RAIRO-Oper. Res. 54 (2020) 1775–1791
https://doi.org/10.1051/ro/2019102

**RAIRO Operations Research**
www.rairo-ro.org

# A GENERALIZED FUZZY COST EFFICIENCY MODEL

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**Abstract.** The concept of cost efficiency has become tremendously popular in data envelopment analysis (DEA) as it serves to assess a decision-making unit (DMU) in terms of producing minimum-cost outputs. A large variety of precise and imprecise models have been put forward to measure cost efficiency for the DMUs which have a role in constructing the production possibility set; yet, there's not an extensive literature on the cost efficiency (CE) measurement for sample DMUs (SDMUs). In an effort to remedy the shortcomings of current models, herein is introduced a generalized cost efficiency model that is capable of operating in a fuzzy environment-involving different types of fuzzy numbers-while preserving the Farrell's decomposition of cost efficiency. Moreover, to the best of our knowledge, the present paper is the first to measure cost efficiency by using vectors. Ultimately, a useful example is provided to confirm the applicability of the proposed methods.

**Mathematics Subject Classification.** 90C70.

Received January 16, 2018. Accepted November 7, 2019.

## 1. Introduction

Data envelopment analysis (DEA), which is known to be a nonparametric (frontier) approach, employs a linear programming (LP) technique to evaluate the best production frontier, and subsequently, measure the efficiency of decision-making units (DMUs). Farrell [16] was the first to suggest estimating the relative (technical) efficiency and cost efficiency (CE) of DMUs. The idea of measuring relative (technical) efficiency and cost efficiency (CE) came from Farrell [16] and was later promoted by Charnes *et al.* [10], who presented the CCR model under constant returns to scale so as to estimate the best-practice frontier of DMUs with multiple inputs and outputs. Banker *et al.* [4] assumed variable returns to scale in order to construct the BCC model, which was a modified version of the CCR model. Farrell [16] also proposed the general concept of cost efficiency for DMUs with known and deterministic input prices, which became the basis of the DEA-based method devised by Färe *et al.* [15] for measuring CE and revenue efficiency (RE). DEA methodology has increasingly been used for CE measurement, seeking to evaluate the ability of a DMU to produce the current outputs at minimal cost. A significant body of research has been conducted in the past two decades, addressing the theoretical and practical aspects of CE measurement (*e.g.* [22,29,32]). Hence, it is not possible to undertake a thorough literature review of DEA and CE. Instead, we intend to focus simply on highly relevant studies to our case. In 1997, Sueyoshi performed production and cost analyses on eight distinct efficiency concepts. Puig-Junoy [29] adopted a practical

*Keywords.* Cost efficiency, sample decision-making unit, fuzzy numbers, $\alpha$-cut.

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two-stage approach to examining operating cost estimates by dint of a regression model. Having realized technical limitations of conventional CE models, Tone proposed a novel model in 2002 for CE measurement. Camanho and Dyson [7] considered elaborate cases of price uncertainty for measuring the cost efficiency of a set of bank branches. In [34], cost efficiency was decomposed by Tone and Tsutsui into technical, price, and allocative efficiencies. In order to reduce computational complexity to a considerable extent, Jahanshahloo *et al.* [20] suggested a streamlined version of the CE model proposed by Camanho and Dyson [7]. Kuosmanen *et al.* [22] resorted to absolute shadow prices for measuring profit efficiency at firm and industry levels and evaluated all firms using the same input-output prices. A new ratio-based DEA model was introduced by Mozaffari *et al.* [27] that could estimate both cost and revenue efficiencies. Two mathematical programming models were designed by Fang and Li [14] in order to compute the upper and lower bounds of efficiency scores in conformity with the law of one price (LOOP). Cesaroni and Giovannola [9] integrated the measurement of scale economies into efficiency analysis by providing a new cost efficiency model based on ray average cost and optimal scale size. Cesaroni [8] worked on the concept of "cost-minimizing industry structure" defined by Baumol *et al.* [5] so as to decompose the industry measure, and therefore, find the association between group and individual measures. A novel stochastic metafrontier Fourier flexible cost function was established by Lee and Huang [24] to make a comparison between the cost efficiencies of banks in Western European countries. Hasançebi [17] strived to measure the cost efficiency of steel frameworks with a focus on the economical design of multi-story buildings. All the above-mentioned studies revolve around those DMUs that are involved in constructing the production possibility set (PPS). Yet, there are some occasions when it becomes necessary to get to grips with sample decision-making units (SDMUs), which are different from DMUs in that they are not engaged in constructing the PSS and may exist either inside or outside the PSS. Therefore, it is worthwhile to approach cost efficiency with a heavy emphasis upon SDMUs. To the best of our knowledge, there is no study in the literature setting out to estimate the cost efficiency of SDMUs. It is often presumed in conventional DEA that DMUs and SDMUs have crisp values. In real-world problems, however, the observed values are sometimes imprecise or ambiguous, in which case interval, ordinal, stochastic, and fuzzy data are potentially beneficial. Given that the fuzzy logic has proven enormously successful in handling data uncertainty and vagueness, we lay particular emphasis on fuzzy DEA models so that we can counter some of the problems associated with estimating the efficiency of units with fuzzy data. As reported by Emrouznejad *et al.* [13] and Hatami-Marbini *et al.* [19], the existing approaches to solving fuzzy DEA models fit into six classification schemes viz. the tolerance approach, the $\alpha$-level based approach, the fuzzy ranking approach, the possibility approach, the fuzzy arithmetic approach, and the type-2 fuzzy random approach. Few studies have been conducted on the estimation of cost/revenue efficiency scores in a fuzzy environment. Bagherzadeh-valami [3] calculated the cost efficiency of DMUs when their input prices are triangular fuzzy numbers. Depending on the fact that inputs, outputs, and price vectors either are fuzzy or lie within intervals, Emrouznejad *et al.* [12] put forward two innovative methods for finding the overall profit Malmquist productivity index. A directional distance function approach was implemented under uncertain conditions by Sahoo *et al.* [30] to determine the cost, revenue, and profit efficiency of DMUs. Aghayi [1] utilized the $\alpha$-level based approach for CE estimation of DMUs with fuzzy data, where the obtained cost efficiency measure was demonstrated to satisfy Farrell's decomposition conditions. Building on the success of the CCR model, a fuzzy sample CCR model-also known as the FSCCR model-was pioneered by Mu *et al.* [28] with a view to calculating the efficiency of a fuzzy sample DMU (FSDMU). Subsequently, five vector-based evaluation techniques for solving the previously proposed FSCCR model were introduced by Ma and Cui [25], who not only generalised fuzzy numbers in their study but also categorized them into three major groups. In an endeavor to overcome the shortcomings of prevailing FDEA models, Ma and Cui [26] presented a generalized fuzzy DEA method which proven particularly useful in measuring the efficiency of FSDMUs. Aghayi and Salehpour [2] implemented a vector-based method for estimating the revenue efficiency of SDMUs. As inferred from the aforementioned review, there is little enthusiasm in the literature for working on cost efficiency models with fuzzy data and research remains still lacking on the techniques of dealing with the generalised membership functions of fuzzy numbers in the presence of SDMUs [6,35,36]. In this paper, we aim to present a generalized fuzzy cost efficiency model, called the GFCE model, to surmount the obstacles that earlier methods put in the way of efficiency estimation. The proposed model makes it possible to calculate the cost efficiency of an FSDMU whose inputs and outputs have volatile membership functions. Moreover, we strive to align five evaluation methods with the GFCE model so that the cost efficiency of an SDMU can be evaluated under precise as well as fuzzy conditions. The GFCE model is also shown to be capable of satisfying Farrell's decomposition for SDMUs in addition to obtaining the CE level of an FSDMU.

The remainder of this paper is structured as follows:

in Section 2, we offer an overview of cost efficiency models; Section 3 deals with defining some concepts of fuzzy numbers; a generalized fuzzy cost efficiency model is introduced in Section 4; five evaluation methods are modified in Section 5 to measure the cost efficiency of FSDMUs; in order to illustrate the applicability of the proposed methods herein, a numerical example is provided in Section 6; finally, a conclusion is drawn in Section 7.

## 2. Cost efficiency

In view of the shortage and expensiveness of resources as well as growing economic competitions during the last few years, having full awareness of costs is a prerequisite to making the best use of available resources. This issue is of such tremendous importance that, in the absence of scrupulous attention, it may raise formidable challenges to the survival of organizations. As a result, cost efficiency models try to find a unit that buys inputs, which are not larger than those of the DMU under evaluation, at minimal cost so as to generate outputs equal to those of the DMU under evaluation. Assume there are n DMUs with m inputs and r outputs such that the price vector of each input is known. Let $x_{ij}$ and $y_{rj}$ be the respective values of the $i$th input and $r$th output of $\text{DMU}_{j_o}(j=1,\ldots,n)$, with $p_{ij_o}$ denoting the price of the $i$th input to $\text{DMU}_{j_o}$. It is presumed that all inputs and outputs are non-negative, with at least one input and one output being positive. The envelopment form of the cost efficiency model proposed by Färe *et al.* [15] is formulated as follows:

$$\text{CE}_{\text{envelopment}}:\begin{cases}\min \sum_{i=1}^{m} p_{ij_o}x_i^o & \\ \text{s.t.}\ \sum_{j=1}^{n}\lambda_j x_{ij}= x_i^o & i=1,2,\ldots,m\\ \qquad \sum_{j=1}^{n}\lambda_j y_{rj}\geq y_{rj_0}, & r=1,2,\ldots,s,\\ \qquad \lambda_j\geq 0, & j=1,2,\ldots,n,\\ \qquad x_i^o\geq 0, & i=1,2,\ldots,m.\end{cases} \tag{2.1}$$

If $x_i^{o*}$ is the optimal solution to model (2.1), then the cost efficiency score of $\text{DMU}_{j_o}$ is given by:

$$\text{CE}_{j_o}=\frac{\sum_{i=1}^{m}p_{ij_o}x_i^{o^*}}{\sum_{i=1}^{m}p_{ij_o}x_{ij_o}}. \tag{2.2}$$

**Definition 2.1.** $\text{DMU}_{j_o}$ is called cost-efficient if and only if $\text{CE}_{j_o}=1$.

Let $v_i$ and $u_r$ be the weights of the $i$th input and $r$th output, respectively. Besides, suppose that $v_{i^a}$ $(v_{i^b})$ represents the weight given to the input $i^a$ $\left(i^b\right)$ and $p_{i^a}$ $(p_{i^b})$ denotes the price of the input $i^a$ $\left(i^b\right)$. With the weight restriction $\frac{v_{i^a}}{v_{i^b}}=\frac{p_{i^a j_o}}{p_{i^b j_o}}$ in place, the multiplier form of the above model can be formulated as follows [7]:

$$\text{CE}_{\text{multiplier}}:\begin{cases}\max \sum_{r=1}^{s}u_r y_{j_0} & \\ \text{s.t.}\ \sum_{i=1}^{m}v_i x_{j_0}=1, & \\ \qquad \sum_{r=1}^{s}u_r y_{rj}-\sum_{i=1}^{m}v_i x_{ij}\leq 0, & j=1,\ldots,n,\\ \qquad v_{i^a}-\dfrac{p_{i^a j_o}}{p_{i^b j_0}}v_i b=0, & i^a<i^b, i^a, i^b=1,\ldots,m,\\ \qquad u_r\geq 0, & r=1,\ldots,s.\end{cases} \tag{2.3}$$

It is worthy of note that the optimal objective value of (2.3) equals the cost efficiency score given by equation (2.2) (see Schaffinit *et al.* [31] for extra details). Given that precise data are a particular kind of fuzzy data, model (2.3) can be altered as follows:

$$\text{FCE}:\begin{cases}\max \sum_{r=1}^{s} u_r \tilde{y}_{j_0} \\ \text{s.t.} \ \ \sum_{i=1}^{m} v_i \tilde{x}_{j_0} = 1, \\ \quad \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0, & j = 1,\dots,n, \\ \quad v_{i^a} - \dfrac{p_{i^a j_o}}{p_{i^b j_o}} v_{i^b} = 0, & i^a < i^b, i^a, i^b = 1,\dots,m, \\ \quad u_r \geq 0, & r = 1,\dots,s. \end{cases} \tag{2.4}$$

where $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ represent the amounts of the $i$th input and $r$th output of $\text{FDMU}_j$, respectively; the price of the $i$th input to $\text{FDMU}_{j_o}$ is denoted by $p_{ij_o}$, $\tilde{y}_{rj_o}$ is the $r$th output produced by $\text{FDMU}_{j_o}$, and the index $j_o$ signifies the unit under evaluation.

## 3. An overview of the concepts of fuzzy numbers

A selection of basic definitions of fuzzy sets theory is offered in this section [11, 23].

**Definition 3.1.** Let $U$ be a universal set. A fuzzy set $\tilde{A}$ in the universal set $U$ is defined by its membership function $\mu_{\tilde{A}}(x) \to [0,\ 1]$, where $\forall x \in U \to \mu_{\tilde{A}}(x)$ is the membership degree of $\tilde{A}$ in $U$.

**Definition 3.2.** A fuzzy subset $\tilde{A}$ proves normal and convex if $\sup_{x\in U} \mu_{\tilde{A}}(x) = 1$ and $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{A}}(y)), \forall x, y \in U, \forall \lambda \in [0,1]$, respectively, where $\wedge$ is the minimum operator.

**Definition 3.3.** A fuzzy set $\tilde{A}$ is a fuzzy number if $\tilde{A}$ is normal and convex.

**Definition 3.4.** A trapezoidal fuzzy number $\tilde{A} = [a_1, a_2,\ a_3, a_4]$ with the membership function $\mu_{\tilde{A}}$ exhibits the following characteristics:

$\mu_{\tilde{A}}$ is a continuous mapping of $\mathbb{R}$ to the closed interval $[0,\ 1]$,

(1) $\mu_{\tilde{A}}(x) = 0, \forall x \in (-\infty,\ \ a_1]$,
(2) $\mu_{\tilde{A}}$ is strictly increasing on$[a_1,\ a_2]$,
(3) $\mu_{\tilde{A}}(x) = 0, \forall x \in [a_2,\ a_3]$,
(4) $\mu_{\tilde{A}}$ is strictly decreasing on$[a_3,\ a_4]$,
(5) $\mu_{\tilde{A}}(x) = 0, \forall x \in [a_4,\ +\infty)$.

It is to be noted that $a_1$, $a_2$, $a_3$, and $a_4$ are real numbers, and the membership function $\mu_{\tilde{A}}$ can be written as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} f_a(x), & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ g_a(x), & a_3 \leq x \leq a_4, \\ 0, & \text{o.w.} \end{cases}$$

where $f_a : [a_1, a_2] \to [0,1]$ and $g_a : [a_3, a_4] \to [0,1]$.

A trapezoidal fuzzy number $\tilde{A} = [a_1, a_2, a_3, a_4]$ can be converted into a real number $A$ when $a_1 = a_2 = a_3 = a_4$. Conversely, a real number $A$ can be written as a trapezoidal fuzzy number $\tilde{A} = [a, a, a, a]$. In the event that $a_2 = a_3$, $\tilde{A} = [a_1, a_2, a_3, a_4]$ is regarded as a triangular fuzzy number.

**Definition 3.5.** The $\alpha$-cut of a fuzzy set $\tilde{A}$, which is represented by $\tilde{A}_\alpha$, is defined as the crisp set $\tilde{A}_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha\}$.

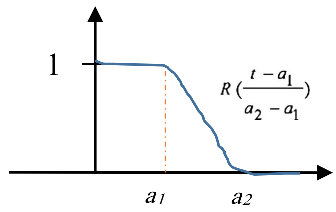


FIGURE 1. A typical representation of a partial small type fuzzy number.

It can be inferred from Definition 6 that the lower end point, *i.e.* $\left[\tilde{A}\right]_{\alpha}^{L}$, and upper end point, *i.e.* $\left[\tilde{A}\right]_{\alpha}^{U}$, of the $\alpha$-cut of a trapezoidal fuzzy set $\tilde{A}_{\alpha} = [a_1, a_2, a_3, a_4]_{\alpha}$ are given by $a_1 + (a_2 - a_1)\,\alpha$ and $a_4 - (a_4 - a_3)\,\alpha$, respectively. There exist a large variety of fuzzy numbers other than trapezoidal (triangular) fuzzy numbers most of which can be classified into three major types, namely partial small type, partial large type, and middle type [25].

Their corresponding membership functions are provided in Appendix A.

The membership function for partial small type fuzzy numbers can be written as follows:

$$\mu_{\tilde{A}}(t) = \begin{cases} 1, & t < a_1, \\ R\left(\dfrac{t - a_1}{a_2 - a_1}\right), & a_1 \le t \le a_2, \\ 0, & t > a_2. \end{cases}$$

This type of fuzzy numbers can generally be represented by $a_1, a_2$, and $a_3$, where $a_3 \in [0,\, a_1)$.

The membership function of partial large type fuzzy numbers is given as follows:

$$\mu_{\tilde{A}}(t) = \begin{cases} 0, & t < a_1, \\ L\left(\dfrac{t - a_1}{a_2 - a_1}\right), & a_1 \le t \le a_2, \\ 1, & t > a_2. \end{cases}$$

Three numbers $a_1, a_2$, and $a_3$, where$a_3 \in (a_2, +\infty)$, are mainly used to describe this type of fuzzy numbers.

For middle type fuzzy numbers, the membership function is defined as follows:

$$\mu_{\tilde{A}}(t) = \begin{cases} 0, & t < a_1, \\ L\left(\dfrac{t - a_1}{a_2 - a_1}\right), & a_1 \le t < a_2, \\ 1, & a_2 \le t < a_3, \\ R\left(\dfrac{a_4 - t}{a_4 - a_3}\right), & a_3 \le t < a_4, \\ 0, & t \ge a_4. \end{cases}$$

This type of fuzzy numbers are mostly characterized by $a_1, a_2, a_3$, and $a_4$.

Typical diagrams of the membership functions for the above-mentioned types of fuzzy numbers are demonstrated in Figure 1.

## 4. GENERALIZED FUZZY COST EFFICIENCY MODEL

Assume DMU is one decision-making unit in a decision-making problem. All units with similar input and output data to the DMU are called sample decision-making units (SDMUs) in this methodology [37]. The efficiency measure of the DMU must be equal to or less than 1, while there is no limitation upon that of

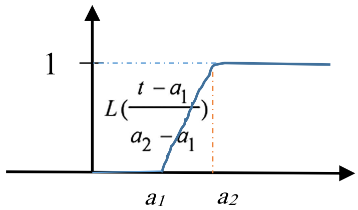


Figure 2. A typical representation of a partial large type fuzzy number.

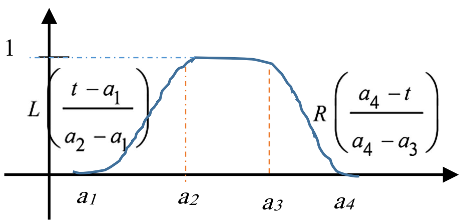


Figure 3. A typical representation of a middle type fuzzy number.

Table 1. Data related to four FDMUs and one FSDMU.

| FDMU | $\text{FDMU}_1$ | $\text{FDMU}_2$ | $\text{FDMU}_3$ | $\text{FDMU}_4$ | FSDMU |
|---|---|---|---|---|---|
| Input | $(0.5, 1.5)$ | $(1.5, 2.5)$ | $(2.5, 3.5)$ | $(3.5, 4.5)$ | $(0.7, 1.4)$ |
| Output | $(2.5, 3.5)$ | $(0.5, 1.5)$ | $(3.5, 4.5)$ | $(1.5, 2.5)$ | $(3.7, 4.4)$ |

a SDMU. A DMU must exist in the constraints of a problem, whilst an SDMU may or may not appear in the constraints. The distinctions between a fuzzy decision making unit (FDMU) and a fuzzy sample decision making unit (FSDMU) are initially presented in this section. Afterward, an analysis of Farrell's decomposition for an FSDMU is given. Then, considering the FSDMU and the difference between an FDMU and an FSDMU, the generalized fuzzy cost efficiency model is formulated besides the FDMU. If fuzzy numbers are positive real numbers, then $\tilde{x}$ and $\tilde{y}$ represent the subsets of $R$, which might be continuous, discontinuous, or mixed continuous-discontinuous sets in themselves. In case of being continuous sets, $\tilde{x}$ and $\tilde{y}$ can generally belong to an interval. Hence, a DMU with fuzzy input and output data can be treated-in bi-dimensional space-as a rectangle whose length and width reflect input and output changes, respectively. As well as an FDMU, an FSDMU may also need to be managed while solving a decision-making problem (Figs. 2 and 3).

**Definition 4.1.** Let DMU be a decision-making unit in a problem. All units with the same input and output data as the DMU are termed sample decision-making units (SDMUs) (see [37]), which do not have a part in constructing the PPS.

The data shown in Table 1 are used to further recognize the distinctions between an FDMU and an FSDMU. As suggested by Figure 4, an FDMU differs from an FSDMU in the following aspects:

- An FDMU must belong to the PPS; however, an FSDMU may be located outside of the PPS.
- The efficiency score of an FDMU must not be greater than 1, while that of an FSDMU can be smaller than, equal to, or larger than 1.
- An FDMU must definitely satisfy the constraints of a problem, while an FSDMU may or may not satisfy the constraints (see Ma and Cui [26] for further details).

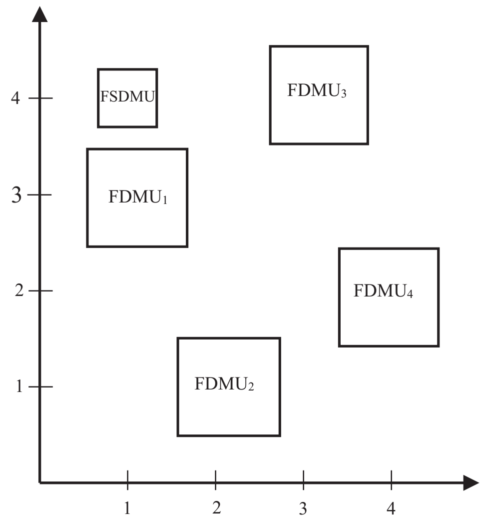


Figure 4. The FDMUs and FSDMU in Table 1.

Table 2. Three FDMUs and one FSDMU, each with two inputs one output.

| | $\text{FDMU}_1$ | $\text{FDMU}_2$ | $\text{FDMU}_3$ | FSDMU |
|---|---|---|---|---|
| Input 1 | $(1.5, 2.5)$ | $(3, 4)$ | $(0.5, 1.5)$ | $(0.4, 0.8)$ |
| Input 2 | $(0.5, 1.5)$ | $(0, 1)$ | $(2.5, 3.5)$ | $(0.5, 1.5)$ |
| Output | 1 | 1 | 1 | 1 |

As for the differences between an FDMU and an FSDMU, it can be stated that they are also different with respect to cost efficiency. Farrell [16] clearly described the original process of decomposing cost efficiency into its technical and allocative efficiency components, which became later acknowledged as Farrell's decomposition. He stated that the cost efficiency of a DMU is determined by the product of technical efficiency and allocative efficiency. At this point, we intent to consider this fundamental decomposition for an FSDMU, according to the data in Table 2.

In order to analyze technical efficiency, we substitute a central DMU for all FDMUs, and accordingly, a central SDMU takes the place of the FSDMU.

Figure 5 illustrates Farrell frontier for the above units in input space.

As implied by Figure 5, the technical efficiency of the SDMU is calculated as:

$$\text{TE}_{\text{SDMU}} = \frac{\text{OS}'}{\text{OS}} \geq 1.$$

To avoid complexity, let us assume that the input price $(P)$ is constant for all units.

Now, let us consider Figure 6 to examine the cost efficiency of the SDMU:

Moreover, it can be deduced from Figure 6 that the cost efficiency of the SDMU can be written as follows:

$$\text{CE}_{\text{SDMU}} = \frac{\text{OS}''}{\text{OS}} \geq 1.$$

Figure 6 obviously illustrates that

$$1 \leq \text{CE}_{\text{SDMU}} \leq \text{TE}_{\text{SDMU}} \to \frac{\text{TE}_{\text{SDMU}}}{\text{CE}_{\text{SDMU}}} \geq 1 \to \frac{\text{CE}_{\text{SDMU}}}{\text{TE}_{\text{SDMU}}} \leq 1$$

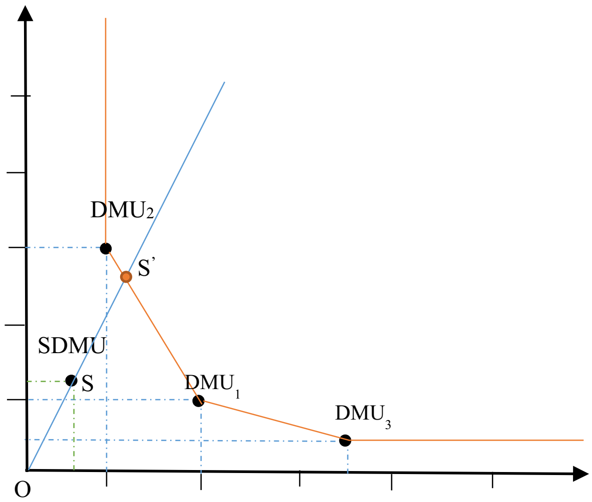


FIGURE 5. Farrell frontier corresponding to the data in Table 2.

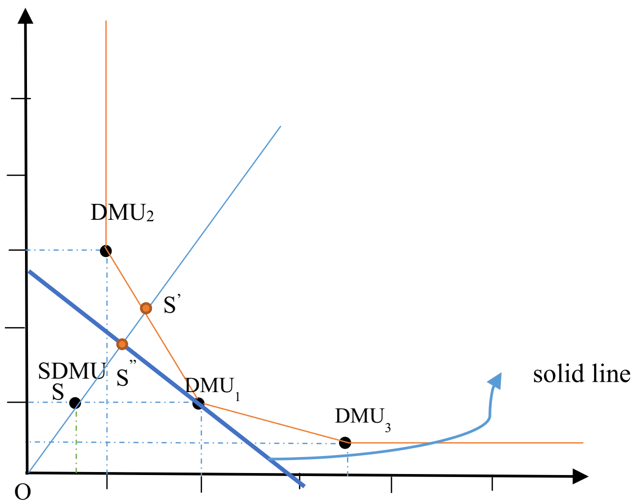


FIGURE 6. Farrell's decomposition based on the data in Table 2.

and thus

$$\mathrm{AE}_{\mathrm{SDMU}} = \frac{\mathrm{CE}_{\mathrm{SDMU}}}{\mathrm{TE}_{\mathrm{SDMU}}} \leq 1 \rightarrow \mathrm{CE}_{\mathrm{SDMU}} = \mathrm{AE}_{\mathrm{SDMU}} \times \mathrm{TE}_{\mathrm{SDMU}}.$$

As shown above, we ended up with Farrell's decomposition, from which it can be concluded that the same decomposition can also be employed for the CE measurement of an SDMU.

TABLE 3. Different cases for DMUs and SDMUs.

| | |
|---|---|
| Best DMU/SDMU | The DMU/SDMU with minimum input and maximum output |
| Worst DMU/SDMU | The DMU/SDMU with maximum input and minimum output |
| Max DMU/SDMU | The DMU/SDMU with maximum input and output |
| Min DMU/SDMU | The DMU/SDMU with minimum input and output |
| Center DMU/SDMU | The DMU/SDMU at the midpoint of the domain |
| 1-cut DMU/SDMU | The DMU/SDMU with the maximum value of the membership function. |
| Vertex DMU/SDMU | The DMU/SDMU whose different input and output have the maximum or minimum value. |

With $\text{FSDMU}_{j_o}$ and $\text{FDMU}_{j_o}$ in mind, we present the following CE model:

$$\text{GFCE}: \begin{cases} \max \sum_{r=1}^{s} u_r \tilde{y}_{i_{so}} \\ \text{s.t.} \ \ \sum_{i=1}^{m} v_i \tilde{x}_{j_{so}} = 1, \\ \qquad \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0, & j = 1, \ldots, n, \\ \qquad v_{i^a} - \dfrac{P_{i^a j_{so}}}{p_{i^b j_{so}}} v_{i^b} = 0 & i^a < i^b, i^a, i^b = 1, \ldots, m, \\ \qquad u_r \geq 0, & r = 1, \ldots, s. \end{cases} \tag{4.1}$$

where $\tilde{x}_{j_{so}}$ and $\tilde{y}_{j_{so}}$ represent the respective input and output values of $\text{FSDMU}_{j_{so}}$, $\tilde{y}_{rj}$ denotes the $r$th output value for $\text{FDMU}_j$, $\tilde{x}_{ij}$ signifies the value of the $i$th input to $\text{FDMU}_j$, $u_r$ is the weight of the $r$th output, $v_i$ represents the weight of the $i$th input, $n$ refers to the number of FDMUs, $m$ denotes the number of inputs, $s$ is the number of outputs, the $\text{FDMU}_j$ under evaluation is designated by index $j_{so}$, $v_{i^a}$ $(v_{i^b})$ denotes the weight given to input $i^a$ $\left(i^b\right)$, and $p_{i^a}$ $(p_{i^b})$ corresponds with the price of input $i^a$ $\left(i^b\right)$ to FSDMU.

## 5. GFCE EVALUATION METHODS

Model (2.3) computes the cost efficiency of DMUs by the approach developed by Camanho and Dyson [7]. If input and output data are fuzzy, model (2.3) will not be able to compute the cost efficiency, for which case we proposed model (2.4). Since the main purpose of this paper is to evaluate the cost efficiency of fuzzy SDMUs, we proposed model (4.1). Therefore, in this section we solve model (4.1) for various cases.

First, we consider the different cases given in Table 3 for DMUs and SDMUs to evaluate cost efficiency for FSDMUs.

If FSDMUs and FDMUs are replaced, respectively, by SDMUs and DMUs in the best case, cost efficiency evaluation is carried out in the optimistic case, which is the Best–Best cost efficiency evaluation. Furthermore, if FSDMUs and FDMUs are replaced, respectively, by SDMUs and DMUs in the worst case, cost efficiency is evaluated in the pessimistic case, which is the Worst–Worst cost efficiency evaluation. Replacing FSDMUs by SDMUs in the worst case and FDMUs by DMUs in the best case yields cost efficiency evaluation in the Worst-Best case; and replacing FSDMUs by SDMUs in the best case and FDMUs by DMUs in the worst case produces the Best–Worst cost efficiency evaluation. Cost efficiency evaluation for the Max–Max, Min–Min, and Center–Center cases is also carried out similarly to what was explained above.

**Definition 5.1.** When the FSDMU is replaced with the worst SDMU and FDMUs are replaced with the best DMUs, the FSDMU is said to be strongly cost-efficient-*i.e.* cost-efficient even in the worst-case scenario-if the cost efficiency score of the SDMU is equal to or larger than 1.

**Definition 5.2.** If the cost efficiency score of the SDMU is equal to or larger than 1, the FSDMU is declared cost-efficient when one of the DMUs is substituted for the FSDMU and FDMUs.

**Definition 5.3.** If the cost efficiency score of the SDMU is equal to or larger than 1, the FSDMU is declared weakly cost-efficient-that is to say cost-efficient merely under the best possible conditions-when the best SDMU is substituted for the FSDMU and FDMUs are replaced with the worst DMUs.

**Definition 5.4.** The FSDMU is declared cost-inefficient if it doesn't prove to be weakly cost-efficient.

In addition to what was explained above, the cost efficiency values for FSDMUs, too, can be obtained by model (4.1). For this purpose, we consider two cases; in the first case, the FSDMU is replaced by one of the cases of the SDMUs in Table 3 and its cost efficiency value is calculated based on the FDMUs. Therefore, model (4.1) will be as follows.

$$\text{SFCE}: \begin{cases} \max \sum_{r=1}^{s} u_r y_{j_s} \\ \text{s.t.} \quad \sum_{i=1}^{m} v_i x_{j_s} = 1, \\ \qquad \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0, & j = 1, \ldots, n, \\ \qquad v_{i^a} - \dfrac{P_{i^a j_s}}{p_{i^b j_s}} v_{i^b} = 0 & i^a < i^b, i^a, i^b = 1, \ldots, m, \\ \qquad u_r \geq 0, & r = 1, \ldots, s. \end{cases} \tag{5.1}$$

where $x_{j_s}$ and $y_{j_s}$ represent the respective input and output data of $\text{SDMU}_{j_s}$, $\tilde{y}_{rj}$ denotes the $r$th output value for $\text{FDMU}_j$, $\tilde{x}_{ij}$ signifies the value of the $i$th input to $\text{FDMU}_j$, $u_r$ is the weight of the $r$th output, $v_i$ represents the weight of the $i$th input, $n$ refers to the number of FDMUs, $m$ denotes the number of inputs, $s$ is the number of outputs, the $\text{SDMU}_j$ under evaluation is designated by index $j_s$, $v_{i^a}$ $(v_{i^b})$ denotes the weight given to input $i^a$ $\left(i^b\right)$, and $p_{i^a}$ $(p_{i^b})$ corresponds with the price of input $i^a$ $\left(i^b\right)$ to $\text{SDMU}_{j_s}$.

**Definition 5.5.** When all $\text{FDMU}_j$s are superseded by the end-point DMUs of vector $\vec{V}_{Wj} + k\left(\vec{V}_{Bj} - \vec{V}_{Wj}\right)$, the FSDMU is said to be $k$-level cost-efficient in the SFCE model if the cost efficiency score of the SDMU is equal to 1.

$k$ always varies in the bounded closed interval $[0, 1]$. From $k = 1$, we can deduce that the best DMU has been substituted for the FDMU, that is to say the cost efficiency score of the FSDMU has been measured in the worst-case scenario. In case of $k = 0$, however, it can be inferred that all FDMUs have been replaced with the worst DMUs-in other words, the cost efficiency score of the FSDMU has been measured in the best-case scenario. Hence, the cost efficiency of the FSDMU improves as the value of $k$ increases in this method, because higher values of $k$ are indicative of the fact that the FSDMU remains cost-efficient in the worst-case scenario.

In the second case, FDMUs are replaced by one of the cases in Table 3 and then the cost efficiency value of the FSDMU is obtained. So, model (4.1) will be rewritten as follows.

$$\text{FSCE}: \begin{cases} \max \sum_{r=1}^{s} u_r \tilde{y}_{j_{so}} \\ \text{s.t.} \quad \sum_{i=1}^{m} v_i \tilde{x}_{j_{so}} = 1, \\ \qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, & j = 1, \ldots, n, \\ \qquad v_{i^a} - \dfrac{P_{i^a j_{so}}}{p_{i^b j_{so}}} v_{i^b} = 0 & i^a < i^b, i^a, i^b = 1, \ldots, m, \\ \qquad u_r \geq 0, & r = 1, \ldots, s. \end{cases} \tag{5.2}$$

where $\tilde{x}_{j_{so}}$ and $\tilde{y}_{j_{so}}$ represent the respective input and output values of $\text{FSDMU}_{j_{so}}$, $y_{rj}$ denotes the $r$th output value for $\text{DMU}_j$, $x_{ij}$ signifies the value of the $i$th input to $\text{DMU}_j$, $u_r$ is the weight of the $r$th output, $v_i$ represents the weight of the $i$th input, $n$ refers to the number of DMUs, $m$ deontes the number of inputs, $s$ is the number of outputs, the $\text{FSDMU}_j$ under evaluation is designated by index $j_{so}$, $v_{i^a}$ $(v_{i^b})$ denotes the weight given to input $i^a$ $\left(i^b\right)$, and $p_{i^a}$ $(p_{i^b})$ corresponds with the price of input $i^a$ $\left(i^b\right)$ to FSDMU.

**Definition 5.6.** If the cost efficiency score of the SDMU is 1, the FSDMU is called $k$-level cost-efficient in the FSCE model when the SDMU at the end point of vector $\vec{V}_{Wj_{so}} + k\left(\vec{V}_{Bj_{so}} - \vec{V}_{Wj_{so}}\right)$ is substituted for the FSDMU.

In this method, $k$ always belongs to the bounded closed interval $[0, 1]$.

In case of $k = 0$, we can draw the conclusion that the worst SDMU has been substituted for the FSDMU, indicating that the CE measurement of the FSDMU has been performed in the worst-case scenario. On the other hand, $k = 1$ indicates that the best SDMU has been substituted for the FSDMU-in other words, the CE measurement of the FSDMU has been carried out in the best-case scenario. Therefore, a lower value of $k$ in this method is associated with a higher cost efficiency score for the FSDMU; that is to say, lower values of $k$ suggest that the FSDMU is more likely to remain cost-efficient in the worst-case scenario.

It must be noted that a different cost efficiency value will be obtained for the FSDMU by using vectors if the FDMUs and FSDMUs are replaced by any arbitrary point other than the cases mentioned in Table 3.

The SDMU at the end point of vector $\vec{V}_{Wj_{so}} + k\left(\vec{V}_{Bj_{so}} - \vec{V}_{Wj_{so}}\right)$ and the end-point DMUs of vector $\vec{V}_{Wj} + \lambda\left(\vec{V}_{Bj} - \vec{V}_{Wj}\right)$ are substituted for the FSDMU and all FDMUs, respectively, in order for the CE measurement of the FSDMU to be performed. This method actually capitalizes on the second and third methods to calculate the cost efficiency of the FSDMU. To be specific, the value of either $k$ or $\lambda$ must be kept constant. If $k$ is assumed constant, the CE measurement model changes into the SFCE model, in which case the FSDMU is assessed as explained in the second method. On the contrary, if $\lambda$ is assumed constant, the CE measurement model changes into the FSCE model, in which case the FSDMU is estimated as per the third method.

**Definition 5.7.** When the SDMU at the end point of vector $\vec{V}_{Wj_{so}} + k\left(\vec{V}_{Bj_{so}} - \vec{V}_{Wj_{so}}\right)$ and the end-point DMUs of vector $\vec{V}_{Wj} + \lambda\left(\vec{V}_{Bj} - \vec{V}_{Wj}\right)$ are substituted for the FSDMU and all FDMUS, respectively, the FSDMU is called $k$-$\lambda$-level cost-efficient in the GFCE model provided that the cost efficiency score of the FSDMU is 1.

In this case, both $k$ and $\lambda$ always belong to the bounded closed interval $[0, 1]$. From $k = 1$, it can be inferred that the best SDMU has been substituted for the FSDMU-that is, the CE measurement of the FSDMU has been performed in the best-case scenario. However, $k = 0$ suggests that the worst SDMU has been substituted for the FSDMU, which means the CE measurement of the FSDMU has been performed in the worst-case scenario. In contrast, $\lambda = 0$ indicates that the worst DMU has been substituted for FDMUs-that is to say, the CE measurement of the FSDMU has been carried out in the best-case scenario. From $\lambda = 1$, we can come to the conclusion that the best DMU has been substituted for all FDMUs, which means the CE measurement of the FSDMU has been performed in the worst-case scenario. Hence, the cost efficiency of the FSDMU improves as the value of $k$ increases and the value of $\lambda$ decreases.

Moreover, if the $\alpha$-cut corresponding to the FSDMUs and FDMUs are obtained, one of the above-mentioned cases will be obtained, in which case cost efficiency can be computed.

## 6. Numerical example

In doing so, we separately apply the first, second, and third methods to the alpha-cuts of the data in Table 4 for the CE measurement of the FSDMU. In harmony with the aforementioned, the input price vector is represented by $p^T = (1, 1)$.

In this table, the serial numbers 0 and $j$ denote the FSDMU and FDMU$_j$, respectively. As for the "input/output" column of Table 4, input and output data are represented by 1 and 2, respectively. The third column of Table 4 involves the types of membership functions, with numbers 1, 2, and 3 representing partial small type, partial large type, and middle type membership functions, respectively.

A variety of scenarios can be imagined for the SDMU and DMU which are to be substituted for the FSDMU and FSDMU$_j$, respectively. In other words, it is possible to suppose that the chosen SDMU and DMU are

Table 4. Data from Ma's study.

| Serial number | Input/output | Membership function type | Membership function | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 3 | Triangular | 3.5 | 4.1 | 4.1 | 4.7 |
| 0 | 1 | 3 | Triangular | 4.9 | 5.6 | 5.6 | 6.1 |
| 0 | 2 | 3 | Triangular | 2.2 | 2.6 | 2.6 | 3.1 |
| 0 | 2 | 3 | Triangular | 1.5 | 1.8 | 1.8 | 2.4 |
| 1 | 1 | 3 | Triangular | 2.9 | 4.3 | 4.6 | 4.8 |
| 1 | 1 | 3 | Triangular | 2.9 | 4.5 | 4.6 | 4.9 |
| 1 | 2 | 3 | Triangular | 2.1 | 2.4 | 2.9 | 3.2 |
| 1 | 2 | 3 | Triangular | 1.9 | 2.7 | 3.3 | 3.7 |
| 2 | 1 | 3 | $\exp(-(x-b)^2)$ | 2.6 | 4.4 | 4.4 | 5.8 |
| 2 | 1 | 3 | $\exp(-(x-b)^2)$ | 2.5 | 3.9 | 3.9 | 6.7 |
| 2 | 2 | 3 | $\exp(-(x-b)^2)$ | 1.1 | 2.4 | 2.4 | 5.7 |
| 2 | 2 | 3 | $\exp(-(x-b)^2)$ | 1.3 | 2.1 | 2.1 | 5.8 |
| 3 | 1 | 3 | $1/(1+(x-b)^2)$ | 1.4 | 5.8 | 5.8 | 6.7 |
| 3 | 2 | 3 | $1/(1+(x-b)^2)$ | 1.8 | 5.3 | 5.3 | 6.7 |
| 3 | 2 | 3 | $1/(1+(x-b)^2)$ | 2.1 | 3.2 | 3.2 | 5.8 |
| 3 | 2 | 3 | $1/(1+(x-b)^2)$ | 2.2 | 3.3 | 3.3 | 5.1 |
| 4 | 1 | 1 | $((d-x)/(d-c))^2$ | – | 4.8 | 5.1 | 5.8 |
| 4 | 1 | 1 | $((d-x)/(d-c))^2$ | – | 3.1 | 3.3 | 3.8 |
| 4 | 1 | 1 | $((d-x)/(d-c))^2$ | – | 1.6 | 2.2 | 2.7 |
| 4 | 1 | 1 | $((d-x)/(d-c))^2$ | – | 1.3 | 1.5 | 2.1 |
| 5 | 1 | 2 | $((d-x)/(d-c))^2$ | 2.7 | 5.8 | 6.2 | – |
| 5 | 1 | 2 | $((d-x)/(d-c))^2$ | 1.1 | 4.6 | 4.7 | – |
| 5 | 1 | 2 | $((d-x)/(d-c))^2$ | 1.8 | 3.1 | 3.5 | – |
| 5 | 2 | 2 | $((d-x)/(d-c))^2$ | 3.6 | 4.2 | 4.4 | – |
| 6 | 1 | 1 | Trapezoidal | – | 3.9 | 4.5 | 6.1 |
| 6 | 1 | 1 | Trapezoidal | – | 3.6 | 3.9 | 6.6 |
| 6 | 2 | 2 | Trapezoidal | 1.6 | 2.8 | 3.1 | - |
| 6 | 2 | 2 | Trapezoidal | 1.1 | 3.3 | 3.9 | – |

Table 5. The CE measure of the FSDMU-calculated by model (4.1).

| $\alpha$ | Worst–Best | Best–Worst | Best–Best | Worst–Worst | Max–Max | Min–Min | Center–Center |
|---|---|---|---|---|---|---|---|
| 0 | 0.1124 | 1.7046 | 0.2036 | 0.9409 | 0.6295 | 0.3991 | 0.58 |
| 0.2 | 0.2942 | 1.5396 | 0.473 | 0.9578 | 0.7307 | 0.7464 | 0.7707 |
| 0.4 | 0.4333 | 1.3514 | 0.6184 | 0.9469 | 0.8366 | 0.8046 | 0.8411 |
| 0.6 | 0.5308 | 1.1844 | 0.6728 | 0.9344 | 0.8595 | 0.8439 | 0.852 |
| 0.8 | 0.6146 | 1.0346 | 0.692 | 0.9189 | 0.813 | 0.7821 | 0.7984 |
| 1 | 0.6485 | 0.8041 | 0.6485 | 0.8041 | 0.7263 | 0.718 | 0.7223 |

respectively Worst–Best, Best–Worst, Best–Best, Worst–Worst, Max–Max, Min–Min, and center-center units. Next, the MATLAB software is run to obtain the CE scores of the FSDMU, as shown in Table 5, under different possible scenarios.

As seen in Table 5, the CE score is larger than 1 merely when the best SDMU is substituted for the FSDMU, indicating that the FSDMU is-according to Definition 5.3 weakly cost-efficient. In other scenarios, however, the FSDMU proves to be cost-inefficient. In view of the last row of Table 5, the FSDMU is expected to prove cost-inefficient at the maximum $\alpha$-cut upon implementation of the next table.

TABLE 6. The CE measure of the FSDMU-calculated by model (5.1).

| $\alpha$ | Best | Worst | Max | Min | Center |
|---|---|---|---|---|---|
| 0 | 0.381-level | Cost-Inefficient | 0.263-level | 0.207-level | 0.245-level |
| 0.2 | 0.481-level | Cost-Inefficient | 0.297-level | 0.201-level | 0.255-level |
| 0.4 | 0.503-level | Cost-Inefficient | 0.252-level | 0.158-level | 0.209-level |
| 0.6 | 0.398-level | Cost-Inefficient | 0.163-level | 0.073-level | 0.121-level |
| 0.8 | 0.084-level | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |
| 1 | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |

TABLE 7. The CE measure of the FSDMU-calculated by model (5.2).

| $\alpha$ | Best | Worst | Max | Min | Center |
|---|---|---|---|---|---|
| 0 | Cost-Inefficient | 0.098-level | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |
| 0.2 | Cost-Inefficient | 0.088-level | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |
| 0.4 | Cost-Inefficient | 0.150-level | Cost-Inefficient | Cost-Inefficient | 0.994-level |
| 0.6 | Cost-Inefficient | 0.283-level | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |
| 0.8 | Cost-Inefficient | 0.712-level | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |
| 1 | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |

Depending on different scenarios where the SDMU that is selected to replace the FSDMU is the best, worst, maximum, minimum, or central unit, the $k$-level cost efficiency of the FSDMU can be measured by the MATLAB software as shown in Table 6.

Larger values of $k$ reveal that closer DMUs to the best DMU have been substituted for FDMUs, and consequently, a higher level of CE is achieved for the FSDMU. As seen in Table 6, when the best SDMU is substituted for the FSDMU, the $k$-level CE is obtained for all $\alpha$-cuts except for the maximum $\alpha$-cut, with $k = 0.503$ being the optimal value in this scenario. When replaced with the worst SDMU, the FSDMU is cost-inefficient. Under the scenarios where the maximum, minimum, and central SDMUs are substituted for the FSDMU, the $k$-level CE is obtained for $\alpha$-cuts equal to 0, 0.2, 0.4, and 0.6, with the optimum values of $k$ being 0.297, 0.207, and 0.255, respectively. It is worth mentioning that the FSDMU is cost-inefficient for the maximum $\alpha$-cut in all scenarios. As described by the results, the optimum cost efficiency is achieved for $\alpha$-cuts greater than 0.4 when the best SDMU is substituted for the FSDMU.

The $k$-level cost efficiency of the FSDMU can be evaluated by the MATLAB software under different scenarios where FDMUs are replaced with the best, worst, maximum, minimum, or central DMU (see Tab. 7).

Smaller values of $k$ reveal that a closer SDMU to the worst DMU has been substituted for the FSDMU, and consequently, a higher level of CE is achieved for the FSDMU-in other words, the FSDMU remains cost-efficient even in the worst-case scenario. According to Table 7, in case the worst DMU is substituted for the FDMU, the $k$-level CE is obtained for all $\alpha$-cuts except the maximum one, with the optimum value of $k$ being 0.088. When the central DMU is substituted for FDMUs, the FSDMU is found cost-inefficient for all $\alpha$-cuts except for an $\alpha$-cut of 0.4, where it proves to be 0.994-level cost efficient. Hence, the optimum cost efficiency is achieved for $\alpha$-cuts greater than 0.2 when the worst DMU is substituted for FDMUs.

### 6.1. Case study

In order to demonstrate the application of the proposed method, we examine the triangular fuzzy data from Melli Bank branches in the Moghan District, Ardabil Province, Iran. These data concern the years 2012 to 2014. For this purpose, the branches in Moghan are considered as DMUs and the branch in Germi is assumed as the SDMU. The indices "non-performing loans" and "sum of four major deposits" are assumed as inputs

Table 8. The cost efficiency of the FSDMU-calculated by model (5).

| $\alpha$ | Worst–Best | Best–Worst | Best–Best | Worst–Worst | Max–Max | Min–Min | Center–Center |
|---|---|---|---|---|---|---|---|
| 0 | 0.5851 | 2.2455 | 1.1438 | 1.0641 | 1.1775 | 1.117 | 1.1916 |
| 0.2 | 0.7143 | 2.0475 | 1.2272 | 1.1104 | 1.2076 | 1.2076 | 1.2269 |
| 0.4 | 0.8699 | 1.8639 | 1.3026 | 1.1596 | 1.2158 | 1.2763 | 1.2645 |
| 0.6 | 1.0304 | 1.6941 | 1.3696 | 1.2165 | 1.2989 | 1.2979 | 1.3044 |
| 0.8 | 1.1831 | 1.537 | 1.3928 | 1.3009 | 1.3441 | 1.3469 | 1.3468 |
| 1 | 1.3921 | 1.3921 | 1.3921 | 1.3921 | 1.3921 | 1.3921 | 1.3921 |

Table 9. The CE measure of the FSDMU-calculated by model (5.1).

| $\alpha$ | Best | Worst | Max | Min | Center |
|---|---|---|---|---|---|
| 0 | Cost-Inefficient | 0.144-level | 0.854-level | 0.772-level | 0.803-level |
| 0.2 | Cost-Inefficient | 0.311-level | 0.956-level | 0.880-level | 0.901-level |
| 0.4 | Cost-Inefficient | 0.599-level | 0.999-level | 0.999-level | 0.999-level |
| 0.6 | Cost-Inefficient | 0.999-level | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |
| 0.8 | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |
| 1 | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |

and "profit" and "the balance of non-governmental deduction" are considered as outputs. To measure the cost efficiency of FSDMUs, first we calculate the input and output values by $\alpha$-cuts. The input cost vector is assumed as $p^T = (1, 1)$.

First we consider the cases in Table 3 for FSDMU and FDMU replacements. Suppose the selected SDMU and DMU are in the Worst–Best, Best–Worst, Best–Best, Worst–Worst, Max–Max, Min–Min, and Center–Center cases, respectively. By MATLAB software, the cost efficiency values are calculated, as demonstrated in Table 8, below.

From the results in Table 8, one can observe that the cost efficiency value for the Germi branch is mostly greater that one. It can, therefore, be said that the branch is cost-efficient.

In what follows, we examine the various cases for FSDMU replacement. If the SDMU that is selected to replace the FSDMU is in the Best, Worst, Max, Min, and Center case, respectively, the $k$-level cost efficiency measure for the FSDMU and the corresponding results are demonstrated in Table 9, below.

Higher values for $k$ indicate that the FDMUs have been replaced by DMUs that are closer to the best DMU; which is more favorable for the cost efficiency level of the FSDMU. With regard to the results in the table, the best level for the Germi branch is 0.999.

If the DMU that is selected to replace the FDMU is in the Best, Worst, Max, Min, and Center case, the $k$-level cost efficiency measure is calculated for the FSDMU, the results of which are given in Table 10.

Lower values for $k$ indicate that the FSDMU has been replaced by the SDMU that is closer to the worst SDMU; which id more favorable for the cost efficiency level of the FSDMU, as it indicates that the FSDMU can have some amount of cost efficiency even in the worst conditions. Regarding the results in the table, 0.002 is the best cost efficiency level for the Germi branch.

## 6.2. Comparison

For purposes of comparison, we consider our proposed method and that of Hatami-Marbini *et al.* [18]. To this end, the fuzzy triangular input and output data for five decision making units are given in Table 11.

Hatami-Marbini *et al.* [18] evaluated the efficiency of five DMUs and calculated the best and worst efficiency values for each DMU. The results from Hatami-Marbini *et al.*'s method are provided in the first and second

Table 10. The cost efficiency of level $k$ for the FSDMU.

| $\alpha$ | Best | Worst | Max | Min | Center |
|---|---|---|---|---|---|
| 0 | 0.811-level | Cost-Inefficient | 0.343-level | 0.282-level | 0.237-level |
| 0.2 | 0.647-level | Cost-Inefficient | 0.232-level | 0.086-level | 0.124-level |
| 0.4 | 0.374-level | Cost-Inefficient | 0.046-level | 0.999-level | 0.002-level |
| 0.6 | 0.002-level | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |
| 0.8 | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |
| 1 | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient | Cost-Inefficient |

Table 11. The input and output data of Hatami-Marbini *et al.* [18].

| DMU | Input 1 | Input 2 | Output 1 | Output 2 |
|---|---|---|---|---|
| A | $(4, 3.5, 4.5)$ | $(2.1, 1.9, 2.3)$ | $(2.6, 2.4, 2.8)$ | $(4.1, 3.8, 4.4)$ |
| B | $(2.9, 2.9, 2.9)$ | $(1.5, 1.4, 1.6)$ | $(2.2, 2.2, 2.2)$ | $(3.5, 3.3, 3.7)$ |
| C | $(4.9, 4.4, 5.4)$ | $(2.6, 2.2, 3)$ | $(3.2, 2.7, 3.7)$ | $(5.1, 4.3, 5.9)$ |
| D | $(4.1, 3.4, 4.8)$ | $(2.3, 2.2, 2.4)$ | $(2.9, 2.5, 3.3)$ | $(5.7, 5.5, 5.9)$ |
| E | $(6.5, 5.9, 7.1)$ | $(4.1, 3.6, 4.6)$ | $(5.1, 4.4, 5.8)$ | $(7.4, 6.5, 8.3)$ |

Table 12. The results of the best and worst efficiency.

| DMU | The Best efficiency scores of the DMUs by method of Hatami-Marbini *et al.* [18] | The Worst efficiency scores of the DMUs by method of Hatami-Marbini *et al.* [18] | The Best CE scores of the DMUs by proposed method | The Worst CE scores of the DMUs by proposed method |
|---|---|---|---|---|
| A | 1 | 0.46 | 1 | 0.30 |
| B | 1 | 0.63 | 1 | 0.37 |
| C | 1 | 0.44 | 1 | 0.31 |
| D | 1 | 0.48 | 1 | 0.36 |
| E | 1 | 0.47 | 1 | 0.37 |

columns of Table 12, and the best and worst cost efficiency values for each DMU are given in the second and fourth columns, respectively. The input cost vector is assumed to be $p^T = (1, 1)$.

The advantage of the proposed method its ability to calculate the cost efficiency level with respect to the viewpoints in Table 3 and to compute the best and worst cost efficiency values. The results for the efficiency measures of FDMUs by the different models are demonstrated in Table 13. Wherever replacement is required, the central DMU is used to serve the purpose.

In the FCE column, the cost efficiency for all DMUs is calculated by the FCE model. It can be observed that units B, D, and E are cost-efficient, whereas by Hatami-Marbini *et al.*'s method, all the DMUs are recognized as being efficient for the best case. In the GFCE column, the unit under evaluation has been considered as an FSDMU and the other units as FDMUs, and they have been replaced by the SDMU and the central DMU, respectively, for cost efficiency evaluation.

In the SFCE column, first the unit under assessment is replaced by the central DMU and then its cost efficiency is calculated based on FDMUs.

In the GSFCE column, the unit under evaluation is considered as an FSDMU and replaced by the central SDMU. Its cost efficiency value is then computed based on the FDMUs.

Table 13. The efficiency value of FDMUs after using different evaluating model.

| DMU | FCE | GFCE | SFCE | GSFCE | FSCE | GFSCE |
|---|---|---|---|---|---|---|
| A | 0.85 | 0.85 | Cost-Inefficien | Cost-Inefficien | 0.89-level | 0.89-0.5-level |
| B | 1 | 1.16 | 0.5-level | 0.5-0.78-level | 0.5-level | 0-0.5-level |
| C | 0.86 | 0.86 | Cost-Inefficien | Cost-Inefficien | 0.78-level | 0.78-0.5-level |
| D | 1 | 1.15 | 0.5-level | 0.5-1-level | 0.5-level | 0.14-0.5-level |
| E | 1 | 1.03 | 0.5-level | 0.5-0.65-level | 0.5-level | 0.44-0.5-level |

In the FSCE column, the unit under assessment remains fuzzy and the other units are replaced by the central DMU; then, the cost efficiency measure is calculated for the unit under evaluation. In the last column, the unit under evaluation id considered as an FSDMU and the other units are replaced by the central DMU; the cost efficiency measure for the unit under assessment is subsequently computed. By the proposed method, the cost efficiency of the units can be considered from different points of view.

## 7. Conclusion

The CE measurement of an FSDMU has emerged recently as a brand-new approach in DEA. All CE models in the existing literature have failed to carry out the CE measurement for an FSDMU.

Bearing that in mind, we presented herein a generalized fuzzy cost efficiency (GFCE) model that is indeed an extension of fuzzy cost efficiency (FCE) models. This model is capable of performing CE measurement for not only a DMU inside the PPS but also any given sample DMU in crisp and fuzzy environments. The presented model also has the advantage over conventional FCE models in that it allows for concurrent implementation and generalization of different fuzzy numbers. Given the fact that the present paper is the first to take advantage of vectors to calculate cost efficiency, the method adopted herein for CE measurement may prove readily applicable to other FCE models. Owing to the implementation of $\alpha$-cuts along with a diversity of fuzzy numbers as inputs and outputs, the proposed model seems to be better founded and more comprehensive than the preceding ones. The FSCE and SFCE models offered in the present paper rely on absolutely innovative methods for CE measurement which can also be employed by other FCE models. Not only do the proposed vector-based methods enhance CE models with fuzzy data, but they may also be advantageous to CE models with interval-probable data. The methodology developed in this paper for CE measurement can also be used to evaluate revenue and profit efficiency scores.

*Acknowledgements.* This research was conducted by Nazila Aghayi whilst on sabbatical at the Universit Catholique de Louvain, who would like to thank Islamic Azad University, Ardabil branch, most sincerely for offering financial support.

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