

OPTIMAL ORDERING AND DISPOSAL DECISIONS FOR PRODUCTS WITH A FIXED SHELF LIFE

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Abstract. In this paper, motivated by the current increasing interest and action on food waste reduction, inventory decisions of a retailer who deals with a product that has a fixed shelf life are studied. Being a common strategy of many retail stores, we assume that at a specific time instant, close to the expiration date, a price markdown is offered in order to increase demand. However, at the same time, due to customers' attention to the freshness of the product, the demand becomes a decreasing function with respect to the time remaining before the expiration date. In accordance with the European Union food donation guidelines, we assume that if at the end of the reorder interval unsold items remain that have not exceeded their expiration date, they can be donated to non-profit organizations for human consumption. The donated products can generate direct revenue from tax deductions and indirect revenue by increasing the company's reputation and gain of goodwill from the customers. If the unsold items have expired, they can be sold at a salvage price to the livestock market. The aim of our model is to determine the reorder interval, the time instant to markdown the product's initial selling price and the quantity that will be donated or sold to the livestock market so that the profit of the system is maximized. Closed form solutions are obtained, which depend on specific parametric conditions, providing managerial insights.

Mathematics Subject Classification. 90B05.

Received October 28, 2018. Accepted September 1, 2019.

1. INTRODUCTION

According to the European Commission, around 88 million tonnes of food are wasted annually in the EU, with associated costs estimated at 143 billion Euro. Specifically, about a third of all food produced globally for human consumption is lost or wasted, which is estimated at around 1.3 billion tonnes per year, according to the Food and Agriculture Organisation. In industrialized countries, over 40% of this food loss occurs at retail and consumer level. On these grounds, the EU and Member States are committed to meeting the Sustainable Development Goals (SDG) target to halve per capita food waste at the retail and consumer level by 2030 and reduce food losses along the food production and supply chains. In addition, if limiting the generation of surplus

Keywords. Inventory, EOQ, expiration date, food donation.

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food at each stage in the food supply chain cannot be achieved the EC advises to redistribute the surplus food for human consumption where safe to do so. Specifically, as part of the Circular Economy Action Plan, the Commission has adopted EU food donation guidelines in order to facilitate the recovery and redistribution of safe, edible food to those in need.

In this context, the objective of this paper is to study the inventory decisions of a retailer who deals with a product that has a fixed shelf life (expiration date). Our goal is to maximize the retailer's profit and prevent product waste. To do so we assume that the retailer offers a price markdown when the product is close to the expiration date in order to increase demand. However, we assume that this also makes the customer aware of the time remaining until the product expires. Hence, the demand becomes a decreasing function with respect to the product's freshness. In addition, if at the end of the replenishment cycle unsold items are leftover that have not exceeded their expiration date, they can be donated to non-profit organizations for human consumption. Fiscal incentives that encourage food donation include tax deductions and tax credits in support of redistribution schemes. According to the EU Food Donation Guidelines: In France 60% and in Spain 35% of the net book value of donated food can be claimed as a corporate tax credit, meaning that food donors are able to deduct that percentage of the value of the donated food from the corporate tax on their revenue. The comparative study carried out by the European Economic and Social Committee (EESC) also shows that in most of the other examined Member States, food donation can be treated as a deductible tax expense and can reduce the taxable income (within certain limits and thresholds depending on the Member States). The EESC specifies that Portugal has in place an enhanced tax deduction, meaning that donors can deduct up to 140% of the value of the food at time of donation, provided that the food will be used for a social purpose. Furthermore, it is logical to assume, that by donating food to those in need a retailer can generate indirect revenue by increasing the company's reputation. This indirect profit can be viewed as a gain of goodwill from the customers. Of course, in order to donate the surplus food, it is required that it has not reached its expiration date. If the unsold items have however expired, instead of wasting them, we assume that they can be sold at a salvage price to the livestock market. In the next section, we present the existing literature that is most relevant to our work.

2. LITERATURE REVIEW

As already mentioned, we assume that we are dealing with a product that has a fixed shelf life. The demand for such products is usually assumed to be affected by their freshness level, which is equivalent to the time remaining until the products reach their expiration date. In this context, Avinadav and Arponen [3] developed an EOQ model for which they proposed a product demand rate that is a linearly decreasing function of time and calculated the optimal cycle length by using numerical approximation method. Using the same demand pattern Demirag *et al.* [8] proposed a closed form solution. They also proposed a second demand pattern that is a polynomial function of time to the expiration date and they used an adaptive heuristic policy to solve the problem. Avinadav *et al.* [4, 5] extended their previous work by assuming that the demand of the product is not only depended on the time to the expiration date but also on the price of the product. First, they assumed a demand pattern that is a linearly decreasing function of price and polynomial function of time to the expiration date, and then a demand function that can be separated into components of price and inventory age either in multiplicative, or additive form. For these models, they maximized the profit function by determining the optimal price and replenishment period. A multiplicative demand pattern of price and time is also used in Chen [7], in order to determine the sale price at each time point. Feng *et al.* [9], proposed an inventory model that stipulates the demand explicitly in a multivariate function of price, freshness and displayed stocks. Due to the stock dependent demand, they assumed a non-negative ending inventory which is sold at a salvage price at the end of the replenishment cycle. Pal *et al.* [14] studied a production inventory system for items with a maximum life-cycle where the production rate of the system is a random variable and the unit production cost depends on production lot size as well as the rate of production. Shortages are allowed and partially backlogged. Recently, Li and Teng [11] developed a joint pricing and lot-sizing

model for retailers selling perishable products, in which the demand depends on not only the selling price and reference price but also the product freshness linked to expiration date and displayed stock level. They discussed the retailer's optimal decisions under two scenarios of demand behaviour: loss neutrality and loss aversion.

A common strategy of many retail stores is to offer a one time price markdown when a product is close to its expiration date, in order to increase demand. In accordance with that, Tsiros and Heilman [17] in their study of the effect of expiration dates on the purchasing behaviour for grocery store perishables, propose that an effective mean of selling ageing inventory is to offer a price discount near the expiration date of the product. They also notice that the effectiveness of price promotions is greater for perishable goods than for other categories. Donselaar *et al.* [18] analysed the impact of relative price discounts on product sales during a promotion and shed light on how to build models to forecast promotional demand for perishable products. Banerjee and Agrawal [6] developed and analysed an inventory model when demand for a deteriorating item depends initially only upon its selling price and later also on the freshness condition. The demand function, as well as deterioration distribution, are deemed to be general. Also, price discounts are used to boost the demand when freshness has declined to some extent and deterioration has already begun. Shortages are allowed and assumed to be lost sales. Herbon [10] studied a product whose demand is price and freshness dependent and the sensitivity to freshness varies across consumers. The extent to which unawareness of the heterogeneity in consumers' sensitivity to freshness is likely to detract from a retailer's profit is evaluated, as well as the extent to which it is likely to affect the price that consumers pay. Unexpectedly, it is proven that the retailer should assign products a lower price at the early stages of their shelf life and then raise the price as the products approach expiration. Satiti *et al.* [15] maximized the retailer's profit through determining the refrigerated display-space allocation and markdown policy based on identified food shelf life, in the agri-food supply chain. More recently, Nagare *et al.* [13] studied a single-period ordering and markdown pricing policy for short lifecycle products by considering differing customer price and time sensitivities. Their study reveals the benefits of market segmentation and markdown pricing which recognise high price-sensitive "bargain hunter" customers and offers deeper discounts that yield greater profits.

Regarding food donations, the only models we came across in the literature as far as an EOQ context is considered, are the models by Aiello *et al.* [1,2] who study a supply chain composed by a retailer and potential recipients of food recovery such as a non-profit Organization and the livestock market. They assume that the retailer deals with k products and maximize the joint profit of the supply chain by assuming that the demand of each product is linearly dependent on the time remaining to its expiration date and tax deductions are granted to the retailer due to the donation of surplus food. A review on the state of the art of food waste/losses issue is presented by Muriana [12].

To the best of our knowledge, the present work is the first to study the optimal replenishment policy of a product with a fixed shelf life from the retailer's perspective assuming freshness dependent demand, price markdown, and food donations, in an EOQ framework. Unlike previous relevant studies that consider freshness dependent demand from the beginning of the replenishment cycle, in this model, the demand is considered freshness dependent after the markdown is offered and the customer becomes aware of the expiration date of the product. Furthermore, we have tried to incorporate the retailer's profit generated by the food donations in a more general way, taking into account not only possible tax deductions, but also the gain of goodwill from the customers. Finally, the closed form solutions and parametric conditions derived, provide insight to retailers confronted with an analogous situation, as well as to further research on this field incorporating more complex assumptions such as stochastic demand, multiple products, etc.

3. NOTATIONS AND ASSUMPTIONS

The notations used throughout the paper are listed in the following table:

D	Demand per time unit
h	Inventory holding cost per unit per time unit, including capital cost
c	Purchase cost per unit
A	Ordering cost per order
p	Selling price per unit
β	Discount percentage on selling price ($0 \leq \beta \leq 1$)
p'	Discounted selling price per unit $p' = (1 - \beta)p$
γ_1	Opportunity gain due to the donation of food to non-profit organizations per unit donated
γ_2	Selling price per unit of expired product to livestock market
e	Time at which the product expires (fixed shelf life of the product)
T	Length of the replenishment cycle (a decision variable)
t_1	The time at which the close to expiration date markdown is offered (a decision variable)
Q	Order quantity (a decision variable)
q	The inventory level at time T (a decision variable)
W	Storage capacity
T_{\min}	The minimum reorder interval
T_d	The maximum time before the product's expiration to donate the product

The mathematical model is developed under the following assumptions:

- (1) The retailer deals with a product that has a fixed shelf life (the product remains safe and suitable for human consumption until the reaching of the self life (expiration date)).
- (2) The planning horizon of the inventory problem is infinite.
- (3) Replenishment rate is infinite.
- (4) The retailer's storage capacity is constant and equal to W . This means that the order quantity can not exceed capacity, *i.e.* $Q \leq W$.
- (5) The demand of the product is constant and equal to D . At time t_1 the retailer offers a price markdown because the product is close to the expiration date, this increases the demand to αD , $\alpha > 1$. However, at the same time, this also makes the customer aware of the expiration date of the product and the demand becomes a decreasing function with respect to the time remaining before the expiration date. Hence the demand of the product is defined as:

$$D(t) = \begin{cases} D, & 0 \leq t < t_1 \\ \alpha \left(\frac{e-t}{e}\right) D & t_1 \leq t \leq e \end{cases}.$$

- (6) We assume that by donating surplus food at the end of the replenishment cycle the retailer generates a direct (such as tax deductions) and indirect profit (gain of goodwill) per unit donated, that is quantified as γ_1 (see Aiello *et al.* [1]).
- (7) We assume that $p > p' > c$ and $\gamma_2 < c$
- (8) For practical purposes we assume that there exists a minimum reorder interval, T_{\min} , as well as a maximum time period, T_d , before the product's expiration date, in which the retailer is able to donate the product.
- (9) We assume that $T \leq e$, to avoid shortages.

4. MODEL FORMULATION

The system operates as follows: At time 0 a new order of Q units arrives. During time period $[0, t_1]$ the inventory level depletes due to constant demand D . At time t_1 the retailer offers a price markdown, because the product is close to the expiration date, reducing the price from p to p' . This increases the demand to αD , $\alpha > 1$. However, this also makes the customer aware of the expiration date of the product and the demand becomes a decreasing function with respect to the time remaining before the expiration date. At time $T \leq e$, the retailer withdraws the remaining q units of product from the shelf and a new order arrives. If the products are withdrawn before the expiration date (*i.e.* $T < e$), they are donated to a non-profit organization. On the

other hand, if the products are withdrawn at their expiration date (*i.e.* $T = e$), the withdrawn items are sold to the livestock market at a salvage price (γ_2). The differential equations that describe the depletion of inventory level during the time intervals $[0, t_1]$ and $[t_1, T]$, are as follows:

$$\begin{aligned}\frac{dI(t)}{dt} &= -D, \quad 0 \leq t < t_1, \\ \frac{dI(t)}{dt} &= -\alpha \left(\frac{e-t}{e} \right) D, \quad t_1 \leq t \leq T, \\ I[T] &= q, \quad I[t_1^+] = I[t_1^-].\end{aligned}$$

By solving the above differential equations we obtain:

$$I(t) = q + \alpha D \left(T - t_1 - \frac{T^2}{2e} + \frac{t_1^2}{2e} \right) + D(t_1 - t), \quad 0 \leq t < t_1 \quad (4.1)$$

$$I(t) = q + \alpha D \left(T - t - \frac{T^2}{2e} + \frac{t^2}{2e} \right), \quad t_1 \leq t \leq T. \quad (4.2)$$

From relation (4.1) we obtain the order quantity

$$Q = I[0] = q + \alpha D \left(T - t_1 - \frac{T^2}{2e} + \frac{t_1^2}{2e} \right) + Dt_1. \quad (4.3)$$

The retailer's profit can generally be expressed as:

$$\pi(T, t_1, q) = \begin{cases} \bar{\pi}(T, t_1, q) + \gamma_1 q, & 0 \leq t_1 \leq T < e, \quad q \geq 0 \\ \bar{\pi}(T, t_1, q) + \gamma_2 q, & T = e, \quad 0 \leq t_1 \leq e, \quad q \geq 0 \end{cases},$$

where

$$\bar{\pi}(T, t_1, q) = pDt_1 + p' \int_{t_1}^T \alpha \left(\frac{e-t}{e} \right) Ddt - cQ - A - h \int_0^{t_1} I(t)dt - h \int_{t_1}^T I(t)dt. \quad (4.4)$$

The terms appearing in $\bar{\pi}(T, t_1, q)$ correspond to (in the order they appear) the sales revenue during the time the product has its original price, the sales revenue after the markdown is offered, the purchasing cost, the ordering cost and the holding cost. The term $\gamma_1 q$ corresponds to the profit generated by the food donation while the term $\gamma_2 q$ by selling the leftover products to the livestock market.

Then the retailer's profit per time unit is defined as:

$$\Pi(T, t_1, q) = \frac{1}{T} \pi(T, t_1, q) = \begin{cases} \Pi_1(T, t_1, q), & T < e \\ \Pi_2(t_1, q), & T = e \end{cases} \quad (4.5)$$

where

$$\begin{aligned}\Pi_1(T, t_1, q) &= \frac{1}{T} \left[(p-c)Dt_1 + (p'-c)\alpha D \left(T - t_1 - \frac{T^2}{2e} + \frac{t_1^2}{2e} \right) + (\gamma_1 - c)q \right. \\ &\quad \left. - A - hqT - \frac{hDt_1^2}{2} - h\alpha D \left(\frac{T^2}{2} - \frac{t_1^2}{2} - \frac{T^3}{3e} + \frac{t_1^3}{3e} \right) \right], \quad T < e\end{aligned} \quad (4.6)$$

$$\begin{aligned}\Pi_2(t_1, q) &= \frac{1}{e} \left[(p-c)Dt_1 + (p'-c)\alpha D \left(\frac{e}{2} - t_1 + \frac{t_1^2}{2e} \right) + (\gamma_2 - c)q \right. \\ &\quad \left. - A - hqe - \frac{hDt_1^2}{2} - h\alpha D \left(\frac{e^2}{6} - \frac{t_1^2}{2} + \frac{t_1^3}{3e} \right) \right].\end{aligned} \quad (4.7)$$

The problem to be solved is:

$$\begin{aligned}\max & \Pi(T, t_1, q) \\ \text{s.t.} & \quad 0 \leq t_1 \leq T \leq e \\ & \quad q \geq 0, Q \leq W.\end{aligned}$$

5. OPTIMIZATION PROCEDURE

In this section results that are required for the derivation of the optimal values for the decision variables are presented. The first order derivative of the profit function for the first branch (*i.e.* relation (4.6)), with respect to q is as follows:

$$\frac{\partial \Pi_1}{\partial q} = \frac{\gamma_1 - c}{T} - h. \tag{5.1}$$

From relation (5.1) we observe that:

- If $\gamma_1 - c \leq 0$, then $\frac{\partial \Pi_1}{\partial q} < 0$ for every $T > 0$ and hence it is optimal to make q as small as possible, *i.e.* set $q = 0$. This means that at the end of the replenishment cycle no inventory should be leftover. This is logical, since $\gamma_1 - c \leq 0$ implies that the gain of donating a unit of product is less or equal to its initial purchasing cost, making the donation of products non-profitable for the retailer.
- If $\gamma_1 - c > 0$ and $hT < \gamma_1 - c$, then $\frac{\partial \Pi_1}{\partial q} > 0$ and hence it is optimal to make q as large as possible, which is equivalent to making the order quantity, Q , as large as possible (according to relation (4.3)), *i.e.* setting $Q = W$. This case implies that the gain of donating a unit of product is greater than the holding cost per unit, urging the retailer to increase his order quantity.
- If $\gamma_1 - c > 0$ and $hT > \gamma_1 - c$, then $\frac{\partial \Pi_1}{\partial q} < 0$ and hence, again, it is optimal to make q as small as possible, *i.e.* set $q = 0$. In this case the holding cost per unit is greater than the gain from donating the surplus quantity. Hence, it is not in the retailer's interest to accumulate inventory.

According to the above, concerning the leftover quantity q , the only two cases that should be examined are to set $q = 0$, or $Q = W$, depending on the value of T . By setting $Q = W$ and solving relation (4.3) with respect to q , we obtain q as a function of (T, t_1) :

$$q = W - \alpha D \left(T - t_1 - \frac{T^2}{2e} + \frac{t_1^2}{2e} \right) - Dt_1. \tag{5.2}$$

We notice that since it must always be $T < e$, if $\gamma_1 - c \geq he$, then $Q = W$. Then two cases are examined depending on the sign of $\gamma_1 - c - he$.

Optimization when $\gamma_1 - c < he$

In this case and according to the above, we distinguish two cases in profit function (4.6), depending on the value of T . If $0 < T \leq \frac{\gamma_1 - c}{h}$, we set q as defined in relation (5.2). Otherwise, if $\frac{\gamma_1 - c}{h} < T < e$, we set $q = 0$. In any case, profit function (4.6) becomes a two variable function as follows:

$$\Pi_1(T, t_1) = \begin{cases} \frac{1}{T} \left\{ (p - \gamma_1 + hT)Dt_1 + (p' - \gamma_1 + hT)\alpha D \left(T - t_1 - \frac{T^2}{2e} + \frac{t_1^2}{2e} \right) + (\gamma_1 - c - hT)W - A - \frac{hDt_1^2}{2} - h\alpha D \left(\frac{T^2}{2} - \frac{t_1^2}{2} - \frac{T^3}{3e} + \frac{t_1^3}{3e} \right) \right\}, & 0 < T \leq \frac{\gamma_1 - c}{h} (Q = W) \\ \frac{1}{T} \left\{ (p - c)Dt_1 + (p' - c)\alpha D \left(T - t_1 - \frac{T^2}{2e} + \frac{t_1^2}{2e} \right) - A - \frac{hDt_1^2}{2} - h\alpha D \left(\frac{T^2}{2} - \frac{t_1^2}{2} - \frac{T^3}{3e} + \frac{t_1^3}{3e} \right) \right\}, & \frac{\gamma_1 - c}{h} < T < e (q = 0) \end{cases}. \tag{5.3}$$

Then the second branch of $\Pi(T, t_1, q)$ is studied. The first order derivative of $\Pi_2(t_1, q)$ (*i.e.* relation (4.7)) with respect to q is:

$$\frac{\partial \Pi_2}{\partial q} = \frac{1}{e}[\gamma_2 - c - he]. \tag{5.4}$$

From relation (5.4) we observe that, since we have assumed that $\gamma_2 < c$, $\frac{\partial \Pi_2}{\partial q} < 0$ for every $q > 0$, it is optimal to make q as small as possible, *i.e.* set $q = 0$. This implies that it is never optimal to keep leftover quantity at the end of the replenishment cycle to be sold at a salvage price to the livestock market.

Hence, profit function (4.7) can be written as:

$$\Pi_2(t_1) = \frac{1}{e} \left\{ (p-c)Dt_1 + (p'-c)\alpha D \left(\frac{e}{2} - t_1 + \frac{t_1^2}{2e} \right) - A - \frac{hDt_1^2}{2} - h\alpha D \left(\frac{e^2}{6} - \frac{t_1^2}{2} + \frac{t_1^3}{3e} \right) \right\}. \quad (5.5)$$

Using the above results and particularly the relations (5.3) and (5.5) the profit function, expressed as function of T, t_1 , is:

$$\Pi(T, t_1) = \begin{cases} \frac{1}{T} \left\{ (p - \gamma_1 + hT)Dt_1 + (p' - \gamma_1 + hT)\alpha D \left(T - t_1 - \frac{T^2}{2e} + \frac{t_1^2}{2e} \right) \right. \\ \left. + (\gamma_1 - c - hT)W - A - \frac{hDt_1^2}{2} \right. \\ \left. - h\alpha D \left(\frac{T^2}{2} - \frac{t_1^2}{2} - \frac{T^3}{3e} + \frac{t_1^3}{3e} \right) \right\}, & 0 < T \leq \frac{\gamma_1 - c}{h} \\ \frac{1}{T} \left\{ (p - c)Dt_1 + (p' - c)\alpha D \left(T - t_1 - \frac{T^2}{2e} + \frac{t_1^2}{2e} \right) - A \right. \\ \left. - \frac{hDt_1^2}{2} - h\alpha D \left(\frac{T^2}{2} - \frac{t_1^2}{2} - \frac{T^3}{3e} + \frac{t_1^3}{3e} \right) \right\}, & \frac{\gamma_1 - c}{h} < T \leq e \end{cases}. \quad (5.6)$$

We will now prove that the profit function (5.6) has no maximum interior point of the feasible area.

Theorem 5.1. *When $\gamma_1 - c < he$, no interior point of the feasible area is an optimal solution to the problem:*

$$\begin{aligned} & \max \Pi(T, t_1) \\ & \text{s.t. } 0 \leq t_1 \leq T \leq e. \end{aligned}$$

Proof. The first order derivatives of the first branch of function (5.6) with respect to t_1 and T are:

$$\frac{\partial \Pi}{\partial t_1} = \frac{D}{T} \left[(p - \gamma_1) + (p' - \gamma_1)\alpha \left(\frac{t_1 - e}{e} \right) - h(1 - \alpha)(T - t_1) + \frac{h\alpha}{e}(t_1 T - t_1^2) \right] \quad (5.7)$$

$$\frac{\partial \Pi}{\partial T} = -\frac{1}{T^2}\pi(T, t_1) + \frac{1}{T} \frac{\partial \pi(T, t_1)}{\partial T}. \quad (5.8)$$

By equating relation (5.7) to zero we obtain the optimal value of t_1 :

$$t_1^* = \frac{eh(\alpha - 1) + \alpha(hT + p' - \gamma_1)}{2\alpha h} + \frac{\sqrt{\Delta_1}}{2\alpha h}, \quad (5.9)$$

where $\Delta_1 = (eh(\alpha - 1) + \alpha(hT + p' - \gamma_1))^2 - 4\alpha h e((\alpha(hT + p' - \gamma_1) - (hT + p - \gamma_1)))$ and which satisfies the second order condition $\frac{\partial^2 \Pi}{\partial t_1^2} \Big|_{t_1=t_1^*} < 0$.

We can easily prove that since $p > p'$, $\Delta_1 > ((eh(\alpha - 1) - \alpha(hT + p' - \gamma_1))^2 \geq 0$. This implies that

$$t_1^* > \frac{eh(\alpha - 1) + \alpha(hT + p' - \gamma_1) + |eh(\alpha - 1) - \alpha(hT + p' - \gamma_1)|}{2\alpha h}.$$

Hence, if $eh(\alpha - 1) - \alpha(hT + p' - \gamma_1) > 0$, i.e. $\frac{e(\alpha - 1)}{a} > \frac{p' - \gamma_1}{h} + T$, then $t_1^* > \frac{e(\alpha - 1)}{\alpha}$. If $eh(\alpha - 1) - \alpha(hT + p' - \gamma_1) < 0$, i.e. $\frac{e(\alpha - 1)}{a} < \frac{p' - \gamma_1}{h} + T$, then $t_1^* > \frac{p' - \gamma_1}{h} + T$. In any case $t_1^* > \frac{p' - \gamma_1}{h} + T$.

If T^* is the solution of equation derived by setting relation (5.8) equal to zero, then the second order condition for T is

$$\frac{\partial^2 \Pi}{\partial T^2} \Big|_{T=T^*} = -\frac{\alpha D}{T e} (p' - \gamma_1 - he + hT^*) < 0,$$

which implies that for T^* to be an optimal solution it must be $T^* > e - \frac{p' - \gamma_1}{h}$. This makes only sense if $\frac{p' - \gamma_1}{h} > 0$, since $T^* < e$. Consequently, $t_1^* > T^* + \frac{p' - \gamma_1}{h}$ is invalid since $t_1^* \leq T^*$.

The first order derivatives of the second branch of function (5.6) with respect to t_1 and T are:

$$\frac{\partial \Pi}{\partial t_1} = \frac{D}{T} \left[(p - c) + (p' - c)\alpha \left(\frac{t_1 - e}{e} \right) - ht_1 - h\alpha \left(\frac{t_1^2}{e} - t_1 \right) \right] \tag{5.10}$$

$$\frac{\partial \Pi}{\partial T} = -\frac{1}{T^2}\pi(T, t_1) + \frac{1}{T} \frac{\partial \pi(T, t_1)}{\partial T}. \tag{5.11}$$

By equating relation (5.10) to zero we obtain the optimal value of t_1 :

$$t_1^* = \frac{\alpha(p' - c) + eh(\alpha - 1) + \sqrt{\Delta_2}}{2\alpha h}, \tag{5.12}$$

where $\Delta_2 = (eh(\alpha - 1) + \alpha(p' - c))^2 - 4\alpha h e((p' - c)\alpha - (p - c))$ and which satisfies the second order condition $\frac{\partial^2 \Pi}{\partial t_1^2} |_{t_1=t_1^*} < 0$.

We can easily prove that since $p > p'$, $\Delta_2 > (eh(\alpha - 1) - \alpha(p' - c))^2 \geq 0$.

This implies that $t_1^* > \frac{\alpha(p' - c) + eh(\alpha - 1) + |eh(\alpha - 1) - \alpha(p' - c)|}{2\alpha h}$. Hence, if $eh(\alpha - 1) - \alpha(p' - c) > 0$, i.e. $\frac{p' - c}{h} < \frac{e(\alpha - 1)}{a}$, then $t_1^* > \frac{e(\alpha - 1)}{a}$. If $eh(\alpha - 1) - \alpha(p' - c) < 0$, i.e. $\frac{p' - c}{h} > \frac{e(\alpha - 1)}{a}$, then $t_1^* > \frac{p' - c}{h}$. Hence, we conclude that in any case: $t_1^* > \frac{p' - c}{h}$.

Then by solving relation (5.11) equal to zero we obtain the first order condition for T :

$$-\frac{1}{T^2}\pi(T, t_1) + \frac{aD}{eT} [(p' - c - hT)(e - T)] = 0.$$

The second order condition for T is:

$$\frac{\partial^2 \Pi}{\partial T^2} |_{T=T^*} = -\frac{\alpha D}{eT^*} (p' - c - hT^*) - \frac{h\alpha D}{eT^*} (e - T^*) < 0.$$

Obviously, for the above equation to have an optimal feasible solution, it must be $T^* \leq \frac{p' - c}{h}$.

Consequently, $t_1^* > \frac{p' - c}{h}$, is invalid, since $t_1^* \leq T^*$. We conclude that since the optimal solution for both cases is $t_1^* > T^*$, which is not an interior point of the feasible area, we must set $T = t_1$. \square

Since we have proven that the optimal solution to the problem lies on the line $T = t_1$, we set $T = t_1$ into relation (5.6) and the profit function becomes a continuous piecewise function of T :

$$\Pi(T) = \begin{cases} (p - \gamma_1)D + \frac{(\gamma_1 - c)W - A}{T} - hW + \frac{hDT}{2}, & 0 < T \leq \frac{\gamma_1 - c}{h} & (5.13a) \\ (p - c)D - \frac{A}{T} - \frac{hDT}{2}, & \frac{\gamma_1 - c}{h} < T \leq e. & (5.13b) \end{cases}$$

The capacity constraint in this case is $q + DT \leq W$. We distinguish two cases depending on the value of $(\gamma_1 - c)W - A$ (this will be clearly seen later).

If $(\gamma_1 - c)W - A \leq 0$, for the first branch of the profit function (5.13a), we obtain:

$$\frac{\partial \Pi(T)}{\partial T} = -\frac{(\gamma_1 - c)W - A}{T^2} + \frac{hD}{2} > 0.$$

For the second branch (5.13b), we obtain:

$$\frac{\partial \Pi(T)}{\partial T} = \frac{A}{T^2} - \frac{hD}{2}$$

and

$$\frac{\partial^2 \Pi(T)}{\partial T^2} = -\frac{2A}{T^3} < 0.$$

Hence, the optimal solution for this case is the EOQ formula $T^* = t_1^* = \sqrt{\frac{2A}{hD}}$, $q^* = 0$, if $\frac{\gamma_1 - c}{h} \leq \sqrt{\frac{2A}{hD}} \leq \min\{e, \frac{W}{D}\}$. Since, the capacity constraint for $q = 0$, becomes $T \leq \frac{W}{D}$. If $\sqrt{\frac{2A}{hD}} < \frac{\gamma_1 - c}{h}$, we must set $T^* = t_1^* = \frac{\gamma_1 - c}{h}$, $q^* = 0$ and if $\sqrt{\frac{2A}{hD}} > \min\{e, \frac{W}{D}\}$, we must set $T^* = t_1^* = \min\{e, \frac{W}{D}\}$, $q^* = 0$.

On the other hand, if $(\gamma_1 - c)W - A > 0$, we observe that when $T \rightarrow 0$, then $\Pi(T) \rightarrow \infty$. Hence, in this case, *i.e.* when the profit that could be accumulated by donating as much product as the storage capacity is greater than the ordering cost, it is optimal to make the cycle length as small as possible. However we have assumed, in order to be realistic, that there exists a minimum reorder interval T_{\min} . So, in order to find the optimal solution for this case, we must compare the profit derived by setting $T^* = t_1^* = T_{\min}$ and $T^* = t_1^* = \sqrt{\frac{2A}{hD}}$ in the first branch (5.13a) and the second branch (5.13b), respectively.

Remarks

- (1) Notice that due to the continuity of relation (5.6), when $T = \frac{\gamma_1 - c}{h}$ the profit remains the same regardless if we set $q = W - DT$ or $q = 0$. This can be explained as follows: $hT = \gamma_1 - c$ implies that the profit of donating a unit of product is equal to the holding cost per unit, this means that the cost of keeping an extra unit at the end of the cycle is compensated by the gain earned by donating this unit, so it is irrelevant how much quantity is leftover at the end of the cycle.
- (2) If $\gamma_1 - c \leq 0$, only the second branch of relation (5.6) is examined.

Optimization when $\gamma_1 - c \geq he$

In this case obviously $\gamma_1 - c > hT$ and we must set $Q = W$. Hence, profit function (4.5) becomes

$$\Pi(T, t_1) = \begin{cases} \frac{1}{T} \left\{ (p - \gamma_1 + hT)Dt_1 + (p' - \gamma_1 + hT)\alpha D \left(T - t_1 - \frac{T^2}{2e} + \frac{t_1^2}{2e} \right) \right. \\ \left. + (\gamma_1 - c - hT)W - A \right. \\ \left. - \frac{hDt_1^2}{2} - h\alpha D \left(\frac{T^2}{2} - \frac{t_1^2}{2} - \frac{T^3}{3e} + \frac{t_1^3}{3e} \right) \right\}, & 0 < T < e \\ \frac{1}{e} \left\{ (p - c)Dt_1 + (p' - c)\alpha D \left(\frac{e}{2} - t_1 + \frac{t_1^2}{2e} \right) - A - \frac{hDt_1^2}{2} \right. \\ \left. - h\alpha D \left(\frac{e^2}{6} - \frac{t_1^2}{2} + \frac{t_1^3}{3e} \right) \right\}, & T = e. \end{cases} \quad (5.14)$$

For the first branch of relation (5.14), we have already proven that we must set $T = t_1$. So we obtain

$$\Pi(T, t_1) = \begin{cases} (p - \gamma_1)D + \frac{(\gamma_1 - c)W - A}{T} - hW + \frac{hDT}{2}, & 0 < T < e \\ \frac{1}{e} \left\{ (p - c)Dt_1 + (p' - c)\alpha D \left(\frac{e}{2} - t_1 + \frac{t_1^2}{2e} \right) - A - \frac{hDt_1^2}{2} \right. \\ \left. - h\alpha D \left(\frac{e^2}{6} - \frac{t_1^2}{2} + \frac{t_1^3}{3e} \right) \right\}, & T = e. \end{cases} \quad (5.15a)$$

$$\Pi(T, t_1) = \begin{cases} (p - \gamma_1)D + \frac{(\gamma_1 - c)W - A}{T} - hW + \frac{hDT}{2}, & 0 < T < e \\ \frac{1}{e} \left\{ (p - c)Dt_1 + (p' - c)\alpha D \left(\frac{e}{2} - t_1 + \frac{t_1^2}{2e} \right) - A - \frac{hDt_1^2}{2} \right. \\ \left. - h\alpha D \left(\frac{e^2}{6} - \frac{t_1^2}{2} + \frac{t_1^3}{3e} \right) \right\}, & T = e. \end{cases} \quad (5.15b)$$

Again, for the first branch (5.15a), if $(\gamma_1 - c)W - A > 0$, the optimal solution is $T^* = t_1^* = T_{\min}$ and $q^* = W - DT^*$. If $(\gamma_1 - c)W - A \leq 0$ we have proven that function (5.15a) is an increasing function with respect to T . Hence, the optimal solution in this case is to set T as close to e as possible ($T \rightarrow e$). In order to be realistic we assume that there is a maximum time period before the expiration of the product, *i.e.* T_d , at which it can be withdrawn from the shelf and donated. So in this case it is optimal to set $T^* = t_1^* = T_d$ and $q^* = W - DT^*$. The optimal value of t_1 for the second branch (5.15b) is obtained by relation (5.12). If $t_1^* < e$

then we must compare the profit obtained by both branches and choose the optimal solution for this case. If $t_1^* > e$, then we must set $t_1^* = e$ and again compare both branches.

In conclusion, we can summarize the above obtained results in the following proposition:

Proposition 5.2. *Depending on the value of $\gamma_1 - c$, we obtain the following optimal solution to the problem:*

- (1) *If $\gamma_1 - c \leq 0$, set $q^* = 0$ and $t_1^* = T^* = \sqrt{\frac{2A}{hD}}$, provided that $\sqrt{\frac{2A}{hD}} < \min\{e, \frac{W}{D}\}$. Else, set $T^* = t_1^* = \min\{e, \frac{W}{D}\}$. Calculate the corresponding profit by using relation (5.13b).*
- (2) *If $0 < \gamma_1 - c < \min\{he, \frac{A}{W}\}$, set $q^* = 0$ and $t_1^* = T^* = \sqrt{\frac{2A}{hD}}$, provided that $\frac{\gamma_1 - c}{h} \leq \sqrt{\frac{2A}{hD}} \leq \min\{e, \frac{W}{D}\}$. If $\sqrt{\frac{2A}{hD}} < \frac{\gamma_1 - c}{h}$, set $T^* = t_1^* = \frac{\gamma_1 - c}{h}$, while if $\sqrt{\frac{2A}{hD}} > \min\{e, \frac{W}{D}\}$, set $T^* = t_1^* = \min\{e, \frac{W}{D}\}$. Calculate the corresponding profit by using relation (5.13b).*
- (3) *If $\frac{A}{W} < \gamma_1 - c < he$, choose the optimal solution by comparing the profit obtained by setting $q = W - DT_{\min}$, $T = t_1 = T_{\min}$ in relation (5.13a) and the profit obtained in (2).*
- (4) *If $\max\{he, \frac{A}{W}\} < \gamma_1 - c$, obtain the optimal solution by comparing the profit derived by: setting $q = W - DT_{\min}$, $T = t_1 = T_{\min}$, in relation (5.15a) and by setting $q = 0$, $T = e$, t_1 as obtained by relation (5.12) provided that $t_1 < e$ (if $t_1 \geq e$, then $t_1 = e$), in relation (5.15b).*
- (5) *If $he \leq \gamma_1 - c \leq \frac{A}{W}$, then obtain the optimal solution by comparing the profit derived: by setting $q = W - DT_d$, $T = t_1 = T_d$, in relation (5.15a) and by setting $q = 0$, $T = e$, t_1 as obtained by relation (5.12) provided that $t_1 < e$ (if $t_1 \geq e$, then $t_1 = e$), in relation (5.15b). Finally, calculate the optimal order quantity Q^* using relation (4.3).*

6. NUMERICAL EXAMPLES

In order to illustrate the results of the theoretical analysis, in this section we present a numerical example and a sensitivity analysis of the basic parameters. For the basic example we use the following parametric values: $D = 100$, $h = 1$, $p = 8$, $\beta = 0.3$, $c = 2$, $A = 150$, $\gamma_1 = 2.5$, $\gamma_2 = 1$, $e = 2$, $\alpha = 1.5$, $W = 250$, $T_{\min} = 0.1$ and $T_d = 0.9e$. For this data it applies that $0 < \gamma_1 - c < \min\{he, \frac{A}{W}\}$. Hence, according to Proposition 5.2 and since $\frac{\gamma_1 - c}{h} < \sqrt{\frac{2A}{hD}} < \min\{e, \frac{W}{D}\}$, the optimal solution is $q^* = 0$ and $t_1^* = T^* = \sqrt{\frac{2A}{hD}} = 1.73$. The corresponding optimal profit is $\Pi(T^*, t_1^*, q^*) = 426.8$ and the optimal order quantity is $Q^* = 173$.

7. MANAGERIAL INSIGHTS

Table 1 presents the effect of the ordering cost on the optimal solution to the problem. We observe that for a low ordering cost it is optimal to have a replenishment cycle as small as possible, *i.e.* to use the minimum reorder interval and to donate as much as possible since the profit from donating is greater than the ordering cost. For higher ordering costs the optimal solution is the EOQ formula or the shelf life of the product. The same effect is observed in Table 2 where the effect of the purchasing cost is examined. In Table 3 we notice that if the opportunity gain obtained by the donation of the product is lower than the purchasing cost, obviously

TABLE 1. Sensitivity analysis with respect to the ordering cost.

A	Parametric relations	T^*	t_1^*	q^*	Π	Q
90	$\frac{A}{W} < \gamma_1 - c < he$	0.1	0.1	240	655	250
120	$\frac{A}{W} < \gamma_1 - c < he$	1.55	1.55	0	455	155
150	$0 < \gamma_1 - c < \min\{eh, \frac{A}{W}\}$	1.73	1.73	0	426.8	173
180	$0 < \gamma_1 - c < \min\{eh, \frac{A}{W}\}$	1.9	1.9	0	410.3	190
210	$0 < \gamma_1 - c < \min\{eh, \frac{A}{W}\}$	2	2	0	395	200

TABLE 2. Sensitivity analysis with respect to the purchase cost.

c	Parametric relations	T^*	t_1^*	q^*	Π	Q
1.5	$\frac{A}{W} < \gamma_1 - c < he$	0.1	0.1	240	1305	250
2	$0 < \gamma_1 - c < \min\{eh, \frac{A}{W}\}$	1.73	1.73	0	426.8	173
2.5	$\gamma_1 - c = 0$	1.73	1.73	0	376.8	173
3	$\gamma_1 - c < 0$	1.73	1.73	0	326.8	173

TABLE 3. Sensitivity analysis with respect to the donation opportunity gain.

γ_1	Parametric relations	T^*	t_1^*	q^*	Π	Q
1	$\gamma_1 - c < 0$	1.73	1.73	0	426.8	173
1.5	$\gamma_1 - c < 0$	1.73	1.73	0	426.8	173
2	$\gamma_1 - c = 0$	1.73	1.73	0	426.8	173
2.5	$0 < \gamma_1 - c < \min\{eh, \frac{A}{W}\}$	1.73	1.73	0	426.8	173
3	$\frac{A}{W} < \gamma_1 - c < he$	0.1	0.1	240	1255	250

TABLE 4. Sensitivity analysis with respect to the expiration date.

e	Parametric relations	T^*	t_1^*	q^*	Π	Q
0.3	$eh < \gamma_1 - c < \frac{A}{W}$	0.27	0.27	223	220.9	250
0.5	$eh = \gamma_1 - c < \frac{A}{W}$	0.5	0.5	0	275	50
0.7	$0 < \gamma_1 - c < \min\{eh, \frac{A}{W}\}$	0.7	0.7	0	350.7	70
1	$0 < \gamma_1 - c < \min\{eh, \frac{A}{W}\}$	1	1	0	400	100
2	$0 < \gamma_1 - c < \min\{eh, \frac{A}{W}\}$	1.73	1.73	0	426.8	173

TABLE 5. Sensitivity analysis with respect to the storage capacity.

W	Parametric relations	T^*	t_1^*	q^*	Π	Q
150	$0 < \gamma_1 - c < \min\{eh, \frac{A}{W}\}$	1.5	1.5	0	425	150
200	$0 < \gamma_1 - c < \min\{eh, \frac{A}{W}\}$	1.73	1.73	0	426.8	173
250	$0 < \gamma_1 - c < \min\{eh, \frac{A}{W}\}$	1.73	1.73	0	426.8	173
300	$\frac{A}{W} = \gamma_1 - c < eh$	1.73	1.73	0	426.8	173
350	$\gamma_1 - c > \frac{A}{W}$	0.1	0.1	340	455	350

it is optimal not to donate any product and to use the EOQ. The effect of the product's expiration date on the optimal solution to the problem is presented in Table 4. We observe that for the specific parameter values a short expiration date leads to taking advantage of full storage capacity, the maximum replenishment cycle and donating the surplus quantity. However, as the product's shelf life increases it is optimal to match the replenishment cycle with the product's shelf life and keep no surplus inventory. Finally, in Table 5 we observe that for the first four cases the optimal solution is the EOQ. However, in the first case due to inadequate storage room the order quantity is limited to the storage capacity. For the last case, due to the increase in storage room, it is optimal to donate as much as possible, *i.e.* use the minimum reorder interval.

8. CONCLUSION

We have studied the optimal replenishment policy for a product with a fixed shelf life and constant demand rate. We assume that when the product is close to its expiration date the retailer offers a price markdown. This has a positive impact on demand but at the same time draws the attention of the customers to the expiration date of the product making the demand a decreasing function with respect to the time remaining to the time it expires (freshness dependent demand). We also assume that any leftover products at the end of the replenishment cycle that have not exceeded their expiration date can be donated to non-profit organizations generating direct (tax deductions) and indirect profit (gain of goodwill) for the retailer. If the leftover products have expired they can be sold at a salvage price to the livestock market. For this problem, we propose a simple optimization procedure that leads to a closed form optimal solution. Interestingly, we have proven that it is never optimal to markdown the product during the replenishment cycle. This we assume is due to the deterministic pattern of the demand rate or due to the consideration of one product. Also, we have proven that depending on the parametric values it is optimal either to have zero leftover quantity at the end of the cycle or to donate as much as possible. We have also proven that it is never optimal selling leftover quantity to the livestock market when the salvage cost is lower than the original purchasing cost of the product.

Notice that the assumption of deterministic demand could be pointed out as a limitation of the proposed model. However, this model could be considered as a good starting point for the study of more complex problems providing important managerial insights through the parametric conditions derived (see also Skouri *et al.* [16]). Further research directions could include the study of different demand patterns for this problem, such as a demand that is dependent on the time of the expiration date from the beginning of its shelf life, stock dependent demand, or a stochastic demand, to examine the case of not only one product but multiple products with different expiration dates and also to examine alternative ways of modelling the gain generated from donating surplus food. The difficulty of quantifying this opportunity gain could be overcome with the assistance of surveys and data analysis.

Acknowledgements. This research is implemented through the Operational Program “Human Resources Development, Education and Lifelong Learning” and is co-financed by the European Union (European Social Fund) and Greek national funds. In addition, the authors wish to thank the two anonymous reviewers for their constructive comments.

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