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PERFORMANCE EVALUATION OF PORTFOLIOS WITH FUZZY RETURNS

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Abstract. The existing literature on DEA (Data Envelopment Analysis) for evaluating fuzzy portfolios usually takes risk as an input and return as an output. This assumption is actually not congruent with the real investment process, where the input is the initial wealth and the output is the corresponding terminal wealth. As for the risk and return, which are essentially two indicators derived from the terminal wealth, both should be regarded as outputs. In addition, few studies have employed the diversification model (nonlinear DEA) to estimate the fuzzy portfolio efficiency (PE), despite the fact that there are many studies available within the framework of classical probability theory. Further, the relationship between DEA and diversification models needs to be defined. In this paper, we take the initial wealth as an input, while the return and risk of terminal wealth are taken as desirable and undesirable outputs, respectively. We construct different evaluation models under the fuzzy portfolio framework. The relationships among the evaluation model based on a real frontier, the diversification model and the DEA model are investigated. We show the convergence of the diversification and DEA models under the fuzzy theory framework. Some simulations as well as empirical analysis are presented to further verify the effectiveness of the proposed models. Finally, we check the robustness of the evaluation results by using the bootstrap re-sampling approach.

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1. Introduction

Since Markowitz [29] proposed the mean-variance portfolio theory, the performance evaluation of portfolios has been a hot research topic in the field of financial studies. There are now many portfolio efficiency evaluation approaches, among which the most famous ones are the three classic indexes presented in Treynor [38], Sharpe [36] and Jensen [20]. The above three indexes are all derived from the Capital Asset Pricing Model (CAPM). However, there are many anomalies that cannot be explained by using the CAPM (single factor model), such as the size of stocks for the fund holdings, book-to-market ratio and momentum characteristics. Therefore, some researchers have turned to the construction of multi-factor models to deal with this dilemma, such as Fama and French [17] and Carhart [10]. Although these multi-factor models can effectively compensate for some weaknesses of earlier single-factor model, the exact number of chosen factors in these models always remains to

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be controversial with the evaluation results being more sensitive to factors. So far, there is no unified standard for choosing factors. Moreover, these evaluation indexes do not consider the effect of market friction factors existed in the practical investment process, such as transaction costs, taxes, trading volume and other friction factors.

Motivated by the seminal work of Markowitz [29], some researchers have turned to using a frontier-based approach to evaluate the portfolio performance recently. This approach uses the distance between the portfolio being evaluated and its projection on the frontier as a measure of portfolio performance. As far as we know, there are mainly two frontier-based approaches which are widely used to assess the portfolio performance. One of them is the diversification model. Morey and Morey [32] first provided a diversification model under mean-variance framework, where the variance and expected values of fund's return were regarded as the input and output, respectively. Under the mean-variance framework, Briec *et al.* [6] established another quadratic nonlinear constrained DEA model, and the efficiency improvement function is introduced in this model. Joro and Na [21] considered the effect of skewness on the portfolio performance, and then established a diversification model under the mean-skewness-variance criterion. By incorporating the Joro and Na [21] mean-variance portfolio theory and stochastic dominance theory, Lozano and Gutiérrez [28] constructed a diversification model with stochastic dominance constraints. Lamb and Tee [23] developed a diversification model with multiple return measures and consistent risk measures. Branda [2] extended the conclusion of Lamb and Tee [23] by replacing the consistent risk measure with a general deviation measure. Branda [4] constructed several kinds of diversification models while allowing inputs and outputs to be negative, by using the direction distance function measure. Among many others, readers may refer to Briec *et al.* [7], Zhao *et al.* [43], Branda [3] and so on. Although the diversification model can effectively diversify risk, its nonlinear restrictions lead to a failure in the large-scale computations. Thus, some researchers have tried to use DEA to evaluate the portfolio performance. Murthi *et al.* [33] first applied DEA into the performance evaluation of portfolios. Basso and Funari [1] used DEA approach to assess fund performance where different risk indicators were regarded as input indicators. For more applications of DEA approach in the performance evaluation of portfolios, readers may refer to Chen and Lin [12], Ding *et al.* [15], Zhou *et al.* [46] and Zhou *et al.* [48] and so on. Compared with diversification model, DEA model is more effective for large-scale computations, since it is essentially a linear programming approach. However, its role in diversification has also been questioned by many researchers. To this problem, Liu *et al.* [27] systematically investigated the theoretical justifications for applying DEA to estimating portfolio performance. These authors have shown that the DEA frontier can converge to the portfolio frontier when adequate portfolio samples exist.

The above portfolio performance evaluation problems are mainly based on classic probability theory to discuss the PE. To be more specific, the uncertainty is assumed to be a stochastic phenomenon (*e.g.*, Huang *et al.* [19], Cao *et al.* [9], Zhou *et al.* [45], Zhou *et al.* [47], Zhou *et al.* [49] and so on). However, there are vast fuzzy phenomena in the financial market, and the role of fuzzy characteristic is very important in some situations (*e.g.*, Cao and Lai [8], Liu *et al.* [26], Zhou *et al.* [44], Liu and Zhang [25] and so on). For some portfolio problems, the classic probability theory lacks flexibility to address the investor's subjective intentions, while the fuzzy theory can describe it effectively. In this situation, the fuzzy theory proposed by Zadeh [40] can address the portfolio optimization problems with fuzzy returns. Zadeh [40] discussed this application of possibilistic theory in portfolio optimization. Huang [18] adopted the fuzzy entropy as the risk measure, and then provided two kinds of portfolio optimization models. Qin *et al.* [34] applied the fuzzy cross entropy measure to investigate the portfolio optimization problem with fuzzy returns. Kamdem *et al.* [22] discussed a generalized portfolio optimization problem under the framework of credibilistic mean-variance-skewness-semi-kurtosis. Zhang *et al.* [44] extended the earlier work into a multi-period possibilistic mean-semivariance-entropy portfolio optimization problem with a constraint on transaction costs. For more literatures, readers may refer to Liu and Liu [24], Zhang and Zhang [41], Liu and Zhang [25] and Mehlawat [31].

Lately, some researchers have made some attempts to combine DEA into fuzzy portfolio optimization problems. Mashayekhi and Omrani [30] incorporated the DEA cross-efficiency into the Markowitz mean–variance model, where the returns of risky assets were described as trapezoidal fuzzy numbers. Chen *et al.* [14] incorporated the mean-semivariance and DEA cross-efficiency models, and then proposed a multi-objective portfolio optimization model in a fuzzy environment. The above literature clearly aims to provide a novel investment strategy that considers DEA cross-efficiencies, while not involving how to assess the performances of portfolios.

Along the aforementioned lines of research, we find that the most of portfolio evaluation models are developed under the framework of classic probability theory. Furthermore, the existing fuzzy portfolio studies mainly focus on portfolio optimization instead of portfolio evaluation. Although Chen *et al.* [13] presented three kinds of DEA models to estimate the fuzzy PE, they still adopted the traditional input-output process (*i.e.*, the risk was an input and the return was an output). More importantly, they do not provide any theoretical foundation to assure that the DEA is a solid method to estimate fuzzy PE. This motivates us to reinvestigate the performance evaluation of portfolios with fuzzy returns. To this end, we redefine the input-output process in accordance with the actual investment process. In this paper, we hold that both return and risk should be treated as outputs, being that they are two derivative indicators from the terminal return of a portfolio. Under this input-output process, we first define the fuzzy PE under the criterion of possibilistic mean-variance-entropy. Subsequent to this, the evaluation model based on real frontier, the diversification model and the traditional DEA model are developed by using directional distance function. This indicates that the diversification and DEA models can be regarded as the nonlinear and linear estimations of the model based on the real frontier, respectively. We show that the efficiencies derived from diversification and DEA models are both convergent on the real PE when the size of portfolio sample size is large enough. We discuss the differences between the DEA model and diversification model by using simulations. These results show that the difference in the convergence rates between traditional DEA model and diversification model is not apparent. For the empirical analysis, we randomly select 50 open-end funds from China fund market to check the feasibility of the above models. Furthermore, the bootstrap re-sampling approach is used to verify the robustness of the above evaluation results.

The remainder of this paper is organized as follows. In Section 2, we introduce some related definitions about fuzzy theory, and then derive the formulation of portfolio with fuzzy returns. In Section 3, we construct three different portfolio evaluation models under the criterion of possibilistic mean-variance-entropy, and then investigate the convergence property of the estimation of PE. We also check the feasibility of the proposed models by some numerical simulations. In Section 4, we carry out an empirical study of the 50 open-end funds from China fund market. Some concluding remarks are summarized in the end.

2. Portfolios with fuzzy returns

2.1. Basic definitions of fuzzy variables

According to the conclusions of, Vercher *et al.* [39], and Saeidifar and Pasha [35], we will introduce some basic definitions of fuzzy variables in this section. The main conclusions are as follows.

Definition 2.1. Let $\mathbf{R}$ denote the set of all real numbers. A fuzzy number A is a fuzzy set of $\mathbf{R}$ with a membership function $\mu_A(y):\ \mathbf{R}\to[0,\ 1]$, and A should satisfy the following conditions:

(1) $\mu_A(y)$ is upper semicontinuous and bounded;
(2) A is normal, $\exists y_0\in\mathbf{R}$ such that $\mu_A(y_0)=1$;
(3) A is fuzzy convex, that is $\forall y_1,\ y_2\in\mathbf{R}$, $\lambda\in[0,\ 1]$

$$\mu_A(\lambda y_1+(1-\lambda)y_2)\geq\min\{\mu_A(y_1),\ \mu_A(y_2)\};$$

(4) Let the γ-level set of A be $[A]^\gamma(0\leq\gamma\leq1)$, among which $[A]^\gamma$ satisfies the following conditions: $[A]^\gamma=\{y|y\in\mathbf{R},\ \mu_A(y)\geq\gamma\}$, and $[A]^\gamma$ can be rewritten as $[A]^\gamma=[\underline{\alpha}(\gamma),\ \bar{\alpha}(\gamma)]$, where $\underline{\alpha}(\gamma)$ and $\bar{\alpha}(\gamma)$ denote the left point and right point of the γ-level of A, respectively.

From the above definitions, for a general LR-type fuzzy number $A=(a,b,c,d)$, the membership function $\mu_A(y)$ can be expressed as

$$\mu_A(y)=\begin{cases}L_A(y) & y\in[a-c,a],\\ 1 & y\in[a,b],\\ R_A(y) & y\in[b,b+d],\\ 0 & \text{otherwise.}\end{cases}\tag{2.1}$$

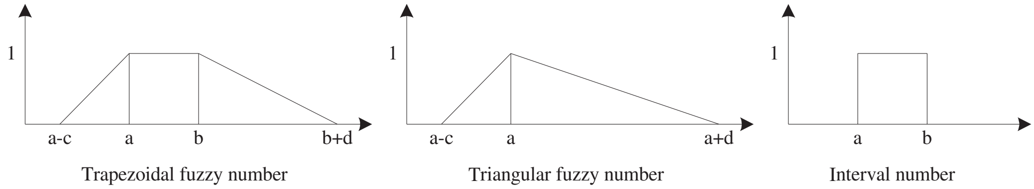

FIGURE 1. The membership functions of different fuzzy numbers.

Here $L_A: [a-c, a] \to [0,1]$ and $R_A: [b, b+d] \to [0,1]$ denote the left and right points of $\mu_A(y)$, respectively, where L_A is a monotone increasing and right-continuous real-valued function, and R_A is a monotone decreasing and left-continuous real-valued function. Therefore, the γ-level set of A can be rewritten as $[A]^\gamma = [L_A^{-1}(\gamma),\ R_A^{-1}(\gamma)]$. In the following, we will show the three common fuzzy numbers popularly used in the existing literature: (i) if L_A and R_A are both linear functions, then A degenerates to the classic trapezoidal fuzzy number; (ii) suppose that L_A and R_A are both linear functions, and $a = b$, then A is triangular fuzzy number; (iii) in addition to condition (i), if $c = d = 0$, then A is interval fuzzy number. The above three situations are shown in Figure 1.

Based on the definition of fuzzy number, we can define the possibilistic expected value, variance, covariance and entropy of fuzzy number A. The main conclusions can be expressed as:

Definition 2.2. Let A be the fuzzy number with $[A]^\gamma = [\underline{\alpha}(\gamma),\ \bar{\alpha}(\gamma)]$, the possibilistic expected value and variance can then be defined.

$$E(A) = \int_0^1 \gamma(\underline{\alpha}(\gamma) + \bar{\alpha}(\gamma))\mathrm{d}\gamma, \tag{2.2}$$

$$\mathrm{Var}(A) = \int_0^1 \gamma\left([E(A) - \underline{\alpha}(\gamma)]^2 + [E(A) - \bar{\alpha}(\gamma)]^2\right)\mathrm{d}\gamma. \tag{2.3}$$

Definition 2.3. For two arbitrary fuzzy numbers A and B, the γ-level sets of A and B are expressed as $[A]^\gamma = [\underline{\alpha}(\gamma),\ \bar{\alpha}(\gamma)]$ and $[B]^\gamma = [\underline{\beta}(\gamma),\ \bar{\beta}(\gamma)]$, respectively. Then, the possibilistic covariance of A and B can be defined.

$$\mathrm{Cov}(A,\ B) = \int_0^1 \gamma\left([E(A) - \underline{\alpha}(\gamma)][E(B) - \underline{\beta}(\gamma)] + [E(A) - \bar{\alpha}(\gamma)][E(B) - \bar{\beta}(\gamma)]\right)\mathrm{d}\gamma. \tag{2.4}$$

Definition 2.4. For an arbitrary fuzzy number A, the membership function $\mu_A(x)$ satisfies (2.1), the possibilistic entropy of A can be defined.

$$H(A) = -\int_{-\infty}^{+\infty}\left[\frac{\mu_A(y)}{2}\ln\frac{\mu_A(y)}{2} + \left(1 - \frac{\mu_A(y)}{2}\right)\ln\left(1 - \frac{\mu_A(y)}{2}\right)\right]\mathrm{d}y. \tag{2.5}$$

2.2. The formulation of portfolio with fuzzy returns

In the following, we assume that the investor joins into the financial market with an initial capital w, and the investor can invest it in n risky assets. Let r_i denote the fuzzy return rate of the ith asset. In addition, let x_i be the investment amount invested in the ith asset. In this case, the investment opportunity set Φ can be represented as

$$\Phi = \left\{\sum_{i=1}^n x_i r_i \,\middle|\, \sum_{i=1}^n x_i = w,\ \ i = 1,\ 2,\ \ldots,\ n\right\} \tag{2.6}$$

Where w is the initial wealth.

Note that if further assume no short-selling, we only need to add an extra constraint $0 \leq x_i \leq w$ ($i = 1,\ 2,\ \ldots,\ n$) into (2.6).

In this paper, we assume that fuzzy return $r_i = (a_i, b_i, c_i, d_i)$ is a trapezoidal fuzzy number. Then the membership function of r_i can be expressed as

$$\mu_{r_i}(y) = \begin{cases} 1 - \dfrac{a_i - y}{c_i} & y \in [a_i - c_i, a_i], \\ 1 & y \in [a_i, b_i], \\ 1 - \dfrac{y - b_i}{d_i} & y \in [b_i, b_i + d_i], \\ 0 & \text{otherwise}. \end{cases} \tag{2.7}$$

For $\forall i,\ j = 1,\ 2,\ \ldots,\ n$, the following conclusions can be derived.

$$E(r_i) = \frac{a_i + b_i}{2} + \frac{d_i - c_i}{6}, \tag{2.8}$$

$$\mathrm{Var}(r_i) = \left(\frac{b_i - a_i}{2} + \frac{d_i + c_i}{6}\right)^2 + \frac{c_i^2}{36} + \frac{d_i^2}{36}, \tag{2.9}$$

$$\begin{aligned}\mathrm{Cov}(r_i,\ r_j) = {} & \frac{a_i a_j}{4} + \frac{b_i b_j}{4} + \frac{c_i c_j}{18} + \frac{d_i d_j}{18} - \frac{a_i}{4}\left(b_j + \frac{c_j}{3} + \frac{d_j}{3}\right) \\ & - \frac{b_i}{4}\left(a_j - \frac{c_j}{3} - \frac{d_j}{3}\right) - \frac{c_i}{12}\left(a_j - b_j - \frac{d_j}{3}\right) - \frac{d_i}{12}\left(a_j - b_j - \frac{c_j}{3}\right),\end{aligned} \tag{2.10}$$

$$H(r_i) = \frac{(c_i + d_i)}{2} - (b_i - a_i)\ln 2. \tag{2.11}$$

Then, we have

$$E\left(\sum_{i=1}^{n} x_i r_i\right) = \sum_{i=1}^{n} x_i E(r_i), \tag{2.12}$$

$$\mathrm{Var}\left(\sum_{i=1}^{n} x_i r_i\right) = \sum_{i=1}^{n} x_i^2 \mathrm{Var}(r_i) + \sum_{i=1}^{n} \sum_{j=1,\ j\neq i}^{n} x_i x_j \mathrm{Cov}(r_i,\ r_j), \tag{2.13}$$

$$H\left(\sum_{i=1}^{n} x_i r_i\right) = \sum_{i=1}^{n} x_i \left[\frac{(c_i + d_i)}{2} - (b_i - a_i)\ln 2\right]. \tag{2.14}$$

In the following, the possibilistic mean is regarded as the return measure, while the possibilistic variance and entropy are treated as the risk measures. Based on (2.12)–(2.14), we will discuss the performance of portfolios with fuzzy returns under the framework of possibilistic mean-variance-entropy.

3. Performance evaluation of portfolios with fuzzy returns

Suppose that there are m portfolios under evaluation. The fuzzy return of the jth portfolio is expressed as Y_j ($Y_j \in \Phi, j = 1, 2, \ldots, m$). The corresponding initial wealth of each portfolio is $w_j (j = 1,\ 2,\ \ldots, m)$. The mean, variance and entropy under the fuzzy criterion are $E(Y_j)$, $\mathrm{Var}(Y_j)$ and $H(Y_j)$, $j = 1, 2, \ldots, m$, respectively. The majority of the existing literature treats the risk as an input and the return as an output in both diversification and DEA models. However, as we explained earlier, the risk and return are indeed two indicators derived from the terminal wealth of the portfolio. Based on the real investment process, we can present the following investment possibility set Ψ.

$$\Psi(w,\ E,\ V,\ H) = \left\{(w,\ E,\ V,\ H) \,\middle|\, \begin{array}{l} w \text{ can produce a return } Y, \text{ which can be} \\ \text{measured by mean, variance and entropy.} \end{array}\right\}$$

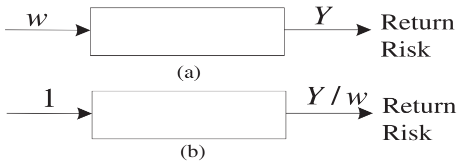

FIGURE 2. The input and output process of portfolio.

As shown in Figure 2a, it is clear that the real input is the investor's initial wealth and the outputs are the return and risk. Note that the return is a desirable output, while the risk is an undesirable one. Thus, for any given initial wealth w, if the strongly free disposability is assumed, the portfolio possibility set $P(w)$ can be expressed as follows:

$$\begin{aligned} P(w) &= \{(E,\ V,\ H)\,|(w,\ E,\ V,\ H) \in \Psi\} \\ &= \left\{ (E,\ V,\ H) \left| \begin{array}{l} E \le E\left(\sum_{i=1}^{n} x_i r_i\right),\ \ V \ge \mathrm{Var}\left(\sum_{i=1}^{n} x_i r_i\right), \\ H \ge H\left(\sum_{i=1}^{n} x_i r_i\right),\ \sum_{i=1}^{n} x_i = w,\ i = 1,\ 2,\ \ldots,\ n. \end{array} \right. \right\} \end{aligned} \tag{3.1}$$

In real applications, the initial wealth is often unknown. However, it can always be rescaled to unity. The terminal wealth is rescaled to the rate of return Y/w accordingly, as shown in Figure 2b. In the following, we assume that the initial wealth $w = 1$. The portfolio possibility set (3.1) can be rewritten as

$$P = \left\{ (E,\ V,\ H) \left| \begin{array}{l} E \le E\left(\sum_{i=1}^{n} x_i r_i\right),\ \ V \ge \mathrm{Var}\left(\sum_{i=1}^{n} x_i r_i\right), \\ H \ge H\left(\sum_{i=1}^{n} x_i r_i\right),\ \sum_{i=1}^{n} x_i = 1,\ i = 1,\ 2,\ \ldots,\ n \end{array} \right. \right\} \tag{3.2}$$

Where r_i denote the fuzzy return rate of ith asset.

Note that (3.2) share the same formulation as that of the output possibility set. In the following, we only discuss the PE under output orientation.

3.1. Portfolio performance evaluation based on the real frontier

In order to address the undesirable output (*i.e.*, risk), we adopt the directional distance function measure to assess the performance of portfolios. Based on the real frontier, for any given direction, we can calculate the PE and the projection on the frontier for each portfolio, as shown in Figure 3.

According to the above projection approach, for each portfolio with fuzzy return $Y_0 \in \Phi$, the following evaluation model can be derived by using the directional distance function measure

$$D_g(Y_0) = \sup\left\{\theta|\,(E(Y_0) + \theta g_E,\ \mathrm{Var}(Y_0) - \theta g_V,\ H(Y_0) - \theta g) \in P\right\}. \tag{3.3}$$

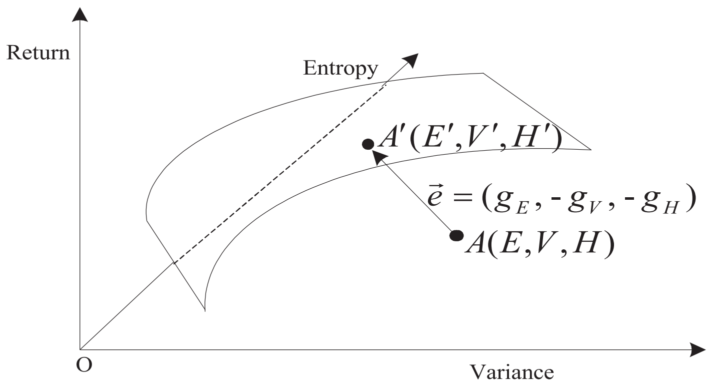

FIGURE 3. The projection of portfolio for a given direction.

This can be rewritten as

$$\theta^1(Y_0) = \max\ \theta$$
$$\text{s.t.} \begin{cases} E\left(\sum_{i=1}^{n} x_i r_i\right) \geq E(Y_0) + \theta g_E \\ \text{Var}\left(\sum_{i=1}^{n} x_i r_i\right) \leq \text{Var}(Y_0) - \theta g_V \\ H\left(\sum_{i=1}^{n} x_i r_i\right) \leq H(Y_0) - \theta g_H \\ \sum_{i=1}^{n} x_i = 1, \;\; i = 1,\ 2,\ \ldots,\ n. \end{cases} \tag{3.4}$$

If all the mean, variance and entropy of portfolios are all positive, we can set the direction as $g_E = E(Y_0)$, $g_V = \text{Var}(Y_0)$ and $g_H = H(Y_0)$. Then, Model (3.4) is consistent with the radial one. If some negative data exists, the direction can be set as $g_E = \max_{Y\in\Phi} E(Y) - E(Y_0)$, $g_V = \text{Var}(Y_0) - \min_{Y\in\Phi} \text{Var}(Y)$, $g_H = H(Y_0) - \min_{Y\in\Phi} H(Y)$, suggested by and Branda [4].

Note that there exist many market friction factors in the practice of investment process, such as transaction costs, taxes and trading volume constraints. In these cases, the real frontier is difficult to derive. It also limits the application of Model (3.4) in the actual performance evaluation of portfolios.

3.2. Portfolio performance evaluation *via* diversification model

In real applications, the data for individual assets are more difficult to obtain than that for portfolios. For example, it is easy to obtain the data for mutual funds, but it is difficult to know all the detailed underlying assets of each mutual fund. In this situation, we cannot use Model (3.4) due to the lack of underlying asset data.

Using the assumption of convexity, we can define a virtual fuzzy portfolio $Y = \sum_{j=1}^{m} \lambda_j Y_j$ with conditions of $\lambda_j \geq 0$ $(j = 1, ..., m)$ and $\sum_{j=1}^{m} \lambda_j = 1$. Motivated by the idea of diversification, we can directly construct the following portfolio probability set P_1 by using these m portfolios under evaluation:

$$P_1 = \left\{ (E,\ V,\ H) \middle| \begin{array}{l} E \leq E\left(\sum_{j=1}^{m} \lambda_j Y_j\right),\ \ V \geq \operatorname{Var}\left(\sum_{j=1}^{m} \lambda_j Y_j\right), \\ H \geq H\left(\sum_{j=1}^{m} \lambda_j Y_j\right),\ \sum_{j=1}^{m} \lambda_j = 1,\ \ \lambda_j \geq 0,\ j = 1,\ 2,\ \ldots, m. \end{array} \right\} \tag{3.5}$$

Comparing (3.2) with (3.5), we can see that P_1 is derived from existing portfolios, where the P is built based on the underlying assets of the portfolios. Evidently, P_1 is a subset of P; that is, P_1 can be regarded as a nonlinear estimation of the portfolio probability set P.

Under the output orientation, for each portfolio being evaluated with a fuzzy return $Y_0 \in \Phi$, supposing that $(E(Y_0) + \theta g_E,\ \operatorname{Var}(Y_0) - \theta g_V,\ H(Y_0) - \theta g) \in P_1$, the diversification model with the directional distance function measure is constructed as

$$\begin{aligned} &\theta^2(Y_0) = \max\ \theta \\ &\text{s.t.} \begin{cases} E\left(\sum_{j=1}^{m} \lambda_j Y_j\right) \geq E(Y_0) + \theta g_E \\ \operatorname{Var}\left(\sum_{j=1}^{m} \lambda_j Y_j\right) \leq \operatorname{Var}(Y_0) - \theta g_V \\ H\left(\sum_{j=1}^{m} \lambda_j Y_j\right) \leq H(Y_0) - \theta g_H \\ \sum_{j=1}^{m} \lambda_j = 1,\ \ \lambda_j \geq 0,\ j = 1,\ 2,\ \ldots, m. \end{cases} \end{aligned} \tag{3.6}$$

As we can see that, Model (3.6) is derived by using the convex combination of the portfolio data rather than that of the individual assets, which is the most significant difference between Models (3.4) and (3.6). In this case, the friction information in the financial market is already contained in the portfolio data. Therefore, we no longer need to consider the extra friction factors in the evaluation process. Additionally, the diversification model remains to be the same form for investment situations with different market frictions. Most importantly, Model (3.6) is a convex quadratic programming problem, and thus we can always find a global solution by using existing optimization algorithms.

3.3. Portfolio performance evaluation via DEA model

Although Model (3.6) is a good approximation of the evaluation model based on the real frontier, it lacks effectiveness in dealing with large-scale portfolio evaluation problems. Under the classic probability theory framework, Branda [4] indicated that DEA approach can perform well when the portfolio samples are large enough. In this section, we will show the conclusion of Liu *et al.* [27] is also valid for the portfolio evaluation under the framework of fuzzy theory. Similarly, we suppose that there exist m portfolios under evaluation, and their fuzzy expected return and risk can be calculated. Under the output orientation, by using the convex combination of the outputs of portfolios rather than that of the fuzzy returns of portfolios, we can construct the following portfolio probability set based on the BCC-DEA approach with three postulate assumptions of convexity, inefficiency and minimum extrapolation.

$$P_2 = \left\{ (E,\ V,\ H) \middle| \begin{array}{l} E \leq \sum_{j=1}^{m} \lambda_j E(Y_j),\ \ V \geq \sum_{j=1}^{m} \lambda_j \mathrm{Var}(Y_j), \\ H \geq \sum_{j=1}^{m} \lambda_j H(Y_j),\ \sum_{j=1}^{m} \lambda_j = 1,\ \ \lambda_j \geq 0,\ j = 1,\ 2,\ \ldots, m. \end{array} \right\} \tag{3.7}$$

From P_1 and P_2, by using the convexity and concavity of the risk and return measures, we can conclude that

$$E\left(\sum_{j=1}^{m} \lambda_j Y_j\right) = \sum_{j=1}^{m} \lambda_j E(Y_j)$$
$$\mathrm{Var}\left(\sum_{j=1}^{m} \lambda_j Y_j\right) \leq \sum_{j=1}^{m} \lambda_j \mathrm{Var}(Y_j)$$
$$H\left(\sum_{j=1}^{m} \lambda_j Y_j\right) \leq \sum_{j=1}^{m} \lambda_j H(Y_j).$$

Obviously, P_2 is a subset of P_1. It indicates that P_2 can be regarded as a linear estimation of P_1. By using the directional distance function measure under the output orientation, for each portfolio being evaluated $Y_0 \in \Phi$, we assume $(E(Y_0) + \theta g_E,\ \mathrm{Var}(Y_0) - \theta g_V,\ H(Y_0) - \theta g) \in P_2$. Then, the following DEA evaluation model can be derived based on P_2:

$$\theta^3(Y_0) = \max\ \theta$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{m} \lambda_j E(Y_j) \geq E(Y_0) + \theta g_E \\ \sum_{j=1}^{m} \lambda_j \mathrm{Var}(Y_j) \leq \mathrm{Var}(Y_0) - \theta g_V \\ \sum_{j=1}^{m} \lambda_j H(Y_j) \leq H(Y_0) - \theta g_H \\ \sum_{j=1}^{m} \lambda_j = 1,\ 0 \leq \lambda_j \leq 1,\ \ i = 1,\ 2,\ \ldots,\ m. \end{cases} \tag{3.8}$$

Evidently, Model (3.8) is a piece-wise linear approximation of diversification model (3.6). Similar to Model (3.6), Model (3.8) can also provide an endogenous benchmark for every inefficient portfolio; however, the output of the benchmark is a linear combination of the outputs of the portfolios under evaluation. Although Model (3.8) does not take into account the diversification role and might also lead to an overestimation of efficiency scores, it is a linear model which can effectively address the large-scale portfolio evaluation problems.

For the above three evaluation models, a portfolio is said to be efficient only when $\theta^i(Y_0) = 0$. Therefore, the efficiency of portfolio being evaluated Y_0 can be expressed as $\theta_i(Y_0) = 1 - \theta^i(Y_0)$, $i = 1,\ 2,\ 3$. Except assessing the performance of portfolios/securities, the above three evaluation models can also provide some advices to further make a better combination of portfolios and securities, *e.g.*, the portfolios/securities with a high score is normally a better choice for the underlying assets of portfolios.

3.4. Theoretical foundations of the proposed approaches: convergence property

According to the portfolio frontier $F(V,\ H) = \sup\{E|(E,\ V,\ H) \in P\}$ generated by Model (3.4), we can conclude the following theorems.

Theorem 3.1. *The portfolio frontier $F(V,\ H)$ generated by Model (3.4) is a concave function.*

Proof. Let $\Omega = \{x| \sum_{i=1}^{n} x_i = 1,\ i = 1,\ 2,\ \ldots,\ n\}$, $f(x) = E(\sum_{i=1}^{n} x_i r_i)$, $g(x) = \mathrm{Var}(\sum_{i=1}^{n} x_i r_i)$ and $h(x) = H(\sum_{i=1}^{n} x_i r_i)$. It is not difficult to find that Ω is a convex set, $f(x)$ and $h(x)$ are both linear function on variable x, while $g(x)$ is a quadratic convex function on x. For $\forall\,(E_1, V_1,\ H_1), (E_2, V_2,\ H_2) \in P$ and $\forall \lambda \in [0, 1]$, $\exists x^1, x^2 \in \Omega$, the following conclusion always holds.

$$f(x^j) \geq E_j, \quad g(x^j) \leq V_j, \quad h(x^j) \leq H_j, \quad j = 1, 2.$$

□

Due to the fact that $\lambda x^1 + (1-\lambda)\, x^2 \in \Omega$ as well as both $f(x)$ and $h(x)$ are linear functions on x, then we have

$$\begin{aligned} f(\lambda x^1 + (1-\lambda)x^2) &= \lambda f(x^1) + (1-\lambda) f(x^2) \\ &= \lambda E_1 + (1-\lambda)E_2. \end{aligned} \tag{3.9}$$

$$\begin{aligned} h(\lambda x^1 + (1-\lambda)x^2) &= \lambda h(x^1) + (1-\lambda) h(x^2) \\ &= \lambda H_1 + (1-\lambda)H_2. \end{aligned} \tag{3.10}$$

Similarly, we obtain that

$$\begin{aligned} g(\lambda x^1 + (1-\lambda)x^2) &\leq \lambda g(x^1) + (1-\lambda) g(x^2) \\ &\leq \lambda V_1 + (1-\lambda)V_2. \end{aligned} \tag{3.11}$$

Then

$$\begin{aligned} &\lambda\,(E_1, V_1,\ H_1) + (1-\lambda)\,(E_2, V_2,\ H_2) \\ &\quad = (\lambda E_1 + (1-\lambda)\,E_2, \lambda V_1 + (1-\lambda)\,V_2,\ \lambda H_1 + (1-\lambda)\,H_2) \in P. \end{aligned} \tag{3.12}$$

Therefore, the portfolio probability set P is a convex set, in other words, $F(V,\ H) = \sup\{E|(E,\ V,\ H) \in P\}$ is a concave function.

Theorem 3.2. *For each portfolio with fuzzy return $Y_0 \in \Phi$, supposing the optimal values of Model (3.4), Model (3.6) and Model (3.8) to be $\theta^1(Y_0)$, $\theta^2(Y_0)$ and $\theta^3(Y_0)$, respectively, we then have*

$$\theta^1(Y_0) \geq \theta^2(Y_0) \geq \theta^3(Y_0).$$

Proof. For $\forall Y_0 \in \Phi$, we can define the corresponding weight vector as $x^0 = (x_1^0,\ x_2^0,\ \ldots x_n^0)'$, then we have

$$\sum_{j=1}^{m} \lambda_j Y_j = \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} x_i^j r_i\right) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \lambda_j x_i^j\right) r_i. \tag{3.13}$$

Due to $\sum\limits_{j=1}^{m} \lambda_j = 1,\ 0 \leq \lambda_j \leq 1,\ \ j = 1,\ 2,\ \ldots,\ m$, we can obtain that

$$\left(\sum_{j=1}^{m} \lambda_j x_1^j,\ \sum_{j=1}^{m} \lambda_j x_2^j,\ \ldots,\ \sum_{j=1}^{m} \lambda_j x_n^j\right) \subseteq \Omega. \tag{3.14}$$

According to the convexity of possibilistic mean and concavities of variance and entropy, we have

$$\max E\left(\sum_{i=1}^{n} x_i r_i\right) \geq \max E\left(\sum_{j=1}^{m} \lambda_j Y_j\right) = \max \sum_{j=1}^{m} \lambda_j E(Y_j), \tag{3.15}$$

TABLE 1. Statistical properties of the selected stocks.

Stock	Sample mean	Sample variance	5th percentile	40th percentile	60th percentile	95th percentile
Sinopec	0.0044	0.0144	−0.1932	−0.0056	0.0227	0.2043
China unicom	0.0052	0.0161	−0.1583	−0.0227	0.0230	0.2008
China life insurance	0.0031	0.0161	−0.2380	−0.0193	0.0248	0.2012
Bank of China	0.0020	0.0061	−0.1333	−0.0130	0.0187	0.1140

$$\min \mathrm{Var}\left(\sum_{i=1}^{n} x_i r_i\right) \le \min \mathrm{Var}\left(\sum_{j=1}^{m} \lambda_j Y_j\right) \le \min \sum_{j=1}^{m} \lambda_j \mathrm{Var}(Y_j), \tag{3.16}$$

$$\min H\left(\sum_{i=1}^{n} x_i r_i\right) \le \min H\left(\sum_{i=1}^{m} \lambda_i Y_i\right) = \min \sum_{i=1}^{m} \lambda_i H(Y_i). \tag{3.17}$$

Therefore, $\theta^1(Y_0) \ge \theta^2(Y_0) \ge \theta^3(Y_0)$, which completes the proof. □

Theorem 3.3. *Suppose that there are m portfolios under evaluation, $\theta^2(Y_0)$ and $\theta^3(Y_0)$ both converge to $\theta^1(Y_0)$ in probability when $m \to \infty$.*

Proof. According to Theorem 3.1, we can find that the portfolio frontier $F(V,\ H)$ generated by Model (3.4) is a concave function. Using the same method in Liu *et al.* [27], we can prove that $\theta^3(Y_0) \to^P \theta^1(Y_0)$ when $m \to \infty$. In addition, due to $\theta^1(Y_0) \ge \theta^2(Y_0) \ge \theta^3(Y_0)$, $\theta^2(Y_0)$ also converges to $\theta^1(Y_0)$ in probability when $m \to \infty$, which completes the proof. □

3.5. Numerical example analysis

To check the feasibility and effectiveness of the proposed models, we select 4 stocks from China stock market: Sinopec, China Unicom, China Life Insurance and Bank of China. The monthly return data from May 2007 to May 2016 are used. The corresponding statistical properties are shown in Table 1.

Similar to Vercher *et al.* [39], we adopt the sample percentiles to approximate the cores and spreads of the trapezoidal fuzzy returns on the assets. This estimation method is widely used in the fuzzy portfolio optimization problem. For more details, readers may refer to Zhang *et al.* [44], Zhang and Zhang [41], Liu and Zhang [25] and so on. For the ith asset, we let the interval $[P_{40},\ P_{60}]$ be the core $[a_j,\ b_j]$ of the fuzzy return r_j, and the quantities $P_{40} - P_5$ and $P_{95} - P_{60}$ be the left (c_j) and right (d_j) spreads, respectively, where $j = 1,\ 2,\ \ldots,\ n$ and P_k be the kth percentile of the sample data. According to (2.12)–(2.14), we can calculate the corresponding possibilistic mean, variance and entropy of the portfolio being evaluated.

We randomly generate different sizes ($m = 200,\ 400,\ 600$) of investment weights, and then calculate the efficiencies and rankings by using Model (3.4), (3.6) and (3.8), respectively. Note that Models (3.4) and (3.6) are solved by using the *trust region reflective algorithm*, while Model (3.8) is solved by using the *simplex algorithm*. The main results are shown in Table 2.

Since the size of portfolios being evaluated is large, we only show the results of the first 15 portfolios in Table 2. As shown in Table 2, we can easily find that the efficiencies of the diversification and DEA models are gradually close to real values with the increase of sample size.

Table 3 shows the P-values of Wilcoxon rank-sum test of scores and rankings by different models. When m choose different values, according to Panel A of Table 3, the Wilcoxon rank sum test accepts that the scores between Models (3.4) and (3.6) and the ones between Models (3.4) and (3.8) have significant difference under the significance level of 5%. However, the Wilcoxon rank sum test rejects the difference of the scores between Models (3.6) and (3.8) under the significance level of 5%. This indicates that Models (3.6) and (3.8) are all

TABLE 2. The efficiencies and rankings of different portfolios being evaluated.

Portfolio	$m = 200$						$m = 400$						$m = 600$					
	Model (3.4)		Model (3.6)		Model (3.8)		Model (3.4)		Model (3.6)		Model (3.8)		Model (3.4)		Model (3.6)		Model (3.8)	
	Score	Ranking	Score	Ranking	Score	Ranking	Score	Ranking	Score	Ranking	Score	Ranking	Score	Ranking	Score	Ranking	Score	Ranking
1	0.2033	151	0.2383	151	0.2383	151	0.2033	284	0.2383	379	0.2383	379	0.2033	430	0.2350	575	0.2371	573
2	0.1427	183	0.2567	183	0.2567	183	0.1427	349	0.2567	364	0.2567	365	0.1427	514	0.2553	548	0.2553	550
3	0.5390	73	0.7651	73	0.8008	73	0.5390	139	0.7651	71	0.7962	67	0.5390	209	0.7623	111	0.7947	108
4	0.3810	104	0.3837	104	0.3992	104	0.3810	196	0.3837	258	0.3940	254	0.3810	290	0.3818	386	0.3915	379
5	0.3502	107	0.5788	107	0.5963	107	0.3502	206	0.5762	149	0.5762	156	0.3502	309	0.5759	221	0.5761	229
6	0.6787	43	0.6795	43	0.7101	43	0.6787	91	0.6793	109	0.7096	105	0.6787	134	0.6790	161	0.7096	155
7	0.7142	35	0.7142	35	0.7425	35	0.7142	74	0.7142	96	0.7423	92	0.7142	113	0.7142	143	0.7423	138
8	0.2539	131	0.4483	131	0.4483	131	0.2539	244	0.4473	214	0.4473	222	0.2539	371	0.4472	317	0.4472	328
9	0.1246	194	0.2274	194	0.2274	194	0.1246	376	0.2274	389	0.2274	390	0.1246	550	0.2245	589	0.2245	591
10	0.6445	50	0.6445	50	0.7300	50	0.6445	100	0.6445	119	0.6760	116	0.6445	147	0.6445	180	0.6741	175
11	0.1518	180	0.3335	180	0.3335	180	0.1518	341	0.3329	297	0.3329	297	0.1518	503	0.3328	445	0.3328	448
12	0.8974	13	0.8974	13	1.0000	13	0.8974	29	0.8974	38	0.9358	36	0.8974	43	0.8974	56	0.9354	54
13	0.2367	140	0.3544	140	0.3544	140	0.2367	258	0.3541	280	0.3541	282	0.2367	392	0.3541	418	0.3541	423
14	0.2130	147	0.2453	147	0.2534	147	0.2130	275	0.2453	373	0.2521	370	0.2130	416	0.2450	563	0.2520	557
15	0.3780	105	0.4234	105	0.4322	105	0.3780	198	0.4230	234	0.4257	236	0.3780	292	0.4230	348	0.4257	352

consistent for this case, which means that the difference of convergence rate between traditional DEA model and diversification model is not apparent. The above simulation results further verify the conclusion of Liu *et al.* [27].

In addition, the investor might also concern these relative rankings in the actual evaluation process. As shown in Panel B of Table 3, the Wilcoxon rank sum test rejects the difference of the rankings from Models (3.4), (3.6) and (3.8) under the significance level of 5% regardless of what the number of portfolios being evaluated. That is, the rankings of portfolios being evaluated derived by Models (3.4), (3.6) and (3.8) are coincident.

In order to distinguish the difference between the existing DEA model and the proposed DEA model in the assessment of the performance of fuzzy portfolios, further simulations are given. To this end, we predominantly compare our model with the one provided by Chen *et al.* [13]. Under the fuzzy mean-variance framework and output orientation, we have the two following DEA models:

$$\theta^4(Y_0) = \max\ \theta$$
$$\text{s.t.}\begin{cases}\sum_{j=1}^{m}\lambda_j E(Y_j) \geq E(Y_0) + \theta g_E\\ \sum_{j=1}^{m}\lambda_j \mathrm{Var}(Y_j) \leq \mathrm{Var}(Y_0)\\ \sum_{j=1}^{m}\lambda_j = 1,\ 0 \leq \lambda_j \leq 1,\ \ i = 1,\ 2,\ \ldots,\ m.\end{cases} \tag{3.18}$$

$$\theta^5(Y_0) = \max\ \theta$$
$$\text{s.t.}\begin{cases}\sum_{j=1}^{m}\lambda_j E(Y_j) \geq E(Y_0) + \theta g_E\\ \sum_{j=1}^{m}\lambda_j \mathrm{Var}(Y_j) \leq \mathrm{Var}(Y_0) - \theta g_V\\ \sum_{j=1}^{m}\lambda_j = 1,\ 0 \leq \lambda_j \leq 1,\ \ i = 1,\ 2,\ \ldots,\ m.\end{cases} \tag{3.19}$$

Model (3.18) is consonant with Chen *et al.* [13], in that it assumes the variance to be an output and the return to be an input. However, Model (3.19) assumes that the fuzzy variance and return are both outputs,

TABLE 3. P-values of Wilcoxon rank-sum test of evaluated results by different models.

Panel A: P-values of Wilcoxon rank-sum test of scores computed by different models			
P-value	$m = 200$	$m = 400$	$m = 600$
Models (3.4), (3.6)	0.0000	0.0000	0.0000
Models (3.4), (3.8)	0.0000	0.0000	0.0000
Models (3.6), (3.8)	0.3649**	0.5510**	0.4716**
Panel B: P-values of Wilcoxon rank-sum test of rankings computed by different models			
Models (3.4), (3.6)	1.0000**	1.0000**	1.0000**
Models (3.4), (3.8)	1.0000**	1.0000**	1.0000**
Models (3.6), (3.8)	1.0000**	1.0000**	1.0000**

Notes. **5% significance level.

TABLE 4. The efficiencies of different portfolios being evaluated.

Portfolio	Score ($m = 200$)		Score ($m = 400$)		Score ($m = 600$)	
	Model (3.18)	Model (3.19)	Model (3.18)	Model (3.19)	Model (3.18)	Model (3.19)
1	0.0756	0.1974	0.0756	0.1974	0.0726	0.1949
2	0.0009	0.1437	0.0009	0.1437	0.0009	0.1435
3	0.1535	0.2816	0.1535	0.2816	0.1522	0.2739
4	0.1372	0.2692	0.1372	0.2692	0.1370	0.2671
5	0.5028	0.6087	0.5028	0.6087	0.4974	0.6073
6	0.3266	0.4675	0.3266	0.4675	0.3179	0.4636
7	0.5175	0.6234	0.5175	0.6234	0.4961	0.6196
8	0.1456	0.2907	0.1456	0.2907	0.1403	0.2905
9	0.0008	0.1266	0.0008	0.1266	0.0008	0.1265
10	0.2731	0.3608	0.2731	0.3608	0.2712	0.3608
11	0.0016	0.2293	0.0016	0.2293	0.0016	0.2283
12	0.3531	0.4419	0.3531	0.4419	0.3515	0.4419
13	0.1025	0.2311	0.1025	0.2311	0.0985	0.2304
14	0.0685	0.1839	0.0685	0.1839	0.0661	0.1805
15	0.2325	0.3739	0.2325	0.3739	0.2312	0.3658

TABLE 5. P-values of Wilcoxon rank-sum test of scores computed by different models.

P-value	$m = 200$	$m = 400$	$m = 600$
Models (3.18), (3.19)	0.0000	0.0000	0.0000

Notes. **5% significance level.

coinciding with the real investment process. By using the data provided in Table 1, we randomly generate m portfolio weights, where the following simulations can be derived (note that we only present the first 15 results in Table 4).

Table 5 shows the P-values of Wilcoxon rank-sum test of scores by Models (3.18) and (3.19). Clearly, we can find that the Wilcoxon rank-sum test accepts that the scores of Models (3.18) and (3.19) are significantly different under the significance level of 5%. This further indicates that the input-output process assumption provided in this paper is more suitable when compared with the existing one presented in Chen *et al.* [13].

4. Empirical study

According to the simulation results presented in Section 3.5, we can find that the diversification and DEA models are both feasible and effective for evaluating the performance of portfolios with fuzzy returns. However, the diversification model is more time-consuming than DEA model. In the following, we will apply these proposed models to assess the performance of 50 open-end funds from China fund market. Based on the monthly net asset values from May 2007 to May 2016, we can obtain the statistical properties of these data, as shown in Table 6.

Since the investor does not know the composition of fund in the actual fund investment process, Model (3.4) is not suitable for fund evaluation in this situation. In the following, we focus on applying the diversification and DEA models. We can calculate the scores and rankings of funds with fuzzy returns by using Models (3.6) and (3.8). The main results are shown in Table 7.

The Wilcoxon rank-sum test results of efficiencies are shown in Table 8. From Table 8, we can easily find that the Wilcoxon rank-sum test rejects the difference of both the scores and rankings between Models (3.6) and (3.8) under the significance level of 5%. The empirical results are coincident with the simulation results presented in Section 4, which further indicate that the DEA approach can be used in the actual evaluation.

Since the observed data only represents a sample of possible realizations values, thus the estimations of fuzzy returns may differ from the true but unknown one. In the following, we will discuss the robustness of the DEA scores and rankings. To this end, we apply the bootstrap approach to achieve this goal. The bootstrap approach mainly focuses on re-sampling the original data with unknown distribution of the returns. Similar to Branda [4], we use the function *datasample* available in MATLAB to obtain the re-sample data. The bootstrap statistics for scores and rankings are shown in Table 9.

Table 9 shows the bootstrap results based on the 125×50 observed data. We employ the *datasample* function available in MATLAB to randomly generate $B = 1000$ group 1000×50 data as the check sample. In this situation, we can derive the corresponding score $\hat{\theta}_i^b(Y_0)$ under the different sample group b and evaluation model i, where $i = 2,\ 3,\ b = 1,\ 2,\ \ldots,\ B$. Thus, the mean bootstrap score, estimated bias and standard error for portfolios with fuzzy return Y_0 are calculated as follows:

$$\hat{\theta}_i^B(Y_0) = \frac{1}{B}\sum_{b=1}^{B}\hat{\theta}_i^b(Y_0), \tag{4.1}$$

$$\text{bias}_i^B(Y_0) = \frac{1}{B}\sum_{b=1}^{B}\hat{\theta}_i^b(Y_0) - \theta_i(Y_0), \tag{4.2}$$

$$s.e._i^B(Y_0) = \sqrt{\frac{1}{B-1}\sum_{b=1}^{B}[\hat{\theta}_i^b(Y_0) - \theta_i(Y_0)]^2}. \tag{4.3}$$

Table 10 shows the P-values of Wilcoxon rank-sum test of rankings by using the bootstrap approach. Here, the Wilcoxon rank-sum test rejects the difference between the new rankings based on mean bootstrap score and the original ranking under the significance level of 5%, both for the diversification model and DEA model. Table 10 also indicates that the evaluation results by the diversification and DEA models have a good robustness.

5. Conclusion

This paper redefines the input-output process in accordance with the actual investment situation, where the initial wealth is taken as an input, while the return and risk are taken as outputs. We first define the efficiencies of portfolios with fuzzy returns under the criterion of possibilistic mean-variance-entropy, by using directional distance function. We distinguish the difference among the model based on real frontier, the diversification model and the DEA model. This indicates that the diversification and DEA models can be regarded as the

Table 6. Statistical properties of the 50 Chinese funds.

Fund	Sample mean	Sample variance	5th percentile	40th percentile	60th percentile	95th percentile
000001	0.0055	0.0083	−0.1538	−0.0037	0.0233	0.1500
020001	0.0085	0.0134	−0.1448	−0.0010	0.0345	0.1710
040001	0.0054	0.0098	−0.1048	−0.0013	0.0293	0.1450
050001	0.0023	0.0084	−0.1233	0.0008	0.0271	0.1233
070003	0.0037	0.0066	−0.1335	−0.0063	0.0246	0.1209
090001	0.0055	0.0108	−0.1409	0.0034	0.0313	0.1391
100020	0.0090	0.0126	−0.1447	0.0058	0.0328	0.1581
110003	0.0070	0.0112	−0.1481	−0.0094	0.0229	0.1808
151001	0.0101	0.0090	−0.1517	0.0066	0.0302	0.1421
160105	0.0053	0.0090	−0.1498	0.0027	0.0296	0.1483
160505	0.0099	0.0092	−0.1328	−0.0006	0.0336	0.1389
160605	0.0076	0.0111	−0.1607	−0.0040	0.0345	0.1729
160706	0.0050	0.0108	−0.1841	−0.0070	0.0298	0.1696
161604	0.0078	0.0091	−0.1718	−0.0049	0.0339	0.1514
161706	0.0090	0.0109	−0.1594	−0.0012	0.0289	0.1551
161903	0.0037	0.0094	−0.1509	−0.0072	0.0288	0.1484
162102	0.0054	0.0083	−0.1503	−0.0050	0.0274	0.1482
162201	0.0069	0.0100	−0.1690	0.0077	0.0380	0.1420
162202	0.0085	0.0113	−0.1557	−0.0001	0.0352	0.1484
162605	0.0053	0.0112	−0.1760	0.0033	0.0343	0.1574
162703	0.0069	0.0121	−0.2064	−0.0032	0.0300	0.1706
163503	0.0005	0.0110	−0.1785	−0.0101	0.0301	0.1278
180003	0.0034	0.0090	−0.1607	−0.0110	0.0253	0.1599
200002	0.0106	0.0122	−0.1522	−0.0021	0.0364	0.1673
210001	0.0056	0.0079	−0.1588	0.0006	0.0262	0.1215
213002	0.0032	0.0125	−0.1873	−0.0042	0.0352	0.1531
217001	0.0004	0.0110	−0.1359	0.0052	0.0298	0.1329
233001	0.0039	0.0088	−0.1398	−0.0038	0.0170	0.1373
240001	0.0111	0.0088	−0.1427	0.0017	0.0312	0.1297
240004	0.0104	0.0125	−0.1862	0.0079	0.0412	0.1626
240005	0.0032	0.0133	−0.1548	0.0070	0.0334	0.1546
257020	0.0076	0.0097	−0.1985	0.0009	0.0263	0.1546
260101	0.0101	0.0085	−0.1422	0.0065	0.0344	0.1398
260104	0.0162	0.0088	−0.1611	0.0065	0.0434	0.1822
270005	0.0088	0.0126	−0.1507	−0.0024	0.0282	0.1627
288002	0.0162	0.0086	−0.1280	−0.0027	0.0394	0.1553
310328	0.0022	0.0104	−0.1889	−0.0062	0.0372	0.1459
320003	0.0071	0.0096	−0.1537	0.0039	0.0360	0.1473
360001	0.0098	0.0115	−0.1609	0.0008	0.0357	0.1568
377010	0.0143	0.0108	−0.1695	0.0003	0.0356	0.1666
398001	0.0019	0.0139	−0.2242	−0.0025	0.0340	0.1359
460001	0.0005	0.0120	−0.1748	−0.0012	0.0331	0.1479
481001	−0.0008	0.0135	−0.1750	−0.0077	0.0346	0.1501
510050	0.0125	0.0094	−0.1465	−0.0085	0.0316	0.1852
510081	0.0067	0.0090	−0.1836	−0.0086	0.0340	0.1455
519001	0.0119	0.0106	−0.1287	0.0012	0.0361	0.1618
519005	0.0013	0.0097	−0.1814	−0.0069	0.0317	0.1402
519180	0.0048	0.0109	−0.1906	−0.0081	0.0309	0.1732
519688	0.0025	0.0110	−0.1563	−0.0106	0.0321	0.1533
519996	0.0051	0.0112	−0.1895	0.0001	0.0227	0.1738

TABLE 7. The scores and rankings of the 50 Chinese funds.

Fund	Model (3.6)		Model (3.8)	
	Score	Rank	Score	Rank
000001	0.1108	16	0.1125	16
020001	0.0866	24	0.0870	25
040001	1.0000	1	1.0000	1
050001	1.0000	1	1.0000	1
070003	0.4374	3	0.4488	3
090001	0.1732	8	0.1769	8
100020	0.1190	15	0.1203	15
110003	0.0779	28	0.0781	28
151001	0.1373	11	0.1400	11
160105	0.1224	14	0.1243	14
160505	0.1837	7	0.1872	7
160605	0.0650	37	0.0654	37
160706	0.0518	47	0.0523	47
161604	0.0675	34	0.0684	33
161706	0.0890	23	0.0901	23
161903	0.1009	20	0.1023	20
162102	0.1091	17	0.1107	17
162201	0.0856	25	0.0871	24
162202	0.0930	22	0.0943	22
162605	0.0669	35	0.0678	35
162703	0.0426	50	0.0430	50
163503	0.0746	31	0.0761	31
180003	0.0758	30	0.0766	30
200002	0.0769	29	0.0774	29
210001	0.1514	10	0.1559	10
213002	0.0546	45	0.0553	45
217001	0.2640	4	0.2717	4
233001	0.2379	5	0.2444	5
240001	0.1861	6	0.1910	6
240004	0.0547	44	0.0554	44
240005	0.1028	19	0.1042	19
257020	0.0556	42	0.0564	42
260101	0.1634	9	0.1668	9
260104	0.0606	40	0.0609	40
270005	0.0938	21	0.0946	21
288002	0.1279	13	0.1286	13
310328	0.0540	46	0.0548	46
320003	0.1030	18	0.1046	18
360001	0.0790	26	0.0799	26
377010	0.0637	38	0.0643	38
398001	0.0428	49	0.0435	49
460001	0.0708	32	0.0719	32
481001	0.0626	39	0.0634	39
510050	0.0683	33	0.0683	34
510081	0.0584	41	0.0592	41

TABLE 7. (Continued).

Fund	Model (3.6)		Model (3.8)	
	Score	Rank	Score	Rank
519001	0.1352	12	0.1357	12
519005	0.0656	36	0.0667	36
519180	0.0462	48	0.0466	48
519688	0.0782	27	0.0791	27
519996	0.0551	43	0.0556	43

TABLE 8. P-values of Wilcoxon rank-sum test of efficiencies by different models.

Panel A: P-values of Wilcoxon rank-sum test of scores computed by different models		
P-value	Model (3.6)	Model (3.8)
Model (3.6)	1.0000**	0.7907**
Model (3.8)	0.7907**	1.0000**
Panel B: P-values of Wilcoxon rank-sum test of rankings computed by different models		
Model (3.6)	1.0000**	1.0000**
Model (3.8)	1.0000**	1.0000**

Notes. **5% significance level.

TABLE 9. Bootstrap statistics for scores and rankings derived by different models.

Fund	Model (3.6)					Model (3.8)				
	Original score	Estimated bias	Standard error	Ranking (Original)		Original score	Estimated bias	Standard error	Ranking (Original)	
000001	0.1108	0.1237	0.1775	23	(16)	0.1125	0.1250	0.1803	23	(16)
020001	0.0866	0.2289	0.2263	14	(24)	0.0870	0.2310	0.2280	14	(25)
040001	1.0000	−0.0563	0.2225	1	(1)	1.0000	−0.0549	0.2204	1	(1)
050001	1.0000	−0.2255	0.3441	2	(1)	1.0000	−0.2174	0.3411	2	(1)
070003	0.4374	0.1673	0.3646	3	(3)	0.4488	0.1596	0.3643	3	(3)
090001	0.1732	0.1544	0.2121	12	(8)	0.1769	0.1550	0.2148	12	(8)
100020	0.1190	0.1578	0.1723	17	(15)	0.1203	0.1601	0.1750	17	(15)
110003	0.0779	0.3142	0.3473	8	(28)	0.0781	0.3166	0.3480	8	(28)
151001	0.1373	0.1091	0.1505	20	(11)	0.1400	0.1095	0.1525	20	(11)
160105	0.1224	0.1071	0.1366	25	(14)	0.1243	0.1080	0.1385	25	(14)
160505	0.1837	0.2620	0.2847	6	(7)	0.1872	0.2615	0.2857	6	(7)
160605	0.0650	0.1480	0.1512	27	(37)	0.0654	0.1492	0.1524	28	(37)
160706	0.0518	0.0997	0.1010	42	(47)	0.0523	0.1004	0.1020	42	(47)
161604	0.0675	0.1035	0.1006	35	(34)	0.0684	0.1037	0.1011	35	(33)
161706	0.0890	0.0978	0.1134	32	(23)	0.0901	0.0988	0.1149	32	(23)
161903	0.1009	0.1359	0.1424	21	(20)	0.1023	0.1368	0.1439	21	(20)
162102	0.1091	0.1271	0.1419	22	(17)	0.1107	0.1278	0.1432	22	(17)
162201	0.0856	0.0812	0.0971	37	(25)	0.0871	0.0814	0.0981	37	(24)
162202	0.0930	0.1212	0.1233	26	(22)	0.0943	0.1219	0.1245	26	(22)
162605	0.0669	0.0935	0.1010	38	(35)	0.0678	0.0943	0.1024	38	(35)
162703	0.0426	0.0692	0.0667	49	(50)	0.0430	0.0699	0.0676	49	(50)
163503	0.0746	0.0828	0.0947	39	(31)	0.0761	0.0824	0.0952	39	(31)
180003	0.0758	0.1198	0.1159	31	(30)	0.0766	0.1206	0.1170	31	(30)

TABLE 9. (Continued).

Fund	Model (3.6)					Model (3.8)				
	Original score	Estimated bias	Standard error	Ranking	(Original)	Original score	Estimated bias	Standard error	Ranking	(Original)
200002	0.0769	0.1794	0.1993	19	(29)	0.0774	0.1806	0.2006	19	(29)
210001	0.1514	0.0788	0.1398	24	(10)	0.1559	0.0772	0.1410	24	(10)
213002	0.0546	0.0759	0.0755	46	(45)	0.0553	0.0761	0.0761	46	(45)
217001	0.2640	0.1011	0.2377	9	(4)	0.2717	0.0991	0.2406	9	(4)
233001	0.2379	0.0999	0.2517	11	(5)	0.2444	0.0981	0.2545	10	(5)
240001	0.1861	0.0992	0.1640	16	(6)	0.1910	0.0977	0.1655	16	(6)
240004	0.0547	0.0990	0.1000	41	(44)	0.0554	0.0998	0.1012	41	(44)
240005	0.1028	0.1039	0.1250	29	(19)	0.1042	0.1049	0.1267	29	(19)
257020	0.0556	0.0642	0.0719	48	(42)	0.0564	0.0648	0.0729	48	(42)
260101	0.1634	0.1335	0.1857	15	(9)	0.1668	0.1341	0.1889	15	(9)
260104	0.0606	0.2781	0.3183	10	(40)	0.0609	0.2794	0.3184	11	(40)
270005	0.0938	0.1679	0.1899	18	(21)	0.0946	0.1701	0.1920	18	(21)
288002	0.1279	0.4593	0.3519	4	(13)	0.1286	0.4594	0.3517	4	(13)
310328	0.0540	0.0850	0.0869	44	(46)	0.0548	0.0850	0.0871	44	(46)
320003	0.1030	0.1098	0.1228	28	(18)	0.1046	0.1104	0.1241	27	(18)
360001	0.0790	0.1202	0.1260	30	(26)	0.0799	0.1210	0.1273	30	(26)
377010	0.0637	0.1050	0.1018	36	(38)	0.0643	0.1058	0.1028	36	(38)
398001	0.0428	0.0511	0.0616	50	(49)	0.0435	0.0511	0.0619	50	(49)
460001	0.0708	0.0856	0.0934	40	(32)	0.0719	0.0859	0.0943	40	(32)
481001	0.0626	0.1100	0.1028	34	(39)	0.0634	0.1102	0.1033	34	(39)
510050	0.0683	0.3706	0.3517	7	(33)	0.0683	0.3731	0.3522	7	(34)
510081	0.0584	0.1194	0.1285	33	(41)	0.0592	0.1197	0.1291	33	(41)
519001	0.1352	0.3960	0.3619	5	(12)	0.1357	0.3989	0.3627	5	(12)
519005	0.0656	0.0851	0.0890	43	(36)	0.0667	0.0851	0.0896	43	(36)
519180	0.0462	0.0840	0.0763	47	(48)	0.0466	0.0847	0.0770	47	(48)
519688	0.0782	0.2445	0.2824	13	(27)	0.0791	0.2445	0.2823	13	(27)
519996	0.0551	0.0766	0.0795	45	(43)	0.0556	0.0778	0.0809	45	(43)

TABLE 10. P-values of Wilcoxon rank-sum test of rankings computed by different models.

P-value	Model (3.6) (Bootstrap)	Model (3.6) (Original)	Model (3.8) (Bootstrap)	Model (3.8) (Original)
Model (3.6) (Bootstrap)	1.0000^{**}	0.9972^{**}	1.0000^{**}	0.9972^{**}
Model (3.6) (Original)	0.9972^{**}	1.0000^{**}	0.9972^{**}	1.0000^{**}
Model (3.8) (Bootstrap)	1.0000^{**}	0.9972^{**}	1.0000^{**}	0.9972^{**}
Model (3.8) (Original)	0.9972^{**}	1.0000^{**}	0.9972^{**}	1.0000^{**}

Notes. **5% significance level.

nonlinear and linear estimations of the model based on real frontier, respectively. We show that the portfolio efficiencies derived from diversification and DEA models can both converge to the real one when the portfolio samples are large enough. We also select 50 Chinese funds to check the feasibility and effectiveness of the DEA and diversification models. These results show that the difference of convergence rates between DEA model and diversification model is not apparent. Finally, the analysis by using the bootstrap re-sampling approach further validates the robustness of the performance evaluation results based on the proposed models.

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