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ANALYSIS OF A GEOMETRIC CATASTROPHE MODEL WITH DISCRETE-TIME BATCH RENEWAL ARRIVAL PROCESS

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Abstract. Discrete-time stochastic models have been extensively studied since the past few decades due to its huge application in areas of computer-communication networks and telecommunication systems. However, the growing use of the internet often makes these systems vulnerable to catastrophe/ virus attack leading to the removal of some or all the elements from the system. Taking note of this, we consider a discrete-time model where the population (in the form of packets, data, etc.) is assumed to grow in batches according to renewal process and is likely to be affected by catastrophes which occur according to Bernoulli process. The catastrophes have a sequential impact on the population and it destroys each individual at a time with probability p. This destruction process stops as soon as an individual survives or when the entire population becomes extinct. We analyze both late and early arrival systems independently and using supplementary variable and shift operator methods obtain explicit expressions of steady-state population size distribution at pre-arrival and arbitrary epochs. We deduce some important performance measures and further show that for both the systems the tail probabilities at pre-arrival epoch can be well approximated using a single root of the characteristic equation. In order to illustrate the computational procedure, we present some numerical results and also investigate the change in the behavior of the model with the change in parameter values.

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1. Introduction

In the field of computers and telecommunication systems the transmission of electronic information in the form of voice calls, data, text, images or video takes place over a long distance. In general, these transmissions occur in the form of packets based on slotted systems. It is well known that such type of systems can be best represented as discrete-time models since the continuous-time counterpart are not suitable enough to handle them. One can refer to the books by Alfa [2], Bruneel and Kim [9], Takagi [28], Woodward [29] to get a clear perception on the importance of discrete-time modeling. In this connection, one may also see the recent papers by Claeys *et al.* [15], Bruneel and Maertens [10], Bruneel *et al.* [11]. With the growing use of internet the enormous network of these systems is at greater risk to large-scale failures due to the external unwanted agent which may result in a huge data loss. Such events can be suitably depicted in the form of catastrophic models, the occurrence of which may cause immediate removal of some or all elements (or packets) from the system.

Keywords. Discrete-time, early arrival, geometric catastrophes, late arrival, population size, renewal batch arrival.

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Dabrowski [16] discussed different catastrophic events experienced in communication networks while Shafer [27], Lee [25], Bartoszynski *et al.* [5], considered catastrophic events in areas of ecology and mathematical biology.

In recent years queueing models with catastrophes (or disasters) have received considerable attention, where the elements (customers) require service from the system and can not only leave the system due to the occurrence of a catastrophe, but also due to the completion of their service (see [3,6,7,24,26,30] and the references therein). However, in the present work we consider a population model where the individuals (elements) of the population abandons the system only due to a catastrophe. Such situation arises in distributed systems which are highly vulnerable to Internet catastrophes (in the form of worm or email virus) that cause wide-spread damage ranging from epidemic traffic to deleting stored data. Meanwhile, if the population is exposed to extreme disastrous conditions, the catastrophe often causes the instantaneous removal of all the elements present in the system and is generally termed as a total catastrophe. In this connection, one may refer to Economou and Fakinos [19], Economou [17]. On the other hand, a catastrophic event resulting in a partial removal of elements from a system was first introduced by Brockwell *et al.* [8] where they coined the term geometric, binomial and uniform catastrophes. A binomial catastrophe model was considered by Economou [18] where he obtained the probability generating function (pgf) of steady-state population size distribution at different epochs. He assumed that the population evolves according to the compound Poisson process and catastrophes occur according to the renewal process. Cairns and Pollett [12] addressed a model with birth, death and catastrophe process wherein the transition rates are allowed to depend on the current population size in an arbitrary manner. A geometric catastrophe model in which population grows according to batch Markovian arrival process and catastrophes occur according to the renewal process was studied by Economou and Gómez-Corral [20]. Along this direction recently Barbhuiya *et al.* [4] considered a continuous time geometric catastrophe model assuming the population growth to be batch renewal process and obtained the population size distribution at pre-arrival and arbitrary epochs.

Throughout the literature, it has been observed that catastrophe population model with discrete time set up has not been investigated as compared to the continuous time. Besides, the assumption of memoryless property of the arrival process may not be always practically useful in modeling many real-life situations, for example, when arrival occurs after a fixed interval of time as in deterministic distribution. This particular distribution has gained importance in recent years due to its substantial application in the telecommunication sector, where information transmitted in the form of packets arrives after a set time period in slots. Thus if the population growth is assumed to be a discrete-time renewal or batch renewal process, it can unify many other arrival processes. Moreover, the total catastrophe model may not be appropriate to represent the situation when the population is vulnerable to moderate catastrophic conditions. In fact, in many cases the catastrophes have a sequential effect on the population which destroys the individuals successively with certain probability until the whole population is eliminated or a first individual survives (see [4, 8]). Such situations can be suitably represented by a geometric catastrophe population model. Furthermore, in discrete-time models the batch arrivals and the occurrence of catastrophes takes place near the slot boundaries. So it becomes very important to keep track of the order in which the arrivals occur, and consequently this gives rise to two different systems namely, late arrival system with delayed access (LAS-DA) and early arrival system (EAS). Throughout the remaining part of the paper we use the term "late arrival system" and "LAS-DA" interchangeably.

In view of all the above, we study a discrete-time model where the population is assumed to grow in batches according to the renewal process and is exposed to catastrophes which occur according to the Bernoulli process. The catastrophe strikes each individual of the population successively with a fixed probability $p \in (0, 1)$, and this sequential process stops when a first individual survives or when the whole population gets destroyed. The main contribution of the paper is twofold. Firstly, for the aforementioned model, we obtain the population size distribution at pre-arrival and arbitrary epochs in a readily tractable form using the supplementary variable and shift operator method, for both LAS-DA and EAS. The methodology used in this paper is quite different from the previously used techniques in analyzing catastrophe population models. Secondly, we obtain some performance measures of the system and present few numerical results which demonstrate the usability of our work. It may be noted that in general, the analysis of late arrival system is considered to be difficult as compared to early arrival system. But the procedure used throughout the paper has made the analysis quite simpler and easy for implementation. Both the systems are studied independently in two different sections for a better understanding of the readers.

The rest of the paper is organized as follows. In Section 2 we give a comprehensive description of the model under consideration. In Sections 3 and 4 we carry out the complete analysis of the model with LAS-DA and EAS policy, respectively. Some useful performance measures are deduced in Section 5. In Section 6 few numerical results are presented along with some graphical comparisons, which are followed by the conclusion in Section 7.

2. Model description

As we discuss the model in discrete-time set-up, let us assume that the time axis is divided into intervals of equal length referred as (time) slot, separated by slot boundaries. We assume that the length of each slot is one unit. We further assume that arrival of batches and occurrence of catastrophes takes place around the slot boundary, say m *i.e.*, in the interval $(m-, m)$ or $(m, m+)$. This gives rise to two possible situations. (i) If the batch arrives in $(m-, m)$ and catastrophe occurs in $(m, m+)$, it is called late arrival system with delayed access (LAS-DA). Under this system, the catastrophe which occurs in $(m, m+)$ effects all the elements present up to that slot, excluding the elements of the batch arriving in $(m-, m)$. (ii) If the catastrophe occurs in $(m-, m)$ and a batch arrives in $(m, m+)$, it is called early arrival system (EAS). Under this system the catastrophe occurring in $(m-, m)$ impacts all the elements present upto that slot. For a point by point discussion about these concepts, see Chaudhry *et al.* [14], Chaudhry [13] and Hunter [23]. In this paper we distinguish both the systems to study the model under consideration.

- The population evolves in batches of random size X with probability mass function (pmf) $g_i = P(X = i)$, $i = 1, 2, 3, \ldots, b$, where $b \in \mathbb{N}$ is the maximum permissible size of the arriving batch. This assumption fits into many practical situations and is also significant from computational perspective. The case when arriving batch size distribution has infinite support can also be suitably tackled by our methodology by truncating the maximum batch size to a sufficiently larger value. The probability generating function (pgf) and mean batch size are respectively $G(z) = \sum_{i=1}^{b} g_i z^i, |z| \leq 1$ and $\overline{g} = \sum_{i=1}^{b} i g_i$.
- Let $T_n, n = 1, 2, \ldots$ be the inter-arrival time between nth and $(n+1)$th batch. We assume that inter-arrival times are independent and identically distributed (i.i.d) discrete random variables (T) with common pmf $a_n = P(T = n),\ n \geq 1$, pgf $A(z) = \sum_{n=1}^{\infty} a_n z^n$ and mean $a = \frac{1}{\lambda} = A'(1) = \sum_{n=1}^{\infty} n a_n$, where λ is the rate of the arriving batch. The inter-arrival times (often referred as inter-renewal times) and the batch sizes are independent.
- Catastrophes occur individually and inter-arrival time (I) between two consecutive catastrophes, referred to as inter-catastrophe time (ICT), are independent and geometrically distributed with parameter μ as $P(I = n) = \overline{\mu}^{n-1}\mu,\ 0 < \mu < 1,\ n \geq 1$ where $\overline{\mu} = (1 - \mu)$. The catastrophes strike the population in a sequential manner and eliminates each individual with a fixed probability $p \in (0, 1)$. This destruction process stops as soon as the first individual survives with probability $q(= 1 - p)$ or the population is destroyed entirely.
- In order to ensure the stability of the system we assume $\lambda\overline{g} < \frac{\mu p}{q}$ (see Appendix A).

In the following sections we give a step-by-step complete analysis of the model under both the systems described above.

3. Analysis of the model with LAS-DA

In this section we study the model subject to LAS-DA policy. The readers are referred to Figure 1 to have a notion of the batch arrivals and catastrophe's occurrence taking place under this system. We begin with the mathematical formulation of the model using supplementary variable technique (SVT) where we treat the remaining inter-arrival time of the next batch to be the supplementary variable. Let us define the following random variables at the instant just before a potential batch arrival *i.e.*, at $m-$.

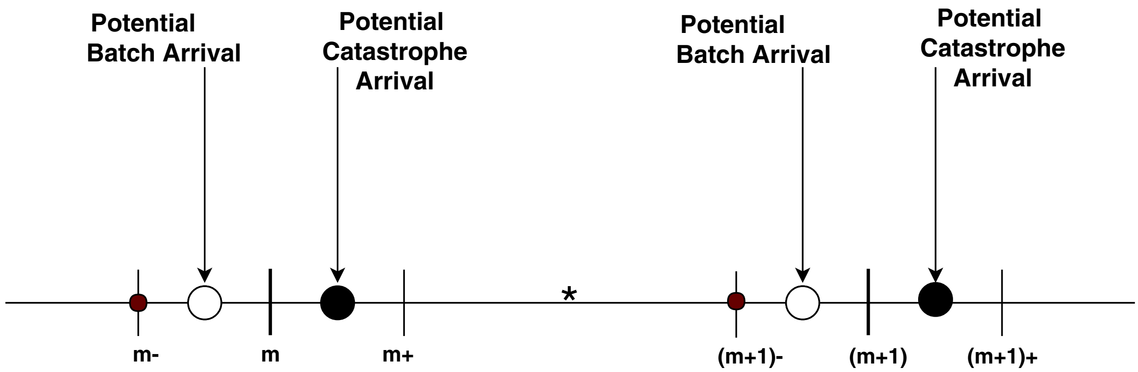

FIGURE 1. A schematic diagram of the model with LAS-DA policy.

- $N_{m-} \equiv$ Population size just before a potential batch arrival.
- $U_{m-} \equiv$ Remaining inter-arrival time of the next batch.

Thus the vector process $\{(N_{m-}, U_{m-}), m = 1, 2, \ldots\}$ becomes a Markov chain in discrete time. We further define the joint probabilities of (N_{m-}, U_{m-}) and the marginal probability of N_{m-} respectively as

$$\widehat{P}_n(m-, u) = \mathrm{Prob}(N_{m-} = n,\ U_{m-} = u), \quad u \geq 0,\ n \geq 0,$$
$$\widehat{P}_n(m-) = \mathrm{Prob}(N_{m-} = n), \quad n \geq 0.$$

Relating the states of the system at two consecutive time epochs $m-$ and $(m+1)-$ and using the arguments of SVT, we obtain (for $u \geq 1$) the following set of governing equations

$$\widehat{P}_0((m+1)-, u-1) = \widehat{P}_0(m-, u)\overline{\mu} + \mu \sum_{k=0}^{\infty} p^k \widehat{P}_k(m-, u), \tag{3.1}$$

$$\widehat{P}_1((m+1)-, u-1) = \widehat{P}_1(m-, u)\overline{\mu} + \mu \sum_{k=1}^{\infty} p^{k-1} q \widehat{P}_k(m-, u) + a_u \overline{\mu} g_1 \widehat{P}_0(m-, 0) + a_u \mu g_1 \sum_{k=0}^{\infty} p^k \widehat{P}_k(m-, 0), \tag{3.2}$$

$$\widehat{P}_n((m+1)-, u-1) = \widehat{P}_n(m-, u)\overline{\mu} + \mu \sum_{k=n}^{\infty} p^{k-n} q \widehat{P}_k(m-, u) + a_u \overline{\mu} \sum_{i=1}^{n} g_i \widehat{P}_{n-i}(m-, 0) + a_u \mu \left(\sum_{i=1}^{n-1} g_i \sum_{k=n-i}^{\infty} p^{k-n+i} q \widehat{P}_k(m-, 0) + g_n \sum_{k=0}^{\infty} p^k \widehat{P}_k(m-, 0) \right), n \geq 2. \tag{3.3}$$

As we discuss the model in steady-state, we define $P_n(u) = \lim_{m- \to \infty} \widehat{P}_n(m-, u), n \geq 0$. Thus, from (3.1) to (3.3), separating the cases for $2 \leq n \leq b$ and $n \geq b+1$, the steady-state equations obtained are

$$P_0(u-1) = P_0(u)\overline{\mu} + \mu \sum_{k=0}^{\infty} p^k P_k(u), \tag{3.4}$$

$$P_1(u-1) = P_1(u)\overline{\mu} + \mu \sum_{k=1}^{\infty} p^{k-1} q P_k(u) + a_u \overline{\mu} g_1 P_0(0) + a_u \mu g_1 \sum_{k=0}^{\infty} p^k P_k(0), \tag{3.5}$$

$$P_n(u-1) = P_n(u)\overline{\mu} + \mu\sum_{k=n}^{\infty} p^{k-n} q P_k(u) + a_u\overline{\mu}\sum_{i=1}^{n} g_i P_{n-i}(0)$$
$$+ a_u\mu\left(\sum_{i=1}^{n-1} g_i \sum_{k=n-i}^{\infty} p^{k-n+i} q P_k(0) + g_n \sum_{k=0}^{\infty} p^k P_k(0)\right), \quad 2 \le n \le b, \tag{3.6}$$

$$P_n(u-1) = P_n(u)\overline{\mu} + \mu\sum_{k=n}^{\infty} p^{k-n} q P_k(u) + a_u\overline{\mu}\sum_{i=1}^{b} g_i P_{n-i}(0)$$
$$+ a_u\mu\left(\sum_{i=1}^{b} g_i \sum_{k=n-i}^{\infty} p^{k-n+i} q P_k(0)\right), \quad n \ge b+1. \tag{3.7}$$

Our objective is to find the steady-state probabilities $P_n, n \ge 0$. But it seems to be difficult to obtain this by directly using (3.4)–(3.7). For this purpose, let us define the pgf of $P_n(u)$ as $P_n^*(z) = \sum_{u=0}^{\infty} P_n(u) z^u$ and hence $P_n = P_n^*(1) = \sum_{u=0}^{\infty} P_n(u)$. Multiplying (3.4)–(3.7) by z^u and summing over u from 1 to ∞ we obtain the following set of transformed equations

$$zP_0^*(z) = \overline{\mu}(P_0^*(z) - P_0(0)) + \mu\sum_{k=0}^{\infty} p^k (P_k^*(z) - P_k(0)), \tag{3.8}$$

$$zP_1^*(z) = \overline{\mu}(P_1^*(z) - P_1(0)) + \mu\sum_{k=0}^{\infty} p^k q (P_{k+1}^*(z) - P_{k+1}(0)) + A(z)\overline{\mu} g_1 P_0(0)$$
$$+ A(z)\mu g_1 \sum_{k=0}^{\infty} p^k P_k(0), \tag{3.9}$$

$$zP_n^*(z) = \overline{\mu}(P_n^*(z) - P_n(0)) + \mu\sum_{k=0}^{\infty} p^k q (P_{k+n}^*(z) - P_{k+n}(0)) + A(z)\overline{\mu}\sum_{i=1}^{n} g_i P_{n-i}(0)$$
$$+ A(z)\mu\left(\sum_{i=1}^{n-1} g_i \sum_{k=0}^{\infty} p^k q P_{k+n-i}(0) + g_n \sum_{k=0}^{\infty} p^k P_k(0)\right), \quad 2 \le n \le b, \tag{3.10}$$

$$zP_n^*(z) = \overline{\mu}(P_n^*(z) - P_n(0)) + \mu\sum_{k=0}^{\infty} p^k q (P_{k+n}^*(z) - P_{k+n}(0)) + A(z)\overline{\mu}\sum_{i=1}^{b} g_i P_{n-i}(0)$$
$$+ A(z)\mu\left(\sum_{i=1}^{b} g_i \sum_{k=0}^{\infty} p^k q P_{k+n-i}(0)\right), \quad n \ge b+1. \tag{3.11}$$

Adding (3.8)–(3.11) for all values of n and taking limit $z \to 1$ we obtain

$$\sum_{n=0}^{\infty} P_n(0) = \frac{1}{a} = \lambda. \tag{3.12}$$

Remark 3.1. For a detailed derivation of (3.12) see Appendix A.

Relation (3.12) suggests that the probability that a batch arrival is about to occur is equal to the rate (λ) of the arrival process which is in accordance with Blackwell's theorem for arithmetic renewal process (see [22]). Thus (3.12) can also be considered as a test for the correctness of the governing equations. We now define P_n^- as the probability that the population size is n at pre-arrival epoch *i.e.* just before the arrival of a batch. As

P_n^- is proportional to $P_n(0)$ and $\sum_{n=0}^{\infty} P_n^- = 1$, both P_n^- and $P_n(0)$ can be connected by the following relation

$$P_n^- = \frac{P_n(0)}{\sum_{k=0}^{\infty} P_k(0)} = aP_n(0), \quad n \geq 0. \tag{3.13}$$

We now discuss the procedure to obtain the state probabilities at pre-arrival (P_n^-) and arbitrary (P_n) epochs.

3.1. Population size distribution at pre-arrival and arbitrary epochs

On the sequence of probabilities $\{P_n(0)\}$ and $\{P_n^*(z)\}$ we define the right shift operator D as $DP_n(0) = P_{n+1}(0)$ and $DP_n^*(z) = P_{n+1}^*(z)$ for all n. So, (3.11) can be rewritten in the form

$$\left(z - \overline{\mu} - \mu q \sum_{k=0}^{\infty} p^k D^k\right) P_n^*(z) = \left(\left(A(z)\sum_{i=1}^{b} g_i D^{b-i} - D^b\right)\left(\overline{\mu} + \mu q \sum_{k=0}^{\infty} p^k D^k\right)\right) P_{n-b}(0), n \geq b+1. \tag{3.14}$$

Substituting $z = \overline{\mu} + \mu q \sum\limits_{k=0}^{\infty} p^k D^k$ in (3.14), we get the following homogeneous difference equation

$$\left(\left(A(H)\sum_{i=1}^{b} g_i D^{b-i} - D^b\right)\left(\overline{\mu} + \mu q \sum_{k=0}^{\infty} p^k D^k\right)\right) P_n(0) = 0, \quad n \geq 1, \tag{3.15}$$

where $H \equiv \overline{\mu} + \mu q \sum\limits_{k=0}^{\infty} p^k D^k$. Throughout the paper, the usual method of solving linear difference equation with constant coefficient has been used for which one can refer to Elaydi [21]. The characteristic equation (c.e.) corresponding to (3.15) is

$$\left(A\left(1 - \frac{\mu p(1-s)}{1-ps}\right)\sum_{i=1}^{b} g_i s^{b-i} - s^b\right)\left(1 - \frac{\mu p(1-s)}{1-ps}\right) = 0, \quad |ps| < 1. \tag{3.16}$$

Since $A\left(1 - \frac{\mu p(1-s)}{1-ps}\right)\sum_{i=1}^{b} g_i s^{b-i} - s^b = 0$ has precisely b roots inside the unit circle (for proof see Appendix A), hence the c.e. have exactly b roots inside $|s| = 1$, denoted by $\beta_1, \beta_2, \ldots, \beta_b$. Thus the general solution of (3.15) is of the form

$$P_n(0) = \sum_{i=1}^{b} d_i \beta_i^n, \quad n \geq 1, \tag{3.17}$$

where $d_1, d_2, \ldots, d_b$ are the arbitrary constants yet to be determined. Now using expression (3.17) in (3.14) we get

$$\begin{aligned}\left(z - \overline{\mu} - \mu q \sum_{k=0}^{\infty} p^k D^k\right) P_n^*(z) = & -\overline{\mu}\sum_{j=1}^{b} d_j \beta_j^n - \mu q \sum_{k=0}^{\infty} p^k \sum_{j=1}^{b} d_j \beta_j^{n+k} + A(z)\overline{\mu}\sum_{i=1}^{b} g_i \sum_{j=1}^{b} d_j \beta_j^{n-i} \\ & + A(z)\mu q \sum_{i=1}^{b} g_i \sum_{k=0}^{\infty} p^k \sum_{j=1}^{b} d_j \beta_j^{n+k-i}, \quad n \geq b+1. \end{aligned} \tag{3.18}$$

Equation (3.18) is a non-homogeneous difference equation with constant coefficient and hence its general solution is given by

$$P_n^*(z) = B\left(\frac{z-1+\mu p}{p(z-\overline{\mu})}\right)^n + \sum_{j=1}^{b} d_j \frac{\psi_j(z)}{\varphi_j(z)} \beta_j^{n-b}, \quad n \geq b+1, \tag{3.19}$$

where

$$\psi_j(z) = \left(A(z)\sum_{i=1}^{b} g_i\beta_j^{b-i} - \beta_j^b\right)\left(\overline{\mu} + \mu q\sum_{k=0}^{\infty} p^k\beta_j^k\right), \tag{3.20}$$

$$\varphi_j(z) = z - \overline{\mu} - \mu q\sum_{k=0}^{\infty} p^k\beta^k. \tag{3.21}$$

The first and second term in the R.H.S of (3.19) are respectively the solution corresponding to the homogeneous part and a particular solution of (3.18), where B in the first term is an arbitrary constant. Now summing over the range of n and taking $\lim_{z\to 1}$ in (3.19), we must have the convergence of (3.19), since $\sum_{n=b+1}^{\infty} P_n^*(1) \le 1$. But as $z \to 1$, $B\sum_{n=b+1}^{\infty}\left(\frac{z-1+\mu p}{p(z-\overline{\mu})}\right)^n \to \infty$. Thus to ensure the convergence of (3.19) we must have $B = 0$. Hence

$$P_n^*(z) = \sum_{j=1}^{b} d_j\frac{\psi_j(z)}{\varphi_j(z)}\beta_j^{n-b}, \quad n \ge b+1. \tag{3.22}$$

Now we find the conditions under which $P_n^*(z)$ is same as (3.22) for $1 \le n \le b$. Taking $n = b$ in (3.10) and using (3.17) we obtain

$$P_0(0) = \sum_{i=1}^{b} d_i\left(1 - \frac{\mu p}{1-p\beta_i}\right)\cdot \tag{3.23}$$

Similarly, substituting in (3.10) the expressions of $P_n(0)$, $n \ge 0$, from (3.17) and (3.23), and using (3.22) we obtain

$$\sum_{j=1}^{b} d_j\beta_j^n\sum_{i=n+1}^{b} g_i\beta_j^{-i}\left(1 - \frac{\mu p(1-\beta_j)}{1-p\beta_j}\right) = 0, \quad 2 \le n \le b-1. \tag{3.24}$$

Taking $n = b-1,\ b-2, ..., 2$ in (3.24) and considering the fact that $g_b \ne 0$, we have

$$\sum_{j=1}^{b}\frac{d_j}{\beta_j^{b-n}}\left(1 - \frac{\mu p(1-\beta_j)}{1-p\beta_j}\right) = 0, \quad 2 \le n \le b-1. \tag{3.25}$$

In a similar manner we obtain the following condition from (3.9)

$$\sum_{j=1}^{b}\frac{d_j}{\beta_j^{b-1}}\left(1 - \frac{\mu p(1-\beta_j)}{1-p\beta_j}\right) = 0. \tag{3.26}$$

Equations (3.25) and (3.26) can be combined together in the form

$$\sum_{j=1}^{b}\frac{d_j}{\beta_j^{n}}\left(1 - \frac{\mu p(1-\beta_j)}{1-p\beta_j}\right) = 0, \ 1 \le n \le b-1. \tag{3.27}$$

Adding (3.23) and (3.17) for all n, and using (3.12) we obtain

$$\sum_{i=1}^{b} d_i\left(\frac{1}{1-\beta_j} - \frac{\mu p}{1-p\beta_j}\right) = \frac{1}{a}\cdot \tag{3.28}$$

Equations (3.27) and (3.28) together constitutes a system of b equations in b unknowns which can be solved to obtain d_j, $1 \le j \le b$. Thus the expression of $P_n(0)$, $n \ge 0$ as given in (3.17) and (3.23) are completely known and $P_n^*(z)$ is given by

$$P_n^*(z) = \sum_{j=1}^{b} d_j\frac{\psi_j(z)}{\varphi_j(z)}\beta_j^{n-b}, \quad n \ge 1. \tag{3.29}$$

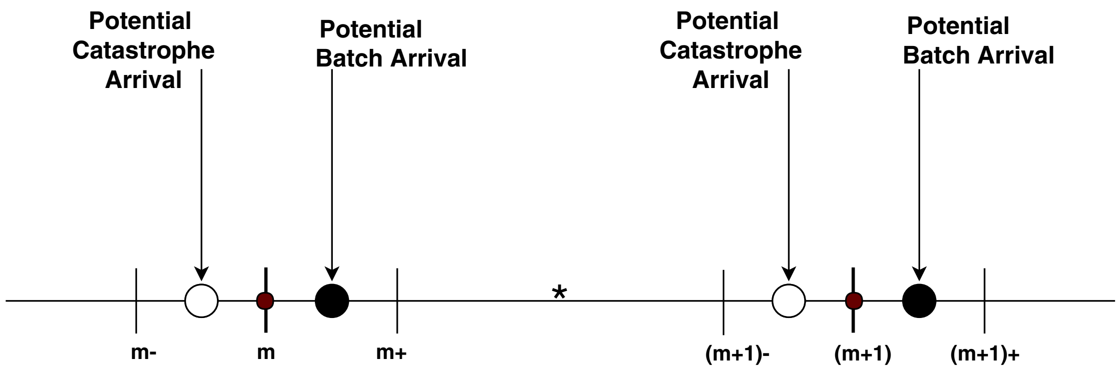

FIGURE 2. A schematic diagram of the model with EAS policy.

Now from the relation (3.13) and (3.29) one can obtain the population size distribution at pre-arrival and arbitrary epochs in explicit form as

$$\begin{aligned}
P_n^- &= a\sum_{i=1}^{b} d_i\beta_i^n, \quad n\geq 1,\\
P_0^- &= a\sum_{i=1}^{b} d_i\left(1-\frac{\mu p}{1-p\beta_i}\right),\\
P_n &= P_n^*(1)=\sum_{j=1}^{b} d_j\frac{\psi_j(1)}{\varphi_j(1)}\beta_j^{n-b}, \quad n\geq 1,\\
P_0 &= 1-\sum_{j=1}^{b} d_j\frac{\psi_j(1)\beta_j^{1-b}}{\varphi_j(1)(1-\beta_j)},
\end{aligned}$$

where $\psi_j(1)$ and $\varphi_j(1)$ can be evaluated from (3.20) and (3.21). This completes the analysis of the model under late arrival system. In the forthcoming section we consider the model with early arrival system. Though the steps involved in the analysis is similar to that of late arrival system, but the results obtained are rather different, and hence for the sake of completeness and the easy understanding of the readers it has been discussed in detail.

4. ANALYSIS OF THE MODEL WITH EAS

One can refer to Figure 2 to have a notion of the arrivals taking place under this system. Similarly as before we consider the governing equations of the model using SVT by treating the remaining inter-arrival of the next batch to be the supplementary variable. We define the following random variables at the instant just before a potential batch arrival *i.e.*, at m.

- $N_m \equiv$ Population size just before a potential batch arrival.
- $U_m \equiv$ Remaining inter-arrival time of the next batch.

Thus the vector process $\{(N_m, U_m), m = 0, 1, 2, \ldots\}$ becomes a Markov chain in discrete time. We also define the joint probabilities of (N_m, U_m) and the marginal probability of N_m respectively as

$$\begin{aligned}
\widehat{Q}_n(m,u) &= \text{Prob}(N_m=n,\ U_m=u), \quad u\geq 0,\ n\geq 0,\\
\widehat{Q}_n(m) &= \text{Prob}(N_m=n), \quad n\geq 0.
\end{aligned}$$

As we discuss the model in steady-state, we define $Q_n(u) = \lim_{m\to\infty} \widehat{Q}_n(m,u), n \geq 0$. Relating the states of the system at two consecutive time epochs m and $m+1$, and using the arguments of SVT, we obtain (for $u \geq 1$) the following steady-state equations

$$Q_0(u-1) = Q_0(u)\overline{\mu} + \mu \sum_{k=0}^{\infty} p^k Q_k(u) + a_u\mu \sum_{i=1}^{b} g_i \sum_{k=0}^{\infty} p^{k+i} Q_k(0), \tag{4.1}$$

$$\begin{aligned} Q_n(u-1) = {} & Q_n(u)\overline{\mu} + \mu \sum_{k=n}^{\infty} p^{k-n} q Q_k(u) + a_u\overline{\mu} \sum_{i=1}^{n} g_i Q_{n-i}(0) + a_u\mu \sum_{i=1}^{n} g_i \sum_{k=n-i}^{\infty} p^{k-n+i} q Q_k(0) \\ & + a_u\mu \sum_{i=n+1}^{b} g_i \sum_{k=0}^{\infty} p^{k-n+i} q Q_k(0), \quad 1 \leq n \leq b-1, \end{aligned} \tag{4.2}$$

$$\begin{aligned} Q_n(u-1) = {} & Q_n(u)\overline{\mu} + \mu \sum_{k=n}^{\infty} p^{k-n} q Q_k(u) + a_u\overline{\mu} \sum_{i=1}^{b} g_i Q_{n-i}(0) \\ & + a_u\mu \left(\sum_{i=1}^{b} g_i \sum_{k=n-i}^{\infty} p^{k-n+i} q Q_k(0) \right), \quad n \geq b. \end{aligned} \tag{4.3}$$

In order to obtain the state probabilities $Q_n, n \geq 0$, define the pgf of $Q_n(u)$ as $Q_n^*(z) = \sum_{u=0}^{\infty} Q_n(u) z^u$ and hence $Q_n = Q_n^*(1) = \sum_{u=0}^{\infty} Q_n(u)$. Multiplying (4.1)–(4.3) by z^u and summing over u from 1 to ∞ we obtain the following set of transformed equations

$$zQ_0^*(z) = \overline{\mu}(Q_0^*(z) - Q_0(0)) + \mu \sum_{k=0}^{\infty} p^k (Q_k^*(z) - Q_k(0)) + A(z)\mu \sum_{i=0}^{b} g_i \sum_{k=0}^{\infty} p^{k+i} Q_k(0),$$

$$\begin{aligned} zQ_n^*(z) = {} & \overline{\mu}(Q_n^*(z) - Q_n(0)) + \mu \sum_{k=0}^{\infty} p^k q (Q_{k+n}^*(z) - Q_{k+n}(0)) + A(z)\overline{\mu} \sum_{i=1}^{n} g_i Q_{n-i}(0) \\ & + A(z)\mu \left(\sum_{i=1}^{n} g_i \sum_{k=0}^{\infty} p^k q Q_{k+n-i}(0) + \sum_{i=n+1}^{b} g_i \sum_{k=0}^{\infty} p^{k-n+i} q Q_k(0) \right), 1 \leq n \leq b-1, \end{aligned} \tag{4.4}$$

$$\begin{aligned} zQ_n^*(z) = {} & \overline{\mu}(Q_n^*(z) - Q_n(0)) + \mu \sum_{k=0}^{\infty} p^k q (Q_{k+n}^*(z) - Q_{k+n}(0)) + A(z)\overline{\mu} \sum_{i=1}^{b} g_i Q_{n-i}(0) \\ & + A(z)\mu \left(\sum_{i=1}^{b} g_i \sum_{k=0}^{\infty} p^k q Q_{k+n-i}(0) \right), \quad n \geq b. \end{aligned} \tag{4.5}$$

Now using the same reasoning as in Section 3 we can obtain the relations

$$\sum_{n=0}^{\infty} Q_n(0) = \frac{1}{a} = \lambda, \tag{4.6}$$

$$Q_n^- = \frac{Q_n(0)}{\sum_{k=0}^{\infty} Q_k(0)} = aQ_n(0), \quad n \geq 0, \tag{4.7}$$

where Q_n^- is the probability that the population size is n just before the arrival of a batch. Below we discuss the procedure to obtain the steady-state probabilities at pre-arrival (Q_n^-) and arbitrary (Q_n) epochs.

4.1. Population size distribution at pre-arrival and arbitrary epochs

As before, we define the displacement operator D on the sequence of probabilities $\{Q_n(0)\}$ and $\{Q_n^*(z)\}$ as $DQ_n(0) = Q_{n+1}(0)$ and $DQ_n^*(z) = Q_{n+1}^*(z)$ for all n. So, (4.5) can be rewritten as

$$\left(z - \overline{\mu} - \mu q \sum_{k=0}^{\infty} p^k D^k\right) Q_n^*(z) = \left(-\overline{\mu} D^b - \mu q \sum_{k=0}^{\infty} p^k D^{k+b} + A(z)\overline{\mu} \sum_{i=1}^{b} g_i D^{b-i} + A(z)\mu q \sum_{i=1}^{b} g_i \sum_{k=0}^{\infty} p^k D^{k+b-i}\right) Q_{n-b}(0), \quad n \geq b. \tag{4.8}$$

Substituting $z = \overline{\mu} + \mu q \sum\limits_{k=0}^{\infty} p^k D^k$ in (4.8), we get the following homogeneous difference equation

$$\left(\left(A(F)\sum_{i=1}^{b} g_i D^{b-i} - D^b\right)\left(\overline{\mu} + \mu q \sum_{k=0}^{\infty} p^k D^k\right)\right) Q_n(0) = 0, \quad n \geq 0, \tag{4.9}$$

where $F \equiv \overline{\mu} + \mu q \sum\limits_{k=0}^{\infty} p^k D^k$. The c.e. corresponding to (4.9) is same as (3.16) and hence using similar arguments as in Section 3.1, we have the general solution of (4.9) in the form

$$Q_n(0) = \sum_{i=1}^{b} c_i r_i^n, \qquad n \geq 0, \tag{4.10}$$

where $r_1, r_2, \ldots, r_b$ are the b roots of the c.e. inside $|s| = 1$, and $c_1, c_2, \ldots, c_b$ are the arbitrary constants yet to be determined. Now using expression (4.10) in (4.8) we obtain

$$\left(z - \overline{\mu} - \mu q \sum_{k=0}^{\infty} p^k D^k\right) Q_n^*(z) = -\overline{\mu} \sum_{j=1}^{b} c_j r_j^n - \mu q \sum_{k=0}^{\infty} p^k \sum_{j=1}^{b} c_j r_j^{n+k} + A(z)\overline{\mu} \sum_{i=1}^{b} g_i \sum_{j=1}^{b} c_j r_j^{n-i} + A(z)\mu q \sum_{i=1}^{b} g_i \sum_{k=0}^{\infty} p^k \sum_{j=1}^{b} c_j r_j^{n+k-i}, \quad n \geq b. \tag{4.11}$$

The general solution of (4.11) is given by

$$Q_n^*(z) = C\left(\frac{z - 1 + \mu p}{p(z - \overline{\mu})}\right)^n + \sum_{j=1}^{b} c_j \frac{\chi_j(z)}{\phi_j(z)} r_j^{n-b}, \quad n \geq b, \tag{4.12}$$

where

$$\chi_j(z) = \left(A(z)\sum_{i=1}^{b} g_i r_j^{b-i} - r_j^b\right)\left(\overline{\mu} + \mu q \sum_{k=0}^{\infty} p^k r_j^k\right), \tag{4.13}$$

$$\phi_j(z) = z - \overline{\mu} - \mu q \sum_{k=0}^{\infty} p^k r_j^k. \tag{4.14}$$

The first and second term in the R.H.S of (4.12) are respectively the solution corresponding to the homogeneous part and a particular solution of (4.11), where C in the first term is an arbitrary constant. Using similar logic as in Section 3.1 we must have $C = 0$ to ensure the convergence of (4.12) and thus

$$Q_n^*(z) = \sum_{j=1}^{b} c_j \frac{\chi_j(z)}{\phi_j(z)} r_j^{n-b}, \quad n \geq b. \tag{4.15}$$

For $Q_n^*(z)$ same as (4.15) for $1 \le n \le b-1$, we obtain the following condition from (4.4)

$$\sum_{j=1}^{b} c_j r_j^n \sum_{i=n+1}^{b} g_i r_j^{-i} \left(1-\mu+\mu q \sum_{k=0}^{\infty} p^k r_j^k\right) - \mu q \sum_{j=1}^{b} c_j \sum_{i=n+1}^{b} g_i \sum_{k=0}^{\infty} p^{k+i-n} r_j^k = 0,\ 1 \le n \le b-1. \quad (4.16)$$

Substituting $n = b-1,\ b-2, ..., 1$, in (4.16) we have a system of $b-1$ equations

$$\sum_{j=1}^{b} \frac{c_j}{r_j^n} = 0, \quad 1 \le n \le b-1. \quad (4.17)$$

Also, summing over n from 0 to ∞ in (4.10) and using (4.6) we obtain

$$\sum_{j=1}^{b} \frac{c_j}{1-r_j} = \frac{1}{a}\cdot \quad (4.18)$$

Equations (4.17) and (4.18) together forms a system of b equations which can be solved for the unknowns $c_j,\ 1 \le j \le b$. This makes the expression of $Q_n(0)$ as given in (4.10) completely known and $Q_n^*(z)$ is given by

$$Q_n^*(z) = \sum_{j=1}^{b} c_j \frac{\chi_j(z)}{\phi_j(z)} r_j^{n-b}, \quad n \ge 1. \quad (4.19)$$

The state probabilities at the pre-arrival and arbitrary epochs are thus obtained in explicit form using (4.7), (4.10) and (4.19) as

$$Q_n^- = a \sum_{i=1}^{b} c_i r_i^n, \qquad n \ge 0,$$

$$Q_n = Q_n^*(1) = \sum_{j=1}^{b} c_j \frac{\chi_j(1)}{\phi_j(1)} r_j^{n-b}, \qquad n \ge 1,$$

$$Q_0 = 1 - \sum_{j=1}^{b} c_j \frac{\chi_j(1)}{\phi_j(1)} \frac{r_j^{1-b}}{1-r_j},$$

where $\chi_j(1),\ \phi_j(1)$ can be evaluated from (4.13) and (4.14). This completes the analysis of the model under EAS policy. In the forthcoming section we work out some performance measures of the system.

5. Performance measures

In this section we mainly focus on deducing some useful performance measures of the model which is based on the analysis done in Sections 3 and 4. Let $L_{\text{EAS}}^-(V_{\text{EAS}}^-)$, $L_{\text{EAS}}(V_{\text{EAS}})$, $L_{\text{LAS}}^-(V_{\text{LAS}}^-)$, $L_{\text{LAS}}(V_{\text{LAS}})$ denote the mean (variance) of the population size distribution at pre-arrival and arbitrary epochs for EAS and LAS-DA policies, respectively. These quantities can be evaluated after generating probability distributions and are given below

$$L_{\text{EAS}}^- = \sum_{n=1}^{\infty} n Q_n^-, \quad V_{\text{EAS}}^- = \sum_{n=1}^{\infty} n^2 Q_n^- - \left(\sum_{n=1}^{\infty} n Q_n^-\right)^2.$$

In order to get $L_{\text{EAS}}(V_{\text{EAS}})$, $L_{\text{LAS}}^-(V_{\text{LAS}}^-)$ and $L_{\text{LAS}}(V_{\text{LAS}})$ simply replace Q_n^- by Q_n, P_n^- and P_n, respectively. We can also find the expressions of mean and variance in explicit form (in terms of roots of the c.e.) without evaluating the probability distributions. These are as follows

$$L^-_{\text{EAS}} = a\sum_{i=1}^{b} c_i \frac{r_i}{(1-r_i)^2}, \quad V^-_{\text{EAS}} = a\sum_{i=1}^{b} c_i \frac{2r_i^2}{(1-r_i)^3} + L^-_{\text{EAS}} - (L^-_{\text{EAS}})^2,$$

$$L_{\text{EAS}} = \sum_{j=1}^{b} c_j \frac{\chi_j(1)}{\phi_j(1)} \frac{r_j^{1-b}}{(1-r_j)^2}, \quad V_{\text{EAS}} = \sum_{j=1}^{b} c_j \frac{\chi_j(1)}{\phi_j(1)} \frac{2r_j^{2-b}}{(1-r_j)^3} + L_{\text{EAS}} - (L_{\text{EAS}})^2,$$

$$L^-_{\text{LAS}} = a\sum_{i=1}^{b} d_i \frac{\beta_i}{(1-\beta_i)^2}, \quad V^-_{\text{LAS}} = a\sum_{i=1}^{b} d_i \frac{2\beta_i^2}{(1-\beta_i)^3} + L^-_{\text{LAS}} - (L^-_{\text{LAS}})^2,$$

$$L_{\text{LAS}} = \sum_{j=1}^{b} d_j \frac{\psi_j(1)}{\varphi_j(1)} \frac{\beta_j^{1-b}}{(1-\beta_j)^2}, \quad V_{\text{LAS}} = \sum_{j=1}^{b} d_j \frac{\psi_j(1)}{\varphi_j(1)} \frac{2\beta_j^{2-b}}{(1-\beta_j)^3} + L_{\text{LAS}} - (L_{\text{LAS}})^2.$$

Furthermore, an important result related to the limiting distributions at pre-arrival epoch is given in the following theorem.

Theorem 5.1. *For the LAS-DA policy, among all the b roots of the equation*

$$A\left(1 - \frac{\mu p(1-s)}{1-ps}\right)\sum_{i=1}^{b} g_i s^{b-i} - s^b = 0, \tag{5.1}$$

lying inside the unit circle subject to the stability condition $\lambda\overline{g} < \frac{\mu p}{q}$, there exists a unique root (β_b) with largest modulus in the interval $(0,1)$. Consequently, the limiting probabilities for large n, often termed as the tail probabilities, at pre-arrival epoch (P_n^-) can be estimated by the single root β_b as $P_n^- \approx a d_b \beta_b^n$. This result also holds true for the model with EAS policy as the root equation given by (5.1) is the same, and the approximation is $Q_n^- \approx a c_b r_b^n$.

The proof of the above theorem goes along the same line as its continuous-time counterpart, for which one may refer to [4]. In the forthcoming section we demonstrate our theoretical work in the form of certain numerical examples.

6. Numerical results

In this section we present some numerical results that has been worked out in order to illustrate the computational procedure. The results presented in tabular form may be used in future by researchers who would like to match their results with ours using some other method or as a special case. All the results are computed upto forty decimal places, but here we have truncated them upto eight decimal places. For both EAS and LAS-DA policy steady-state population size distribution at pre-arrival and arbitrary epochs has been obtained for different inter-renewal distributions namely, geometric, deterministic and arbitrary (non-geometric). The results for geometric inter-renewal distribution is shown in Table 1 where the parameters taken are $a = 2.5$, $\mu = 0.8$, $p = 0.7$ and $G(z) = 0.4z + 0.3z^3 + 0.2z^6 + 0.1z^{10}$. It can be observed that the distribution at pre-arrival and arbitrary epochs are the same for both EAS and LAS-DA which proves the accuracy of our analysis. In Table 2 the results for deterministic inter-renewal distribution has been displayed for which the input parameters are $a = 5$, $\mu = 0.8$, $p = 0.6$ and $G(z) = 0.3z + 0.3z^3 + 0.2z^6 + 0.1z^{10} + 0.1z^{12}$. Similarly, Table 3 displays the results for an arbitrarily chosen non-geometric inter-renewal distribution with the input parameters being $A(z) = 0.5z^3 + 0.3z^6 + 0.2z^{10}$, $\mu = 0.9$, $p = 0.6$ and $G(z) = 0.4z + 0.3z^5 + 0.2z^9 + 0.1z^{12}$. The fourth and seventh column of each of the Tables 1–3 shows the ratio of the distribution at pre-arrival epoch for EAS and LAS-DA policy, respectively. One may observe that after certain large value of n, this ratio

Table 1. Population size distribution at various epochs for geometric inter-renewal distribution.

	EAS			LAS-DA		
n	Q_n^-	Q_n	Q_{n+1}^-/Q_n^-	P_n^-	P_n	P_{n+1}^-/P_n^-
0	0.22727273	0.22727273	0.23116883	0.13636364	0.13636364	0.49783550
1	0.05253837	0.05253837	0.73678681	0.06788666	0.06788666	0.46595144
2	0.03870958	0.03870958	1.23617953	0.03163189	0.03163189	1.96565744
3	0.04785199	0.04785199	0.76156229	0.06217745	0.06217745	0.57619415
4	0.03644227	0.03644227	0.98649769	0.03582628	0.03582628	0.89448409
5	0.03595022	0.03595022	1.17427045	0.03204604	0.03204604	1.71644464
6	0.04221528	0.04221528	0.77521812	0.05500526	0.05500526	0.63568859
7	0.03272605	0.03272605	0.94802593	0.03496621	0.03496621	0.89406431
8	0.03102514	0.03102514	1.00515275	0.03126204	0.03126204	1.04180705
70	0.00041291	0.00041291	0.93170026	0.00046393	0.00046393	0.93170026
71	0.00038471	0.00038471	0.93170026	0.00043224	0.00043224	0.93170026
72	0.00035844	0.00035844	0.93170026	0.00040272	0.00040272	0.93170026
73	0.00033395	0.00033395	0.93170026	0.00037522	0.00037522	0.93170026
150	0.00000144	0.00000144	0.93170026	0.00000162	0.00000162	0.93170026
200	0.00000004	0.00000004	0.93170026	0.00000005	0.00000005	0.93170026
$\geq$250	0.00000000	0.00000000	0.93170026	0.00000000	0.00000000	0.93170026
Sum	1.00000000	1.00000000	–	1.00000000	1.00000000	–
Mean	11.60000000	11.60000000	–	13.0000000	13.0000000	–
Variance	195.22285714	195.22285714	–	201.38285714	201.38285714	–

Table 2. Population size distribution at various epochs for deterministic inter-renewal distribution.

	EAS			LAS-DA		
n	Q_n^-	Q_n	Q_{n+1}^-/Q_n^-	P_n^-	P_n	P_{n+1}^-/P_n^-
0	0.33961693	0.21635249	0.14292489	0.28242504	0.14842910	0.18432827
1	0.04853971	0.05927009	0.93289531	0.05205892	0.06993916	0.91249298
2	0.04528247	0.04617590	0.96543941	0.04750340	0.04003179	0.99111959
3	0.04371748	0.05641964	0.92450408	0.04708155	0.07077011	0.88766519
4	0.04041699	0.04023631	0.97791113	0.04179265	0.03768834	1.00291230
5	0.03952422	0.04230588	0.95371116	0.04191437	0.03954301	0.97183507
6	0.03769469	0.04805906	0.92369530	0.04073385	0.05909930	0.89564231
7	0.03481841	0.03601495	0.96617469	0.03648296	0.03567956	0.99067445
8	0.03364067	0.03626422	0.94421390	0.03614274	0.03580795	0.97113720
80	0.00003534	0.00004367	0.90927435	0.00003909	0.00004830	0.90927435
81	0.00003213	0.00003971	0.90927435	0.00003554	0.00004391	0.90927435
82	0.00002922	0.00003610	0.90927435	0.00003231	0.00003993	0.90927435
83	0.00002657	0.00003283	0.90927435	0.00002938	0.00003631	0.90927435
160	0.00000002	0.00000002	0.90927435	0.00000002	0.00000002	0.90927435
$\geq$180	0.00000000	0.00000000	0.90927435	0.00000000	0.00000000	0.90927435
Sum	1.00000000	1.00000000	–	1.00000000	1.00000000	–
Mean	7.74240657	9.43251171	–	8.517708683	10.35251171	–
Variance	105.3281712	114.6456098	–	110.0629619	117.6094164	–

TABLE 3. Population size distribution at various epochs for arbitrary inter-renewal distribution.

n	EAS Q_n^-	EAS Q_n	EAS Q_{n+1}^-/Q_n^-	LAS-DA P_n^-	LAS-DA P_n	LAS-DA P_{n+1}^-/P_n^-
0	0.39985871	0.30678144	0.11768653	0.33725265	0.23133640	0.15776322
1	0.04705798	0.05825179	0.88521289	0.05320606	0.07955094	0.75511632
2	0.04165633	0.03962938	1.03084046	0.04017677	0.03532124	1.07425265
3	0.04294103	0.04274837	1.00338262	0.04316000	0.03779016	1.06747682
4	0.04308629	0.04642884	0.95804718	0.04607230	0.04154019	1.03973819
5	0.04127870	0.05260402	0.86563506	0.04790313	0.07070089	0.76408197
6	0.03573229	0.03708117	0.98557523	0.03660192	0.03611826	1.01757380
7	0.03521686	0.03772476	0.96362337	0.03724515	0.03613475	1.01909669
8	0.03393579	0.03893316	0.92052360	0.03795641	0.03761868	0.99560604
80	0.00001111	0.00001337	0.89510898	0.00001266	0.00001523	0.89510898
81	0.00000994	0.00001196	0.89510898	0.00001133	0.00001363	0.89510898
82	0.00000890	0.00001071	0.89510898	0.00001014	0.00001220	0.89510898
83	0.00000797	0.00000959	0.89510898	0.00000908	0.00001092	0.89510898
140	0.00000001	0.00000002	0.89510898	0.00000002	0.00000002	0.89510898
≥160	0.00000000	0.00000000	0.89510898	0.00000000	0.00000000	0.89510898
Sum	1.00000000	1.00000000	–	1.00000000	1.00000000	–
Mean	6.24249314	7.41065218	–	7.043292968	8.335180485	–
Variance	76.24236545	83.35057643	–	81.41680186	87.59998259	–

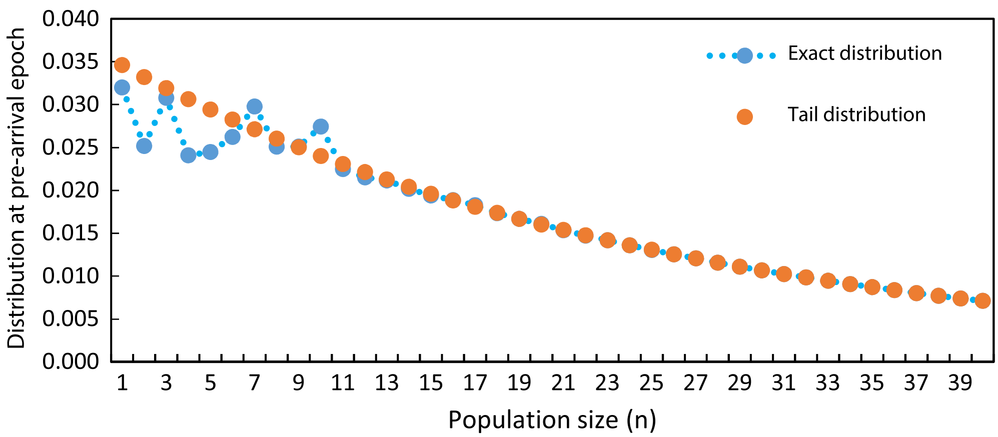

FIGURE 3. Exact and tail probabilities of geometric inter-renewal distribution with EAS policy.

coincides with the root having largest modulus of (5.1), which validates our theoretical work. The average population size (mean) and the variance are respectively given in the last row of each table. One significant observation is that the mean and variance is less in EAS as compared to the LAS-DA policy. This justifies that in discrete time set-up, the order of arrivals plays an important role in determining the performance of the system.

Further, in Figure 3 the exact and tail probabilities at pre-arrival epoch (Q_n^-) for EAS policy have been plotted with geometric inter-renewal distribution. The parameters taken are $a = 3.33$, $\mu = 0.8$, $p = 0.65$, $G(z) = 0.35z + 0.3z^3 + 0.2z^7 + 0.15z^{10}$, $\bar{g} = 4.15$. One may observe that the probabilities approximated by the

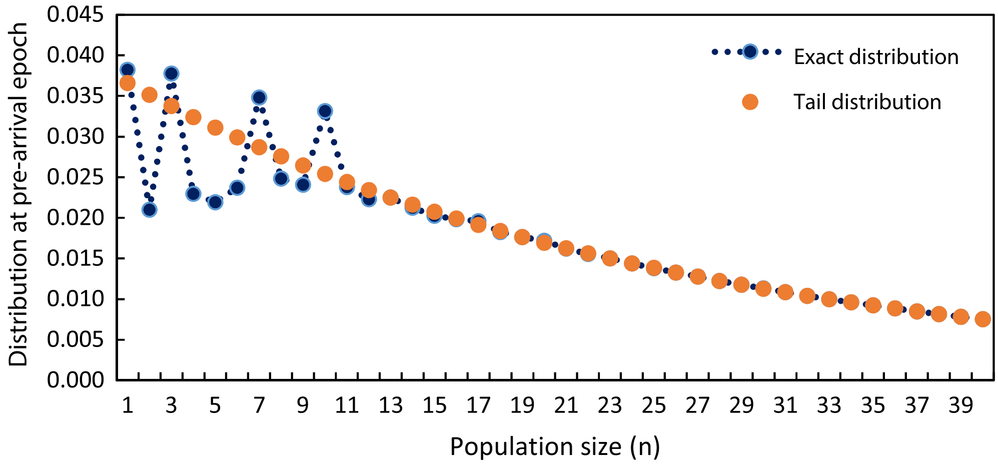

FIGURE 4. Exact and tail probabilities of geometric inter-renewal distribution with LAS-DA policy.

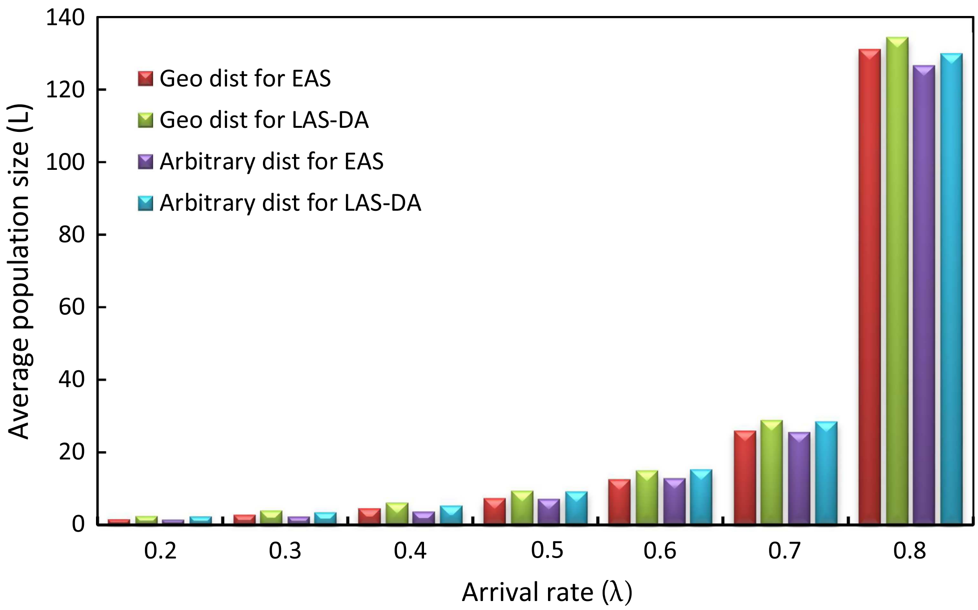

FIGURE 5. Effect of inter-renewal distribution on average population size.

single root r_b initially shows significant divergence, but coincides with the exact probabilities after certain large value of n (here $n = 13$). Thus we can conclude that the tail probabilities Q_n^- can be estimated well using the single largest modulus root of the root equation (5.1). The same has been repeated for the LAS-DA policy in Figure 4 with the same parameters as mentioned above, and thus a similar conclusion can be drawn for the distribution P_n^-.

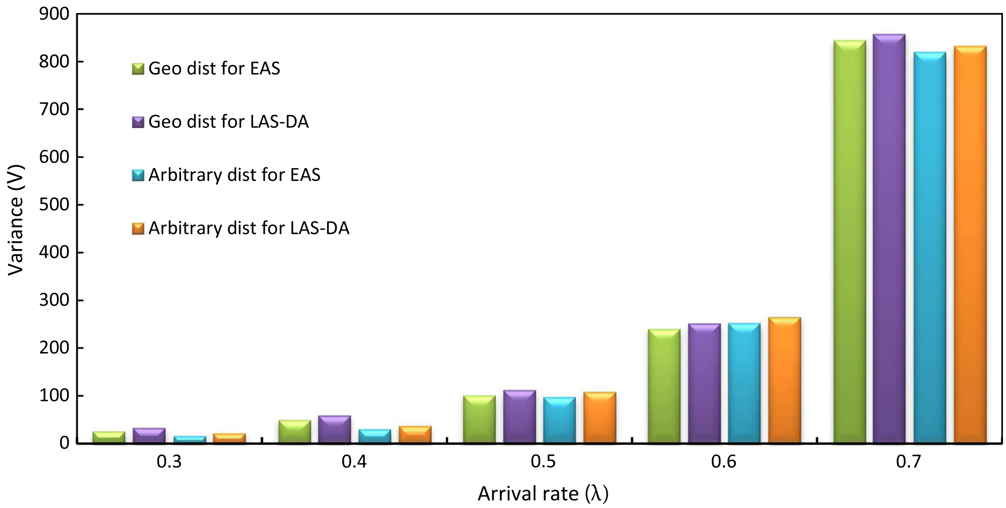

FIGURE 6. Effect of inter-renewal distribution on variance of population size.

Figure 5 displays the impact of inter-renewal distribution on the average population size at arbitrary epoch (L). The input parameters taken are $\mu = 0.85$, $p = 0.8$, $G(z) = 0.4z + 0.3z^3 + 0.2z^8 + 0.1z^{12}$, with inter-renewal distributions as geometric and arbitrary (non-geometric). One may observe that for a fixed arrival rate (λ), L is greater for geometric distribution than the non-geometric distribution, for both EAS and LAS-DA policies. Also L increases with the increase in λ, but after certain λ (here $\lambda = 0.7$), the increase in L is substantial. Similarly, the impact on the variance of population size at arbitrary epoch (V) with the change in inter-renewal distribution is shown in Figure 6 for both EAS and LAS-DA policies with the same parameters as given above. One may note that for this case as well we experience the same behavior as given in the description of Figure 5 for L. We can thus conclude that the choice of inter-renewal distribution have a major impact on the performance of the system.

7. Conclusion

In this paper, we have considered a discrete-time catastrophic model where the population is assumed to grow according to batch renewal arrival process and is subject to geometric catastrophes. Keeping in mind the order of arrival of batches and the occurrence of catastrophes, which plays a significant role in discrete-time models, we have studied both late arrival system with delayed access (LAS-DA) and early arrival system (EAS). The procedure used throughout the paper is based on the supplementary variable technique and difference equation method. The complete analysis has been carried out in such a manner that it does not involve much complexity and is easily tractable for the readers. We have obtained steady-state population size distribution at pre-arrival and arbitrary epochs in a readily presentable form and deduced some important performance measures related to the system. We have presented some numerical results and discussed the behavior of the tail distribution at pre-arrival epoch. We finally hope that the analysis done throughout the paper will be useful for the researchers as well as system designers.

Appendix A.

Theorem A.1. $\sum_{n=0}^{\infty} P_n(0) = \frac{1}{a} = \lambda.$

Proof. Adding (3.8)–(3.11) for all values of n, we get

$$z\sum_{n=0}^{\infty} P_n^*(z) = \overline{\mu}\sum_{n=0}^{\infty}(P_n^*(z) - P_n(0)) + \mu\sum_{k=0}^{\infty} p^k(P_k^*(z) - P_k(0)) + \mu\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} p^k q(P_{k+n}^*(z) - P_{k+n}(0)) + A(z)\overline{\mu}\left(\sum_{n=1}^{b}\sum_{i=1}^{n} g_i P_{n-i}(0) + \sum_{n=b+1}^{\infty}\sum_{i=1}^{b} g_i P_{n-i}(0)\right) + A(z)\mu\left(\sum_{n=1}^{b} g_n \sum_{k=0}^{\infty} p^k P_k(0) + \sum_{n=2}^{b}\sum_{i=1}^{n-1} g_i \sum_{k=0}^{\infty} p^k q P_{k+n-i}(0) + \sum_{n=b+1}^{\infty}\sum_{i=1}^{b} g_i \sum_{k=0}^{\infty} p^k q P_{k+n-i}(0)\right).$$

Readjusting the range of summation in 3rd and 5th term, we obtain

$$z\sum_{n=0}^{\infty} P_n^*(z) = \overline{\mu}\sum_{n=0}^{\infty}(P_n^*(z) - P_n(0)) + \mu\sum_{k=0}^{\infty} p^k(P_k^*(z) - P_k(0)) + \mu\sum_{n=1}^{\infty}\sum_{k=n}^{\infty} p^{k-n} q(P_k^*(z) - P_k(0)) + A(z)\overline{\mu}\left(\sum_{n=1}^{b}\sum_{i=1}^{n} g_i P_{n-i}(0) + \sum_{n=b+1}^{\infty}\sum_{i=1}^{b} g_i P_{n-i}(0)\right) + A(z)\mu\left(\sum_{n=1}^{b} g_n \sum_{k=0}^{\infty} p^k P_k(0) + \sum_{n=2}^{b}\sum_{i=1}^{n-1} g_i \sum_{k=n-i}^{\infty} p^{k-n+i} q P_k(0) + \sum_{n=b+1}^{\infty}\sum_{i=1}^{b} g_i \sum_{k=n-i}^{\infty} p^{k-n+i} q P_k(0)\right).$$

Changing the order of summations, we get

$$z\sum_{n=0}^{\infty} P_n^*(z) = \overline{\mu}\sum_{n=0}^{\infty}(P_n^*(z) - P_n(0)) + \mu\sum_{k=0}^{\infty} p^k(P_k^*(z) - P_k(0)) + \mu\sum_{k=1}^{\infty}\left(\sum_{n=1}^{k} p^{k-n} q\right)(P_k^*(z) - P_k(0)) + A(z)\overline{\mu}\left(\sum_{i=1}^{b}\sum_{n=i}^{b} g_i P_{n-i}(0) + \sum_{n=b+1}^{\infty}\sum_{i=1}^{b} g_i P_{n-i}(0)\right) + A(z)\mu\left(\sum_{n=1}^{b} g_n \sum_{k=0}^{\infty} p^k P_k(0) + \sum_{i=1}^{b}\sum_{n=i+1}^{b+1} g_i \sum_{k=n-i}^{\infty} p^{k-n+i} q P_k(0) + \sum_{n=b+2}^{\infty}\sum_{i=1}^{b} g_i \sum_{k=n-i}^{\infty} p^{k-n+i} q P_k(0)\right).$$

Using $p^0 = 1$ and $\sum_{i=1}^{b} g_i = 1$, we obtain

$$z\sum_{n=0}^{\infty} P_n^*(z) = \overline{\mu}\sum_{n=0}^{\infty}(P_n^*(z) - P_n(0)) + \mu\sum_{k=0}^{\infty} p^k(P_k^*(z) - P_k(0))$$

$$+ \mu\sum_{k=0}^{\infty}(1-p^k)(P_k^*(z) - P_k(0)) + A(z)\overline{\mu}\left(\sum_{i=1}^{b}\sum_{n=i}^{\infty} g_i P_{n-i}(0)\right)$$

$$+ A(z)\mu\left(\sum_{k=0}^{\infty} p^k P_k(0) + \sum_{i=1}^{b}\sum_{n=i+1}^{\infty} g_i \sum_{k=n-i}^{\infty} p^{k-n+i} q P_k(0)\right).$$

Combining the first three terms and readjusting the range of summations, we get

$$z\sum_{n=0}^{\infty} P_n^*(z) = \sum_{n=0}^{\infty}(P_n^*(z) - P_n(0)) + A(z)\overline{\mu}\left(\sum_{i=1}^{b} g_i \sum_{k=0}^{\infty} P_k(0)\right)$$

$$+ A(z)\mu\left(\sum_{k=0}^{\infty} p^k P_k(0) + \sum_{i=1}^{b} g_i \sum_{l=1}^{\infty}\sum_{k=l}^{\infty} p^{k-l} q P_k(0)\right).$$

After simplification, we obtain

$$z\sum_{n=0}^{\infty} P_n^*(z) = \sum_{n=0}^{\infty}(P_n^*(z) - P_n(0)) + A(z)\overline{\mu}\left(\sum_{k=0}^{\infty} P_k(0)\right)$$

$$+ A(z)\mu\left(\sum_{k=0}^{\infty} p^k P_k(0) + \sum_{k=0}^{\infty}(1-p^k) P_k(0)\right).$$

Combining the coefficient of $A(z)$, we get

$$z\sum_{n=0}^{\infty} P_n^*(z) = \sum_{n=0}^{\infty}(P_n^*(z) - P_n(0)) + A(z)\sum_{k=0}^{\infty} P_k(0),$$

which leads to

$$\sum_{n=0}^{\infty} P_n^*(z) = \frac{A(z)-1}{z-1}\sum_{k=0}^{\infty} P_k(0).$$

Taking limit $z \to 1$ and applying L'Hôspital's rule in the above equation, we finally get the desired result. □

Theorem A.2. *The equation* $A\Big(1 - \frac{\mu p(1-s)}{1-ps}\Big)\sum_{i=1}^{b} g_i s^{b-i} - s^b = 0$ *has exactly* b *roots inside the unit circle* $|s| = 1$.

Proof. Let us assume $g(s) = A\Big(1 - \frac{\mu p(1-s)}{1-ps}\Big)\sum_{i=1}^{b} g_i s^{b-i}$ and $f(s) = -s^b$. Let $K(s) = A\Big(1 - \frac{\mu p(1-s)}{1-ps}\Big)$ which can be expressed as $K(s) = \sum_{i=0}^{\infty} k_i s^i$, where the coefficients are such that $k_i \geq 0 \ \forall\ i \geq 0$. Consider the circle

$|s| = 1-\delta$ with $\delta(>0)$ is sufficiently small, we have

$$\begin{aligned}
|f(s)| &= (1-\delta)^b = 1 - b\delta + o(\delta),\\
|g(s)| &= \big|K(s)\big|\Big|\sum_{i=1}^{b} g_i s^{b-i}\Big|\\
&\le K(|s|)\sum_{i=1}^{b} g_i|s|^{b-i} \;=\; K(1-\delta)\sum_{i=1}^{b} g_i(1-\delta)^{b-i},\\
&= 1 - b\delta + \left(\bar{g} - \frac{\mu a p}{q}\right)\delta + o(\delta).
\end{aligned}$$

Since $\lambda\bar{g} < \frac{\mu p}{q}$ and δ is a very small quantity, we have $|g(s)| < |f(s)|$ on the circle $|s| = 1-\delta$. Thus, from Rouché's theorem we can say that $f(s)$ and $f(s)+g(s)$ have exactly b zeroes inside the unit circle. Moreover, from the results given in Abolnikov and Dukhovny [1] it can be shown that if $g'(1) > b$, then the function $s^b - g(s)$ have exactly b roots inside the open unit disk. From this the condition $\bar{g}q < a\mu p$ can be derived which thus confirms that it is necessary as well as sufficient condition for the stability of the system. □

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