

PRESERVATION TECHNOLOGY INVESTMENT, TRADE CREDIT AND PARTIAL BACKORDERING MODEL FOR A NON-INSTANTANEOUS DETERIORATING INVENTORY

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Abstract. In a perfectly transparent and competitive market, suppliers must provide a competitive pricing and service for their customers. The aim of this study is to provide an insight into how preservation technology and credit financing could be used both to reduce the deterioration rate as well as to provide flexible financing for retailers. The methodology is to optimize the cycle length, selling price, the amount of preservation technology and credit financing using inventory theory. The result derived is an optimal total profit per unit time for the system. Finally, using MATLAB 2017a, it is shown graphically that the profit function is concave. The sensitivity analysis is illustrated using Lingo 17. The study not only provides insights to business managers in making wise managerial decisions, it also enables them to weigh the pro and con of implementing preservation technology and credit financing.

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1. INTRODUCTION

Inventory is defined as idle stocks in a supply chain. The management of inventory is critical factor in the day-to-day operation of an enterprise, therefore a proper system to manage inventory is critical. One of the key factors to consider in inventory modeling is deterioration. Deterioration is the evaporation, decay or damage of inventory. Inventory has been the focus of many researches for over a century. Mishra [17] projected an inventory model for controllable deterioration with time-varying demand and time-varying holding cost. Mishra [16] presented an inventory model with controllable deterioration rate and time-dependent demand. Pandey *et al.* [22] established an inventory model with negative exponential demand and probabilistic deterioration under backlogging. Nabil *et al.* [20] developed a model for single machine lot scheduling problem for negative exponential deteriorating items. Nabil *et al.* [21] projected a model for economic lot size problem for cleaner manufacturing system with another interesting parameter warm-up period. A two-warehouse model with increasing demand under time-varying deterioration is developed by Sett *et al.* [26]. Sarkar and Sarkar [25] developed an inventory model with partial backlogging and stock dependent demand. Palanivel and Uthayakumar [24] presented an inventory

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TABLE 1. A comparison of the present study with the previous researches.

Authors	Demand pattern	Deterioration	Preservation technology	Shortages	Trade credit
Dye and Hsieh [7]	Constant	Time-dependent	Yes	Partial Back-ordering	No
Hsieh and Dye [10]	Time-dependent	Constant	Yes	No	No
He and Huang [9]	Price-dependent	Constant	Yes	No	No
Dye [6]	Constant	Non-instantaneous	Yes	Partial back-ordering	No
Zhang <i>et al.</i> [36]	Price dependent	Constant	Yes	No	No
Liu <i>et al.</i> [13]	Price dependent	Constant	Yes	No	No
Lu <i>et al.</i> [14]	Price and stock dependent	Constant	No	No	No
Jaggi <i>et al.</i> [11]	Price-dependent	Non-instantaneous	No	Complete backordering	Yes
Li <i>et al.</i> [12]	Price-dependent	Non-instantaneous	Yes	No	No
Tiwari <i>et al.</i> [33]	Price-dependent	Non-instantaneous	No	Partial back-ordering	Yes
Mishra <i>et al.</i> [18]	Price-dependent	Constant	Yes	No	Yes
This paper	Price dependent	Non-instantaneous	Yes	Partial back-ordering	Yes

model for imperfect items with stock dependent demand under permissible delay in payments. Shaikh *et al.* [28] presented a model considering stock and price sensitive demand under fully backlogged shortages. Mashud *et al.* [15] developed an inventory model for non-instantaneous inventory model with same demand. Nobil *et al.* [19] developed an economic production quantity model with and without shortages for imperfect products. Dey *et al.* [5] projected an inventory model considering price dependent demand with setup cost reduction.

In terms of preservation of products, Dye and Hsieh [7] developed an inventory model considering the investment in preservation technology. Dye [6] also developed a non-instantaneous deteriorating inventory model with preservation technology. Singh *et al.* [29] developed an EOQ model for stock-dependent demand deteriorating items with preservation technology; while Yang *et al.* [35] introduced optimal dynamic trade credit and preservation in their model.

Goyal [8] was one of the first researchers who developed an EOQ model under permissible delays in payments. Aggarwal and Jaggi [1] introduced an ordering policy for deteriorating items under permissible delays in payments. Tsao and Sheen [34] considered deteriorating items inventory model with price and time-dependent demand under permissible delays in payments. Pal [23] developed a production inventory model for imperfect products with permissible delay in payments under shortages. Yang *et al.* [35] considered the preservation technology investment to control the deterioration for a time-dependent demand rate under trade credit policy. Teng *et al.* [30] developed an EOQ model with trade credit financing for non-decreasing demand. Chen and Teng [3] modified this model by introducing upstream and downstream trade credit policy in his time-varying deteriorating model with cash flow. Lastly, Tiwari *et al.* [31, 32] developed a non-instantaneous deteriorating inventory model considering inflation and trade credit policy in a two-warehouse environment. A comparison of the present study with the previous researches is shown in Table 1.

The objective of this paper is to optimize the cycle length, selling price considering preservation technology. Three numerical examples were used to illustrate the theory. Graphical representations are used to illustrate the results.

TABLE 2. Notations.

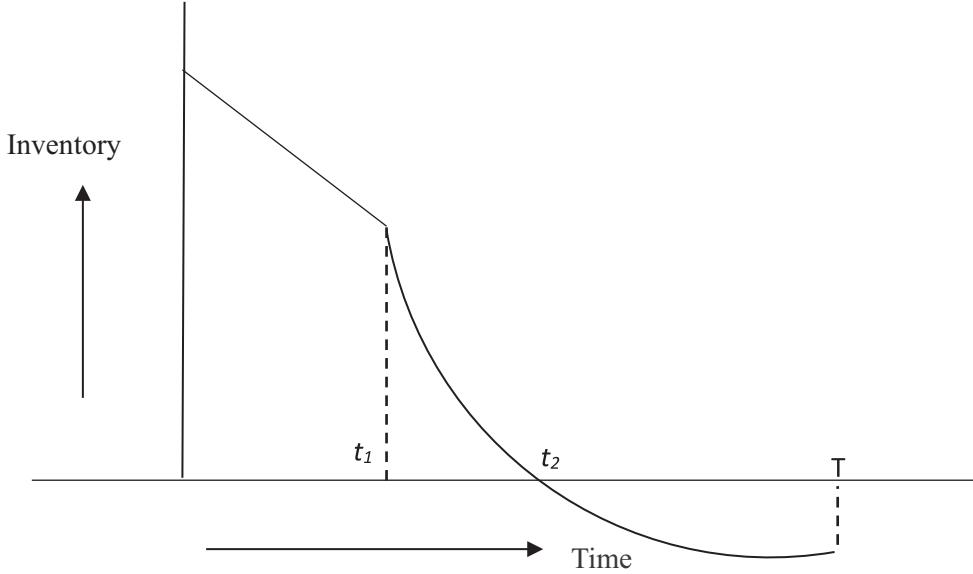
Notations	Units	Description
c	\$/order	Purchasing cost per order
c_1	\$/unit	Holding cost per unit
c_2	\$/unit	Shortage cost per unit
c_3	\$/unit	Lost sale cost per unit
A	\$/unit	Replenishment cost per order
δ	\$/unit	Backlogging parameter
θ	Constant	Deterioration rate
M	Month	Period of permissible delay in payments offered by the supplier
a		All factors affecting price other than costs (e.g. income, fashion)
b		The slope of the demand curve
I_e	\$/unit	The rate of interest earned by the retailer
I_c	\$/unit	The rate of interest payable by the supplier
R	units	Maximum shortage quantity per cycle
S	units	Initial inventory level
Q	units	Order size per cycle
X	units	Total inventory cost
t_1	month	The time point at which the deterioration starts
$\pi(p, \xi, T)$	\$/month	The total profit per unit time
<i>Dependent Variable</i>		
t_2	month	The time point at which the shortages are allowed
<i>Decision variables</i>		
p	month	Unit selling price
T	month	The total length of the inventory cycle
ξ	\$/unit	Preservation technology cost

2. NOTATIONS AND ASSUMPTIONS

The notations as seen in Table 2 followed by the assumptions in Section 2.1 were utilized to develop the inventory models.

2.1. Assumptions

- (1) The model considered a linear price-dependent demand pattern $D(p) = a - bp$ (i.e. demand function depends on price for a single deteriorating item).
- (2) The considered deterioration rate $\theta (0 < \theta \ll 1)$ is constant and depends on the stock amount.
- (3) No replacement or repairs on deteriorating products were considered for the whole period.
- (4) Lead-time is negligible and the replenishment rate is infinite.
- (5) The total planning horizon considered in the inventory system is infinite.
- (6) The relationship between the deterioration rate and the preservation technology investment parameter always satisfies the conditions $\frac{\partial m(\xi)}{\partial \xi} < 0, \frac{\partial^2 m(\xi)}{\partial \xi^2} > 0$. The research considered that $m(\xi) = e^{-a_1 \xi}$; where $m(\xi)$ is the deterioration rate with the investment of preservation technology, θ is the deterioration rate without the preservation technology investment, and a_1 is the sensitive parameter of investment to the deterioration rate which is similar as Mishra *et al.* [18] and He and Huang [9].
- (7) The backlogging rate during the stock out period is considered as a variable dependent on the length of waiting time for the next replenishment. Therefore, the backlogging rate for negative inventory is given by $\frac{1}{1+\delta(T-t)}$, where δ is a backlogging parameter, and $(T - t)$ is the waiting time.

FIGURE 1. Graphical presentation of inventory *vs.* time.

3. MATHEMATICAL FORMULATIONS

Figure 1 shows the graphical demonstration of inventory *vs.* time wherein the inventory level at any time interval between $0 \leq t \leq T$ is represented by $I(t)$. It is important to present that the level of inventory $I(t)$ was portrayed due to demand in the period $[0, t_1]$ and it dropped to zero at $t = t_2$ owing to the deterioration and demand in the period $[t_1, t_2]$. Afterwards, shortages were permitted to take place and the total demand in the period $[t_2, T]$ is partially backordered.

The following differential equations illustrate the inventory model:

Change in inventory level,

$$\frac{dI_1(t)}{dt} = -(a - bp) \quad 0 < t \leq t_1 \quad (3.1)$$

with the initial and boundary conditions as $I_1(t) = S$ at $t = 0$.

The solution for equation (3.1) is as follows:

$$I_1(t) = S - (a - bp)t, \quad 0 < t \leq t_1 \quad (3.2)$$

and

$$\frac{dI_2(t)}{dt} + \theta m(\xi) I_2(t) = -(a - bp) \quad t_1 < t \leq t_2 \quad (3.3)$$

with the boundary condition $I_2(t) = 0$ at $t = t_2$, the solution for equation (3.3) is as follows:

$$I_2(t) = \frac{(a - bp)}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2 - t)} - 1 \right) \quad t_1 < t \leq t_2 \quad (3.4)$$

where $I(t)$ is continuous at $t = t_1$, hence $I_1(t_1) = I_2(t_1)$ resulting in,

$$S = (a - bp)t_1 + \frac{(a - bp)}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2 - t_1)} - 1 \right). \quad (3.5)$$

At time, $t = t_2$, the inventory level is zero. For $t > t_2$, shortage occurs, the inventory level at any time t is directed by the differential equation

$$\frac{dI_3(t)}{dt} = \frac{-(a-bp)}{1+\delta(T-t)} \quad t_2 < t \leq T \quad (3.6)$$

with the boundary condition $I_3(t) = 0$ at $t = t_2$, one has:

$$I_3(t) = \frac{(a-bp)}{\delta} \{ \log(1+\delta(T-t)) - \log(1+\delta(T-t_2)) \} \quad t_2 < t \leq T. \quad (3.7)$$

The maximum amount of demand, which is the backlogged is given as follows:

$$R = -I_3(T) = \frac{(a-bp)}{\delta} \{ \log(1+\delta(T-t_2)) \}. \quad (3.8)$$

Hence, the total order quantity per cycle is:

$$\begin{aligned} Q &= S + R \\ &= (a-bp)t_1 + \frac{(a-bp)}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2-t_1)} - 1 \right) + \frac{(a-bp)}{\delta} \{ \log(1+\delta(T-t_2)) \}. \end{aligned} \quad (3.9)$$

The holding cost in the entire cycle is set as

$$\begin{aligned} \text{IHC} &= c_1 \int_0^{t_1} I_1(t) dt + c_1 \int_{t_1}^{t_2} I_2(t) dt \\ &= c_1 \left[St_1 - \frac{(a-bp)t_1^2}{2} - \frac{(a-bp)}{\theta^2 m(\xi)^2} \left(1 - e^{\theta m(\xi)(t_2-t_1)} + \theta m(\xi)(t_2-t_1) \right) \right]. \end{aligned} \quad (3.10)$$

The backorder cost in the entire cycle is set as

$$\begin{aligned} \text{BC} &= C_2 D \frac{1}{\delta} \left[\int_{t_2}^T \{ \log(1+\delta(T-t_2)) - \log(1+\delta(T-t)) \} dt \right] \\ &= c_2 \left[\left(R + \frac{D}{\delta} (T-t_2) \right) - \frac{D}{\delta^2} (1+\delta(T-t_2)) \log(1+\delta(T-t_2)) \right]. \end{aligned} \quad (3.11)$$

The opportunity cost due to lost sales in the entire cycle is specified in the form

$$\begin{aligned} \text{LSC} &= c_3 \int_{t_2}^T \left\{ 1 - \frac{1}{1+\delta(T-t)} \right\} (a-bp) dt \\ &= \frac{c_3 (a-bp)}{\delta} \{ \delta(T-t_2) - \log(1+\delta(T-t_2)) \}. \end{aligned} \quad (3.12)$$

The total purchase cost per cycle is given in the form

$$\text{PC} = cQ = c \left[(a-bp)t_1 + \frac{(a-bp)}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2-t_1)} - 1 \right) + \frac{(a-bp)}{\delta} \{ \log(1+\delta(T-t_2)) \} \right]. \quad (3.13)$$

The total sales revenue in the entire cycle is known in the form

$$\text{SR} = p \left[(a-bp)t_2 + \frac{(a-bp)}{\delta} \{ \log(1+\delta(T-t_2)) \} \right]. \quad (3.14)$$

The preservation technology cost in the entire cycle is

$$PTC = \xi T. \quad (3.15)$$

For the supplier, the period of permissible delay in payment offered (M) has different scenarios which are shown as:

Scenario 1 $0 < M \leq t_1$.

Scenario 2 $t_1 < M \leq t_2$.

Scenario 3 $t_2 < M \leq T$.

Equation (3.16) displays the interest earned bearing in mind scenario 1, when $0 < M \leq t_1$.

$$\begin{aligned} IE_1 &= pI_e \int_0^{t_2} \int_0^t (a - bp) du dt + \frac{pI_e (a - bp) t_2}{\delta} \log(1 + \delta(T - t_2)) \\ &= \frac{pI_e (a - bp) t_2^2}{2} + \frac{pI_e (a - bp) t_2}{\delta} \log(1 + \delta(T - t_2)). \end{aligned} \quad (3.16)$$

The interest paid to the supplier for this case is

$$\begin{aligned} IC_1 &= cI_c \int_M^{t_1} I_1(t) dt + cI_c \int_{t_1}^{t_2} I_2(t) dt \\ &= cI_c \left[S(t_1 - M) - (a - bp) \frac{(t_1^2 - M^2)}{2} \right] - \frac{cI_c (a - bp)}{(\theta m(\xi))^2} \left[\left(1 - e^{\theta m(\xi)(t_2 - t_1)} \right) + \theta m(\xi) (t_2 - t_1) \right]. \end{aligned} \quad (3.17)$$

Therefore, the total profit is

$$\pi^1(p, \xi, T) = \frac{X}{T},$$

where

$$\begin{aligned} X &= (SR - OC - PC - IHC - BC - LSC - PTC + IE_1 - IC_1) \\ &= p \left[(a - bp) t_2 + \frac{(a - bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] - A \\ &\quad - c \left[(a - bp) t_1 + \frac{(a - bp)}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2 - t_1)} - 1 \right) + \frac{(a - bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] \\ &\quad - c_1 \left[St_1 - \frac{(a - bp) t_1^2}{2} - \frac{(a - bp)}{\theta^2 m(\xi)^2} \left(1 - e^{\theta m(\xi)(t_2 - t_1)} + \theta m(\xi) (t_2 - t_1) \right) \right] \\ &\quad - c_2 \left[\left(R + \frac{D}{\delta} (T - t_2) \right) - \frac{D}{\delta^2} (1 + \delta(T - t_2)) \log(1 + \delta(T - t_2)) \right] \\ &\quad - \frac{c_3 (a - bp)}{\delta} \left\{ \delta(T - t_2) - \log(1 + \delta(T - t_2)) \right\} + \frac{pI_e (a - bp) t_2^2}{2} + \frac{pI_e (a - bp) t_2}{\delta} \log(1 + \delta(T - t_2)) \\ &\quad - \xi T - cI_c \left[S(t_1 - M) - (a - bp) \frac{(t_1^2 - M^2)}{2} \right] + \frac{cI_c (a - bp)}{(\theta m(\xi))^2} \left[\left(1 - e^{\theta m(\xi)(t_2 - t_1)} \right) + \theta m(\xi) (t_2 - t_1) \right]. \end{aligned} \quad (3.18)$$

The corresponding optimization problem is as follows:

Problem 3.1. $\pi^1(p, \xi, T) = \frac{X}{T}$, subject to constraints $0 < M \leq t_1$.

For scenario 2, the interest earned is measured when $t_1 < M \leq t_2$ as seen in equation (3.19).

$$\begin{aligned} IE_2 &= pI_e \int_0^{t_2} \int_0^t (a - bp) du dt + \frac{pI_e (a - bp) t_2}{\delta} \log(1 + \delta(T - t_2)) \\ &= \frac{pI_e (a - bp) t_2^2}{2} + \frac{pI_e (a - bp) t_2}{\delta} \log(1 + \delta(T - t_2)). \end{aligned} \quad (3.19)$$

For this case, the interest paid to the supplier is

$$\begin{aligned} \text{IC}_2 &= cI_c \int_M^{t_2} I_2(t) dt \\ &= cI_c \left[\frac{(a-bp)}{(\theta m(\xi))^2} \left(e^{\theta m(\xi)(t_2-M)} - 1 \right) - \frac{(a-bp)(t_2-M)}{\theta m(\xi)} \right]. \end{aligned} \quad (3.20)$$

Hence, the total profit is

$$\pi^2(p, \xi, T) = \frac{X}{T},$$

where

$$\begin{aligned} X &= (\text{SR} - \text{OC} - \text{PC} - \text{IHC} - \text{BC} - \text{LSC} - \text{PTC} + \text{IE}_2 - \text{IC}_2) \\ &= p \left[(a-bp)t_2 + \frac{(a-bp)}{\delta} \{ \log(1 + \delta(T-t_2)) \} \right] - A \\ &\quad - c \left[(a-bp)t_1 + \frac{(a-bp)}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2-t_1)} - 1 \right) + \frac{(a-bp)}{\delta} \{ \log(1 + \delta(T-t_2)) \} \right] \\ &\quad - c_1 \left[St_1 - \frac{(a-bp)t_1^2}{2} - \frac{(a-bp)}{\theta^2 m(\xi)^2} \left(1 - e^{\theta m(\xi)(t_2-t_1)} + \theta m(\xi)(t_2-t_1) \right) \right] \\ &\quad - c_2 \left[\left(R + \frac{D}{\delta}(T-t_2) \right) - \frac{D}{\delta^2} (1 + \delta(T-t_2)) \log(1 + \delta(T-t_2)) \right] \\ &\quad - \xi T \frac{c_3(a-bp)}{\delta} \{ \delta(T-t_2) - \log(1 + \delta(T-t_2)) \} + \frac{pI_e(a-bp)t_2^2}{2} \\ &\quad + \frac{pI_e(a-bp)t_2}{\delta} \log(1 + \delta(T-t_2)) - cI_c \left[\frac{(a-bp)}{(\theta m(\xi))^2} \left(e^{\theta m(\xi)(t_2-M)} - 1 \right) + \frac{(a-bp)(t_2-M)}{\theta m(\xi)} \right]. \end{aligned} \quad (3.21)$$

Thus, the corresponding optimization problem is as follows:

Problem 3.2. $\pi^2(p, \xi, T) = \frac{X}{T}$, subject to constraints $t_1 < M \leq t_2$.

For scenario 3, the interest earned is measured when $t_2 < M \leq T$ as seen in equation (3.22).

$$\begin{aligned} \text{IE}_3 &= pI_e \int_0^{t_2} \int_0^t (a-bp) du dt + \frac{pI_e(a-bp)t_2}{\delta} \log(1 + \delta(T-t_2)) \\ &= \frac{pI_e(a-bp)t_2^2}{2} + \frac{pI_e(a-bp)t_2}{\delta} \log(1 + \delta(T-t_2)). \end{aligned} \quad (3.22)$$

For this case, the interest paid to the supplier is zero.

Therefore, the total profit is

$$\pi^3(p, \xi, T) = \frac{X}{T},$$

where

$$\begin{aligned}
X &= (\text{SR} - \text{OC} - \text{PC} - \text{IHC} - \text{BC} - \text{LSC} - \text{PTC} + \text{IE}_3) \\
&= p \left[(a - bp) t_2 + \frac{(a - bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] - A \\
&\quad - c \left[(a - bp) t_1 + \frac{(a - bp)}{\theta m(\xi)} (e^{\theta m(\xi)(t_2 - t_1)} - 1) + \frac{(a - bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] \\
&\quad - c_1 \left[St_1 - \frac{(a - bp)t_1^2}{2} - \frac{(a - bp)}{\theta^2 m(\xi)^2} (1 - e^{\theta m(\xi)(t_2 - t_1)} + \theta m(\xi)(t_2 - t_1)) \right] \\
&\quad - c_2 \left[\left(R + \frac{D}{\delta} (T - t_2) \right) - \frac{D}{\delta^2} (1 + \delta(T - t_2)) \log(1 + \delta(T - t_2)) \right] - \frac{c_3 (a - bp)}{\delta} \left\{ \frac{\delta(T - t_2)}{\log(1 + \delta(T - t_2))} \right\} \\
&\quad - \xi T + \frac{pI_e (a - bp) t_2^2}{2} + \frac{pI_e (a - bp) t_2}{\delta} \log(1 + \delta(T - t_2)). \tag{3.23}
\end{aligned}$$

Thus, the corresponding optimization problem is as follows:

Problem 3.3. $\pi^3(p, \xi, T) = \frac{X}{T}$, subject to constraints $t_2 < M \leq T$.

4. OPTIMAL SOLUTIONS AND THEORETICAL RESULTS

The profit function considers sales revenue, ordering cost, holding cost, purchasing cost, backordering cost, lost sale cost, preservation technology cost, interest earned, and the interest payable. The average profit per unit time for the retailer can be expressed as:

$$\pi(p, \xi, T) = \frac{1}{T} [\text{SR} - \text{OC} - \text{PC} - \text{IHC} - \text{BC} - \text{LSC} - \text{PTC} + \text{IE} - \text{IP}] \tag{4.1}$$

$$\pi(p, \xi, T) = \begin{cases} \pi^1(p, \xi, T), & \text{when } 0 < M \leq t_1 \\ \pi^2(p, \xi, T), & \text{when } t_1 < M \leq t_2 \\ \pi^3(p, \xi, T), & \text{when } t_2 < M \leq T \end{cases} \tag{4.2}$$

The total profit function $\pi(p, \xi, T)$ takes three branches function in which the maximum values of those three branch functions will be the required solution.

From the above expression, some continuity relations will arise given as $\pi^1(p, \xi, T) = \pi^2(p, \xi, T)$, which is continuous at point t_1 and $\pi^2(p, \xi, T) = \pi^3(p, \xi, T)$, which is continuous at point t_2 .

Examining for the points (p, ξ, T) directly, the maximum value of $\pi(p, \xi, T)$ is calculated by the local maximum points or the boundary points of (p, ξ, T) , wherein it would be bounded within the valid ranges (*i.e.* $0 < M \leq t_1, t_1 < M \leq t_2, t_2 < M \leq T$). The optimal form will be measured, (p^*, ξ^*, T^*) such that $\pi^*(p, \xi, T) = \max \{ \pi^{*1}(p, \xi, T), \pi^{*2}(p, \xi, T), \pi^{*3}(p, \xi, T) \}$.

The optimal values of p, ξ and T which maximize the profit function $\pi^3(p, \xi, T)$ for the given data set are the decision variables of the problem. The necessary conditions for maximizing the total profit function $\pi^3(p, \xi, T)$ can be attained by setting the first order derivatives with respect to decision variables equal to zero. Therefore, the necessary conditions are:

$$\frac{\partial \pi^3(p, \xi, T)}{\partial p} = 0, \frac{\partial \pi^3(p, \xi, T)}{\partial \xi} = 0 \text{ and } \frac{\partial \pi^3(p, \xi, T)}{\partial T} = 0. \tag{4.3}$$

The first order derivatives of $\pi^3(p, \xi, T)$ with respect to the decision variables p , ξ and T are

$$\frac{\partial \pi^3(p, \xi, T)}{\partial p} = \frac{1}{T} \left[\begin{array}{l} (a - 2bp)t_2 + \frac{(a-2bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \\ + bc \left[t_1 + \frac{1}{\theta m(\xi)} (e^{\theta m(\xi)(t_2 - t_1)} - 1) + \frac{1}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] \\ - c_1 \left[\frac{1}{2}bt_1^2 + \frac{b}{\theta^2 m(\xi)^2} \{ 1 - e^{\theta m(\xi)(t_2 - t_1)} + \theta m(\xi)(t_2 - t_1) \} \right] \\ - c_2 \left[\left(\frac{b}{\delta} (t_2 - T) \right) + \frac{b}{\delta^2} (1 + \delta(T - t_2)) \log(1 + \delta(T - t_2)) \right] \\ + \frac{c_3 b}{\delta} \{ \delta(T - t_2) - \log(1 + \delta(T - t_2)) \} - \frac{I_e(a-2bp)t_2^2}{2} \\ + \frac{I_e(a-2bp)t_2}{\delta} \log(1 + \delta(T - t_2)) \end{array} \right] \quad (4.4)$$

$$\frac{\partial \pi^3(p, \xi, T)}{\partial \xi} = \frac{1}{T} \left[\begin{array}{l} -ca_1(a - bp) \left[\frac{e^{\theta m(\xi)(t_2 - t_1)}}{\theta m(\xi)} - e^{\theta m(\xi)(t_2 - t_1)}(t_2 - t_1) \right] \\ - c_1 a_1 \left[-\frac{2(a-bp)}{\theta^2} \frac{1}{m(\xi)^2} - \frac{(a-bp)}{\theta} e^{\theta m(\xi)(t_2 - t_1)} + e^{\theta m(\xi)(t_2 - t_1)} \frac{2}{\theta^2} \frac{1}{m(\xi)^2} \right] - T \\ + \frac{(a-bp)}{\theta m(\xi)} (t_2 - t_1) \end{array} \right] \quad (4.5)$$

and

$$\frac{\partial \pi^3}{\partial T} = -\frac{1}{T} \pi^3(p, \xi, T) + \frac{1}{T} \left[\begin{array}{l} \frac{(p-c)(a-bp)}{1+\delta(T-t_2)} + \frac{c_2(a-bp)}{\delta} \log(1 + \delta(T - t_2)) \\ - c_3(a - bp) \left[1 - \frac{1}{1+\delta(T-t_2)} \right] + \frac{pI_e(a-bp)t_2}{1+\delta(T-t_2)} - \xi \end{array} \right]. \quad (4.6)$$

Due to the high non-linearity of the profit function, the optimality of the equations could be demonstrated mathematically from the objective functions by using the following corollaries and theorems.

Corollary 4.1. *The objective function $\pi^3(p, \xi, T)$ is maximum with respect to T when the decision variable p and ξ are fixed and when $\frac{1}{T} \frac{\partial^2 X}{\partial T^2} + \frac{2X}{T^3} < \frac{2}{T^2} \frac{\partial X}{\partial T}$ (Appendix B).*

Corollary 4.2. *The objective function $\pi^3(p, \xi, T)$ is maximum with respect to p when the decision variable T and ξ are fixed and when $\frac{\delta I_e t_2 (t_2 - 4)}{2I_e t_2 + 4} < \log(1 + \delta(T - t_2))$ (Appendix B).*

To prove concavity, we use the theorems from Cambini and Martein [4] which is also used by Dye [6].

Lemma 4.3. *If $\phi_1(x)$ is non-negative, differentiable and (strictly) concave, and $\phi_2(x)$ is positive, differentiable and convex, then the real-value function $\chi(x) = \frac{\phi_1(x)}{\phi_2(x)}$, $x \in \mathbb{R}$ is (strictly) pseudo-concave.*

Proof. See Cambini and Martein ([4], p. 245) for details. \square

For any given p and T , applying Lemma 4.3, it could easily prove that the total profit $\pi^3(p, \xi, T)$ is strictly pseudo-concave in ξ . As a result, for any given p and T , there exists a unique global optimal solution ξ^* such that $\pi^3(p, \xi, T)$ is maximized.

Theorem 4.4. *The value of the objective function $\pi^3(p, \xi, T)$ attains its global maximum with respect to ξ when other parameters are fixed.*

Proof. We define the profit function as

$$\phi_1(\xi) = \begin{cases} p \left[(a - bp) t_2 + \frac{(a - bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] - A \\ - c \left[(a - bp) t_1 + \frac{(a - bp)}{\theta m(\xi)} (e^{\theta m(\xi)(t_2 - t_1)} - 1) + \frac{(a - bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] \\ - c_1 \left[St_1 - \frac{(a - bp)t_1^2}{2} - \frac{(a - bp)}{\theta^2 m(\xi)^2} (1 - e^{\theta m(\xi)(t_2 - t_1)} + \theta m(\xi)(t_2 - t_1)) \right] \\ - c_2 \left[(R + \frac{D}{\delta}(T - t_2)) - \frac{D}{\delta^2} (1 + \delta(T - t_2)) \log(1 + \delta(T - t_2)) \right] - \frac{c_3(a - bp)}{\delta} \left\{ \frac{\delta(T - t_2)}{\log(1 + \delta(T - t_2))} \right\} \\ - \xi T + \frac{pI_e(a - bp)t_2^2}{2} + \frac{pI_e(a - bp)t_2}{\delta} \log(1 + \delta(T - t_2)) \end{cases} \quad (4.7)$$

and $\phi_2(\xi) = T > 0$.

From the above equation, it is clear that $\phi_2(\xi)$ is non-negative, differentiable and (strictly) concave. We only need to proof that $\phi_1(\xi)$ is positive, differentiable and convex. For any fixed p and T , the second order derivative of the objective function $\pi^3(p, \xi, T)$ with respect to the decision variables ξ is

$$\phi'_1(\xi) = \begin{cases} -c(a - bp) \left[e^{\theta m(\xi)(t_2 - t_1)} \cdot \frac{a_1 e^{-a_1 \xi}}{\theta m(\xi)^2} - \frac{a_1 e^{-a_1 \xi}}{\theta m(\xi)} \cdot e^{\theta m(\xi)(t_2 - t_1)} \theta(t_2 - t_1) \right] \\ -c_1 \left[-\frac{2a_1 e^{-a_1 \xi}(a - bp)}{\theta^2} \frac{1}{m(\xi)^3} - \frac{(a - bp)a_1 e^{-a_1 \xi}}{\theta m(\xi)} e^{\theta m(\xi)(t_2 - t_1)} \right. \\ \left. + e^{\theta m(\xi)(t_2 - t_1)} \frac{2}{\theta^2} \frac{a_1 e^{-a_1 \xi}}{m(\xi)^3} + \frac{(a - bp)a_1 e^{-a_1 \xi}}{\theta m(\xi)^2} (t_2 - t_1) \right] - T \end{cases}. \quad (4.8)$$

Using the necessary condition $\frac{\partial \pi^3(p, \xi, T)}{\partial \xi} = 0$, equation (4.8) becomes

$$\phi''_2(\xi) = \begin{cases} c(a - bp)a_1^2 e^{\theta m(\xi)(t_2 - t_1)}(t_2 - t_1) \left[1 - \theta(t_2 - t_1)e^{-a_1 \xi} \right] - c(a - bp)a_1 \frac{e^{\theta m(\xi)(t_2 - t_1)}}{\theta m(\xi)^2} \\ - c_1 a_1^2 \left[-\frac{4(a - bp)}{\theta^2} \frac{1}{m(\xi)^2} + (a - bp)e^{\theta m(\xi)(t_2 - t_1)}(t_2 - t_1)e^{-a_1 \xi} + \frac{4}{\theta^2} e^{\theta m(\xi)(t_2 - t_1)} \frac{1}{m(\xi)^2} \right] \\ + \frac{2}{\theta} \frac{1}{m(\xi)} e^{\theta m(\xi)(t_2 - t_1)}(t_2 - t_1) + (a - bp)(t_2 - t_1) \frac{1}{\theta m(\xi)} \end{cases} \quad (4.9)$$

$$\phi''_2(\xi) = \begin{cases} c a_1^2 (a - bp) e^{\theta m(\xi)(t_2 - t_1)} (t_2 - t_1) \left[1 - e^{-a_1 \xi} (1 + \theta(t_2 - t_1)) \right] - c(a - bp)a_1 \frac{e^{\theta m(\xi)(t_2 - t_1)}}{\theta m(\xi)^2} \\ - c_1 a_1^2 \left[\frac{4}{\theta^2} \frac{1}{m(\xi)^2} (e^{\theta m(\xi)(t_2 - t_1)} - (a - bp)) + \frac{1}{\theta} \frac{1}{m(\xi)} (t_2 - t_1) (2e^{\theta m(\xi)(t_2 - t_1)} + a - bp) \right] \end{cases}. \quad (4.10)$$

Now we need to prove $\phi''_2(\xi)$ is less than zero:

$$\begin{cases} c a_1^2 (a - bp) e^{\theta m(\xi)(t_2 - t_1)} (t_2 - t_1) \left[1 - e^{-a_1 \xi} (1 + \theta(t_2 - t_1)) \right] - c(a - bp)a_1 \frac{e^{\theta m(\xi)(t_2 - t_1)}}{\theta m(\xi)^2} \\ - c_1 a_1^2 \left[\frac{4}{\theta^2} \frac{1}{m(\xi)^2} (e^{\theta m(\xi)(t_2 - t_1)} - (a - bp)) + \frac{1}{\theta} \frac{1}{m(\xi)} (t_2 - t_1) (2e^{\theta m(\xi)(t_2 - t_1)} + a - bp) \right] \end{cases} < 0. \quad (4.11)$$

This means,

$$\begin{aligned} & \Rightarrow [c a_1^2 (a - bp) e^{\theta m(\xi)(t_2 - t_1)} (t_2 - t_1) \left[1 - e^{-a_1 \xi} (1 + \theta(t_2 - t_1)) \right]] < c(a - bp)a_1 \frac{e^{\theta m(\xi)(t_2 - t_1)}}{\theta m(\xi)^2} \\ & + c_1 a_1^2 \left[\frac{4}{\theta^2} \frac{1}{m(\xi)^2} (e^{\theta m(\xi)(t_2 - t_1)} - (a - bp)) + \frac{1}{\theta} \frac{1}{m(\xi)} (t_2 - t_1) (2e^{\theta m(\xi)(t_2 - t_1)} + a - bp) \right] \end{aligned} \quad (4.12)$$

$$\begin{aligned} & \Rightarrow a_1 (t_2 - t_1) \theta (a - bp) m(\xi)^2 e^{\theta m(\xi)(t_2 - t_1)} [1 - m(\xi) [1 + \theta(t_2 - t_1)]] \\ & < [e^{\theta m(\xi)(t_2 - t_1)} [(a - bp) + \frac{4a_1}{\theta}] + m(\xi)(t_2 - t_1) [(a - bp) + 2e^{\theta m(\xi)(t_2 - t_1)}]]. \end{aligned} \quad (4.13)$$

To prove that $[(a - bp) + \frac{4a_1}{\theta}]$ and $[(a - bp) + 2e^{\theta m(\xi)(t_2 - t_1)}]$ are greater than zero, consider that demand is always positive and that deterioration lies between 0 to 1, the right-hand side of the equation mentioned above will satisfy the equation above. From the equation, the right-hand side is greater than zero. If the left side is less than zero, it satisfies the condition. That is

$$a_1(t_2 - t_1)\theta(a - bp)m(\xi)^2e^{\theta m(\xi)(t_2 - t_1)}[1 - m(\xi)[1 + \theta(t_2 - t_1)]] < 0. \quad (4.14)$$

Moreover, $[1 - m(\xi)[1 + \theta(t_2 - t_1)]] < 0$ as $(t_2 - t_1) \geq 0$ and the deterioration is positive, the value of $\theta(t_2 - t_1)$ is also positive. It could be written as $[1 - m(\xi)[1 + k_1]] < 0$. One can deduce that $[1 - m(\xi)k_2] < 0$, since $[1 + k_1] = k_2$. The deterioration rate $m(\xi)$ with the investment of preservation technology and having the original deterioration from 0 to 1, the value of $m(\xi)k_2$ is then positive. Therefore, $a_1(t_2 - t_1)\theta(a - bp)m(\xi)^2e^{\theta m(\xi)(t_2 - t_1)}[1 - m(\xi)[1 + \theta(t_2 - t_1)]] < 0$; thus completing the proof. \square

Theorem 4.5. *The value of the objective function $\pi^3(p, \xi, T)$ attains its global maximum with respect to p when other parameters are fixed.*

Proof. The proof is omitted because it is similar to the proof of Theorem 4.4. \square

Theorem 4.6. *The value of the objective function $\pi^3(p, \xi, T)$ attains its global maximum with respect to T when other parameters are fixed.*

Proof. The proof is omitted because it is similar to the proof of Theorem 4.4. \square

Theorem 4.7. *The value of the objective function $\pi^3(p, \xi, T)$ attains its global maximum with respect to p and T when other parameters are fixed and hence there exists a unique maximum solution at (p^*, T^*) .*

Proof. For convenience, let us define the profit function as

$$\phi_1(p, T) = \begin{bmatrix} p \left[(a - bp) t_2 + \frac{(a - bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] - A \\ -c \left[(a - bp) t_1 + \frac{(a - bp)}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2 - t_1)} - 1 \right) + \frac{(a - bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] \\ -c_1 \left[St_1 - \frac{(a - bp)t_1^2}{2} - \frac{(a - bp)}{\theta^2 m(\xi)^2} \left(1 - e^{\theta m(\xi)(t_2 - t_1)} + \theta m(\xi)(t_2 - t_1) \right) \right] \\ -c_2 \left[(R + \frac{D}{\delta}(T - t_2)) - \frac{D}{\delta^2}(1 + \delta(T - t_2)) \log(1 + \delta(T - t_2)) \right] - \frac{c_3(a - bp)}{\delta} \left\{ \frac{\delta(T - t_2)}{-\log(1 + \delta(T - t_2))} \right\} \\ -\xi T + \frac{p I_e (a - bp) t_2^2}{2} + \frac{p I_e (a - bp) t_2}{\delta} \log(1 + \delta(T - t_2)) \end{bmatrix} \quad (4.15)$$

and $\phi_2(p, T) = T > 0$.

From the results in Cambini and Martein ([4], p. 245), we prove that $\phi_1(p, T)$ is a differentiable, negative and strictly joint concave function with respect to the decision variables p and T . To generate the Hessian matrix for the function $f_1(P, T)$, we calculate all the second-order partial derivatives with respect to the decision variables p and T with a Hessian matrix for the function $\phi_1(p, T)$

$$\frac{\partial^2 \phi_1(p, T)}{\partial p^2} = - \left[2bt_2 + \frac{2b}{\delta} \{ \log(1 + \delta(T - t_2)) \} - \frac{I_e b t_2^2}{2} + \frac{I_e b t_2}{\delta} \log(1 + \delta(T - t_2)) \right] \quad (4.16)$$

$$\frac{\partial^2 \phi_1(p, T)}{\partial T^2} = - \frac{D}{(1 + \delta(T - t_2))^2} [\delta(p - c) + c_2 \delta + c_2 (1 + \delta(T - t_2)) + c_3 \delta + p \delta I_e t_2] \quad (4.17)$$

$$\begin{aligned} \frac{\partial^2 \phi_1(p, T)}{\partial p \partial T} = & \left[\frac{b}{\delta} (1 + \log(1 + \delta(T - t_2)) + c_2) + \frac{1}{(1 + \delta(T - t_2))} (c_2 b + b(c - p) + c_3 b \delta(T - t_2)) \right. \\ & \left. + I_e D t_2 - p b I_e t_2 \right]. \end{aligned} \quad (4.18)$$

Therefore, the Hessian matrix for the function $\phi_1(p, T)$ is

$$H_{ij} = \begin{bmatrix} \frac{\partial^2 \phi_1}{\partial p^2} & \frac{\partial^2 \phi_1}{\partial p \partial T} \\ \frac{\partial^2 \phi_1}{\partial T \partial p} & \frac{\partial^2 \phi_1}{\partial T^2} \end{bmatrix}. \quad (4.19)$$

Now, considering the first principal minor:

$$|H_{11}| = \frac{\partial^2 \phi_1(p, T)}{\partial p^2} = - \left[2bt_2 + \frac{2b}{\delta} \{ \log(1 + \delta(T - t_2)) \} - \frac{I_e b t_2^2}{2} + \frac{I_e b t_2}{\delta} \log(1 + \delta(T - t_2)) \right]. \quad (4.20)$$

It is clearly seen that the above $|H_{11}|$ is less than zero.

Then, to calculate Hessian matrix H_{ij} we need to check the second principal minor

$$\begin{aligned} |H_{22}| &= \frac{\partial^2 \phi_1(p, T)}{\partial p^2} \frac{\partial^2 \phi_1(p, T)}{\partial T^2} - \frac{\partial^2 \phi_1(p, T)}{\partial p \partial T} \\ &= \left(2bt_2 + \frac{2b}{\delta} \{ \log(1 + \delta(T - t_2)) \} - \frac{I_e b t_2^2}{2} + \frac{I_e b t_2}{\delta} \log(1 + \delta(T - t_2)) * \frac{D}{(1 + \delta(T - t_2))^2} (\delta(p - c) \right. \\ &\quad \left. + c_2 \delta + c_2 (1 + \delta(T - t_2)) + c_3 \delta + p \delta I_e t_2) \right) - \left[\frac{b}{\delta} (1 + \log(1 + \delta(T - t_2)) + c_2) \right. \\ &\quad \left. + \frac{1}{(1 + \delta(T - t_2))} (c_2 b + b(c - p) + c_3 b \delta(T - t_2) + I_e D t_2 - p b I_e t_2) \right] \\ &= \frac{1}{(1 + \delta(T - t_2))} \begin{bmatrix} \left(2bt_2 - \frac{I_e b t_2^2}{2} + \left(\frac{2b}{\delta} + \frac{I_e b t_2}{\delta} \right) (\log(1 + \delta(T - t_2))) \right) \\ * \frac{D}{(1 + \delta(T - t_2))} (\delta(p - c) + c_2 \delta + c_2 (1 + \delta(T - t_2)) + c_3 \delta + p \delta I_e t_2) \\ - \left[\frac{b}{\delta} (1 + \delta(T - t_2)) (1 + \log(1 + \delta(T - t_2)) + c_2) + (c_2 b + b(c - p) \right. \\ \left. + c_3 b \delta(T - t_2)) + I_e D t_2 - p b I_e t_2 \right] \end{bmatrix} \end{aligned} \quad (4.21)$$

Since the lost sale cost is greater than the purchase cost of the products, one can conclude from the above equation that without loss of generality the principal minor $|H_{22}|$ is always less than zero.

It is seen from the above manipulation that first principle minor is negative and second one is positive so the Hessian matrix for $\phi_1(p, T)$ are negative, so the Hessian matrix is also treat as negative definite matrix. Therefore, one can conclude that the function $\phi_1(p, T)$ is differentiable, and (strictly) concave function with respect to the decision variables p and T . Likewise, the total profit function per unit time $\pi^3(p, \xi, T)$ is pseudo-concave with respect to decision variables p and T . Consequently, the objective function $\pi^3(p, \xi, T)$ attains its global maximum value at the point (p^*, T^*) . \square

Theorem 4.8. *The value of the objective function $\pi^3(p, \xi, T)$ attains its global maximum with respect to p and ξ when other parameters are fixed and hence there exists a unique maximum solution at (p^*, ξ^*) .*

Proof. The proof is omitted because it is similar to the proof of Theorem 4.7 \square

Theorem 4.9. *The value of the objective function $\pi^3(p, \xi, T)$ attains its global maximum with respect to ξ and T when other parameters are fixed and hence there exists a unique maximum solution at (ξ^*, T^*) .*

Proof. The proof is omitted because it is similar to the proof of Theorem 4.7 \square

5. SPECIAL CASES OF THE PROPOSED MODEL

- (i) When $\delta \rightarrow \infty$ (i.e. no shortages) then the proposed model here will become an EOQ model with no shortages.
- (ii) For the non-deterioration period $t_1 = 0$ with $t_2 = 0$ and $\delta \rightarrow \infty$, then the model is reduced to the instantaneous EOQ model of Mishra *et al.* [18].
- (iii) If $\delta \rightarrow 0$ then the proposed model will be reduce to an EOQ model of fully backlogged shortages.
- (iv) Taking into account $t_2 = T$ and $\delta \rightarrow \infty$, then the model becomes that of Zhang *et al.* [36], an instantaneous deterioration with no shortages.
- (v) If $t_1 = 0$ and $b = 0$ (i.e. demand function is constant), then the proposed model is reduced to the model of Dye and Hsieh [7].

6. FLOWCHART FOR THE SOLUTION SYSTEM

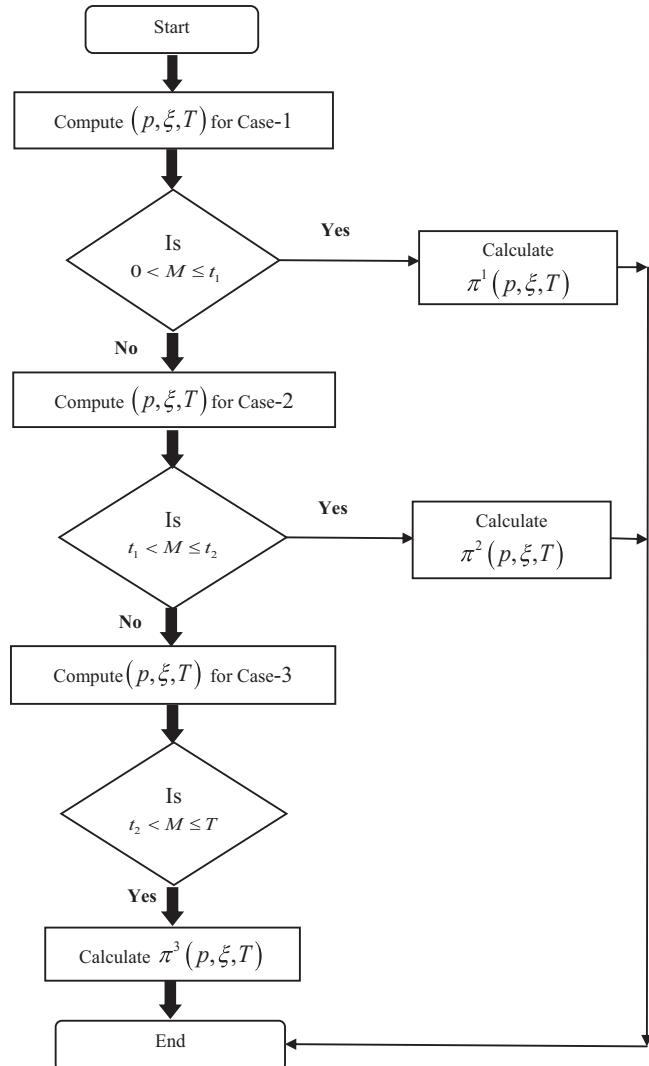


FIGURE 2. Flowchart of the solution system.

7. NUMERICAL EXAMPLES

The solution of the proposed inventory model is discussed in this part, specifically deteriorating products like vegetables, fruit, sweets, etc. This is because in any season, the demand for this item increases linearly depending on the selling price. Initially, no deterioration in the initial stage is considered as the items are of high quality. After some time, deterioration begins, and the stocks deplete. To demonstrate the inventory model, three numerical examples are given with different parameters as shown:

Example 1

Given: ordering cost is $OC = \$1500$ per order, $a = 300$; $b = 5$, holding cost rates for inventory of different items $C_1 = \$5$, backorder cost rate for inventory $C_2 = \$5$, lost sale cost for the inventory $C_3 = \$27$, purchase cost of different item $C = \$8$, $a_1 = \$0.8$ per unit, deterioration rate $\theta = 0.5$, backlogging parameter for the inventory is $\delta = 1.5$, time after deterioration will start for the inventory $t_1 = 0.3$, percentage of interest charged $I_c = \$0.06$ percentage of interest earned $I_e = \$0.12$, and permissible delay period from supplier to retailers is $M = 0.1$ yr.

Example 2

Given: Similar to example 1, the permissible delay period from the supplier to the retailers is $M = 0.8$ yr instead of 0.1 yr.

Example 3

Given: Similar to example 1, the permissible delay period from the supplier to the retailers is $M = 2.5$ yr instead of 0.1 yr (Tab. 3).

Therefore, a unique optimal solution is illustrated in Figures 3–5, showing that the profit function is concave.

To clarify the graph of “Total profit” vs. P vs. T , a two-dimensional graph of “Total profit” vs. P and “Total profit” vs. T is provided as seen in Figures 6 and 7.

Example 4

In this example, consider the same parametric values for the inventory model of Dye and Hsieh [7] given that:

$c_2 = \$4$ /per order; $c = \$20$ /per order; $OC = \$120$ /per order; $p = \$35$ /per order; $c_1 = \$3$ /per order; $D = 1000$; $c_3 = \$5$ /per order; $t_1 = 0$; $\delta = 2$; $a_1 = 0.01$

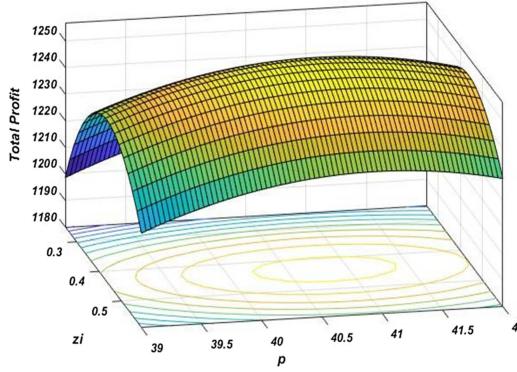
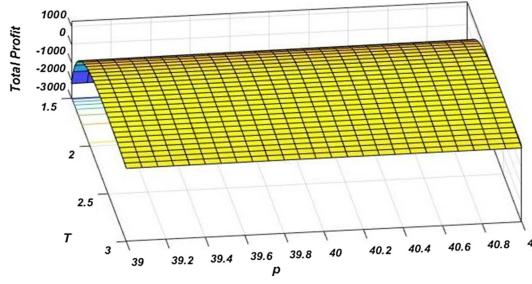
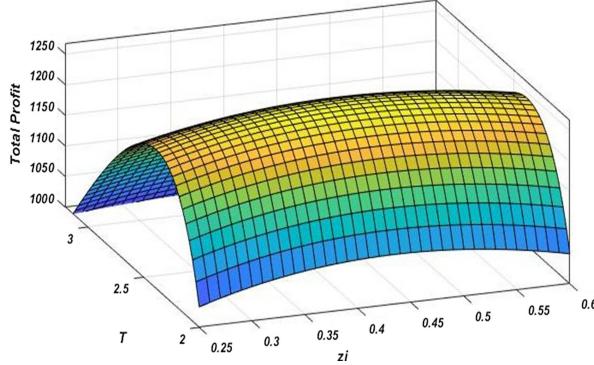
and if one assumes that the trade credit period $M = 0$ and the interest earn is $I_e = 0$, the example is identical with that of Dye and Hsieh [7], but shows an optimal value is \$11908.4 compared with Dye’s \$13919.3.

However, the trade credit period and interest earned should be: $M = 3.2$; $I_e = 0.06$;

Therefore, the optimal solution is $T^* = 3.522897$, $p^* = 35$, $\xi^* = 89.16101$, $\Pi^*(p^*, \xi^*, T^*) = 14076.86$, and the values of M to satisfy the condition must be $t_2 < M \leq T$.

TABLE 3. Results for each case of the inventory model.

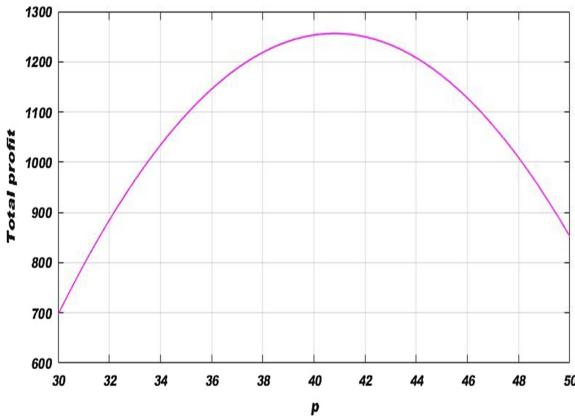
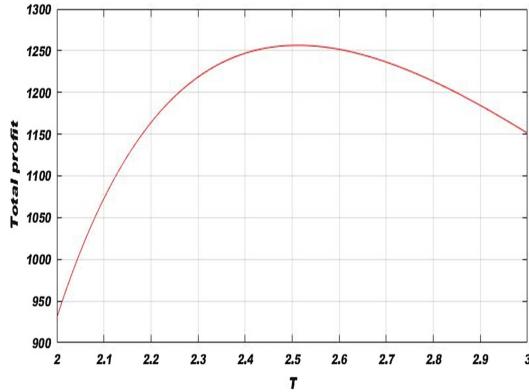
Examples	Results						
	Condition	S	R	p	ξ	T	$\pi^*(p, \xi, T)$
1	$0 < M \leq t_1$	170.067	29.664	39.952	0.367	1.796	1184.30
2	$t_1 < M \leq t_2$	94.304	32.355	37.913	0.269	1.167	1066.879
3	$t_2 < M \leq T$	258.123	32.648	40.963	0.453	2.50	1255.818

FIGURE 3. Total profit function with respect to p and zi .FIGURE 4. Total profit with respect to p and T .FIGURE 5. Total profit function with respect to T and zi .

Observation

It is noted that when the interest earned is zero, the total profit is decreasing with the increase of trade credit policy. Similarly, it happens when interest earned is $I_e = 0.06$ and 0.08 .

From Table 4, it is seen that with the increase of trade credit period, the total cycle length will also increase. Therefore, the retailer consequently needs to preserve products over that time. As a result, the retailer needs to invest more money in preservation technology, which will decrease his total profit. It also noted that after a certain period the facility of trade credit policy would become a burden for the retailer.

FIGURE 6. Total profit with p .FIGURE 7. Total profit with T .

Example 5

In this example, we consider the same parametric values for the inventory system of Zhang *et al.* [36], given $OC = k = \$80/\text{per order}$; $a = \alpha = 100$; $b = \beta = 5$; $c_1 = h = \$1/\text{per order}$; $a_1 = \gamma = 0.05$; $c = \$5/\text{per order}$; $\theta = 0.5$ with the trade credit period $M = 0$ and Interest earn $I_e = 0$.

The optimal solution is $T^* = 1.585916$, $p^* = 13.22326$, $\xi^* = 12.19286$, $\pi^*(p^*, \xi^*, T^*) = 166.7125$. While the value of Zhang *et al.* [36] is 153.8268 (Tab. 5).

Observation

When the values of interest earned are $I_e = 0, 0.06, 0.08$, it could be readily observed that with the increase of trade credit period M the total profit would decrease. Like the trade, credit period and selling price increased. This shows that the retailer would want to sell more products to increase the profit. However, with the increase of total cycle length, the retailer needs more time to preserve products which will cost more money in the investments of preservation technology in order to fully preserve the products.

For different types of deterioration, the following were considered

Case 1 Considering the Weibull distribution deterioration rate.

TABLE 4. Computational values for different M and I_e .

M	I_e	$t_2 < M \leq T$	p^*	ξ^*	T^*	π^*
2.5	0	No	35.00	73.62892	2.532359	11162.31
	0.06	No	35.00	79.20355	2.891557	13885.10
	0.08	No	35.00	79.22611	2.894031	14816.65
3.2	0	Yes	35.00	83.20161	3.200000	10958.48
	0.06	Yes	35.00	89.16101	3.522897	14076.86
	0.08	Yes	35.00	90.71324	3.621504	15251.42
4.5	0	Yes	35.00	100.1261	4.50000	9769.280
	0.06	Yes	35.00	103.4082	4.50000	13755.44
	0.08	Yes	35.00	104.1227	4.50000	15177.86
26	0	Yes	35.00	—	—	Non-optimal
	0.06	Yes	35.00	244.0023	26.0000	116.7840
	0.08	Yes	35.00	255.4683	26.0000	1060.789

TABLE 5. Computational values for different M and I_e .

M	I_e	p^*	ξ^*	T^*	π^*
0	0	13.22326	12.19286	1.585916	166.7125
	0.06	13.65001	13.60593	2.500000	155.7524
	0.08	13.39491	13.63798	2.500000	188.6056
3	0	13.31763	13.64722	2.500000	199.6978
	0.06	13.89860	14.37439	3.000000	144.8509
	0.08	13.57610	14.41559	3.000000	183.5771
		13.48015	14.42709	3.000000	196.7103

Taking all the values (same as that of Example 3), except the deterioration rate, the Weibull distribution deterioration is suitable to be used when there is a minimal life, $t_1(> 0)$. Considering a deterioration-free period offered in the model, the details of which are available from the study of Begum *et al.* [2]. Taking the same Weibull distribution deterioration rate of Shah *et al.* [27], $\theta(t) = \alpha\beta(t - \gamma)^{\beta-1}$, but having the deterioration free time to be ($\gamma = t_1$), α and β are 0.04 and 2.4, respectively and γ is 0.3, the optimal solution is

$$T^* = 3.349560, p^* = 42.56130, \xi^* = 0.4622905, \pi^*(p^*, \xi^*, T^*) = 1056.650.$$

The following managerial insights would be obtained from the results (Tab. 6).

Observation 1

When the values of α and β are fixed, the profit increases with the increase of the non-deterioration period. It is also observed that when the retailer gets more time to sell his products with a lower rate, the products would get better profit due to lower deterioration period. This is because as the non-deterioration period increased then the retailer does not need to invest more money in preservation technology to reduce deterioration.

Observation 2

When the values of α is fixed, increasing β and the non-deterioration period t_1 will increase the total profit. However, the total cycle time, as well as the selling price of the products will decrease with the increase of non-deterioration period together with β .

TABLE 6. Computational results for different values of α and β for fixed $M = 3.2$.

α	β	t_1	p^*	ξ^*	T^*	π^*
0.04	2.4	0.1	43.11616	0.7700489	3.868266	1083.132
		0.3	42.96688	0.7057555	3.875276	1109.910
		0.5	42.78921	0.6386975	3.886757	1142.537
0.04	2.6	0.1	42.04866	0.9092852	3.592346	1249.924
		0.3	41.96194	0.8412425	3.622224	1270.096
		0.5	41.84539	0.7709814	3.658458	1296.912
0.06	2.4	0.1	41.72899	0.8850493	3.411081	1288.146
		0.3	41.61300	0.8155702	3.427674	1312.834
		0.5	41.46695	0.7430020	3.449885	1344.411
0.06	2.6	0.1	40.86074	1.022808	3.200147	1430.049
		0.3	40.80553	0.9492549	3.236880	1447.581
		0.5	40.71807	0.8731135	3.280793	1472.921

Case 2 Considering variable deterioration rate.

Taking all the values (same as that of Example 3), except the deterioration rate; also taking the same time varying deterioration rate of Dye and Hsieh [7] as $\theta(t) = 0.2 + 0.1t$, and adapting it to the model shows that the optimal solution is $T^* = 3.00$, $p^* = 41.86581$, $\xi^* = 0.5119012$, $\pi^*(p^*, \xi^*, T^*) = 1190.987$.

8. SENSITIVITY ANALYSIS

Sensitivity analysis was executed on the optimal result for the given data set (example 3 from the numerical example) in order to study the influence of under or overestimation of the inventory system factors on the optimal values of the cycle length (T^*), preservation cost (ξ^*), selling price (p^*), initial stock (S^*), maximum shortage (R^*), along with the maximum profit (π^*) of the system. The percentage variations in the aforementioned optimal values were taken as processes of sensitivity. The analysis was carried out by changing the key parameters from -20% to $+20\%$, one parameter at a time and keeping all other parameter fixed, the result shown in Table 7.

From Table 7, the following observations were found.

Observations 1

When the value of the parameters $b, c_p, c_1, c_2, c_3, \delta, M$, decreases and the value of the parameters a, θ, t_1, I_e increases, the optimum total profit function π^* decreases. It is observed that when the replenishment cost per order increases, the values of ξ^*, p^* will increase while the total profit π^* decreases. The result is analogous to the study of Zhang *et al.* [36]. This indicates that if the replenishment cost per order becomes higher, the enterprise will try to lengthen the replenishment cycle as well as ordering more quantity; this will result in more time to recover the cost of preservation technology. In order to increase profit, it is necessary for the enterprise to increase the selling price of goods.

Observations 2

S^* increases when the value of the parameters b, δ, t_1, c_p, c_1 decreases and $a, c_2, c_3, \theta, I_e, M$ increases while S^* decreases. Moreover, there is no effect on S^* if the value of the parameters OC, a_1 changes. The retailer would want to store more products when the cost of purchasing is available at a lower rate. Consequently, when the manufacturer gives more significant trade credit period to retailers, the retailers would tend to increase the product inventory in order to attract more buyers and sell more products.

TABLE 7. Sensitivity analysis with respect to different parameters.

Parameter	% Change in parameters	% Change in				
		π^*	S^*	R^*	p^*	ξ^*
b	-20	82.35	15.62	-1.82	18.62	1.99
	-10	35.94	7.6	-0.26	8.28	0.9
	10	-28.43	-7.22	-0.84	-6.77	-0.92
	20	-51.22	-14.1	-2.65	-12.42	-1.78
a	-20	-78.32	-33.51	-22.8	-14.82	-1.7
	-10	-42.87	-17.2	-10.89	-7.45	-1.01
	10	50.35	17.58	9.78	7.45	0.9
	20	108.2	35.47	18.6	14.9	1.7
C_p	-20	9.3	4.6	4.19	-1.47	-15.95
	-10	4.51	2.18	2.09	-0.72	-7.59
	10	-4.27	-1.99	-2.09	0.69	6.95
	20	-8.34	-3.82	-4.17	1.36	13.37
c_1	-20	25.81	13.39	-15.54	-3.4	19.6
	-10	12.27	6.57	-7.33	-1.64	9.4
	10	-11.13	-6.29	6.51	1.53	-8.78
	20	-21.23	-12.28	12.28	2.94	-17.06
c_2	-20	0.22	-0.25	0	-0.06	-0.19
	-10	0.11	-0.12	0	-0.03	-0.09
	10	-0.11	0.12	0	0.03	0.09
	20	-0.21	0.24	0	0.06	0.19
c_3	-20	1.91	-2.21	10.07	-0.53	-1.69
	-10	0.91	-1.04	4.79	-0.25	-0.8
	10	-0.82	0.94	-4.36	0.22	0.72
	20	-1.58	1.8	-8.36	0.42	1.38
a_1	-20	0.04	0	0	0	25
	-10	0.04	0	0	0	11.11
	10	0	0	0	0	-9.09
	20	0.1	0	0	0	-16.67
θ	-20	-13.36	-9.69	2.27	1.98	-29.99
	-10	-6.13	-4.56	1.02	0.9	-14.34
	10	5.25	4.1	-0.83	-0.76	13.26
	20	9.8	7.81	-1.5	-1.41	25.63
δ	-20	3.44	-3.38	20.08	-0.67	-2.43
	-10	1.57	-1.55	9.12	-0.31	-1.12
	10	-1.34	1.33	-7.7	0.27	0.97
	20	-2.49	2.48	-14.29	0.5	1.81
OC	-20	9.55	0	0	0	0
	-10	4.78	0	0	0	0
	10	-4.78	0	0	0	0
	20	-9.55	0	0	0	0
t_1	-20	-2.67	0.75	-0.61	0.41	1.08
	-10	-1.34	0.37	-0.3	0.21	0.55
	10	1.35	-0.36	0.29	-0.21	-0.56
	20	2.7	-0.7	0.57	-0.41	-1.12
I_e	-20	-4.17	-1.36	1.2	0.31	-0.35
	-10	-2.09	-0.67	0.6	0.15	-0.17
	10	2.1	0.67	0.59	-0.15	0.17
	20	4.2	1.33	1.18	-0.3	0.34

TABLE 7. continued.

Parameter	% Change in parameters	% Change in				
		π^*	S^*	R^*	p^*	ξ^*
M	-20	2.28	-21.26	-12.52	-2.33	-13.29
	-10	1.64	-10.56	-6.23	-1.14	-6.59
	10	-2.34	10.35	6.13	1.11	6.47
	20	-5.2	20.41	12.11	2.2	12.82

Observations 3

R^* increases when the value of the parameters $\theta, I_e, \delta, c_p, c_3$ decreases and a, c_1, b, t_1, M increases while R^* decreases. Moreover, there is no effect on R if the value of the parameters OC, a_1, c_2 changes. It is observed that with the increase in δ , the amount of shortages decreases and when it approaches infinity, then there will be no shortages. This means that the optimal value of R^* will be zero. The amount of shortages increases when the value of holding cost increases which is analogous to the study by Li *et al.* [12].

Observations 4

p^* increases when the value of the parameters b, θ, a_1, I_e, c_t decreases and b, I_e, t_1, θ increases while p^* decreases. Moreover, there is no effect on p^* if the value of the parameters OC, a_1 changes, but there is a proportional relationship between p^* and the value of the parameters $a, \delta, c_p, c_2, c_3, c_1, M$. However, when the selling price increases, the retailer would tend to sell more products so that the total cycle time will be longer and reduce the shortage in order to get maximum profit. When purchase cost increases, then optimal total profit (π^*) and shortage (R^*) decreases together with the increase in selling price. The result of which is analogous to the study by Tiwari *et al.* [33].

Observations 5

ξ^* increases when the value of the parameters a, b, t_1, c_1 increases and a, b, t_1, c_1 increases while ξ^* decreases. Moreover, there is no effect on R if the value of the parameters OC changes, but there is a proportional relationship between p^* and the value of the parameters $a, \delta, c_p, c_2, c_3, I_e, M, \theta$. When the value of the simulation coefficient a_1 increases, then the value of preservation investment decreases. It is also observed that the investment on preservation technology lengthened the non-deterioration period and reduced the deterioration, which is observed in the study by Li *et al.* [12].

Observations 6

T^* increases when the value of the parameters a, b decreases. Moreover, there is no effect on R when the value of the parameters OC, $a_1, c_2, c_3, c_p, c_1, I_e, t_1, \theta, \delta$ changes and there is a proportional relationship between total cycle length and trade credit period. This means that when the retailer gets more trade credit period, the retailer would lengthen the cycle length.

9. CONCLUSIONS AND COMMENTS

In this study, we have derived a deteriorating inventory model with price dependent demand, partial backlogging and trade credit considering preservation technology for perishable goods. It has been shown that the investment in preservation technology has lengthened the freshness of goods. Furthermore, to increase competitiveness, this model also takes into account of credit financing as a strategy that benefits retailers. We have derived the optimal cycle length, optimal selling price and investments in preservation technology in order to maximize the total profit. Numerical analysis is carried out to show the importance of preservation technology investment and trade credit policy. Finally, with the help of MATLAB 2017a software, it is shown graphically

that the profit function is concave, and sensitivity analysis is done using Lingo17. The study provides insight for managers to make smart decisions especially in the scope of applying preservation technology and credit financing. The main limitation in the study is the assumption of fixed trade-credit period. For future research, this paper can be extended to consider some features such as probabilistic demand rate, variable trade-credit period, variable deterioration rate, quantity discounts, and multiple products.

APPENDIX A.

From equation (3.9):

$$R = -I_3(T) = \frac{(a-bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \}$$

which implies that

$$t_2 = T + \frac{1}{\delta} \left(1 - e^{D \frac{R}{\delta}} \right) \quad (\text{A.1})$$

which can be written in the form

$$t_2 = T + f(D).$$

Now putting the value of t_2 in equation (3.23) then it reduces to

$$\pi^3(p, \xi, T) = \frac{1}{T} \left[\begin{array}{l} p \left[(a-bp)(T + f(D)) + \frac{(a-bp)}{\delta} \{ \log(1 + \delta(T - T - f(D))) \} \right] - A \\ - c \left[(a-bp)t_1 + \frac{(a-bp)}{\theta m(\xi)} (e^{\theta m(\xi)(T+f(D)-t_1)} - 1) + \frac{(a-bp)}{\delta} \{ \log(1 + \delta(T - T - f(D))) \} \right] \\ - c_1 \left[St_1 - \frac{(a-bp)t_1^2}{2} - \frac{(a-bp)}{\theta^2 m(\xi)^2} (1 - e^{\theta m(\xi)(T+f(D)-t_1)} + \theta m(\xi)(T + f(D) - t_1)) \right] \\ - c_2 \left[(R + \frac{D}{\delta}(T - T - f(D))) - \frac{D}{\delta^2} (1 + \delta(T - T - f(D))) \log(1 + \delta(T - T - f(D))) \right] \\ - \frac{c_3(a-bp)}{\delta} \{ \delta(T - T - f(D)) - \log(1 - \delta(T - T - f(D))) \} + \frac{pI_e(a-bp)(T+f(D))^2}{2} \\ + \frac{pI_e(a-bp)(T+f(D))}{\delta} \log(1 + \delta(T - T - f(D))) \end{array} \right]$$

$$\begin{aligned} &= \frac{1}{T} \left[\begin{array}{l} p \left[(a-bp)(T + f(D)) + \frac{(a-bp)}{\delta} \{ \log(1 - \delta f(D)) \} \right] - A \\ - c \left[(a-bp)t_1 + \frac{(a-bp)}{\theta m(\xi)} (e^{\theta m(\xi)(T+f(D)-t_1)} - 1) + \frac{(a-bp)}{\delta} \{ \log(1 - \delta f(D)) \} \right] \\ - c_1 \left[St_1 - \frac{(a-bp)t_1^2}{2} - \frac{(a-bp)}{\theta^2 m(\xi)^2} (1 - e^{\theta m(\xi)(T+f(D)-t_1)} + \theta m(\xi)(T + f(D) - t_1)) \right] \\ - c_2 \left[(R - \frac{D}{\delta}(f(D))) - \frac{D}{\delta^2} (1 - \delta(f(D))) \{ \log(1 - \delta f(D)) \} \right] \\ - \frac{c_3(a-bp)}{\delta} \{ -\delta(f(D)) - \{ \log(1 - \delta f(D)) \} \} + \frac{pI_e(a-bp)(T+f(D))^2}{2} \\ + \frac{pI_e(a-bp)(T+f(D))}{\delta} \{ \log(1 - \delta f(D)) \} \end{array} \right] \\ &= \frac{1}{T} G(p, \xi, T), \quad \text{since } D = a - bp \end{aligned} \quad (\text{A.2})$$

Differentiate X with respect to ξ we have

$$\begin{aligned}
 \text{SR} &= p \left[(a - bp)t_2 + \frac{(a - bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right], \frac{\partial \text{SR}}{\partial \xi} = \frac{\partial^2 \text{SR}}{\partial \xi^2} = 0 \\
 \text{OC} &= A, \frac{\partial \text{OC}}{\partial \xi} = 0 \\
 \text{PC} &= c \left[(a - bp)t_1 + \frac{(a - bp)}{\theta m(\xi)} \left\{ e^{\theta m(\xi)(t_2 - t_1)} - 1 \right\} \right] \\
 \frac{\partial \text{PC}}{\partial \xi} &= c(a - bp) \left[-e^{\theta m(\xi)(t_2 - t_1)} \cdot \frac{1}{\theta m(\xi)^2} \cdot \frac{dm(\xi)}{d\xi} + \frac{1}{\theta m(\xi)} \cdot e^{\theta m(\xi)(t_2 - t_1)} \theta(t_2 - t_1) \cdot \frac{dm(\xi)}{d\xi} \right] \\
 &= c(a - bp)a_1 \left[e^{\theta m(\xi)(t_2 - t_1)} \cdot \frac{1}{\theta m(\xi)} - e^{\theta m(\xi)(t_2 - t_1)}(t_2 - t_1) \right] \\
 \frac{\partial^2 \text{PC}}{\partial \xi^2} &= c(a - bp)a_1^2 e^{\theta m(\xi)(t_2 - t_1)}(t_2 - t_1) [1 - \theta(t_2 - t_1)e^{-a_1 \xi}] - c(a - bp)a_1 \frac{e^{\theta m(\xi)(t_2 - t_1)}}{\theta m(\xi)^2} \quad (\text{A.3}) \\
 \text{BC} &= c_2 \left(R + \frac{D}{\delta} (T - t_2) \right) - \frac{D}{\delta^2} (1 + \delta(T - t_2)) \log(1 + \delta(T - t_2)) \\
 \frac{\partial \text{BC}}{\partial \xi} &= \frac{\partial^2 \text{BC}}{\partial \xi^2} = 0 \\
 \text{LSC} &= \frac{c_3 (a - bp)}{\delta} \{ \delta(T - t_2) - \log(1 + \delta(T - t_2)) \} \Rightarrow \frac{\partial \text{LSC}}{\partial \xi} = \frac{\partial^2 \text{LSC}}{\partial \xi^2} = 0 \\
 \text{IE}_3 &= \frac{p I_e (a - bp) t_2^2}{2} + \frac{p I_e (a - bp) t_2}{\delta} \log(1 + \delta(T - t_2)), \frac{\partial \text{IE}_3}{\partial \xi} = \frac{\partial^2 \text{IE}_3}{\partial \xi^2} = 0 \\
 \text{PTC} &= \xi T \Rightarrow \frac{\partial \text{PTC}}{\partial \xi} = T \text{ and } \frac{\partial^2 \text{PTC}}{\partial \xi^2} = 0.
 \end{aligned}$$

Differentiating X with respect to “ p ” shows

$$\begin{aligned}
 \text{SR} &= p \left[(a - bp)t_2 + \frac{(a - bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right], \\
 \frac{\partial^2 \text{SR}}{\partial p^2} &= \left[-2bt_2 - \frac{2b}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] \quad (\text{A.4})
 \end{aligned}$$

$$\begin{aligned}
 \text{OC} &= A, \frac{\partial \text{OC}}{\partial p} = \frac{\partial^2 \text{OC}}{\partial p^2} = 0 \\
 \text{PC} &= cQ = c \left[(a - bp)t_1 + \frac{(a - bp)}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2 - t_1)} - 1 \right) + \frac{(a - bp)}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] \quad (\text{A.5})
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial \text{PC}}{\partial p} &= -bc \left[t_1 + \frac{1}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2 - t_1)} - 1 \right) + \frac{1}{\delta} \{ \log(1 + \delta(T - t_2)) \} \right] \Rightarrow \frac{\partial^2 \text{PC}}{\partial p^2} = 0 \\
 \text{IHC} &= c_1 \left[st_1 - \frac{(a - bp)t_1^2}{2} - \frac{(a - bp)}{\theta^2 m(\xi)^2} \left\{ 1 - e^{\theta m(\xi)(t_2 - t_1)} + \theta m(\xi)(t_2 - t_1) \right\} \right] \\
 \frac{\partial \text{IHC}}{\partial p} &= c_1 \left[\frac{1}{2}bt_1^2 + \frac{b}{\theta^2 m(\xi)^2} \left\{ 1 - e^{\theta m(\xi)(t_2 - t_1)} + \theta m(\xi)(t_2 - t_1) \right\} \right] \Rightarrow \frac{\partial^2 \text{IHC}}{\partial p^2} = 0 \\
 \text{BC} &= c_2 \left[\left(R + \frac{a - bp}{\delta} (T - t_2) \right) - \frac{a - bp}{\delta^2} (1 + \delta(T - t_2)) \log(1 + \delta(T - t_2)) \right] \\
 \frac{\partial \text{BC}}{\partial p} &= c_2 \left[\left(\frac{b}{\delta} (t_2 - T) \right) + \frac{b}{\delta^2} (1 + \delta(T - t_2)) \log(1 + \delta(T - t_2)) \right] \Rightarrow \frac{\partial^2 \text{BC}}{\partial p^2} = 0 \quad (\text{A.6})
 \end{aligned}$$

$$\begin{aligned}
\text{LSC} &= \frac{c_3(a-bp)}{\delta} \{ \delta(T-t_2) - \log(1+\delta(T-t_2)) \} \\
\frac{\partial \text{LSC}}{\partial p} &= -\frac{c_3b}{\delta} \{ \delta(T-t_2) - \log(1+\delta(T-t_2)) \} \Rightarrow \frac{\partial^2 \text{LSC}}{\partial p^2} = 0 \\
\text{IE}_3 &= \frac{pI_e(a-bp)t_2^2}{2} + \frac{pI_e(a-bp)t_2}{\delta} \log(1+\delta(T-t_2)) \\
\frac{\partial E_3}{\partial p} &= \frac{I_e(a-2bp)t_2^2}{2} + \frac{I_e(a-2bp)t_2}{\delta} \log(1+\delta(T-t_2)) \\
\frac{\partial^2 E_3}{\partial p^2} &= -\frac{I_ebt_2^2}{2} - \frac{I_ebt_2}{\delta} \log(1+\delta(T-t_2)). \tag{A.7}
\end{aligned}$$

Differentiating X with respect to “ T ” shows

$$\begin{aligned}
\text{SR} &= p \left[(a-bp)t_2 + \frac{(a-bp)}{\delta} \{ \log(1+\delta(T-t_2)) \} \right] \\
\frac{\partial \text{SR}}{\partial T} &= \frac{p(a-bp)}{(1+\delta(T-t_2))}; \text{OC} = A, \frac{\partial \text{OC}}{\partial T} = 0 \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
\text{PC} &= c \left[(a-bp)t_1 + \frac{(a-bp)}{\theta m(\xi)} \left\{ e^{\theta m(\xi)(t_2-t_1)} - 1 \right\} \right] \\
\frac{\partial \text{PC}}{\partial T} &= \frac{c(a-bp)}{(1+\delta(T-t_2))} \tag{A.9}
\end{aligned}$$

$$\begin{aligned}
\text{IHC} &= c_1 \left[st_1 - \frac{(a-bp)t_1^2}{2} - \frac{(a-bp)}{\theta^2 m(\xi)^2} \left\{ 1 - e^{\theta m(\xi)(t_2-t_1)} + \theta m(\xi)(t_2-t_1) \right\} \right], \frac{\partial \text{IHC}}{\partial T} = 0 \\
\text{BC} &= \frac{c_2(a-bp)}{\delta} (1+\delta(T-t_2)(T-t_2) + c_2 \frac{a-bp}{\delta^2} [(1+\delta(T-t_2) \{ \log(1+\delta(T-t_2)) - 1 \} + 1] \\
\frac{\partial \text{BC}}{\partial T} &= \frac{c_2(a-bp)(1+2\delta(T-t_2))}{\delta} - c_2 \frac{a-bp}{\delta} [1 + \{ \log(1+\delta(T-t_2)) - 1 \}] \\
\text{LSC} &= \frac{c_3(a-bp)}{\delta} \{ \delta(T-t_2) - \log(1+\delta(T-t_2)) \} \\
\frac{\partial \text{LSC}}{\partial T} &= c_3(a-bp) \left[1 - \frac{1}{(1+\delta(T-t_2))} \right] \\
\text{IE}_3 &= \frac{pI_e(a-bp)t_2^2}{2} + \frac{pI_e(a-bp)t_2}{\delta} \log(1+\delta(T-t_2)) \Rightarrow \frac{\partial E_3}{\partial T} = \frac{pI_e(a-bp)t_2^2}{(1+\delta(T-t_2))} \\
\text{PTC} &= \xi T, \Rightarrow \frac{\partial \text{PTC}}{\partial T} = \xi. \tag{A.10}
\end{aligned}$$

APPENDIX B.

Proof of Corollary 4.1

The objective function is

$$\pi^3(p, \xi, T) = \frac{X}{T}. \tag{B.1}$$

Differentiating the equation (B.1) with respect to T ,

$$\frac{\partial \pi^3(p, \xi, T)}{\partial T} = \frac{T \frac{\partial X}{\partial T} - X}{T^2}. \quad (\text{B.2})$$

Differentiating the equation (B.2) with respect to T ,

$$\begin{aligned} \frac{\partial^2 \pi^3(p, \xi, T)}{\partial T^2} &= \frac{T^2 \left\{ \frac{\partial X}{\partial T} + T \frac{\partial^2 X}{\partial T^2} - \frac{\partial X}{\partial T} \right\} - 2T \left\{ T \frac{\partial X}{\partial T} - X \right\}}{T^3} \\ &= \frac{T^2 \frac{\partial^2 X}{\partial T^2} - 2T \frac{\partial X}{\partial T} + 2X}{T^3}. \end{aligned} \quad (\text{B.3})$$

From equation (B.3) we observed that

$$\frac{\partial^2 \pi^3(p, \xi, T)}{\partial T^2} < 0,$$

which means

$$\begin{aligned} \frac{T^2 \frac{\partial^2 X}{\partial T^2} - 2T \frac{\partial X}{\partial T} + 2X}{T^3} &< 0, \Rightarrow \frac{1}{T} \frac{\partial^2 X}{\partial T^2} - \frac{2}{T^2} \frac{\partial X}{\partial T} + \frac{2X}{T^3} < 0 \\ &\Rightarrow \frac{1}{T} \frac{\partial^2 X}{\partial T^2} + \frac{2X}{T^3} < \frac{2}{T^2} \frac{\partial X}{\partial T}. \end{aligned} \quad (\text{B.4})$$

Proof of Corollary 4.2

For any fixed ξ and T , the second order derivative of the objective function $\pi^3(p, \xi, T)$ with respect to the decision variables p is

$$\frac{\partial^2 \pi^3(p, \xi, T)}{\partial p^2} = -\frac{1}{T} \left[2bt_2 + \frac{2b}{\delta} \{ \log(1 + \delta(T - t_2)) \} - \frac{I_e b t_2^2}{2} + \frac{I_e b t_2}{\delta} \log(1 + \delta(T - t_2)) \right] < 0 \quad (\text{B.5})$$

which implies that, $\left[2bt_2 + \frac{2b}{\delta} \{ \log(1 + \delta(T - t_2)) \} - \frac{I_e b t_2^2}{2} + \frac{I_e b t_2}{\delta} \log(1 + \delta(T - t_2)) \right] > 0$, since T is the total cycle length

$$\begin{aligned} &\Rightarrow 4t_2 + \left(\frac{4}{\delta} + \frac{2I_e t_2}{\delta} \right) \log(1 + \delta(T - t_2)) > I_e t_2^2 \\ &\Rightarrow \left(\frac{4}{\delta} + \frac{2I_e t_2}{\delta} \right) \log(1 + \delta(T - t_2)) > I_e t_2^2 - 4t_2 \\ &\Rightarrow \log(1 + \delta(T - t_2)) > \frac{I_e \delta(t_2 - 4)t_2}{2I_e t_2 + 4} W. \end{aligned} \quad (\text{B.6})$$

Now the mixed derivatives with respect to the decision variables

$$\begin{aligned} \frac{\partial^2 \pi^3(p, \xi, T)}{\partial \xi \partial p} &= \frac{1}{T} \left[a_1 bc \left[\frac{e^{\theta m(\xi)(t_2 - t_1)}}{\theta m(\xi)} + e^{\theta m(\xi)(t_2 - t_1)}(t_2 - t_1) \right] \right. \\ &\quad \left. - c_1 a_1 \left[\frac{2b}{\theta^2} \frac{1}{m(\xi)^2} + \frac{1}{\theta} e^{\theta m(\xi)(t_2 - t_1)} - \frac{b}{\theta m(\xi)}(t_2 - t_1) \right] \right] \\ \frac{\partial^2 \pi^3(p, \xi, T)}{\partial \xi \partial T} &= -\frac{1}{T^2} \left[\begin{aligned} &-ca_1(a - bp) \left[\frac{e^{\theta m(\xi)(t_2 - t_1)}}{\theta m(\xi)} - e^{\theta m(\xi)(t_2 - t_1)}(t_2 - t_1) \right] \\ &-c_1 a_1 \left[-\frac{2(a-bp)}{\theta^2} \frac{1}{m(\xi)^2} - \frac{(a-bp)}{\theta} e^{\theta m(\xi)(t_2 - t_1)} + e^{\theta m(\xi)(t_2 - t_1)} \frac{2}{\theta^2} \frac{1}{m(\xi)^2} \right. \\ &\quad \left. + \frac{(a-bp)}{\theta m(\xi)}(t_2 - t_1) \right] - 1 \end{aligned} \right] \end{aligned}$$

$$\frac{\partial^2 \pi^3(p, \xi, T)}{\partial p \partial \xi} = \frac{1}{T} \left[\begin{array}{l} a_1 b c \left[\frac{1}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2 - t_1)} - 1 \right) - e^{\theta m(\xi)(t_2 - t_1)} (t_2 - t_1) \right] \\ - c_1 b a_1 \left[\frac{2}{\theta^2 m(\xi)^2} \left\{ 1 - e^{\theta m(\xi)(t_2 - t_1)} + \theta m(\xi)(t_2 - t_1) \right\} + \frac{(t_2 - t_1)}{\theta m(\xi)} \left\{ e^{\theta m(\xi)(t_2 - t_1)} - 1 \right\} \right] \\ \left[(a - 2bp)t_2 + \frac{(a-2bp)}{\delta} \left\{ \log(1 + \delta(T - t_2)) \right\} + bc \left[\begin{array}{l} t_1 + \frac{1}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2 - t_1)} - 1 \right) \\ + \frac{1}{\delta} \left\{ \log(1 + \delta(T - t_2)) \right\} \end{array} \right] \right] \\ - c_1 \left[\frac{1}{2} b t_1^2 + \frac{b}{\theta^2 m(\xi)^2} \left\{ 1 - e^{\theta m(\xi)(t_2 - t_1)} + \theta m(\xi)(t_2 - t_1) \right\} \right] \\ - c_2 \left[\left(\frac{b}{\delta} (t_2 - T) \right) + \frac{b}{\delta^2} (1 + \delta(T - t_2)) \log(1 + \delta(T - t_2)) \right] \\ + \frac{c_3 b}{\delta} \left\{ \delta(T - t_2) - \log(1 + \delta(T - t_2)) \right\} - \frac{I_e(a-2bp)t_2^2}{2} \\ + \frac{I_e(a-2bp)t_2}{\delta} \log(1 + \delta(T - t_2)) \end{array} \right] \quad (B.7)$$

$$\begin{aligned} \frac{\partial^2 \pi^3(p, \xi, T)}{\partial p \partial T} = & -\frac{1}{T^2} \left[\begin{array}{l} \frac{(a-2bp)}{1+\delta(T-t_2)} + bc \left[\frac{1}{1+\delta(T-t_2)} \right] - c_2 \left[\left(-\frac{b}{\delta} \right) + \frac{b}{\delta} \log(1 + \delta(T - t_2)) + \delta \right] \\ + c_3 b \left\{ 1 - \frac{1}{1+\delta(T-t_2)} \right\} + \frac{I_e(a-2bp)t_2}{1+\delta(T-t_2)} \\ \left[(a - 2bp)t_2 + \frac{(a-2bp)}{\delta} \left\{ \log(1 + \delta(T - t_2)) \right\} \right. \\ \left. + bc \left[t_1 + \frac{1}{\theta m(\xi)} \left(e^{\theta m(\xi)(t_2 - t_1)} - 1 \right) + \frac{1}{\delta} \left\{ \log(1 + \delta(T - t_2)) \right\} \right] \right] \\ - c_1 \left[\frac{1}{2} b t_1^2 + \frac{b}{\theta^2 m(\xi)^2} \left\{ 1 - e^{\theta m(\xi)(t_2 - t_1)} + \theta m(\xi)(t_2 - t_1) \right\} \right] \\ - c_2 \left[\left(\frac{b}{\delta} (t_2 - T) \right) + \frac{b}{\delta^2} (1 + \delta(T - t_2)) \log(1 + \delta(T - t_2)) \right] \\ + \frac{c_3 b}{\delta} \left\{ \delta(T - t_2) - \log(1 + \delta(T - t_2)) \right\} - \frac{I_e(a-2bp)t_2^2}{2} \\ + \frac{I_e(a-2bp)t_2}{\delta} \log(1 + \delta(T - t_2)) \end{array} \right] \\ + & \frac{1}{T} \left[\begin{array}{l} \frac{a+bc-2bp}{1+\delta(T-t_2)} - \frac{c_2 b}{\delta} \log(1 + \delta(T - t_2)) \\ - c_3 b \left[1 - \frac{1}{1+\delta(T-t_2)} \right] - \frac{I_e(a-2bp)t_2}{1+\delta(T-t_2)} \\ \left[-ca_1(a - bp) \left[\frac{e^{\theta m(\xi)(t_2 - t_1)}}{\theta m(\xi)} - e^{\theta m(\xi)(t_2 - t_1)} (t_2 - t_1) \right] \right. \\ \left. - c_1 a_1 \left[\begin{array}{l} -\frac{2(a-bp)}{\theta^2} \frac{1}{m(\xi)^2} - \frac{(a-bp)}{\theta} e^{\theta m(\xi)(t_2 - t_1)} + e^{\theta m(\xi)(t_2 - t_1)} \frac{2}{\theta^2} \frac{1}{m(\xi)^2} \\ + \frac{(a-bp)}{\theta m(\xi)} (t_2 - t_1) \end{array} \right] \right] \end{array} \right] \quad (B.8) \end{aligned}$$

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