])}\{\sigma\sqrt{L} + (n-1)\sigma\sqrt{L}\} \begin{bmatrix} B & C & 0 \\ -\frac{D}{nQ^2(1-E[y_c])} & 0 & \frac{b}{S^2} \\ -\frac{D\pi(\Phi(k)-1)}{Q^2(1-E[y_c])}\{\sigma\sqrt{L} + (n-1)\sigma\sqrt{L}\} & 0 & 0 \end{bmatrix}$$

$$+ \left[\frac{D\pi\sigma\sqrt{L}}{Q(1-E[y_c])} + \frac{D\pi(n-1)\sigma\sqrt{T}}{Q(1-E[y_c])}\right]\phi(k)|H_{33}|$$

$$= \left(\frac{D\pi(\Phi(k)-1)}{Q^2(1-E[y_c])}\{\sigma\sqrt{L} + (n-1)\sigma\sqrt{L}\}\right)^2 C\frac{b}{S^2}$$

$$+ \left[\frac{D\pi\sigma\sqrt{L}}{Q(1-E[y_c])} + \frac{D\pi(n-1)\sigma\sqrt{T}}{Q(1-E[y_c])}\right]\phi(k)|H_{33}|.$$

Here, the first part is positive and already we have proved that $|H_{33}|$ is positive.

Therefore, $|H_{44}| > 0$.

From the above derivations, all the principal minors of the Hessian matrix is positive. Hence, the given Hessian matrix H is positive definite at (Q, P, S, k).

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