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FACETS BASED ON CYCLES AND CLIQUES FOR THE ACYCLIC COLORING POLYTOPE

Mónica Braga* and Javier Marenco

Abstract. A *coloring* of a graph is an assignment of colors to its vertices such that any two vertices receive distinct colors whenever they are adjacent. An *acyclic coloring* is a coloring such that no cycle receives exactly two colors, and the *acyclic chromatic number* $\chi_A(G)$ of a graph G is the minimum number of colors in any such coloring of G. Given a graph G and an integer k, determining whether $\chi_A(G) \leq k$ or not is NP-complete even for $k = 3$. The acyclic coloring problem arises in the context of efficient computations of sparse and symmetric Hessian matrices *via* substitution methods. In a previous work we presented facet-inducing families of valid inequalities based on induced even cycles for the polytope associated to an integer programming formulation of the acyclic coloring problem. In this work we continue with this study by introducing new families of facet-inducing inequalities based on combinations of even cycles and cliques.

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1. Introduction

A *coloring* of a graph G is an assignment of colors to the vertices of G such that any two vertices receive distinct colors whenever they are adjacent. An *acyclic coloring* of a graph G is a coloring such that no cycle of G receives exactly two colors, *i.e.*, such that the subgraph of G induced by any two color classes is acyclic. The *acyclic chromatic number* $\chi_A(G)$ of a graph G is the minimum number of colors in any such coloring of G. Given a graph G, the *acyclic coloring problem* consists in finding $\chi_A(G)$, and this problem has been shown to be NP-hard [6]. Kostochka [12] proved that even deciding whether the acyclic chromatic number of a graph is at most 3 is NP-complete.

The acyclic coloring problem arises in the context of matrix partitioning for the estimation of the Hessian matrix associated to numerical optimization problems and the solution of systems of non linear equations [7,9]. Many previous research efforts on this problem consisted in finding bounds on $\chi_A(G)$ for particular classes of graphs [2,3,8]. Efficient heuristic algorithms for the acyclic coloring problem were developed in [10,11]. However, not too many approaches in order to solve this problem in practice exist.

We presented in a previous work [5] an integer programming model for the acyclic coloring problem based on an existing formulation for the classical vertex coloring problem. We initiated the study of the polyhedron

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Sciences Institute, National University of General Sarmiento, J.M. Gutierrez 1150, B1613 GSX Los Polvorines, Argentina.
*Corresponding author: mbraga@campus.ungs.edu.ar

associated with this formulation by introducing six facet-inducing families of valid inequalities. In [4] we also studied their disjunctive rank, as defined by Balas *et al.* [1], and we explored a dual concept, which we called the *disjunctive anti-rank* of a valid inequality.

In this work we present advances in the polyhedral study of this problem, by introducing new families of facet-inducing inequalities based on combinations of cycles and cliques. We are particularly interested in valid inequalities involving cliques in the graph, since clique-based inequalities have been shown to be very effective for the classical vertex coloring problem within cutting-plane environments [13, 14], and a similar behaviour might be expected for the acyclic coloring problem.

This paper is organized as follows. In Section 2 we introduce the integer programming formulation for the acyclic coloring problem and give some definitions. In Section 3 we introduce facet-inducing families of clique-based valid inequalities, and Section 4 closes the paper with conclusions and final remarks.

2. Integer programming formulation

We now present the integer programming model for the acyclic coloring problem considered in [5]. Let $G = (V, E)$ be an undirected graph with no isolated vertex, and let $\mathcal{C}$ be the set of available colors. For $i \in V$ and $c \in \mathcal{C}$, we define the *assignment variable* x_{ic} to be $x_{ic} = 1$ if the vertex i is assigned the color c, and $x_{ic} = 0$ otherwise. For every $c \in \mathcal{C}$ we define the binary *color variable* w_c to be $w_c = 1$ if some vertex uses the color c.

Denote by $\mathbf{C}(G) \subseteq 2^V$ the set of all cycles of G. The acyclic coloring problem can be formulated in terms of the assignment variables and the color variables in the following way:

$$\min \sum_{c\in\mathcal{C}} w_c$$

$$\text{s.t.} \quad \sum_{c\in\mathcal{C}} x_{vc} = 1 \qquad \forall v \in V \tag{2.1}$$

$$x_{uc} + x_{vc} \le w_c \qquad \forall uv \in E, \qquad \forall c \in \mathcal{C} \tag{2.2}$$

$$\sum_{v\in\mathbf{C}} (x_{vc} + x_{vc'}) \le |\mathbf{C}| - 1 \qquad \forall\, \mathbf{C} \in \mathbf{C}(G), \qquad \forall c, c' \in \mathcal{C}, c \ne c' \tag{2.3}$$

$$x_{vc} \in \{0,1\} \qquad \forall v \in V, \qquad \forall c \in \mathcal{C} \tag{2.4}$$

$$w_c \in \{0,1\} \qquad \forall c \in \mathcal{C}. \tag{2.5}$$

The objective function asks to minimize the number of used colors. Constraints (2.1) impose that each vertex must receive exactly one color, while constraints (2.2) prevent two adjacent vertices from receiving the same color. Note that constraints (2.2) suffice to properly define the w-variables, as we assume that G has no isolated vertex. Finally, constraints (2.3) prevent a cycle from receiving exactly two colors. These constraints need not to be enforced over every cycle in $\mathbf{C}(G)$, as odd cycles always receive at least three colors in any feasible coloring. Hence, we can restrict this constraint to operate on the induced even cycles of G only.

We define $P_S(G, \mathcal{C}) \subseteq \mathbb{R}^{|V||\mathcal{C}|+|\mathcal{C}|}$ to be the convex hull of the vectors (x, w) satisfying constraints (2.1)–(2.5).

Theorem 2.1 ([4])**.** *If $|\mathcal{C}| > \chi_A(G)$, then $\dim(P_S(G, \mathcal{C})) = |V|(|\mathcal{C}| - 1) + |\mathcal{C}|$. Furthermore, constraints (2.1) provide a minimal system of equations for $P_S(G, \mathcal{C})$.*

3. New valid inequalities

Since clique-based inequalities have been shown to be very effective for the classical vertex coloring problem within cutting-plane environments, in this section we introduce families of valid inequalities based on combinations of cycles and cliques. The idea is to capture the acyclicity properties of the feasible solutions in the clique-based inequalities.

In this work we consider a clique to be a (not necessarily maximal) complete subgraph of G. Let V_G be the vertex set and E_G be the edge set of the graph G.

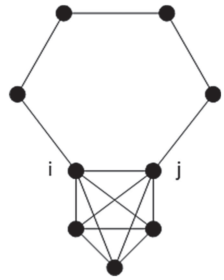

FIGURE 1. Structure for the basket inequalities.

3.1. Basket inequalities

Definition 3.1. Let $\mathbf{C}$ be an even induced cycle of G and $K \subseteq V$ be a clique such that $\mathbf{C} \cap K = \{i, j\}$ (see Fig. 1). Let $c_0, c_1 \in \mathcal{C}$, with $c_0 \neq c_1$ and $\mathcal{D} \subset \mathcal{C}$ such that $|\mathcal{C}| - |K| \leq |\mathcal{D}| \leq |\mathcal{C}| - 2$ and $c_0, c_1 \notin \mathcal{D}$. We define

$$\sum_{v \in \mathbf{C} \setminus \{i,j\}} (x_{vc_0} + x_{vc_1}) + \sum_{v \in \mathbf{C} \setminus \{i,j\}} \sum_{c \in \mathcal{D}} x_{vc} - \sum_{v \in K \setminus \{i,j\}} (x_{vc_0} + x_{vc_1}) \leq |\mathbf{C}| - 3 + \sum_{c \in \mathcal{D}} w_c \tag{3.1}$$

to be the *basket inequality* associated with the cycle $\mathbf{C}$, the clique K, the vertices i and j, the color set $\mathcal{D}$, and the colors c_0 and c_1.

Theorem 3.2. *The basket inequalities are valid for $P_S(G, \mathcal{C})$.*

Proof. Let $z = (x, w)$ be a feasible solution. If $\sum_{c \in \mathcal{D}} w_c \geq 1$, then the right hand side of (3.1) is greater than or equal to $|\mathbf{C}| - 2$. As the first two terms of the left hand side of (3.1) involve the vertices of $\mathbf{C} \setminus \{i, j\}$, then $\sum_{v \in \mathbf{C} \setminus \{i,j\}} (x_{vc_0} + x_{vc_1} + \sum_{c \in \mathcal{D}} x_{vc})$ is less than or equal to $|\mathbf{C}| - 2$, and the inequality is satisfied. Now assume that $\sum_{c \in \mathcal{D}} w_c = 0$. Then, the second term in the left hand side of (3.1) is null. Besides, $|\mathcal{D}| = |\mathcal{C}| - |K|$ (if $|\mathcal{D}| > |\mathcal{C}| - |K|$ then at least one vertex in the clique must use a color in $\mathcal{D}$, contradicting the assumption) and $\sum_{v \in K} (x_{vc_0} + x_{vc_1}) = 2$. Consider the following cases:

(1) If $\sum_{v \in K \setminus \{i,j\}} (x_{vc_0} + x_{vc_1})$ is greater than or equal to 1 then the left hand side of (3.1) is less than or equal to $|\mathbf{C}| - 3$, and the inequality is satisfied.
(2) If $\sum_{v \in K \setminus \{i,j\}} (x_{vc_0} + x_{vc_1})$ is equal to 0, then vertices i and j use colors c_0 and c_1. Since z is an acyclic coloring, then the first term of the left hand side of (3.1) is less than or equal to $|\mathbf{C}| - 3$, and the inequality is satisfied.

Since in all the cases the basket inequality (3.1) is satisfied and z is an arbitrary solution, we conclude that this inequality is valid for $P_S(G, \mathcal{C})$. □

Let $G_B = (\mathbf{C} \cup K, E_{\mathbf{C}} \cup E_K)$ be a graph where $\mathbf{C}$ is an even induced cycle and K is a clique such that $\mathbf{C} \cap K = \{i, j\}$, with i and j two adjacent vertices in the cycle. Note that $\chi_A(G_B) = \max\{3, |K|\}$.

The following result studies the facetness properties of the basket inequalities. It is interesting to note that this theorem provides necessary and sufficient conditions for (3.1) to induce a facet of $P_S(G_B, \mathcal{C})$, given suitable bounds on $|\mathcal{C}|$.

In the proof of the theorem we use the following solutions $z = (x, w)$ in the face of $P_S(G_B, \mathcal{C})$ defined by (3.1):

- *Type I solutions*: solutions with $w_d = 0$ for all $d \in \mathcal{D}$ (so no vertex uses colors in $\mathcal{D}$), and such that all the vertices in $\mathbf{C} \setminus \{i\}$ (or in $\mathbf{C} \setminus \{j\}$) and exactly one vertex in $K \setminus \{i, j\}$ use colors c_0 and c_1.

– *Type II solutions*: solutions with $w_d = 0$ for all $d \in \mathcal{D}$, and such that all the vertices in $\mathbf{C} \setminus \{v\}$, with $v \neq i, j$, use colors c_0 and c_1, (so no vertex in $K \setminus \{i, j\}$ uses colors c_0 and c_1), and vertex v uses a color in $\mathcal{C} \setminus (\mathcal{D} \cup \{c_0, c_1\})$.
– *Type III solutions*: solutions having exactly one w_d-coordinate, $d \in \mathcal{D}$, equal to 1. So, no vertex in $K \setminus \{i, j\}$ uses colors c_0 and c_1 and all the vertices in $\mathbf{C} \setminus \{i, j\}$ use at least two colors in the set $\{c_0, c_1, d\}$.

Theorem 3.3. *Assume that $|\mathcal{C}| > |K|$. The basket inequalities are facet-defining for $P_S(G_B, \mathcal{C})$ if and only if $|\mathcal{D}| = |\mathcal{C}| - |K|$ and $|K| \geq 3$.*

Proof. Let $\mathcal{F}$ be the face of $P_S(G_B, \mathcal{C})$ defined by (3.1).

By definition, $|\mathcal{D}| \geq |\mathcal{C}| - |K|$. If $|\mathcal{D}| \geq |\mathcal{C}| - |K| + 2$, then the face $\mathcal{F}$ is empty as $\sum_{c \in \mathcal{D}} w_c > 1$ in any feasible solution.

On the other hand, if $|\mathcal{D}| = |\mathcal{C}| - |K| + 1$, then every solution in $\mathcal{F}$ also satisfies the additional equality $\sum_{c \in \mathcal{D}} w_c = 1$, which is not a linear combination of (2.1) and (3.1) at equality. Then $\mathcal{F}$ is not a facet of $P_S(G_B, \mathcal{C})$ in this case.

Note that, if $K = \{i, j\}$ then $|\mathcal{D}| = |\mathcal{C}| - 2$ and the inequality (3.1) states that $\sum_{c \in \mathcal{D}} w_c \geq 1$. So, every solution in the face $\mathcal{F}$ also satisfies the equality $w_{c_0} + w_{c_1} = 2$, since it represents an acyclic coloring. Then $\mathcal{F}$ is not a facet of $P_S(G_B, \mathcal{C})$.

Now we prove that (3.1) defines a facet of $P_S(G_B, \mathcal{C})$ if $|\mathcal{D}| = |\mathcal{C}| - |K|$ and $|K| \geq 3$. Since $|\mathcal{C}| > |K|$, we have that $|\mathcal{D}| \geq 1$ in this case. Besides, we can always get an acyclic coloring of G_B without using the colors in $\mathcal{D}$. Let $\lambda x + \mu w = \lambda_0$ be an equality that is satisfied by every solution $z = (x, w)$ in the face $\mathcal{F}$. We shall prove that (λ, μ) is a linear combination of the coefficient vector of (3.1) and the coefficient vectors of the model constraints (2.1). In other words, we shall find scalars α and β_i, $i \in V$, such that

$$(\lambda, \mu) = \alpha \pi + \sum_{i \in V} \beta_i \gamma^i, \tag{3.2}$$

where π is the coefficient vector of (3.1) and γ^i is the coefficient vector of the model constraint (2.1) corresponding to the vertex i, for $i \in \mathbf{C} \cup K$ (recall that (2.1) defines a minimal system of equations for $P_S(G, \mathcal{C})$).

Claim 1: $\mu_c = 0$, for every $c \notin \mathcal{D}$

Let $c \notin \mathcal{D}$. Let $z = (x, w)$ be a solution in $\mathcal{F}$ such that $w_c = 0$. This solution exists since $|\mathcal{C}| > |K|$. So we have enough colors for coloring K without using color c. As $|K| \geq 3$, we can extend the coloring of K to G_B. In particular, if $c = c_0$, then the vertices in $\mathbf{C} \setminus \{i, j\}$ use colors c_1 and d, where $d \in \mathcal{D}$ is the only color used in $\mathcal{D}$, and vertex i or j uses a color in $\mathcal{C} \setminus \{c_0, c_1, d\}$. This solution is of type III (as defined before). Let $z' = (x, w')$ be the solution obtained from z by replacing $w_c = 0$ by $w'_c = 1$. The solution z' is also of type III. Since $z, z' \in \mathcal{F}$, then $\lambda x + \mu w = \lambda_0 = \lambda x + \mu w'$. As these solutions only differ in the w_c-coordinate, then $\mu_c = 0$.

Claim 2: $\lambda_{vc} = \lambda_{vc'}$, for every $v \in \mathbf{C} \setminus \{i, j\}$, and $c, c' \notin \mathcal{D} \cup \{c_0, c_1\}$

This claim involves the cases in which $|K| \geq 4$, since $c, c' \in \mathcal{C} \setminus (\mathcal{D} \cup \{c_0, c_1\})$ and $|\mathcal{C}| = |\mathcal{D}| + |K|$. Let $v \in \mathbf{C} \setminus \{i, j\}$ and $c, c' \notin \mathcal{D} \cup \{c_0, c_1\}$ be two different colors. Let $z = (x, w)$ be a solution in $\mathcal{F}$ such that $x_{vc} = 1$ and $w_{c'} = 1$. This solution is of type II. Since the RHS of (3.1) is equal to $|\mathbf{C}| - 3$, then v is the only vertex in the cycle $\mathbf{C}$ that uses a color in $\mathcal{C} \setminus (\mathcal{D} \cup \{c_0, c_1\})$. Let $z' = (x', w)$ be the solution obtained from z by only replacing $x_{vc} = 1$ and $x_{vc'} = 0$ by $x'_{vc} = 0$ and $x'_{vc'} = 1$. The solution z' is of type II too. Since $z, z' \in \mathcal{F}$, then $\lambda x + \mu w = \lambda_0 = \lambda x' + \mu w$. As these solutions only differ in the x_{vc}- and $x_{vc'}$-coordinates, then $\lambda_{vc} = \lambda_{vc'}$.

Claim 3: $\lambda_{vc} = \lambda_{vd}$, for every $v \in \mathbf{C} \setminus \{i, j\}$, $c \in \{c_0, c_1\}$, and $d \in \mathcal{D}$

Let $v \in \mathbf{C} \setminus \{i, j\}$, and $c \in \{c_0, c_1\}$ and $d \in \mathcal{D}$ be two colors. Let $z = (x, w)$ be a solution in $\mathcal{F}$ such that $x_{vc} = 1$, $x_{kd} = 1$ for $k \in K \setminus \{i, j\}$ and no vertex in $\mathbf{C} \setminus \{i, j\}$ uses a color in $\mathcal{D}$. Let $z' = (x', w)$ be the solution obtained from z by only replacing $x_{vc} = 1$ and $x_{vd} = 0$ by $x'_{vc} = 0$ and $x'_{vd} = 1$. Both solutions are of type III. As $z, z' \in \mathcal{F}$, then we obtain that $\lambda_{vc} = \lambda_{vd}$.

Claim 4: $\lambda_{vc_0} = \lambda_{vc_1}$, for every $v \in \mathbf{C} \setminus \{i,j\}$

Let $v \in \mathbf{C} \setminus \{i,j\}$ and $d \in \mathcal{D}$. By Claim 3, $\lambda_{vc_0} = \lambda_{vd}$ and $\lambda_{vc_1} = \lambda_{vd}$. Hence, $\lambda_{vc_0} = \lambda_{vc_1}$.

Claim 5: $\lambda_{vd} = \lambda_{vd'}$, for every $v \in \mathbf{C} \setminus \{i,j\}$, and $d, d' \in \mathcal{D}$

Let $v \in \mathbf{C} \setminus \{i,j\}$ and $d, d' \in \mathcal{D}$ two different colors. By Claim 3, $\lambda_{vc_0} = \lambda_{vd}$ and $\lambda_{vc_0} = \lambda_{vd'}$. Hence, $\lambda_{vd} = \lambda_{vd'}$.

Claim 6: $\mu_d = \lambda_{kc} - \lambda_{kd}$, for every $k \in K \setminus \{i,j\}, c \in \{c_0, c_1\}$, and $d \in \mathcal{D}$

Let $k \in K \setminus \{i,j\}$, $c \in \{c_0, c_1\}$ and $d \in \mathcal{D}$. Let $z = (x,w)$ be a solution in $\mathcal{F}$ such that $x_{kc} = 1$ and $w_d = 0$ for all d in $\mathcal{D}$. This solution is of type I. Note that in this case the colors used in K were enough to additionally color $\mathbf{C}$. Let $z' = (x', w')$ be the solution obtained from z by only replacing $x_{kc} = 1$, $x_{kd} = 0$ and $w_d = 0$ by $x'_{kc} = 0$, $x'_{kd} = 1$ and $w'_d = 1$. This solution is of type III. As $z, z' \in \mathcal{F}$, then we obtain that $\mu_d + \lambda_{kd} = \lambda_{kc}$.

Claim 7: $\lambda_{kc_0} = \lambda_{kc_1}$, for every $k \in K \setminus \{i,j\}$

Let $k \in K$ and $d \in \mathcal{D}$ ($\mathcal{D} \neq \emptyset$). By Claim 6, $\mu_d = \lambda_{kc_0} - \lambda_{kd}$ and $\mu_d = \lambda_{kc_1} - \lambda_{kd}$. Hence, $\lambda_{kc_0} = \lambda_{kc_1}$.

Claim 8: $\mu_d = \mu_{d'}$, for every $d, d' \in \mathcal{D}$

Let $v \in \mathbf{C} \setminus \{i,j\}$ and $d, d' \in \mathcal{D}$ be two different colors. Let $z = (x,w)$ be a solution in $\mathcal{F}$ such that $x_{vd} = 1$ and no other vertex in $\mathbf{C} \cup K$ uses that color. Let $z' = (x', w')$ be the solution obtained from z by only replacing $x_{vd} = 1$, $x_{vd'} = 0$, $w_d = 1$ and $w_{d'} = 0$ by $x'_{vd} = 0$, $x'_{vd'} = 1$, $w'_d = 0$ and $w'_{d'} = 1$. Both solutions are of type III. As $z, z' \in \mathcal{F}$, then we get that $\lambda_{vd} + \mu_d = \lambda_{vd'} + \mu_{d'}$. By Claim 5, we conclude $\mu_d = \mu_{d'}$.

Claim 9: $\lambda_{kd} = \lambda_{kd'}$, for every $k \in K \setminus \{i,j\}$, and $d, d' \in \mathcal{D}$

Let $k \in K \setminus \{i,j\}$ and $d, d' \in \mathcal{D}$ be two different colors. By Claim 6, $\mu_d = \lambda_{kc_0} - \lambda_{kd}$ and $\mu_{d'} = \lambda_{kc_0} - \lambda_{kd'}$. By Claim 8 we obtain $\lambda_{kd} = \lambda_{kd'}$.

Claim 10: $\lambda_{kc} = \lambda_{kc'}$, for every $k \in K \setminus \{i,j\}$, and $c, c' \notin \mathcal{D} \cup \{c_0, c_1\}$

Let $k \in K \setminus \{i,j\}$ and $c, c' \notin \mathcal{D} \cup \{c_0, c_1\}$ two different colors. Let $z = (x,w)$ be a solution in $\mathcal{F}$ such that $k \in K$ is the only vertex in $\mathbf{C} \cup K$ that uses color c, $x_{vc'} = 0$ for all $v \in \mathbf{C} \cup K$ and $w_{c'} = 1$. Let $z' = (x', w)$ be the solution obtained from z by only replacing $x_{kc} = 1$ and $x_{kc'} = 0$ by $x'_{kc} = 0$ and $x'_{kc'} = 1$. As $z, z' \in \mathcal{F}$, the claim follows.

Claim 11: $\lambda_{kc} = \lambda_{kd}$, for every $k \in K \setminus \{i,j\}$, $c \notin \mathcal{D} \cup \{c_0, c_1\}$, and $d \in \mathcal{D}$

Let $k \in K \setminus \{i,j\}$, $c \notin \mathcal{D} \cup \{c_0, c_1\}$ and $d \in \mathcal{D}$. Let $z = (x,w)$ be a solution in $\mathcal{F}$ such that the color d is used by at least one vertex in $\mathbf{C} \setminus \{i,j\}$ and $x_{kc} = 1$. Let $z' = (x', w)$ be the solution obtained from z by only replacing $x_{kc} = 1$ and $x_{kd} = 0$ by $x'_{kc} = 0$ and $x'_{kd} = 1$. Both solutions are of type III. As $z, z' \in \mathcal{F}$, the claim follows.

Claim 12: $\mu_d = \lambda_{vc} - \lambda_{vd}$, for every $v \in \mathbf{C} \setminus \{i,j\}$, $c \notin \mathcal{D} \cup \{c_0, c_1\}$, and $d \in \mathcal{D}$

Let $v \in \mathbf{C} \setminus \{i,j\}$, $c \notin \mathcal{D} \cup \{c_0, c_1\}$ and $d \in \mathcal{D}$. Let $z = (x,w)$ be a solution in $\mathcal{F}$ such that $x_{vc} = 1$ and $w_d = 0$ for all d in $\mathcal{D}$. This solution is of type II. Let $z' = (x', w')$ be the solution obtained from z by only replacing $x_{vc} = 1$, $x_{vd} = 0$ and $w_d = 0$ by $x'_{vc} = 0$, $x'_{vd} = 1$ and $w'_d = 1$. This solution is of type III. As $z, z' \in \mathcal{F}$, then we get $\lambda_{vc} = \lambda_{vd} + \mu_d$.

Claim 13: $\lambda_{uc} = \lambda_{uc'}$, for every $u \in \{i,j\}$, and $c, c' \in \mathcal{C} \setminus \mathcal{D}$

Let $u \in \{i,j\}$, $c \notin \mathcal{D} \cup \{c_0, c_1\}$ and $c' \in \{c_0, c_1\}$. Let $z = (x,w)$ be a solution in $\mathcal{F}$ such that $x_{uc'} = 1$, $w_c = 1$ and the color c is not used. Let $z' = (x', w)$ be the solution obtained from z by only replacing $x_{uc'} = 1$ and $x_{uc} = 0$ by $x'_{uc'} = 0$ and $x'_{uc} = 1$. Both solutions are of type III. As $z, z' \in \mathcal{F}$, then we get $\lambda_{uc} = \lambda_{uc'}$. Also, as a result of this, $\lambda_{uc_0} = \lambda_{uc_1}$ and, if $c'' \notin \mathcal{D} \cup \{c_0, c_1\}$, we get $\lambda_{uc} = \lambda_{uc''}$.

Claim 14: $\lambda_{ud} = \lambda_{ud'}$, for every $u \in \{i,j\}$, and $d, d' \in \mathcal{D}$

Let $u \in \{i,j\}$, $v \in \mathbf{C} \setminus \{i,j\}$ at even distance of u and $d \in D$. Let $z = (x,w)$ be a solution in $\mathcal{F}$ such that $x_{uc_0} = 1$, $x_{vd} = 1$ and v is the only vertex in $\mathbf{C} \setminus \{i,j\}$ that uses the color d and no other color in $\mathcal{D}$ is used. Let $z' = (x', w)$ be the solution obtained from z by only replacing $x_{uc_0} = 1$, $x_{ud} = 0$, $x_{vc_0} = 0$ and $x_{vd} = 1$ by $x'_{uc_0} = 0$, $x'_{ud} = 1$, $x'_{vc_0} = 1$ and $x'_{vd} = 0$. Both solutions are of type III. As $z, z' \in \mathcal{F}$, then we get $\lambda_{uc_0} = \lambda_{ud}$. Also, if $d' \in \mathcal{D}$, we conclude $\lambda_{ud} = \lambda_{ud'}$.

We define $\alpha = -\mu_d$ with $d \in \mathcal{D}$. Note that the choice of d does not affect the definition of α, by Claim 8. Furthermore, we define $\beta_v = \lambda_{vc}$ for all $v \in \mathbf{C} \cup K$ and any $c \in \mathcal{C} \setminus (\mathcal{D} \cup \{c_0, c_1\})$. There exist such a c because $|K| \geq 3$. The definition of β_v is independent of the choice of the color c, by Claims 2, 10 and 13. From the definitions of α and β_v we obtain:

- $\lambda_{vc} = \alpha + \beta_v$, for $v \in \mathbf{C} \setminus \{i,j\}$ and $c \in \mathcal{D} \cup \{c_0, c_1\}$, by Claims 12 and 3,
- $\lambda_{kc} = -\alpha + \beta_k$, for $k \in K \setminus \{i,j\}$ and $c \in \{c_0, c_1\}$, by Claims 6 and 11,
- $\lambda_{kd} = \beta_k$, for $k \in K \setminus \{i,j\}$ and $d \in \mathcal{D}$, by Claim 11 and
- $\lambda_{uc} = \beta_u$, for $u \in \{i,j\}$ and $c \in \mathcal{D} \cup \{c_0, c_1\}$, by Claims 13 and 14.

Under these definitions we conclude that (λ, μ) is obtained by multiplying the coefficient vector of (3.1) by $-\mu_d$, with $d \in \mathcal{D}$ and the coefficient vector of (2.1) by λ_{vc}, for $c \notin \mathcal{D} \cup \{c_0, c_1\}$ and for all $v \in \mathbf{C} \cup K$. ☐

Theorem 3.3 provides necessary and sufficient conditions for (3.1) to induce a facet of $P_S(G_B, \mathcal{C})$, namely if the graph does not have any additional vertex. Additional hypotheses must be considered in order to ensure that (3.1) induces a facet of $P_S(G, \mathcal{C})$, with G a general graph. To this end, let $G = (V, E)$ such that $\mathbf{C} \cup K \subseteq V$. We say that $\mathbf{C} \cup K$ is a *binding basket* if G does not admit any cycle $\mathbf{C}' \subseteq G$ such that $\mathbf{C}' \setminus (\mathbf{C} \cup K)$ and $\mathbf{C}' \cap (\mathbf{C} \cup K)$ are independent sets. In this way we can acyclically color $\mathbf{C} \cup K$ and $G \setminus (\mathbf{C} \cup K)$ separately, thus obtaining an acyclic coloring of G. With techniques similar to those in the proof of Theorem 3.3, it can be verified that if (a) $\mathbf{C} \cup K$ is a binding basket, (b) $|\mathcal{C}| > \chi_A(G \setminus (\mathbf{C} \cup K)) + |K|$, and (c) for each $v \in V \setminus (\mathbf{C} \cup K)$ there exists $k \in \mathbf{C} \cup K$ such that $vk \notin E$, then the basket inequalities are facet-defining for $P_S(G, \mathcal{C})$.

3.2. Basket cycle inequalities

The next family of facet-inducing inequalities is a generalization of (3.1), defined over a structure that involves more than one clique. First we introduce the definition of such a structure.

Definition 3.4. Let $\mathbf{C}$ be an even induced cycle of G, and let $K_i \subseteq V$ with $i \in I = \{1, \ldots, q\}$ be q cliques with at least three vertices such that $|E_{\mathbf{C}} \cap E_{K_i}| = 1$ for all $i \in I$ and $(K_i \cap K_j) \setminus \mathbf{C} = \emptyset$ for all $i, j \in I$, $i \neq j$ (see Fig. 2). Let $Q = \bigcup_{i=1}^{q} K_i$ and $t = |\mathbf{C} \cap Q|$. Let $c_0, c_1 \in \mathcal{C}$, with $c_0 \neq c_1$, and $\mathcal{D} \subset \mathcal{C}$ such that $|\mathcal{C}| - \min\{|K_i|\}_{i \in I} \leq |\mathcal{D}| \leq |\mathcal{C}| - 2$ and $c_0, c_1 \notin \mathcal{D}$. We define

$$\sum_{v \in \mathbf{C} \setminus Q} (x_{vc_0} + x_{vc_1}) + \sum_{v \in \mathbf{C} \setminus Q} \sum_{c \in \mathcal{D}} x_{vc} - \sum_{i=1}^{q} \sum_{v \in K_i \setminus \mathbf{C}} (x_{vc_0} + x_{vc_1}) \leq |\mathbf{C}| - 1 - t + \sum_{c \in \mathcal{D}} w_c \tag{3.3}$$

to be the *basket cycle inequality* associated with the cycle $\mathbf{C}$, the cliques K_i, with $i \in I$, the colors set $\mathcal{D}$, and the colors c_0 and c_1.

Theorem 3.5. *The basket cycle inequalities are valid for $P_S(G, \mathcal{C})$.*

Proof. Let $z = (x, w)$ be a feasible solution. If $|\mathcal{D}| > |\mathcal{C}| - \min\{|K_i|\}_{i \in I}$ or there exists $j \in I$ such that $|K_j| > \min\{|K_i|\}_{i \in I}$, then there always exists a vertex in a clique that uses a color in $\mathcal{D}$. Since $\sum_{c \in \mathcal{D}} w_c \geq 1$, the right hand side of (3.3) is greater than or equal to $|\mathbf{C}| - t$. As the left hand side is less than or equal to this value, the inequality (3.3) is trivially satisfied. Otherwise, *i.e.*, $|\mathcal{D}| = |\mathcal{C}| - \min\{|K_i|\}_{i \in I}$ and all the cliques K_i,

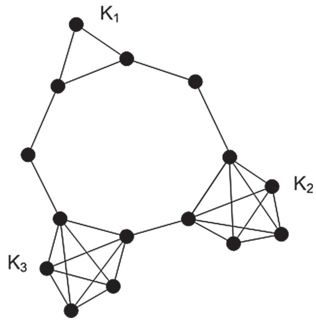

FIGURE 2. Example of the support of a basket cycle inequality.

with $i \in I$, are of the same size, we consider the following cases:

(1) If $\sum_{c\in\mathcal{D}} w_c \geq 1$, the inequality (3.3) is satisfied.
(2) If $\sum_{c\in\mathcal{D}} w_c = 0$ then there exists a vertex $v \in \mathbf{C}$ that uses a color $c \in \mathcal{C} \setminus (\mathcal{D} \cup \{c_0, c_1\})$, as z is an acyclic coloring. If $v \in \mathbf{C}\setminus Q$, then $\sum_{v\in\mathbf{C}\setminus Q}(x_{vc_0} + x_{vc_1}) \leq |\mathbf{C}|-1-t$ and the inequality (3.3) is satisfied. Otherwise, $v \in \mathbf{C} \cap K_i$, for some $i \in I$. Since $\sum_{c\in\mathcal{D}} w_c = 0$ and $|\mathcal{D}| = |\mathcal{C}| - |K_i|$, there exists a vertex in $K_i \setminus \mathbf{C}$ that uses the color c_0 or c_1. Therefore, $\sum_{i=1}^{q}\sum_{v\in K_i\setminus\mathbf{C}}(x_{vc_0} + x_{vc_1}) \geq 1$ and the inequality is satisfied.

Since in all the cases the basket cycle inequality (3.3) is satisfied and z is an arbitrary solution of $P_S(G, \mathcal{C})$, we conclude that this inequality is valid for $P_S(G, \mathcal{C})$. □

Let $G_Q = (\mathbf{C}\cup(\cup_{i\in I}K_i), E_{\mathbf{C}\cup(\cup_{i\in I}K_i)})$ be the graph exclusively composed by the structure described in Figure 2. The following theorem provides the conditions for the basket cycle inequality (3.3) to induce a facet of $P_S(G_Q, \mathcal{C})$. We omit the proof of Theorem 3.6 since it is similar to the previous proof.

Theorem 3.6. *If $|\mathcal{C}| > \chi_A(G_Q)$, $\{K_i\}_{i\in I}$ is a set of q pairwise disjoint cliques such that $|K_i| = |K_j|$ for all $i, j \in I$ and $|\mathcal{D}| = |\mathcal{C}| - |K_i|$, then the basket cycle inequality is facet-defining for $P_S(G_Q, \mathcal{C})$.*

3.3. 1-replicated two-color inequalities

Before presenting the new family of inequalities we give a preliminary definition. If $\mathbf{C}$ is an even cycle of G and $j \in \mathbf{C}$, we denote by $\mathbf{C}_j$ the set of all vertices in $\mathbf{C}$ located at even distance in $\mathbf{C}$ from j.

In [5] we presented the following family of facet-defining inequalities. Let $\mathbf{C}$ be an even cycle of G. Let $c_0, c_1 \in \mathcal{C}$ with $c_0 \neq c_1$. We define

$$\sum_{v\in\mathbf{C}}(x_{vc_0} + x_{vc_1}) \leq 1 + \left(\frac{|\mathbf{C}|-2}{2}\right)(w_{c_0} + w_{c_1}) \tag{3.4}$$

to be the *two-color inequality* associated with the cycle $\mathbf{C}$ and the colors c_0 and c_1. Under technical hypotheses, these inequalities are facet-defining for $P(G, \mathcal{C})$ for any graph G, as the following result shows.

Theorem 3.7 ([4]). *Assume that $|\mathcal{C}| \geq \chi_A(G \setminus \mathbf{C}) + 4$ and G does not admit any cycle $\mathbf{C}' \subseteq G$ such that $\mathbf{C}' \setminus \mathbf{C}$ and $\mathbf{C}' \setminus (G \setminus \mathbf{C})$ are independent sets. The two-color inequality (3.4) is facet-defining for $P_S(G, \mathcal{C})$ if and only if for each vertex $i \in V \setminus \mathbf{C}$ there exists $j \in \mathbf{C}$ such that $ik \notin E$ for all $k \in \mathbf{C}_j \setminus \{j\}$.*

We now generalize these inequalities by replacing a vertex of the cycle by a clique in the following way.

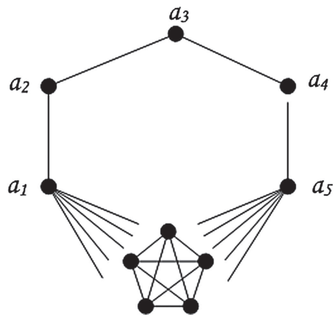

FIGURE 3. Structure for the 1-replicated two-color inequalities.

Definition 3.8. Let P and K be two disjoint sets of vertices of V, with K a clique and $P = \{a_1, \ldots, a_p\}$ a path with p odd and $|P| \geq 3$, such that $K \cup \{a_1\}$ and $K \cup \{a_p\}$ are cliques and $a_1 a_p \notin E$ (see Fig. 3). Let $c_0, c_1 \in \mathcal{C}$, with $c_0 \neq c_1$. We define

$$\sum_{v \in P \cup K} (x_{vc_0} + x_{vc_1}) \leq 1 + \frac{|P| - 1}{2}(w_{c_0} + w_{c_1}) \tag{3.5}$$

to be the *1-replicated two-color inequality* associated with the path P, the clique K, and the colors c_0 and c_1.

Theorem 3.9. *The 1-replicated two-color inequalities are valid for* $P_S(G, \mathcal{C})$.

Proof. Let $z = (x, w)$ be a feasible solution, and consider the following cases:

(1) If z uses colors c_0 and c_1, then we divide the analysis into the following sub-cases:
   (a) If no vertex in the clique K uses colors c_0 and c_1, then all the vertices in the path P can use these colors.
   (b) If exactly one vertex in the clique K uses color c_0 or c_1, then at most $|P| - 1$ vertices in the path P can use the colors c_0 and c_1, as z represents an acyclic coloring.
   (c) If two vertices in the clique K use colors c_0 and c_1, then the vertices a_1 and a_p cannot use those colors.

   In all the cases the left hand side of the inequality (3.5) is less than or equal to $|P|$. Note that the right hand side of the inequality is equal to $|P|$, as the colors c_0 and c_1 are used.
(2) If z uses the color c_0 and does not use the color c_1, then the left hand side of (3.5) is less than or equal to $\frac{|P| + 1}{2}$. The right hand side of the inequality is greater than or equal to this value as $w_{c_0} = 1$. A similar argument can be applied if z uses color c_1 and does not use color c_0.
(3) If colors c_0 and c_1 are not used by z, then the left hand side of (3.5) is equal to zero, and the inequality is trivially satisfied as the RHS is at least 1.

Since in the three cases the 1-replicated two-color (3.5) is satisfied and z is an arbitrary solution, we conclude that the inequality is valid for $P_S(G, \mathcal{C})$. □

We analize the facetness properties of the 1-replicated two-color inequalities. Let $G_{P \cup K} = (P \cup K, E_{P \cup K})$ be the graph exclusively composed by the structure described in Figure 3. For $j \in P \cup K$, let $P_j \subseteq P \cup K$ be the following set of vertices:

$$P_j = \{v \in P : v \text{ is at even distance of } j\} \cup \begin{cases} \{k_0\}, & \text{if the vertices of } K \text{ are at even distance of } j; \\ \emptyset, & \text{otherwise.} \end{cases}$$

where k_0 is an arbitrary vertex of K.

In the following we prove that (3.5) is facet-defining for $P_S(G_{P \cup K}, \mathcal{C})$.

Theorem 3.10. *If* $|\mathcal{C}| > \chi_A(G_{P\cup K})$*, then the 1-replicated two-color inequality* (3.5) *defines a facet of* $P_S(G_{P\cup K},\mathcal{C})$*.*

Proof. Let $\mathcal{F}$ be the face of $P_S(G_{P\cup K},\mathcal{C})$ defined by the inequality (3.5), *i.e.*,

$$\mathcal{F}=\left\{(x,w)\in P_S(G_{P\cup K},\mathcal{C}):\sum_{v\in P\cup K}(x_{vc_0}+x_{vc_1})=1+\frac{|P|-1}{2}(w_{c_0}+w_{c_1})\right\}.$$

Let $\lambda^T x+\mu^T w=\lambda_0$ be an equality that is satisfied by every solution $z=(x,w)$ in the face $\mathcal{F}$. We shall prove that (λ,μ) is a linear combination of the coefficient vector of (3.5) and the coefficient vectors of the model constraints (2.1). In other words, we shall find scalars α and $\beta_i, i\in P\cup K$, such that

$$(\lambda,\mu)=\alpha\pi+\sum_{i\in V}\beta_i\gamma^i, \tag{3.6}$$

where π is the coefficient vector of (3.5) and γ^i is the coefficient vector of the model constraint (2.1) corresponding to the vertex i, for $i\in P\cup K$.

Claim 1: $\boldsymbol{\mu_c=0 \qquad \forall c\in\mathcal{C}\setminus\{c_0,c_1\}}$

Let $c\in\mathcal{C}$ with $c\neq c_0,c_1$. Let $z=(x,w)$ be a feasible solution in $\mathcal{F}$ such that $w_c=0$. The graph $G_{P\cup K}$ can be acyclically colored in this way since $|\mathcal{C}|\geq\chi_A(G_{P\cup K})+1$. Let $z'=(x,w')$, with $w'=w+e_c$, where e_c is the unit vector associated to the variable w_c. The solution w' is obtained from w by replacing $w_c=0$ by $w'_c=1$. Note that both z and z' satisfy (3.5) at equality, hence $z,z'\in F$ and $\lambda^T x+\mu^T w=\lambda_0=\lambda^T x'+\mu^T w'$. Since z and z' only differ in the w_c-coordinate, then $\mu_c w_c=0=\mu_c w'_c$. Hence, $\mu_c=0$.

Claim 2: $\boldsymbol{\lambda_{vc}=\lambda_{vc'} \qquad \forall v\in P\cup K, \qquad \forall c,c'\in\mathcal{C}\setminus\{c_0,c_1\}}$

Let $v\in P\cup K$ and $c,c'\in\mathcal{C}\setminus\{c_0,c_1\}$. Let $z=(x,w)$ be a feasible solution in $\mathcal{F}$ such that $x_{vc}=1$ and no vertex uses color c'. Let $z'=(x',w)$ be the feasible solution in $\mathcal{F}$ obtained from z by replacing $x_{vc}=1$ and $x_{vc'}=0$ by $x'_{vc}=0$ and $x'_{vc'}=1$. Since z and z' only differ in the x_{vc}- and $x_{vc'}$-coordinates and $\lambda^T x+\mu^T w=\lambda_0=\lambda^T x'+\mu^T w$, then $\lambda_{x_{vc}}x_{vc}=\lambda_{x_{vc'}}x'_{vc'}$. Hence $\lambda_{x_{vc}}=\lambda_{x_{vc'}}$.

Claim 3: $\boldsymbol{\lambda_{vd}-\lambda_{vc}=\lambda_{v'd}-\lambda_{v'c} \qquad \forall v,v'\in P\cup K}$ **at even distance in** $\boldsymbol{P\cup K}$**,** $\boldsymbol{\forall c\in\mathcal{C}\setminus\{c_0,c_1\}}$**,** $\boldsymbol{d\in\{c_0,c_1\}}$

Let $v\in P\cup K$, and let $v'\in P_v\setminus\{v\}$. Let $z=(x,w)$ be a feasible solution in $\mathcal{F}$ such that all the vertices in $P_v\setminus\{v'\}$ get color c_0 and v' is the only vertex in $P\cup K$ that uses color c. Let $z'=(x',w)$ be the feasible solution in $\mathcal{F}$ obtained from z by replacing $x_{vc_0}=1$, $x_{v'c}=1$, $x_{vc}=0$ and $x_{v'c_0}=0$ by $x'_{vc_0}=0$, $x'_{v'c}=0$, $x'_{vc}=1$ and $x'_{v'c_0}=1$. Since both z and z' satisfy (3.5) at equality, then $\lambda^T x+\mu^T w=\lambda_0=\lambda^T x'+\mu^T w$. So, $\lambda_{v'c_0}+\lambda_{vc}=\lambda_{v'c}+\lambda_{vc_0}$, implying $\lambda_{v'c_0}-\lambda_{v'c}=\lambda_{vc_0}-\lambda_{vc}$. Since v is an arbitrary vertex, then this equality holds for every $v,v'\in P\cup K$ at even distance in $P\cup K$. A symmetrical argument can be given for $d=c_1$.

Claim 4: $\boldsymbol{\lambda_{vd}-\lambda_{vc}=\lambda_{v'd'}-\lambda_{v'c} \qquad \forall v\in P, v'\in P\cup K}$ **at odd distance in** $\boldsymbol{P\cup K}$**,** $\boldsymbol{d,d'\in\{c_0,c_1\}, d\neq d', \quad \forall c\in\mathcal{C}\setminus\{c_0,c_1\}}$

Let $v\in P$, and let v' be a vertex at odd distance in $P\cup K$. Let $z=(x,w)$ be a feasible solution in $\mathcal{F}$ in which all vertices in P_v get color c_0, all vertices in $P_{v'}\setminus\{v'\}$ get color c_1 and v' is the only vertex in $P\cup K$ that uses color c. Let $z'=(x',w)$ be the feasible solution in $\mathcal{F}$ obtained from z by replacing $x_{vc_0}=1$, $x_{v'c}=1$, $x_{vc}=0$ and $x_{v'c_1}=0$ by $x'_{vc_0}=0$, $x'_{v'c}=0$, $x'_{vc}=1$ and $x'_{v'c_1}=1$. Since both z and z' satisfy (3.5) at equality, then $\lambda^T x+\mu^T w=\lambda_0=\lambda^T x'+\mu^T w$. So, $\lambda_{vc_0}+\lambda_{v'c}=\lambda_{vc}+\lambda_{v'c_1}$, implying $\lambda_{vc_0}-\lambda_{vc}=\lambda_{v'c_1}-\lambda_{v'c}$. As this is valid for any vertex v' at odd distance in $P\cup K$ and v is an arbitrary vertex, the claim follows. A symmetrical argument can be given if v gets color c_1.

Claim 5: $\boldsymbol{\mu_d = \left(\frac{|P|-1}{2}\right)(\lambda_{vc} - \lambda_{vd}) \qquad \forall v \in P \cup K, \quad d \in \{c_0, c_1\}, \quad \forall c \in \mathcal{C} \setminus \{c_0, c_1\}}$

Let $v, v' \in P \cup K$, not both in K, with $vv' \in E_{P\cup K}$ and $c, c' \in \mathcal{C} \setminus \{c_0, c_1\}$. Let $z = (x, w)$ be the feasible solution in $\mathcal{F}$ such that vertex v gets color c', the vertices in $P_v \setminus \{v\}$ get color c_0 and the vertices in $P_{v'}$ get color c_1. No vertex in $P \cup K$ gets color c. Let $z' = (x', w')$ be the feasible solution in $\mathcal{F}$ such that the vertices that received color c_0 in z now get color c and $w'_{c_0} = 0$. Since z and z' satisfy (3.5) at equality, then $\lambda^T x + \mu^T w = \lambda_0 = \lambda^T x' + \mu^T w$. Then, $\mu_{c_0} + \sum_{u \in P_v \setminus \{v\}} \lambda_{uc_0} = \sum_{u \in P_v \setminus \{v\}} \lambda_{uc}$, implying $\mu_{c_0} = \sum_{u \in P_v \setminus \{v\}} (\lambda_{uc} - \lambda_{uc_0})$. By Claim 3, we pull $(\lambda_{vc} - \lambda_{vc_0})$ out as common factor and as v is an arbitrary vertex, we obtain $\mu_{c_0} = \left(\frac{|P|-1}{2}\right)(\lambda_{vc} - \lambda_{vc_0})$. A symmetrical argument can be given for $d = c_1$.

Claim 6: $\boldsymbol{\mu_{c_0} = \mu_{c_1}}$

By Claim 5, $\mu_{c_0} = \left(\frac{|P|-1}{2}\right)(\lambda_{vc} - \lambda_{vc_0})$ for any vertex $v \in P \cup K$ and color $c \in \mathcal{C} \setminus \{c_0, c_1\}$. Let $v' \in P \cup K$ located at odd distance of v in $P \cup K$, not both in K. Claim 4 implies $\lambda_{v'c_1} - \lambda_{v'c} = \lambda_{vc_0} - \lambda_{vc}$, so, $\mu_{c_0} = \left(\frac{|P|-1}{2}\right)(\lambda_{v'c} - \lambda_{v'c_1})$. By Claim 5, this last term equals μ_{c_1}, hence, $\mu_{c_0} = \mu_{c_1}$.

Claim 7: $\boldsymbol{\lambda_{vc_0} = \lambda_{vc_1} \qquad \forall v \in P \cup K}$

By Claims 6 and 5, $\left(\frac{|P|-1}{2}\right)(\lambda_{vc'} - \lambda_{vc_0}) = \left(\frac{|P|-1}{2}\right)(\lambda_{vc'} - \lambda_{vc_1})$ for any $c' \in \{c_0, c_1\}$. Hence, $\lambda_{vc_0} = \lambda_{vc_1}$.

We define $\alpha = \lambda_{vc_0} - \lambda_{vc}$, for any $v \in P \cup K$, $c \in \mathcal{C} \setminus \{c_0, c_1\}$. Note that the choice of v and c does not alter the definition of α, by Claim 4 and Claim 7. Furthermore, we define $\beta_v = \lambda_{vc}$, for any $v \in P \cup K$, $c \in \mathcal{C} \setminus \{c_0, c_1\}$. By Claim 2 the choice of c does not alter the definition of β_v. From the definition of α and β we obtain $\lambda_{vc_0} = \alpha + \beta_v$. By Claim 7, $\lambda_{vc_1} = \alpha + \beta_v$.

By Claims 6 and 5, $\mu_{c_1} = \mu_{c_0} = \left(\frac{|P|-1}{2}\right)(\lambda_{vc} - \lambda_{vc_0})$, for any $c' \in \{c_0, c_1\}$. The definition of α implies $\mu_{c_0} = \left(\frac{|P|-1}{2}\right)(-\alpha) = \left(\frac{1-|P|}{2}\right)\alpha$. By Claim 5, $\mu_{c_1} = \left(\frac{|P|-1}{2}\right)(-\alpha) = \left(\frac{1-|P|}{2}\right)\alpha$.

Under these definitions we conclude that the equality (3.6) is satisfied. Therefore, λ is indeed a linear combination of the coefficient vector of the inequality (3.5) and the coefficient vectors of the model constraints (2.1), hence (3.5) induces a facet of $P_S(G_{P\cup K}, \mathcal{C})$. □

Theorem 3.10 provides necessary and sufficient conditions for (3.5) to induce a facet of $P_S(G_{P\cup K}, \mathcal{C})$. Hypotheses ensuring that (3.5) induces a facet of $P_S(G, \mathcal{C})$ for a general graph G can be given as follows. Let $G = (V, E)$ such that $P \cup K \subseteq V$. We say that $P \cup K$ is *binding* if G does not admit any cycle $\mathbf{C}' \subseteq G$ such that $\mathbf{C}' \setminus (P \cup K)$ and $\mathbf{C}' \cap (P \cup K)$ are independent sets. If (a) $P \cup K$ is binding, (b) $|\mathcal{C}| > \chi_A(G \setminus (P \cup K)) + \chi_A(P \cup K)$, and (c) for each $v \in V \setminus (P \cup K)$ there exists $j \in P \cup K$ such that $vk \notin E$ for all $k \in P_j \setminus \{j\}$, then a similar proof shows that the 1-replicated two-color inequalities are facet-defining for $P_S(G, \mathcal{C})$.

3.4. Replicated two-color inequalities

Continuing with the idea of introducing families of valid inequalities based on combinations of cycles and cliques, we generalize the 1-replicated two-color inequalities (3.5) over structures that involve more than one clique. First we introduce some preliminary definitions.

Definition 3.11. A *K-cycle* is a set $\mathbf{C} = \bigcup_{i=1}^{n} K_i$ where $K_1, \ldots, K_n$ are n cliques and the subgraph induced by $K_i \cup K_{i+1}$ is a clique, for $i = 1, \ldots, n$ (the indices are taken modulo n). If n is even, then we say that $\mathbf{C}$ is an *even* K-cycle.

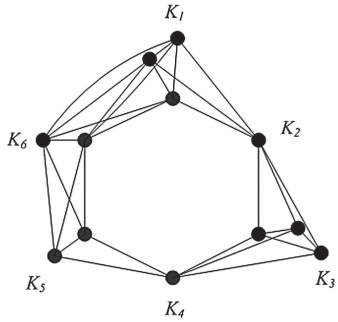

FIGURE 4. K-cycle.

Figure 4 shows a K-cycle, which can be informally considered as a "cycle of cliques". The generalization of the inequality (3.5) arises from this definition in the following way.

Definition 3.12. Let $\mathbf{C} \subseteq G$ be an even K-cycle that involves n cliques $K_1, \ldots, K_n$ such that $|K_i| = |K_p| = 1$ for some $i, p \in \{1, \ldots, n\}$ with i an odd index and p an even index. Let $c_0, c_1 \in \mathcal{C}$, with $c_0 \neq c_1$. We define

$$\sum_{v \in \mathbf{C}} (x_{vc_0} + x_{vc_1}) \leq 1 + \left(\frac{n}{2} - 1\right)(w_{c_0} + w_{c_1}) \tag{3.7}$$

to be the *replicated two-color inequality* associated with the K-cycle $\mathbf{C}$ and the colors c_0 and c_1.

Theorem 3.13. *The replicated two-color inequalities are valid for $P_S(G, \mathcal{C})$.*

Proof. Let $z = (x, w)$ be a feasible solution. Consider the following cases:

(1) If $w_{c_0} = 1$ and $w_{c_1} = 1$ in z, then the right hand side of the inequality (3.7) is equal to $n - 1$. Note that at most $n - 2$ vertices of $\mathbf{C} \setminus (K_i \cup K_p)$ can use colors c_0 and c_1. If exactly $n - 2$ vertices in $\mathbf{C} \setminus (K_i \cup K_p)$ use colors c_0 and c_1, then at most only one vertex in $K_i \cup K_p$ can use these colors, because z represents an acyclic coloring. Therefore, the left hand side of (3.7) is less than or equal to $n - 1$. If less than $n - 2$ vertices in $\mathbf{C} \setminus (K_i \cup K_p)$ use colors c_0 and c_1, then the inequality (3.7) is trivially satisfied.
(2) If $w_{c_0} = 1$ and $w_{c_1} = 0$, then the left hand side of (3.7) is less than or equal to $\frac{n}{2}$, and the right hand side equals this value. A similar argument can be applied if z uses color c_1 and does not use color c_0.
(3) If $w_{c_0} = 0$ and $w_{c_1} = 0$ in z, then the left hand side of (3.7) is null and the inequality is trivially satisfied as the right hand side is equal to 1.

Since in the three cases the replicated two-color inequality (3.7) is satisfied and z is an arbitrary solution, we conclude that this inequality is valid for $P_S(G, \mathcal{C})$. □

We omit the proof of facetness for the replicated two-color inequalities since it involves similar arguments as the ones used in the previous proofs. Let $G_{\mathbf{C}}$ be the graph exclusively composed by a K-cycle $\mathbf{C}$.

Theorem 3.14. *If $|\mathcal{C}| > \chi_A(G_{\mathbf{C}})$, then the replicated two-color inequality (3.7) induces a facet of $P_S(G_{\mathbf{C}}, \mathcal{C})$.*

4. CONCLUSIONS

In this work we have continued the polyhedral study of the acyclic coloring problem. Since inequalities based on cliques have shown to be effective for the classic coloring problem within a cutting-plane algorithm, we introduce new families of facet-inducing inequalities involving even cycles and cliques for the acyclic coloring problem. The introduction of cliques into existing cycle-based inequalities did not follow the same pattern in all cases. In some cases (as, *e.g.*, the basket and the basket cycle inequalities) we replace one or several edges of the original even cycle by cliques, whereas in other cases (as, *e.g.*, the replicated two-color inequalities) single vertices are replaced by cliques and all the vertices in each clique are adjacent to the original neighbors of the originating vertex.

As future work, we plan to study the computational complexity of the separation problems associated with the families presented in this work, eventually considering restricted subfamilies if the general separation problem turns out to be intractable. This is an important step prior to the incorporation of these inequalities into a procedure based on cutting planes.

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