

## REGULARITY CONDITIONS AND FARKAS-TYPE RESULTS FOR SYSTEMS WITH FRACTIONAL FUNCTIONS

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**Abstract.** This paper deals with some new versions of Farkas-type results for a system involving cone convex constraint, a geometrical constraint as well as a fractional function. We first introduce some new notions of regularity conditions in terms of the epigraphs of the conjugate functions. By using these regularity conditions, we obtain some new Farkas-type results for this system using an approach based on the theory of conjugate duality for convex or DC optimization problems. Moreover, we also show that some recently obtained results in the literature can be rediscovered as special cases of our main results.

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### 1. INTRODUCTION

Farkas-type results is a fundamental tool in the development of optimality and duality theory for many mathematical optimization problems. The classical Farkas-type results [9] characterizes families of linear inequalities which are consequences of a consistent linear inequality system. Over the years, many interesting results on characterizations of Farkas-type results of various types of systems have been obtained; see, for example, convex inequality systems in [1, 13, 17], conic-convex systems in [8, 18], systems with composite functions in [2, 10, 12, 29], and non-convex systems in [6, 26, 27]. For more details, one can refer to the recent survey on the generalizations of the Farkas-type results [5].

Recently, an optimization problem where the objective is a fraction of two functions is attracting a great deal of interest, since many decision-making problems arising in practice need turn out to be of fractional type, such as minimizing cost/time, and maximizing return/investment, etc. A variety of interesting and important theoretical results as well as applications have been established in the last decades, see, *e.g.*, [3, 16, 19, 23, 24, 28, 30, 31, 33–35, 37] and references therein. For the up-to-date bibliography of fractional programming, we refer the reader to the literature [25]. Particular interest is in the characterization of Farkas-type results for systems with fractional functions. Using an interior-point regularity condition, Farkas-type results for systems with fractional functions are obtained in [3, 33, 37]. By employing the properties of the epigraph of the conjugate functions, some

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new Farkas-type results are obtained in [28] for systems with DC and fractional functions. Using a parametrized Lagrange duality approach, Farkas-type results are also investigated in [31] for systems with fractional functions.

We observe that Farkas-type results for systems with fractional functions appeared in the literature were done under the lower semicontinuous assumptions. However, to the best of our knowledge, the Farkas-type result has not been considered in the case where the involved functions are not lower semicontinuous. This is the main motivation to consider Farkas-type results for systems with fractional functions. On the other hand, regularity condition is an essential ingredient for guaranteeing Farkas-type result, duality theory and optimality condition, see [4, 7, 11, 14, 15, 20, 21, 32] and the references therein. It is worth mentioning that among them, regularity conditions which involve epigraphs is proving to be a fruitful tool in the study of Farkas-type result. So, our primary goal of the paper is to introduce some new weaker regularity conditions to establish new versions of Farkas-type results for systems with fractional functions in locally convex vector space. In general, we do not impose any topological assumptions on the constraint set and the involved functions; that is, the constraint set is not necessarily closed and the involved functions are not necessarily lower semicontinuous.

Our contributions can be more specifically stated as follows. We first introduce, by using the epigraphs of the conjugate functions, some new regularity conditions for systems with fractional functions. These new regularity conditions are the generalizations of the regularity conditions used in [3, 28, 33, 37]. Based on the new regularity conditions, we present some new Farkas-type results for systems with fractional functions which generalize the results in [3, 28, 33, 37]. We also show that the results obtained here underline the connections that exist between Farkas-type results and alternative type theorems. Furthermore, we show that the obtained results for cone convex system in the reference are actually particular instances of our results.

The paper is organized as follows. In Section 2, we recall some notions and give some preliminary results. In Section 3, we first introduce some new regularity conditions. Then, based on the theory of conjugate duality for convex or DC optimization problems, we establish some Farkas-type results for systems with fractional functions. In Section 4, we show that the obtained results for cone convex system in the reference are actually particular instances of our results.

## 2. MATHEMATICAL PRELIMINARIES

In this paper, we use the standard notation, please see [22, 36]. Unless otherwise specified, let  $X$  be a real locally convex Hausdorff topology vector space. The continuous dual space of  $X$  is denoted by  $X^*$ . Let  $\langle x^*, x \rangle = x^*(x)$  be the value of the continuous linear functional  $x^* \in X^*$  at  $x \in X$ . Given a set  $D \subseteq X \times \mathbb{R}$ , the closure of  $D$  is denoted by  $\text{cl } D$ . For a nonempty set  $C \subseteq X$ , the indicator function  $\delta_C$  of  $C$  is defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases}$$

Let  $f : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$  be an extended real valued function. The effective domain and the epigraph are defined by  $\text{dom } f = \{x \in X : f(x) < +\infty\}$  and  $\text{epi } f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ , respectively. We say  $f$  is proper, iff its effective domain is nonempty and  $f(x) > -\infty$ , for all  $x \in X$ . We say that  $f$  is lower semicontinuous, iff  $\text{epi } f$  is a closed set and that  $f$  is convex, iff  $\text{epi } f$  is a convex set. Moreover, we say that  $f$  is concave, iff  $-f$  is convex. The lower semicontinuous hull  $\text{cl } f : X \rightarrow \overline{\mathbb{R}}$  of  $f$  is defined by

$$\text{epi } (\text{cl } f) := \text{cl } (\text{epi } f),$$

where the topological closure is taken with respect to the product topology. The conjugate function  $f^* : X^* \rightarrow \overline{\mathbb{R}}$  of  $f$  is defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}.$$

Moreover, for any  $p \in X^*$  and  $\alpha \in \mathbb{R}$ , we have

$$(f + p + \alpha)^*(x^*) = f^*(x^* - p) - \alpha, \text{ for each } x^* \in X^*, \quad (2.1)$$

and

$$\text{epi} (f + p + \alpha)^* = \text{epi} f^* + (p, -\alpha). \tag{2.2}$$

Let  $Y$  be another locally convex Hausdorff topology vector space with its dual space  $Y^*$ , endowed with the weak\* topology  $w(Y^*, Y)$ . Let  $S \subseteq Y$  be a nonempty convex cone. We consider the partial order “ $\leq_S$ ” of  $Y$ , induced by  $S$ , defined by

$$y_1 \leq_S y_2 \iff y_2 - y_1 \in S, \text{ for any } y_1, y_2 \in Y.$$

Let  $Y^\bullet = Y \cup \{\infty_Y\}$ , where  $\infty_Y \notin Y$  is a greatest element with respect to “ $\leq_S$ ”. Then, for any  $y \in Y^\bullet$ , one has  $y \leq_S \infty_Y$  and we define the following operations on  $Y^\bullet$ :

$$y + (\infty_Y) = (\infty_Y) + y = \infty_Y \text{ and } t(\infty_Y) = \infty_Y, \text{ for any } y \in Y \text{ and } t \geq 0.$$

The (positive) dual cone  $S^*$  is defined by  $S^* = \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \forall y \in S\}$ . By convention,  $\langle y^*, \infty_Y \rangle = +\infty$ , for any  $y^* \in Y^*$ .

Let  $h : X \rightarrow Y^\bullet$  be an extended vector valued function. The domain and the  $S$ -epigraph of  $h$  are defined by  $\text{dom } h = \{x \in X : h(x) \in Y\}$ , and  $\text{epi}_S h = \{(x, y) \in X \times Y : y \in h(x) + S\}$ , respectively.  $h$  is said to be proper, iff  $\text{dom } h \neq \emptyset$ .  $h$  is said to be  $S$ -epi-closed, iff  $\text{epi}_S h$  is closed.  $h$  is said to be a  $S$ -convex function, iff for any  $x, y \in X$  and  $t \in [0, 1]$ , we have  $h(tx + (1 - t)y) \leq_S th(x) + (1 - t)h(y)$ . Moreover, let  $\lambda \in S^*$ . The function  $(\lambda h) : X \rightarrow \overline{\mathbb{R}}$  is defined by

$$(\lambda h)(x) = \begin{cases} \langle \lambda, h(x) \rangle, & \text{if } x \in \text{dom } h, \\ +\infty, & \text{otherwise.} \end{cases}$$

Obviously,  $h$  is  $S$ -convex if and only if  $(\lambda h)$  is convex, for each  $\lambda \in S^*$ .

### 3. FARKAS TYPE RESULTS

Throughout this and the next sections, let  $X$  and  $Y$  be separated locally convex vector spaces,  $C \subseteq X$  be a nonempty convex subset,  $S \subseteq Y$  be a nonempty closed convex cone,  $f, -g : X \rightarrow \overline{\mathbb{R}}$  be two proper convex functions, and  $h : X \rightarrow Y^\bullet$  be a proper  $S$ -convex function. Let  $\mu \in \mathbb{R}$ . We consider the solvability issue for the system

$$x \in C, h(x) \leq_S 0, \frac{f(x)}{g(x)} < \mu. \tag{3.1}$$

Obviously, the insolvability of (3.1) is equivalent to saying that  $\frac{f(x)}{g(x)} \geq \mu$  is a consequence of the system

$$x \in C, h(x) \leq_S 0, \tag{3.2}$$

that is, the insolvability of (3.1) is equivalent to

$$\frac{f(x)}{g(x)} \geq \mu, \text{ for any } x \in C, h(x) \leq_S 0.$$

In this section, we give some new characterizations of Farkas type results for systems with fractional functions. Our approach makes use of the powerful conjugate analysis. To do this, we consider the following fractional programming problem

$$(P) \quad \inf_{x \in \mathcal{A}} \frac{f(x)}{g(x)}.$$

Here, the constraint set of (P) is defined by  $\mathcal{A} := \{x \in C, h(x) \leq_S 0\}$ .

**Assumption 3.1.** *Throughout this section, we assume that the function  $g(x) > 0$ , for all  $x \in \mathcal{A}$ .*

We associate  $(P)$  with the following optimization problem

$$(P_\mu) \quad \inf_{x \in \mathcal{A}} \{f(x) - \mu g(x)\}.$$

Now, we first construct the dual problems of  $(P_\mu)$ , and present the duality assertions. Then, by using the duality assertions, we investigated some Farkas-type results for systems with fractional functions. It is obvious that there exists the following relation between the optimal value  $\text{val}(P)$  of  $(P)$  and the optimal value  $\text{val}(P_\mu)$  of  $(P_\mu)$ .

**Lemma 3.2.** *The inequality  $\text{val}(P) \geq \mu$  holds if and only if the inequality  $\text{val}(P_\mu) \geq 0$  holds.*

Since the objective function of  $(P_\mu)$  depends on the sign of  $\mu$ , we have to treat two different cases. We first assume that  $\mu$  is a non-negative value. In this case, the objective function  $f - \mu g$  is a convex function and then the convex optimization theory can be used. In the case that  $\mu$  is a negative real number, the objective function  $f - \mu g$  is the difference of two convex functions and therefore we can use an approach inspired from DC optimization problems.

### 3.1. The case $\mu \geq 0$

For any real number  $\mu \geq 0$ , it is easy to see that the objective function of the problem  $(P_\mu)$  is a convex function. Then, the problem  $(P_\mu)$  is a convex programming problem. By using the similar approach presented by Boç et al. in Section 3 of [3], we can establish the following dual problem

$$(D_\mu) \quad \sup_{\substack{\lambda \in S^*, x^* \in \text{dom } f^*, \\ y^* \in \text{dom } (-g)^*, z^* \in \text{dom } (\lambda h)^*}} \{-f^*(x^*) - \mu(-g)^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-x^* - \mu y^* - z^*)\}.$$

We denote by  $\text{val}(P_\mu)$  and  $\text{val}(D_\mu)$  the optimal objective values of  $(P_\mu)$  and  $(D_\mu)$ , respectively. Obviously,  $\text{val}(P_\mu) \geq \text{val}(D_\mu)$ ; that is, the weak duality holds between  $(P_\mu)$  and  $(D_\mu)$ . Now, we give sufficient conditions in order to achieve strong duality between  $(P_\mu)$  and  $(D_\mu)$ , i.e., the situation when there is no duality gap between  $(P_\mu)$  and  $(D_\mu)$  (that is,  $\text{val}(P_\mu) = \text{val}(D_\mu)$ ), and the dual problem  $(D_\mu)$  has an optimal solution.

In the spirit of the condition introduced in [13], we first introduce a new regularity condition which play an important role in the study of Farkas-type results.

**Definition 3.3.** Let  $\mu \geq 0$ . The family  $(f, g, \delta_C, h)$  is said to satisfy the new regularity condition  $(\text{NRC})_\mu^\geq$ , iff

$$\text{epi } (f + \mu(-g) + \delta_{\mathcal{A}})^* \subseteq \text{epi } f^* + \mu \text{epi } (-g)^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*. \quad (3.3)$$

**Remark 3.4.** (i) Let  $\mu \geq 0$ . It is easy to see that

$$\text{epi } f^* + \mu \text{epi } (-g)^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^* \subseteq \text{epi } (f + \mu(-g) + \delta_{\mathcal{A}})^*. \quad (3.4)$$

Indeed, let  $(x^*, r) \in \text{epi } f^* + \mu \text{epi } (-g)^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*$ . Then, there exist  $(u^*, r_1) \in \text{epi } f^*$ ,  $(v^*, r_2) \in \text{epi } (-g)^*$ ,  $(w^*, r_3) \in \text{epi } (\bar{\lambda} h)^*$  with  $\bar{\lambda} \in S^*$ , and  $(q^*, r_4) \in \text{epi } \delta_C^*$ , such that  $x^* = u^* + \mu v^* + w^* + q^*$

and  $r = r_1 + \mu r_2 + r_3 + r_4$ . Moreover, since  $\bar{\lambda} \in S^*$  and  $h(x) \leq_S 0$ ,  $(\bar{\lambda}h)(x) \leq 0$ . Hence,

$$\begin{aligned}
(f + \mu(-g) + \delta_{\mathcal{A}})^*(x^*) &= \sup_{x \in \mathcal{A}} \{ \langle x^*, x \rangle - (f + \mu(-g))(x) \} \\
&\leq \sup_{x \in \mathcal{A}} \{ \langle x^*, x \rangle - (f + \mu(-g) + \bar{\lambda}h)(x) \} \\
&\leq \sup_{x \in C} \{ \langle x^*, x \rangle - (f + \mu(-g) + \bar{\lambda}h)(x) \} \\
&= \sup_{x \in X} \{ \langle u^* + \mu v^* + w^* + q^*, x \rangle - (f + \mu(-g) + \bar{\lambda}h)(x) + \delta_C(x) \} \\
&\leq \sup_{x \in X} \{ \langle u^*, x \rangle - f(x) \} + \mu \sup_{x \in X} \{ \langle v^*, x \rangle - (-g)(x) \} \\
&\quad + \sup_{x \in X} \{ \langle w^*, x \rangle - (\bar{\lambda}h)(x) \} + \sup_{x \in X} \{ \langle q^*, x \rangle - \delta_C(x) \} \\
&= f^*(u^*) + \mu(-g)^*(v^*) + (\bar{\lambda}h)^*(w^*) + \delta_C^*(q^*) \\
&\leq r_1 + \mu r_2 + r_3 + r_4 \\
&= r.
\end{aligned}$$

Thus,  $(x^*, r) \in \text{epi}(f + \mu(-g) + \delta_{\mathcal{A}})^*$ , and so (3.4) holds. Consequently, the family  $(f, g, \delta_C, h)$  satisfies the new regularity condition  $(\text{NRC})_{\mu}^{\geq}$  if and only if

$$\text{epi}(f + \mu(-g) + \delta_{\mathcal{A}})^* = \text{epi} f^* + \mu \text{epi} (-g)^* + \bigcup_{\lambda \in S^*} \text{epi} (\lambda h)^* + \text{epi} \delta_C^*.$$

- (ii) In the case when  $\mu = 0$ , the new regularity condition  $(\text{NRC})_{\mu}^{\geq}$  collapses to the regularity condition introduced in Definition 6.3 (a) of [13].
- (iii) Note that if, in addition,  $f, -g$  are l.s.c., and  $h$  is  $S$ -epi-closed, then, by Proposition 6.4 of [13], the new regularity condition  $(\text{NRC})_{\mu}^{\geq}$  is equivalent to the following qualification condition  $(\text{CC})_1$  introduced in [28],

$$\text{epi} f^* + \mu \text{epi} (-g)^* + \bigcup_{\lambda \in S^*} \text{epi} (\lambda h)^* + \text{epi} \delta_C^* \text{ is a weak}^* \text{ closed set.}$$

By using the new qualification condition, we obtain the strong duality between  $(P_{\mu})$  and  $(D_{\mu})$ .

**Proposition 3.5.** *Let  $\mu \geq 0$ . Suppose that the family  $(f, g, \delta_C, h)$  satisfies the new regularity condition  $(\text{NRC})_{\mu}^{\geq}$ . Then the strong duality between  $(P_{\mu})$  and  $(D_{\mu})$  holds.*

*Proof.* It is known that weak duality for the problems  $(P_{\mu})$  and  $(D_{\mu})$  always holds, namely

$$\text{val}(P_{\mu}) \geq \text{val}(D_{\mu}). \quad (3.5)$$

If  $\text{val}(P_{\mu}) = -\infty$ , the strong duality for  $(P_{\mu})$  and  $(D_{\mu})$  holds. So, we assume that

$$\text{val}(P_{\mu}) = \alpha \in \mathbb{R}. \quad (3.6)$$

On the other hand, it is easy to see that

$$\text{val}(P_{\mu}) = \inf_{x \in X} \{ (f(x) + \mu(-g)(x) + \delta_{\mathcal{A}})(x) \} = -(f + \mu(-g) + \delta_{\mathcal{A}})^*(0). \quad (3.7)$$

By (3.6) and (3.7), we have

$$(0, -\alpha) \in \text{epi}(f + \mu(-g) + \delta_{\mathcal{A}})^*.$$

Since the family  $(f, g, \delta_C, h)$  satisfies  $(\text{NRC})_{\mu}^{\geq}$ , we can show that

$$(0, -\alpha) \in \text{epi } f^* + \mu \text{epi } (-g)^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*.$$

This implies that there exist  $(u^*, r) \in \text{epi } f^*$ ,  $(v^*, s) \in \text{epi } (-g)^*$ , and  $(w^*, t) \in \text{epi } (\lambda h)^*$  with  $\bar{\lambda} \in S^*$ , such that  $(-u^* - \mu v^* - w^*, -\alpha - r - \mu s - t) \in \text{epi } \delta_C^*$ . Then,

$$\begin{aligned} \text{val}(D_{\mu}) &\geq -f^*(u^*) - \mu(-g)^*(v^*) - (\bar{\lambda}h)^*(w^*) - \delta_C^*(-u^* - \mu v^* - w^*) \\ &\geq -r - \mu s - t - (-\alpha - r - \mu s - t) \\ &= \alpha \\ &= \text{val}(P_{\mu}). \end{aligned}$$

Thus,  $\text{val}(P_{\mu}) = \text{val}(D_{\mu})$  and  $(D_{\mu})$  has an optimal solution. This completes the proof □

Now, we prove the main result of this subsection.

**Theorem 3.6.** *Let  $\mu \geq 0$ . Suppose that the family  $(f, g, \delta_C, h)$  satisfies the new regularity condition  $(\text{NRC})_{\mu}^{\geq}$ . Then, the following statements are equivalent:*

- (i)  $x \in C, h(x) \leq_S 0 \implies \frac{f(x)}{g(x)} \geq \mu$ ;
- (ii)  $(0, 0) \in \text{epi } f^* + \mu \text{epi } (-g)^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*$ ;
- (iii) *There exist  $x^* \in \text{dom } f^*, y^* \in \text{dom } (-g)^*, z^* \in \text{dom } (\lambda h)^*$  with  $\lambda \in S^*$ , such that*

$$f^*(x^*) + \mu(-g)^*(y^*) + (\lambda h)^*(z^*) + \delta_C^*(-x^* - \mu y^* - z^*) \leq 0.$$

*Proof.* (i)  $\implies$  (ii). Suppose that (i) holds. Then, for any  $x \in \mathcal{A}$ ,  $f(x) \geq \mu g(x)$ . It follows that  $(f + \mu(-g) + \delta_{\mathcal{A}})(x) \geq 0$ . Then,

$$(0, 0) \in \text{epi}(f + \mu(-g) + \delta_{\mathcal{A}})^*.$$

Since the family  $(f, g, \delta_C, h)$  satisfies  $(\text{NRC})_{\mu}^{\geq}$ , we obtain that

$$(0, 0) \in \text{epi } f^* + \mu \text{epi } (-g)^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*,$$

and (ii) holds.

(ii)  $\implies$  (iii). Suppose that (ii) holds. Then, there exist  $(x^*, r) \in \text{epi } f^*$ ,  $(y^*, s) \in \text{epi } (-g)^*$ ,  $(z^*, t) \in \text{epi } (\lambda h)^*$  with  $\lambda \in S^*$ , and  $(p^*, p) \in \text{epi } \delta_C^*$  such that

$$0 = x^* + \mu y^* + z^* + p^* \tag{3.8}$$

and

$$0 = r + \mu s + t + p. \tag{3.9}$$

Since  $f^*(x^*) \leq r$ ,  $(-g)^*(y^*) \leq s$ ,  $(\lambda h)^*(z^*) \leq t$  and  $\delta_C^*(p^*) \leq p$ , it follows from (3.8) and (3.9) that

$$f^*(x^*) + \mu(-g)^*(y^*) + (\lambda h)^*(z^*) + \delta_C^*(-x^* - \mu y^* - z^*) \leq 0,$$

and (iii) holds.

(iii)  $\implies$  (i). Suppose that (iii) holds. Then, there exist  $x^* \in \text{dom } f^*$ ,  $y^* \in \text{dom } (-g)^*$ ,  $z^* \in \text{dom } (\lambda h)^*$ , and  $\lambda \in S^*$  such that

$$-f^*(x^*) - \mu(-g)^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-x^* - \mu y^* - z^*) \geq 0$$

which implies that

$$\max_{\substack{\lambda \in S^*, x^* \in \text{dom } f^*, \\ y^* \in \text{dom } (-g)^*, z^* \in \text{dom } (\lambda h)^*}} \{-f^*(x^*) - \mu(-g)^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-x^* - \mu y^* - z^*)\} \geq 0.$$

Then,

$$\text{val}(D_\mu) \geq 0.$$

By Proposition 3.5,

$$\text{val}(P_\mu) \geq 0.$$

This means that  $\text{val}(P) \geq \mu$ , and the proof is complete. □

By Remark 3.4 (iii), the result in the corollary below can be easily established.

**Corollary 3.7** ([28], Thms. 4.3 and 4.4). *Let  $\mu \geq 0$ . Suppose that the functions  $f, -g$  are lower semicontinuous, and  $h$  is  $S$ -epi-closed. If the new qualification condition  $(CC)_1$  holds, then, the conclusions of Theorem 3.6 hold.*

**Remark 3.8.** Obviously, Theorem 3.6 improves the corresponding results in Theorems 4.3 and 4.4 of [28] as, here, we do not impose any topological assumptions on  $f, g$ , and  $h$ .

Now, we present an example to explain Theorem 3.6 for which the  $(NRC)_\mu^{\geq}$  holds but the functions  $f$  and  $-g$  are not lower semi-continuous.

**Example 3.9.** Let  $X = Y = C = \mathbb{R}$ , and  $\mu = 1$ . Let  $h(x) = -x$ , for any  $x \in \mathbb{R}$ . Then,  $\mathcal{A} = [0, +\infty)$ . Let  $f, -g : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  be defined respectively by

$$f(x) = \begin{cases} x + 2, & \text{if } x > 0, \\ \frac{5}{2}, & \text{if } x = 0, \\ +\infty, & \text{if } x < 0, \end{cases} \text{ and } g(x) = \begin{cases} x + 1, & \text{if } x > 0, \\ \frac{1}{2}, & \text{if } x = 0, \\ -\infty, & \text{if } x < 0. \end{cases}$$

Obviously,  $f$  and  $-g$  are not lower semicontinuous functions. Clearly, for any  $x^* \in \mathbb{R}$ ,

$$\begin{aligned} f^*(x^*) &= \begin{cases} -2, & \text{if } x^* \leq 1, \\ +\infty, & \text{if } x^* > 1, \end{cases} & (-g)^*(x^*) &= \begin{cases} 1, & \text{if } x^* \leq -1, \\ +\infty, & \text{if } x^* > -1, \end{cases} \\ (\lambda h)^*(x^*) &= \begin{cases} 0, & \text{if } x^* = -\lambda, \\ +\infty, & \text{if } x^* \neq -\lambda, \end{cases} & \delta_C^*(x^*) &= \begin{cases} 0, & \text{if } x^* = 0, \\ +\infty, & \text{if } x^* \neq 0. \end{cases} \end{aligned}$$

Then,

$$\text{epi } f^* + \mu \text{epi } (-g)^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^* = (-\infty, 0] \times [-1, +\infty),$$

and

$$(0, 0) \in \text{epi } f^* + \mu \text{epi } (-g)^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*.$$

Note that

$$(f - \mu g + \delta_{\mathcal{A}})(x) = \begin{cases} 1, & \text{if } x > 0, \\ 2, & \text{if } x = 0, \\ +\infty, & \text{if } x < 0, \end{cases}$$

Then, for any  $x^* \in \mathbb{R}$ ,

$$(f - \mu g + \delta_{\mathcal{A}})^*(x^*) = \begin{cases} -1, & \text{if } x^* \leq 0, \\ +\infty, & \text{if } x^* > 0. \end{cases}$$

Clearly,

$$\text{epi } (f - g + \delta_{\mathcal{A}})^* = (-\infty, 0] \times [-1, +\infty).$$

Thus, the new regularity condition  $(\text{NRC})_{\mu}^{\geq}$  holds.

On the other hand, for instance, there exist  $x^* = 1, y^* = -3, z^* = -2$ , and  $\lambda = 2$ , such that

$$f^*(x^*) + \mu(-g)^*(y^*) + (\lambda h)^*(z^*) + \delta_C^*(-x^* - \mu y^* - z^*) = -1 \leq 0.$$

Then, all statements in Theorem 3.6 hold. Thus, Theorem 3.6 is applicable.

The previous result can be reformulated as a theorem of the alternative in the following way.

**Corollary 3.10.** *Let  $\mu \geq 0$ . Suppose that the family  $(f, g, \delta_C, h)$  satisfies the new regularity condition  $(\text{NRC})_{\mu}^{\geq}$ . Then, either the inequality system*

$$x \in C, h(x) \leq_S 0, \frac{f(x)}{g(x)} < \mu$$

has a solution, or the system

$$\begin{aligned} f^*(x^*) + \mu(-g)^*(y^*) + (\lambda h)^*(z^*) + \delta_C^*(-x^* - \mu y^* - z^*) &\leq 0, \\ x^* \in \text{dom } f^*, y^* \in \text{dom } (-g)^*, z^* \in \text{dom } (\lambda h)^*, \lambda \in S^* \end{aligned}$$

has a solution, but never both.

### 3.2. The case $\mu < 0$

If  $\mu$  is a negative real number, the objective function  $f - \mu g$  of the problem  $(P_{\mu})$  can be viewed as the difference of two convex functions  $f$  and  $\mu g$ . Since the difference of two convex functions is not necessarily a convex function, the problem  $(P_{\mu})$  is a nonconvex programming problem. Then, using the approach presented in [26], we can establish the following dual problem

$$(D_{\mu}) \quad \inf_{x^* \in \text{dom } (-g)^*} \sup_{\substack{y^* \in \text{dom } f^*, \\ \lambda \in S^*, z^* \in \text{dom } (\lambda h)^*}} \left\{ -\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-\mu x^* - y^* - z^*) \right\}.$$

In order to characterize the Farkas type results of systems with fractional functions, we need to introduce a new regularity condition. To this aim, we will make use of the following set  $\Lambda$  defined by

$$\Lambda := \bigcap_{x^* \in \text{dom } (-g)^*} \left( \text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^* + \mu(x^*, (-g)^*(x^*)) \right).$$

Let  $\bar{f} := f + \mu \text{cl } (-g)$ . If  $\text{cl } (-g)$  is a proper function, then,  $\bar{f}$  is proper. By using the similar approach of Lemma 3.1 of [26], we can easily get

$$\Lambda = \text{epi } (\bar{f})^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*. \tag{3.10}$$

On the other hand, it is easy to see that

$$\text{epi } (\bar{f})^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^* \subseteq \text{epi } (\bar{f} + \delta_A)^*.$$

So, when  $-g$  is lower semicontinuous, the following assertion holds:

$$\Lambda \subseteq \text{epi } (f - \mu g + \delta_A)^*. \tag{3.11}$$

Moreover, by simply modifying, [29], Example 3.2, we can show that the inclusion (3.11) may not hold in general.

The following proposition describes the relationship among the set  $\Lambda$  and  $(D_{\mu})$ .

**Proposition 3.11.** *Let  $\mu < 0$  and  $\alpha \in \mathbb{R}$ . Then,  $(0, \alpha) \in \Lambda$  if and only if  $\text{val}(D_\mu) \geq -\alpha$ , and for any  $x^* \in \text{dom}(-g)^*$ , there exist  $\lambda \in S^*$ ,  $y^* \in \text{dom} f^*$ , and  $z^* \in \text{dom}(\lambda h)^*$  such that*

$$-\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-\mu x^* - y^* - z^*) \geq -\alpha. \tag{3.12}$$

*Proof.* ( $\Rightarrow$ ) Let  $(0, \alpha) \in \Lambda$ . Then, for any  $x^* \in \text{dom}(-g)^*$ , we have

$$(-\mu x^*, -\mu(-g)^*(x^*) + \alpha) \in \text{epi} f^* + \bigcup_{\lambda \in S^*} \text{epi}(\lambda h)^* + \text{epi} \delta_C^*,$$

Then, there exist  $\lambda \in S^*$ ,  $(y^*, \alpha_1) \in \text{epi} f^*$ ,  $(z^*, \alpha_2) \in \text{epi}(\lambda h)^*$ , and  $(q^*, \alpha_3) \in \delta_C^*$  such that

$$(-\mu x^*, -\mu(-g)^*(x^*) + \alpha) = (y^*, \alpha_1) + (z^*, \alpha_2) + (q^*, \alpha_3),$$

which means that

$$-\mu x^* = y^* + z^* + q^* \tag{3.13}$$

and

$$-\mu(-g)^*(x^*) + \alpha = \alpha_1 + \alpha_2 + \alpha_3. \tag{3.14}$$

Since  $f^*(y^*) \leq \alpha_1$ ,  $(\lambda h)^*(z^*) \leq \alpha_2$ , and  $\delta_C^*(q^*) \leq \alpha_3$ , it follows from (3.13) and (3.14) that

$$-\mu(-g)^*(x^*) + \alpha \geq f^*(y^*) + (\lambda h)^*(z^*) + \delta_C^*(-\mu x^* - y^* - z^*).$$

Thus,

$$-\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-\mu x^* - y^* - z^*) \geq -\alpha.$$

It follows that  $\text{val}(D_\mu) \geq -\alpha$ , and  $\lambda \in S^*$ ,  $y^* \in \text{dom} f^*$ ,  $z^* \in \text{dom}(\lambda h)^*$  satisfy (3.12).

( $\Leftarrow$ ) Assume that  $\text{val}(D_\mu) \geq -\alpha$ , and for any  $x^* \in \text{dom}(-g)^*$ , there exist  $\lambda \in S^*$ ,  $y^* \in \text{dom} f^*$ , and  $z^* \in \text{dom}(\lambda h)^*$  such that (3.12) holds. Then,

$$\delta_C^*(-\mu x^* - y^* - z^*) \leq -\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) + \alpha,$$

which means that

$$(-\mu x^* - y^* - z^*, -\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) + \alpha) \in \text{epi} \delta_C^*. \tag{3.15}$$

Moreover, it is easy to see that

$$0 = y^* + z^* + (-\mu x^* - y^* - z^*) + \mu x^*, \tag{3.16}$$

and

$$\alpha = f^*(y^*) + (\lambda h)^*(z^*) + (-\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) + \alpha) + \mu(-g)^*(x^*). \tag{3.17}$$

By (3.15)–(3.17), we get

$$(0, \alpha) \in \text{epi} f^* + \bigcup_{\lambda \in S^*} \text{epi}(\lambda h)^* + \text{epi} \delta_C^* + \mu(x^*, (-g)^*(x^*)).$$

Since  $x^* \in \text{dom}(-g)^*$  is arbitrary, we obtain that

$$(0, \alpha) \in \bigcap_{x^* \in \text{dom}(-g)^*} \left( \text{epi} f^* + \bigcup_{\lambda \in S^*} \text{epi}(\lambda h)^* + \text{epi} \delta_C^* + \mu(x^*, (-g)^*(x^*)) \right) = \Lambda,$$

and the proof is complete. □

Now, in order to characterize the Farkas type results, we introduce the following regularity condition by using the possible relationships between  $\Lambda$  and  $\text{epi}(f - \mu g + \delta_{\mathcal{A}})^*$ .

**Definition 3.12.** Let  $\mu < 0$ . The family  $(f, g, \delta_C, h)$  is said to satisfy new regularity condition  $(\text{NRC})_{\mu}^<$ , iff

$$\text{epi}(f + \mu(-g) + \delta_{\mathcal{A}})^* = \Lambda.$$

**Remark 3.13.** It should be noted that new regularity condition  $(\text{NRC})_{\mu}^<$  introduced in Definition 3.12 is inspired by similar concepts in the context of DC optimization problems in [14, 15, 26, 29]. Moreover, in the case that  $\mu = -1$ , the new regularity condition  $(\text{NRC})_{\mu}^<$  collapses to the closedness condition  $(\text{CC})_1$  introduced in Definition 5.1 of [26].

Now, we first give the strong duality regarding an optimization problem of the type  $(P_{\mu})$  and its dual problem  $(D_{\mu})$ .

**Proposition 3.14.** Let  $\mu < 0$ . Suppose that the family  $(f, g, \delta_C, h)$  satisfies the new regularity condition  $(\text{NRC})_{\mu}^<$ . Then the strong duality between  $(P_{\mu})$  and  $(D_{\mu})$  holds.

*Proof.* Suppose that there exist  $\alpha \in \mathbb{R}$  such that

$$\text{val}(P_{\mu}) < -\alpha < \text{val}(D_{\mu}). \quad (3.18)$$

By Proposition 3.11, we get  $(0, \alpha) \in \Lambda$ . Since the family  $(f, g, \delta_C, h)$  satisfies the  $(\text{NRC})_{\mu}^<$ , one has

$$(0, \alpha) \in \text{epi}(f + \mu(-g) + \delta_{\mathcal{A}})^*.$$

Then,

$$(f + \mu(-g) + \delta_{\mathcal{A}})^*(0) \leq \alpha.$$

This, together with

$$\text{val}(P_{\mu}) = \inf_{x \in X} \{f(x) - \mu g(x) + \delta_{\mathcal{A}}\} = -(f + \mu(-g) + \delta_{\mathcal{A}})^*(0), \quad (3.19)$$

yields  $\text{val}(P_{\mu}) \geq -\alpha$ . This contradicts (3.18), and hence

$$\text{val}(P_{\mu}) \geq \text{val}(D_{\mu}).$$

Now, it suffices to show that  $\text{val}(D_{\mu}) \geq \text{val}(P_{\mu})$ , and for any  $x^* \in \text{dom}(-g)^*$ , there exist  $\lambda \in S^*$ ,  $y^* \in \text{dom} f^*$ , and  $z^* \in \text{dom}(\lambda h)^*$  such that (3.12) holds. In fact, if  $\text{val}(P_{\mu}) = -\infty$ , then the strong duality between  $(P_{\mu})$  and  $(D_{\mu})$  holds naturally. So, we assume that  $\text{val}(P_{\mu}) = -\alpha \in \mathbb{R}$ . This, together with (3.19), yields

$$(0, \alpha) \in \text{epi}(f + \mu(-g) + \delta_{\mathcal{A}})^*. \quad (3.20)$$

On the other hand, as the family  $(f, g, \delta_C, h)$  satisfies the  $(\text{NRC})_{\mu}^<$ , it follows from (3.20) that

$$(0, \alpha) \in \Lambda.$$

Combining this and Proposition 3.11, one gets  $\text{val}(D_{\mu}) \geq -\alpha$  and for any  $x^* \in \text{dom}(-g)^*$ , there exist  $\lambda \in S^*$ ,  $y^* \in \text{dom} f^*$ , and  $z^* \in \text{dom}(\lambda h)^*$  such that (3.12) holds. This means that the strong duality between  $(P_{\mu})$  and  $(D_{\mu})$  holds. The proof is complete.  $\square$

Now, we present the main result in this subsection.

**Theorem 3.15.** Let  $\mu < 0$ . If the family  $(f, g, \delta_C, h)$  satisfies the  $(\text{NRC})_{\mu}^<$ , then, the following statements are equivalent:

- (i)  $x \in C, h(x) \leq_S 0 \implies \frac{f(x)}{g(x)} \geq \mu;$
- (ii) For any  $x^* \in \text{dom } (-g)^*$ ,

$$-\mu(x^*, (-g)^*(x^*)) \in \text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*;$$

- (iii) For any  $x^* \in \text{dom } (-g)^*$ , there exist  $y^* \in \text{dom } f^*, z^* \in \text{dom } (\lambda h)^*$ , and  $\lambda \in S^*$  such that

$$-\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-\mu x^* - y^* - z^*) \geq 0.$$

*Proof.* (i)  $\implies$  (ii). Suppose that (i) holds. Then, for any  $x \in \mathcal{A}, f(x) \geq \mu g(x)$ . It follows that  $(f + \mu(-g) + \delta_{\mathcal{A}})(x) \geq 0$ . Then,

$$(0, 0) \in \text{epi}(f + \mu(-g) + \delta_{\mathcal{A}})^*.$$

Since the family  $(f, g, \delta_C, h)$  satisfies  $(FRC)_{\mu}^<$ , we obtain that

$$(0, 0) \in \bigcap_{x^* \in \text{dom } (-g)^*} \left( \text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^* + \mu(x^*, (-g)^*(x^*)) \right).$$

Thus, for any  $x^* \in \text{dom } (-g)^*$ ,

$$-\mu(x^*, (-g)^*(x^*)) \in \text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*.$$

(ii)  $\implies$  (iii). Suppose that (ii) holds. Then, for any  $x^* \in \text{dom } (-g)^*$ , there exist  $(y^*, r) \in \text{epi } f^*, (z^*, s) \in \text{epi } (\lambda h)^*$  with  $\lambda \in S^*$ , and  $(w^*, t) \in \text{epi } \delta_C^*$  such that

$$-\mu(x^*, (-g)^*(x^*)) = (y^*, r) + (z^*, s) + (w^*, t). \tag{3.21}$$

Since  $f^*(y^*) \leq r, (\lambda h)^*(z^*) \leq s$  and  $\delta_C^*(w^*) \leq t$ , it follows from (3.21) that

$$-\mu(-g)^*(x^*) \geq f^*(y^*) + (\lambda h)^*(z^*) + \delta_C^*(-\mu x^* - y^* - z^*).$$

Thus, (iii) holds.

(iii)  $\implies$  (i). Suppose that (iii) holds. Then, for any  $x^* \in \text{dom } (-g)^*$ , there exist  $\lambda \in S^*, y^* \in \text{dom } f^*$ , and  $z^* \in \text{dom } (\lambda h)^*$  such that

$$-\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-\mu x^* - y^* - z^*) \geq 0.$$

which implies that

$$\inf_{x^* \in \text{dom } (-g)^*} \sup_{\substack{y^* \in \text{dom } f^*, \\ \lambda \in S^*, z^* \in \text{dom } (\lambda h)^*}} \{-\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-\mu x^* - y^* - z^*)\} \geq 0.$$

Then,

$$\text{val}(D_{\mu}) \geq 0.$$

By Proposition 3.14,

$$\text{val}(P_{\mu}) \geq 0.$$

This means that  $\text{val}(P) \geq \mu$ , and the proof is complete. □

**Remark 3.16.** When  $f, -g$  are lower semicontinuous, and  $h$  is  $S$ -epi-closed, Farkas-type result similar to the one in Theorem 3.15 is obtained in Theorems 4.7 and 4.8 of [28] under a closeness condition. Note that Farkas-type result similar to the one in Theorem 3.15 also appeared in [14,26] with different kinds of regularity conditions. Clearly, our result covers the corresponding results in [14,26].

Now, we give an example to explain Theorem 3.15 for which the  $(\text{NRC})_\mu^<$  holds and the function  $-g$  is proper convex but not lower semi-continuous.

**Example 3.17.** Let  $X = Y = \mathbb{R}$ , and  $\mu = -1$ . Let  $C = (0, 1)$  and  $h(x) = x - 1$ , for any  $x \in \mathbb{R}$ . Then,  $\mathcal{A} = (0, 1)$ . Let  $f, -g : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  be defined respectively by

$$f(x) = \begin{cases} x^2, & \text{if } x > 1, \\ 1, & \text{if } 0 \leq x \leq 1, \\ +\infty, & \text{if } x < 0, \end{cases} \text{ and } g(x) = \begin{cases} -x^2 + 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -\infty, & \text{if } x < 0. \end{cases}$$

Obviously,  $f$  and  $-g$  are proper convex functions. Clearly, for any  $x^* \in \mathbb{R}$ ,

$$f^*(x^*) = \begin{cases} \frac{(x^*)^2}{4}, & \text{if } x^* > 2, \\ x^* - 1, & \text{if } 0 \leq x^* \leq 2, \\ -1, & \text{if } x^* < 0, \end{cases} \quad (-g)^*(x^*) = \begin{cases} \frac{(x^*)^2}{4} + 1, & \text{if } x^* > 0, \\ 1, & \text{if } x^* \leq 0, \end{cases}$$

$$(\lambda h)^*(x^*) = \begin{cases} \lambda, & \text{if } x^* = \lambda, \\ +\infty, & \text{if } x^* \neq \lambda, \end{cases} \quad \delta_C^*(x^*) = \begin{cases} x^*, & \text{if } x^* \geq 0, \\ 0, & \text{if } x^* < 0. \end{cases}$$

Consequently,

$$\text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^* = \{(x^*, r) \in \mathbb{R}^2 : x^* \leq 0, r \geq -1\} \cup \{(x^*, r) \in \mathbb{R}^2 : x^* > 0, r \geq x^* - 1\}.$$

Then, for any  $x^* \in \text{dom } (-g)^*$  and  $\mu = -1$ ,

$$-\mu(x^*, (-g)^*(x^*)) \in \text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*.$$

Note that

$$(f - \mu g + \delta_{\mathcal{A}})(x) = \begin{cases} -x^* + 2, & \text{if } 0 \leq x \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

Then, for any  $x^* \in \mathbb{R}$ ,

$$(f - \mu g + \delta_{\mathcal{A}})^*(x^*) = \begin{cases} x^* - 1, & \text{if } x^* \geq -1, \\ -2, & \text{if } x^* < -1. \end{cases}$$

Clearly,

$$\text{epi } (f - \mu g + \delta_{\mathcal{A}})^* = \{(x^*, r) \in \mathbb{R}^2 : x^* \leq -1, r \geq -2\} \cup \{(x^*, r) \in \mathbb{R}^2 : x^* > -1, r \geq x^* - 1\}.$$

Moreover, note that

$$\bar{f}(x) = (f - \text{cl}(-g))(x) = \begin{cases} 1, & \text{if } x > 1, \\ -x^2 + 2, & \text{if } 0 \leq x \leq 1, \\ +\infty, & \text{if } x < 0. \end{cases}$$

Then, for any  $x^* \in \mathbb{R}$ ,

$$(\bar{f})^*(x^*) = \begin{cases} +\infty, & \text{if } x^* > 0, \\ x^* - 1, & \text{if } -1 \leq x^* \leq 0, \\ -2, & \text{if } x^* < -1. \end{cases}$$

This, together with (3.10), implies that

$$\begin{aligned} \Lambda &= \text{epi}(\bar{f})^* + \bigcup_{\lambda \in S^*} \text{epi}(\lambda h)^* + \text{epi}\delta_C^* \\ &= \{(x^*, r) \in \mathbb{R}^2 : x^* \leq -1, r \geq -2\} \cup \{(x^*, r) \in \mathbb{R}^2 : x^* > -1, r \geq x^* - 1\}. \end{aligned}$$

Thus, the new regularity condition  $(\text{NRC})_\mu^<$  holds.

Note that  $\text{dom}(-g)^* = \mathbb{R}$ . We consider two cases:

- (i) If  $x^* \leq 0$ , there exist  $y^* = 0, z^* = \lambda = \frac{1}{2}$ , such that

$$-\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-\mu x^* - y^* - z^*) = \frac{3}{2} \geq 0.$$

- (ii) If  $x^* > 0$ , there exist  $y^* = 0, z^* = \lambda = 0$ , such that

$$-\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-\mu x^* - y^* - z^*) = \frac{(x^*)^2}{4} + 2 - x^* \geq 0.$$

Then, all statements in Theorem 3.15 hold. Thus, Theorem 3.15 is applicable.

The previous result can be reformulated as a theorem of the alternative in the following way.

**Corollary 3.18.** *Let  $\mu < 0$ . If the family  $(f, g, \delta_C, h)$  satisfies the  $(\text{NRC})_\mu^<$ , then, either the inequality system*

$$x \in C, h(x) \leq_S 0, \frac{f(x)}{g(x)} < \mu$$

*has a solution, or for any  $x^* \in \text{dom}(-g)^*$ , the system*

$$\begin{aligned} -\mu(-g)^*(x^*) - f^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-\mu x^* - y^* - z^*) &\geq 0, \\ y^* \in \text{dom} f^*, z^* \in \text{dom}(\lambda h)^*, \lambda \in S^* & \end{aligned}$$

*has a solution, but never both.*

#### 4. A SPECIAL CASE

In this section, as a special case of our general result, we provide a unify Farkas-type results for a cone-convex system which does not depend on the sign of  $\mu$ . Moreover, the assertions within this section generalize some recently results obtained in [3, 27, 28].

In this section,  $X, Y, S, C, f$ , and  $h$  are considered as before, while the function  $g$  is taken constant  $g(x) \equiv 1$ . So, one has

$$(-g)^*(y^*) = \begin{cases} 1, & \text{if } y^* = 0, \\ +\infty, & \text{if } y^* \neq 0. \end{cases}$$

Then, taking (2.2) into account, one has  $(\text{NRC})_\mu^>$  and  $(\text{NRC})_\mu^<$  are equivalent to the following regularity condition which has been studied in [13].

**Definition 4.1** ([13], Def. 6.3). The family  $(f, \delta_C, h)$  is said to satisfy the condition  $C_1(f, \mathcal{A})$ , iff

$$\text{epi}(f + \delta_{\mathcal{A}})^* = \text{epi} f^* + \bigcup_{\lambda \in S^*} \text{epi}(\lambda h)^* + \text{epi} \delta_C^*. \tag{4.1}$$

**Remark 4.2.** If, in addition,  $f$  is l.s.c., and  $h$  is  $S$ -epi-closed, then, by Proposition 6.4 of [13], the condition  $C_1(f, \mathcal{A})$  is equivalent to the following qualification condition  $(CQC)_1$ ,

$$\text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^* \text{ is a weak}^* \text{ closed set.}$$

Note that the qualification condition  $(CQC)_1$  was renamed as  $(CCCQ)$  or  $(CC)$  in other papers, such as [6, 7, 28].

**Theorem 4.3.** *Let  $\mu \geq 0$ . Suppose that the family  $(f, \delta_C, h)$  satisfies the condition  $C_1(f, \mathcal{A})$ . Then, the following statements are equivalent:*

- (i)  $x \in C, h(x) \leq_S 0 \implies f(x) \geq \mu$ ;
- (ii)  $(0, -\mu) \in \text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*$ ;
- (iii) *There exist  $x^* \in \text{dom } f^*, z^* \in \text{dom } (\lambda h)^*$ , and  $\lambda \in S^*$  such that*

$$f^*(x^*) + (\lambda h)^*(z^*) + \delta_C^*(-x^* - z^*) \leq -\mu.$$

*Proof.* Since  $\text{epi } (-g)^* = \{0\} \times [1, +\infty)$ , it follows from Theorem 3.6 that (i) is equivalent to

$$\begin{aligned} (0, 0) &\in \text{epi } f^* + \mu(\{0\} \times [1, +\infty)) + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^* \\ &= \text{epi } f^* + (0, \mu) + \{0\} \times [0, +\infty) + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*. \end{aligned}$$

That can be equivalently written as (ii).

On the other hand, from  $g(x) \equiv 1$ , we get  $\text{dom } (-g)^* = \{0\}$  and  $(-g)^*(0) = 1$ . By Theorem 3.6, (ii) is fulfilled if and only if there exist  $x^* \in \text{dom } f^*, z^* \in \text{dom } (\lambda h)^*$ , and  $\lambda \in S^*$  such that

$$f^*(x^*) + \mu + (\lambda h)^*(z^*) + \delta_C^*(-x^* - z^*) \leq 0,$$

and the proof is complete. □

**Theorem 4.4.** *Let  $\mu < 0$ . If the family  $(f, \delta_C, h)$  satisfies the condition  $C_1(f, \mathcal{A})$ , then, the following statements are equivalent:*

- (i)  $x \in C, h(x) \leq_S 0 \implies f(x) \geq \mu$ ;
- (ii)  $(0, -\mu) \in \text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*$ ;
- (iii) *There exist  $y^* \in \text{dom } f^*,$  and  $z^* \in \text{dom } (\lambda h)^*$  with  $\lambda \in S^*$ , such that*

$$f^*(y^*) + (\lambda h)^*(z^*) + \delta_C^*(-y^* - z^*) \leq -\mu.$$

*Proof.* Since  $\text{dom } (-g)^* = \{0\}$  and  $(-g)^*(0) = 1$ , Theorem 3.15 ensures that (i) is equivalent to

$$-\mu(0, 1) \in \text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*.$$

That can be equivalently written as (ii).

On the other hand, by Theorem 3.15, (ii) is fulfilled if and only if there exist  $y^* \in \text{dom } f^*$ , and  $z^* \in \text{dom } (\lambda h)^*$  with  $\lambda \in S^*$  such that

$$-\mu - f^*(y^*) - (\lambda h)^*(z^*) - \delta_C^*(-y^* - z^*) \geq 0,$$

and the proof is complete. □

The following results unify the previous two results.

**Theorem 4.5.** *Let  $\mu \in \mathbb{R}$ . If the family  $(f, \delta_C, h)$  satisfies the condition  $C_1(f, \mathcal{A})$ , then, the following statements are equivalent:*

- (i)  $x \in C, h(x) \leq_S 0 \implies f(x) \geq \mu$ ;
- (ii)  $(0, -\mu) \in \text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*$ ;
- (iii) *There exist  $x^* \in \text{dom } f^*$ , and  $y^* \in \text{dom } (\lambda h)^*$  with  $\lambda \in S^*$ , such that*

$$f^*(x^*) + (\lambda h)^*(y^*) + \delta_C^*(-x^* - y^*) \leq -\mu.$$

**Corollary 4.6** ([27], Thm. 4.6). *Let  $\mu \in \mathbb{R}$ . Suppose that  $f$  is lower semicontinuous and  $h$  is  $S$ -epi-closed. If the condition  $(\text{CQC})_1$  is satisfied, then, the conclusions of Theorem 4.5 hold.*

**Remark 4.7.** By virtue of the new regularity condition  $C_1(f, \mathcal{A})$ , we succeeded to eliminate superfluous lower semicontinuous functions in Theorem 4.5. Obviously, Theorem 4.5 modifies and improves Theorem 4.6 in [27].

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