RAIRO-Oper. Res. 53 (2019) 1563–1580
https://doi.org/10.1051/ro/2019068

RAIRO Operations Research
www.rairo-ro.org

DEVELOPMENT OF A NEW COST PPS AND DECOMPOSITION OF OBSERVED ACTUAL COST FOR DMU IN A NON-COMPETITIVE SPACE IN DEA

Elham Rezaei Hezaveh[1], Reza Fallahnejad[1,*], Masoud Sanei[2] and Mohammad Izadikhah[3]

Abstract. Data Envelopment Analysis (DEA) is an appropriate tool for estimating various types of efficiency such as cost efficiency. There are two different sates in cost spaces; in the first space prices are equal for all Decision Making Units (DMUs) which is competitive space, and in the second space prices are different form one DMU to another; this is known as non-competitive space. The present paper introduces a new method to assess Cost Efficiency (CE), Revenue Efficiency (RE) and Profit Efficiency (PE) in a non-competitive space. The present paper also proposes a Production Possibility Set (PPS) in which DMUs are evaluated based on both their own prices and the prices of other DMUs in non-competitive space. Moreover, a new decomposition is provided for observed actual cost DMUs based on the cost efficiency model and the proposed PPS, thus the observed actual cost can be shown by summation of several technical, price and allocative efficiency (AE) losses. The biggest advantage of this method comparing to the previous methods is that passive the developed cost efficiency and the cost Production Possibility Set has been developed and the performed decomposition is more accurate; this is because the new inefficiency sources are defined and added to this new decomposition. Therefore, it includes more inefficient sources.

Mathematics Subject Classification. 90B99, 90C99.

Received October 13, 2018. Accepted June 20, 2019.

1. Introduction

Data Envelopment Analysis (DEA) was introduced in Charnes *et al.* [7] as a powerful tool for measuring the relative efficiency of a set of Decision Making Units (DMUs). These DMUs receive the input (x_i) with price (c_i) and produce the output (y_r) with price (p_r). Using various mathematical models, DEA can evaluate the efficiencies such as the technical efficiency, cost efficiency, revenue efficiency, profit efficiency, and allocative efficiency [10]. The price information for inputs and outputs are occasionally available. Which should be included

Keywords. Data Envelopment Analysis (DEA), Cost Efficiency (CE), competitive space, non-competitive space.

[1] Department of Mathematics, Khorramabad Branch, Islamic Azad University, Khorramabad, Iran.
[2] Department of Mathematics, Tehran Center Branch, Islamic Azad University, Tehran, Iran.
[3] Department of Mathematics, Arak Branch, Islamic Azad University, Arak, Iran.
*Corresponding author: r.fallahnejad@khoiau.ac.ir, rezafallahnejad120@gmail.com

Article published by EDP Sciences

in evaluations; otherwise the estimates will be incomplete in terms of price. Generally DMUs can be evaluated in both competitive and non-competitive environments. In competitive environments, prices of all DMUs are the same (or very close), but in non-competitive environments prices can have minor or even major differences in one or more indicators. Even prices can be different in all indicators; and each input or output has a separate price for itself.

Feasibility studies which was conducted in the field of Cost Efficiency (CE) and its equivalent versions, like Revenue Efficiency (RE) and Profitability Efficiency (PE), in DEA literature can be considered from different aspects. The concept of CE can be traced back to Farrell [16], and for the first time Färe and Grosskopf [15] formulated a Linear Programming (LP) model for CE assessment.

In the methodology proposed in Färe and Grosskopf [15], a Production Possibility Set typical DEA and its frontier were used to estimate the cost efficiency so that a point on the efficiency frontier with the lowest cost and with the minimal output was used by the considered unit. This model was used as the basis for many applications and the following future theoretical developments. In the case of theoretical development and eliminating some of the shortcomings of the original CE model, some studies have been carried out, which can be divided into two categories based on the following two assumptions. First, the same prices are considered for all the units, which indicates a highly competitive atmosphere and the law of one price. Second, prices are non-identical, which indicates a non-competitive atmosphere so that the place at which prices are presented by the units under evaluation is known, and therefore, units within it are not necessarily price takers and thus there is a possibility change in improvement, and modification of prices.

In the first category of the studies, Camanho and Dyson [5] developed a method for estimating the bounds of the CE in situations where only a maximal and minimal bounds of input prices can be estimated for each DMU. Their methods were based on incorporating input cone assurance regions weight restrictions. While the Camanho and Dyson [5]'s approach can handle only the DMUs with one output, Fang and Li [13] proposed a model which can obtain the Pessimistic CE measure in situations of multiple outputs and multiple inputs by using the weight restrictions in the form of input cone assurance to calculate the lower bound of CE efficiency. Jahanshahloo *et al.* [20] proposed a simplified version of Camanho and Dyson [5]'s DEA cost efficiency model of with fewer numbers of constraints and variables. Portella and Thanassoulis [19] provided a method for cost efficiency analysis which deals with ordinal data. Mozaffari *et al.* [23] formulated an original DEA-R cost and revenue efficiency models in case of same price vector for ratio quantities of inputs to outputs. In [17] the inverse DEA problem when price information is available was considered.

In the second category of studies, Tone [29] pointed out the necessity of identifying the cheaper input mix in cost efficiency analysis and showed that if prices be different between units, the cost and allocation efficiencies of the Farrell model in multiple output mode might be inaccurate. He stated that defections are due to the structure of the production conditions used to calculate the cost efficiency and introduced a set of cost-based manufacturing options. Tone and Tsutsui [30] decomposed observed total cost into the global optimal (minimum) cost and loss due to technical inefficiency, which is measured by using the traditional CCR model within the technical production possibility set, input price difference and inefficient cost mix, which are measured in the cost based production possibility set. Input price difference and inefficient cost mix are related to the price efficiency index and the global allocative efficiency index. The former reflects the differences in input unit prices, while the latter evaluates the efficiency of the input cost mix. Camanho and Dyson [4] introduced the concept of economic efficiency as a development for Farrell cost efficiency in the non-competitive environment that includes components of Farrell cost efficiency and market efficiency. They defined economic efficiency as a measurement of ability to produce current outputs with the lowest possible cost, in which all DMUs are priced by the same cost sets; Farrell cost efficiency is introduced as the ability to produce current outputs at a minimum cost at its current price level of the DMU, and introduced the market efficiency as the success of the DMUs in addressing the minimum input prices in the market. They proposed two alternative procedures to identify the potential minimum prices. Farrell efficiency model can be used in two modes; first, with a shared virtual price vector between all the units, which is obtained from the minimum component vector of the observed price, and second, the use of all the observed price vectors for all units and finding the lowest Farrell cost efficiency among all the measured cost efficiencies for each unit. The calculated efficiency in the first model is not necessarily similar to that of the latter one, because the first one is based on an ideal price vector that even in most cases is not available in the convex combination of the observed price vectors and the latter is based on the observe price vectors. Sahoo *et al.* [25] states that in a non-competitive market with different input prices, it would be appropriate to use a value-based technology as a reference technology, in which the performance of units can be evaluated in comparison with it. Therefore, they developed value based models of Tone [29] and their respective value-based technical and allocative efficiencies in a directional DEA environment.

Portella and Thanassoulis [19] presented a nonlinear model for evaluation of economic efficiency when the prices are not the same between the units so that their model optimizes the prices and quantities concurrently. Their model includes a price possibility constraint and convexity assumption for the observed prices, which implies that convex combination of the observed prices are also acceptable. Although their proposed model had some superiorities to the former available ones, its non-linearity provides no guarantee to get the lowest cost. Secondly, in the described example, the introduced price target vector is not available in the convex set of the observed vectors and in fact, the obtained point is the same positive ideal price point derived from the minimal component of the observed price vectors and indeed, it does not satisfy principles of price observations, convexity and price feasibility, which are introduced implicitly in the introduced model. About other developments about cost efficiency and its applications refer to [1–3, 6, 8, 9, 11, 12, 14, 18, 21, 22, 26–28, 31].

When the price is fixed for all units, the cost inefficiencies factors are only dependent on quantities therefore the cost frontier like a technical efficiency frontier can be considered. To this end, studies in this area deal with the Production Possibility Set typical DEA, *i.e.*, Production Possibility Set quantities. However, in the second category, cost efficiency should be based on a frontier in which one can consider also inefficiencies caused by the use of the wrong price vectors. Considering DEA in which the production function that the technical efficiency is estimated based on distance of the vector, the quantities of the unit under evaluation have been estimated is the frontier of a production possibility set resulted from the quantities, it will be expected that the cost efficiency of the evaluated unit being estimated on the basis of its position to the frontier of a possible production set resulted from the possible cost vectors. This frontier should be in a way that one can calculate all the possible inefficiencies based on it. In other words, if a unit placed over it, then, no inefficiency like inefficiency in choosing the wrong price vector would be available for it. In the literature of the cost efficiency, to the best knowledge of the authors, studies [25, 29, 30] have only introduced a cost Production Possibility Set.

The paper's structure is as follow: In section two, the present study reviews the Cost Efficiency, which were provided in Tone [29], along with the decomposition of observed initial cost introduced in Tone and Tsutsui [30]. Section three first introduces a PPS in which all DMUs were evaluated by their price data in addition to price data of other DMUs. After wards, the section provides the equations of a new Production Possibility Set. The model which were formerly used to evaluate the cost efficiency, is rewritten in the new Production Possibility Set and provides a decomposition will be observed actual cost. Finally, it compares the decomposition of new observed actual cost with observed cost in section two. Section four will provided an applied example, and finally section five will present a final conclusion.

2. Background

Assume that there is n DMUs ($\text{DMU}_j, j = 1, \ldots, n$) each of which have m inputs $(x_{ij}), i = 1, \ldots, m$ for production of s outputs $(y_{rj}), r = 1, \ldots, s$. Furthermore, assume that $c = (c_1, \ldots, c_m)^T$ is the price of x_{ij}, and $p = (p_1, \ldots, p_s)^T$ is the price of y_{rj}. According to Tone [29], if we use the initial model of cost efficiency by Farrell [16] for evaluation of DMUs in PPS (2.1), which have equal input and output, but different input prices, then the cost efficiency can be equal indicating the weakness of that method. Therefore, Tone [29] provides the PPS (2) as follows:

$$T = \{ (x, y) |\, x \geq X\lambda,\ y \leq Y\lambda,\ \lambda \geq 0 \} \tag{2.1}$$

$$\bar{T} = \{ (\bar{x}, y) |\, \bar{x} \geq \bar{X}\lambda,\ y \leq Y\lambda,\ \lambda \geq 0 \} \tag{2.2}$$

where, $\bar{X}=(\bar{x}_1,\ldots,\bar{x}_n)$ with $\bar{x}_j=(c_{1j}x_{1j},\ldots,c_{mj}x_{mj})^T$ (it is the multiplication of price by input values). Here it is assumed that the matrices X and C are non-negative. $(\bar{x}_o,y_o)$ Corresponds to the evaluated observed unit of (x_o,y_o) with input price vector of c_o in the PPS (2.1).

Tone [29] used the Input-Oriented CCR[4] model for evaluation of technical efficiency of $(\bar{x}_o,y_o)$ in the $\bar{T}$ set. Technical efficiency of evaluated unit of $(\bar{x}_o,y_o)$ can be evaluated by the following model:

$$\begin{aligned} \bar{\theta}^* &= \min_{\bar{\theta},\lambda} \bar{\theta} \\ \text{s.t.}\quad \bar{\theta}\bar{x}_o &\geq \bar{X}\lambda \\ y_o &\leq Y\lambda \\ \lambda &\geq 0 \end{aligned} \tag{2.3}$$

$\bar{\theta}^*$, which refers to the technical efficiency in the PPS (2), is equivalent to the price efficiency in $\bar{T}$.

$\bar{x}_o^*$ (Minimum cost for DMU_o) is the optimal solution of the LP given below:

$$\begin{aligned} e\bar{x}_o^* &= \min_{\bar{x},\lambda} e\bar{x} \\ \text{s.t.}\quad \bar{x} &\geq \bar{X}\lambda \\ y_o &\leq Y\lambda \\ \lambda &\geq 0 \end{aligned} \tag{2.4}$$

where, $\bar{\gamma}^*=\frac{e\bar{x}_o^*}{e\bar{x}_o}$ is used as the cost efficiency. The allocative efficiency $(\bar{\alpha}^*)$ is then introduced by dividing $\bar{\gamma}^*$ by $\bar{\theta}^*$.

In order to develop their previous research Tone and Tsutsui [30] developed a new study in which they performed a decomposition of cost efficiency and applied it for comparison of electric tools of Japan and America. They also provided decomposition of observed costs. The decomposition was based on the observed input cost of DMU_o or $C_o=\sum_{i=1}^m c_{io}x_{io}\,(o=1,\ldots,n)$ and the Cost of input technical efficiency of DMU_o in the PPS $\bar{T}$ as $C^*_{o_{\bar{T}}}$ which is obtained from the following equation.

The technically efficient input Cost for DMU_o in $\bar{T}$:

$$C^*_{o_{\bar{T}}}=\sum_{i=1}^m c_{io}x^*_{io}\,(o=1,\ldots,n) \tag{2.5}$$

where, x^*_{io} is the input i of technically efficient for DMU_o and it can be measured by solving the CCR model in the PPS (2.1).

Accordingly, the loss cost due to technical inefficiency in $\bar{T}$ is shown by $L^*_{o_{\bar{T}}}=C_o-C^*_{o_{\bar{T}}}\,(\geq 0)$.

Using the following model, a strong efficient cost can be defined, which is in fact the technical and price efficient cost by $C^{**}_{o_{\bar{T}}}=\sum_{i=1}^m\left(\rho^*\bar{x}_{io}-t^{-*}_{io}\right)$.

$$\begin{aligned} \rho^* &= \min_{\rho,\mu,t^-,t^+} \rho \\ \text{s.t.}\quad \rho\bar{x}_o &= \bar{X}\mu+t^- \\ y_o &= Y\mu-t^- \\ \mu &\geq 0, t^-\geq 0, t^+\geq 0. \end{aligned} \tag{2.6}$$

In this respect the corresponding loss for the price technical inefficiency in $\bar{T}$ is shown by $L^{**}_{o_{\bar{T}}}=C^*_{o_{\bar{T}}}-C^{**}_{o_{\bar{T}}}$.

[4]Charnes Cooper Rhodes.

Finally, the allocative efficiency is according to $\alpha^* = \frac{C^{***}_{o_{\bar{T}}}}{C^{**}_{o_{\bar{T}}}}$ (≤ 1) where $C^{***}_{o_{\bar{T}}}$ is obtained by the following model.

$$\begin{aligned} C^{***}_o &= \min_{\bar{x},\mu} e\bar{x} \\ \text{s.t.} \quad \bar{x} &\geq \bar{X}\mu \\ y_o &\leq Y\mu \\ \mu &\geq 0. \end{aligned} \tag{2.7}$$

The corresponding loss for the cost allocative inefficiency in $\bar{T}$ is also shown by $L^{***}_{o_{\bar{T}}} = C^{**}_{o_{\bar{T}}} - C^{***}_{o_{\bar{T}}}$.

Ultimately, the decomposition of observed input cost for DMU_o in $\bar{T}$ is provided according to the following equation:

$$C_o = L^{*}_{o_{\bar{T}}} + L^{**}_{o_{\bar{T}}} + L^{***}_{o_{\bar{T}}} + C^{***}_{o_{\bar{T}}}. \tag{2.8}$$

3. Introduction of the new cost Production Possibility Set

3.1. Input cost efficiency

In this section, the researchers proposed methods is described. in order to build the input vectors of new DMUs, the input vector of DMU_1 is multiplied to its own and other DMUs' input price vectors. So that the input vector of $\bar{x}'_1 = (c_{11}x_{11}, c_{21}x_{21}, \ldots, c_{m1}x_{m1})$ is built according to the multiplication of input vectors of DMU_1 by input price of DMU_1. Accordingly, $\bar{X}'_1 = (\bar{x}'_1, \ldots, \bar{x}'_n)$ the vector is built from multiplication of input vectors of DMU_1 by input of other DMUs. Each of $\bar{x}'_1, \ldots, \bar{x}'_n$ is generated as follows:

$$\begin{array}{ll} \text{Input of new DMU}_1: & \bar{x}'_1 = (c_{11}x_{11}, c_{21}x_{21}, \ldots, c_{m1}x_{m1}) \to c_{i1}x_{i1} \\ \text{Input of new DMU}_2: & \bar{x}'_2 = (c_{12}x_{11}, c_{22}x_{21}, \ldots, c_{m2}x_{m1}) \to c_{i2}x_{i1} \\ \vdots & \vdots \\ \text{Input of new DMU}_\text{n}: & \bar{x}'_n = (c_{1n}x_{11}, c_{2n}x_{21}, \ldots, c_{mn}x_{m1}) \to c_{in}x_{i1} \end{array}. \tag{3.1}$$

All of these DMUs correspond with DMU_1, and are obtained from multiplication of input vector of DMU_1 by price vectors of other DMUs. Similarly, for other DMUs, we have:

$$\begin{array}{ll} \text{Input of new DMU}_{n+1}: & \bar{x}'_{n+1} = (c_{11}x_{12}, c_{21}x_{22}, \ldots, c_{m1}x_{m2}) \to c_{i1}x_{i2} \\ \vdots & \vdots \\ \text{Input of new DMU}_\text{2n}: & \bar{x}'_{2n} = (c_{1n}x_{12}, c_{2n}x_{22}, \ldots, c_{mn}x_{m2}) \to c_{ij}x_{i2} \\ \vdots & \vdots \\ \text{Input of new DMU}^2_\text{n}: & \bar{x}'_{n^2} = (c_{1n}x_{1n}, c_{2n}x_{2n}, \ldots, c_{mn}x_{mn}) \to c_{in}x_{in} \end{array}.$$

Therefore, the Production Possibility Set of $\bar{T}'$ is as follows:

$$\bar{T}' = \left\{ \left(\bar{x'}, y\right) \middle| \, \bar{x'} \geq \sum_{j=1}^{n^2} \lambda_j \bar{x'}_j, y \leq \sum_{j=1}^{n^2} \lambda_j y_j, \lambda_j \geq 0 \right\}. \tag{3.2}$$

If the input quantities of an observed DMU is considered and the observed price vectors of other DMU is used then another price vectors which can be considered as the corresponding virtual cost vectors of that unit together with the output of that unit under evaluation as virtual units in cost Production Possibility ($\bar{T}'$) can be obtained.

Since the number of DMUs in the new Production Possibility Set is increased, they can be classified into two groups:

$P =$ (observed DMUs) or the set of actual DMUs
$Q =$ (new DMUs) or the set of virtual DMUs.

Since the Production Possibility Set of $\bar{T}'$ is built by n^2 points including n building points of $\bar{T}$, It can be conclude that $\bar{T} \subseteq \bar{T}'$. Therefore, the amounts of efficiency of actual units in $\bar{T}'$ are not greater than the amounts of efficiency of the same units in $\bar{T}$. In other words, it is expected that the amounts of efficiency of projection point of DMUs corresponding with the observed cost vector in $\bar{T}'$ are lower or equal to the amounts of their efficiency in $\bar{T}$. Therefore, because of this difference, the difference between the scores of efficiency in $\bar{T}$ and $\bar{T}'$ can be defined as a new inefficiency sources.

Now, we can define the technical, cost and allocative efficiency according to Production Possibility Set of $\bar{T}'$.

Definition 3.1. The new technical efficiency [*NTech*] is calculated in $\bar{T}'$ by the following model:

$$\begin{aligned} &[NTech]\ \bar{\theta'}^{*} = \min \bar{\theta'} \\ &\text{s.t.} \quad \sum_{j=1}^{n^2} \lambda_j \bar{x'}_j \leq \bar{\theta'} \bar{x'}_o \\ &\qquad \sum_{j=1}^{n^2} \lambda_j y_j \geq y_o \\ &\qquad \lambda_j \geq 0. \end{aligned} \tag{3.3}$$

In order to calculation the new cost efficiency in $\bar{T}'$, the following model should be solved first:

$$\begin{aligned} &[NCost]\ e\bar{x'}^{*}_o = \min e\bar{x'} \\ &\text{s.t.} \quad \sum_{j=1}^{n^2} \lambda_j \bar{x'}_j \leq \bar{x'} \\ &\qquad \sum_{j=1}^{n^2} \lambda_j y_j \geq y_o \\ &\qquad \lambda_j \geq 0. \end{aligned} \tag{3.4}$$

Definition 3.2. The new cost efficiency [*NCost*] is calculated by the following relation:

$$\bar{\gamma'}^{*} = \frac{e\bar{x'}^{*}}{e\bar{x'}_o} \tag{3.5}$$

where $e\bar{x'}^{*}$ is the optimal solution of (3.4) and $e\bar{x'}_o$ is the observed cost for DMU_o.

Theorem 3.3. *The amount of new cost efficiency [NCost] is not higher than the new technical efficiency [NTech].*

Proof. Assume that $(\bar{\theta'}^{*}, \lambda^{*})$ is the optimal solution of model (3.3), then $(\bar{\theta'}^{*} \bar{x'}_o, \lambda^{*})$ is a feasible solution for model (3.4).

Therefore, we have:

$$e\bar{\theta'}^{*} \bar{x'}_o \geq e\bar{x'}^{*}_o \Rightarrow \bar{\theta'}^{*} \geq \frac{e\bar{x'}^{*}_o}{e\bar{x'}_o} = \bar{\gamma'}^{*}. \tag{3.6}$$

□

And finally

Definition 3.4. The new allocative efficiency based on $\bar{T}'$ can be calculated by the following formula:

$$\bar{\alpha'}^{*} = \frac{\bar{\gamma'}^{*}}{\bar{\theta'}^{*}}. \tag{3.7}$$

3.2. New decomposition of cost efficiency in $\bar{T}'$

In the Production Possibility Set with a constant scale (T_C), there are n DMUs each of which have m inputs for production of s inputs. Assume that x_{io} and y_{ro} are respectively the ith input and rth output of DMU$_o$; and c_{io} is the price of x_{io}. It should be noted that our evaluations are done in a non-competitive space (there are not equal different input prices for DMUs).

The observed cost (actual cost) for DMU$_o$ is as follows:

$$C_o = \sum_{i=1}^{m} c_{io}x_{io} \ (o = 1, \ldots, n). \tag{3.8}$$

Let the technical efficient input for DMU$_o$ be x_o^* in the PPS (2.1), which can be obtained by solving the CCR model, then the cost of this technical efficiency input will be as follows:

$$C^*_{o_{\bar{T}'}} = \sum_{i=1}^{m} c_{io}x^*_{io}\,, (o = 1, \ldots, n) \tag{3.9}$$

$C^*_{o_{\bar{T}'}}$ is the cost of this technical efficiency input for DMU$_o$ in the PPS $\bar{T}'$.

Since the projected point dominates the point itself, all of the components of the projection input vector is less than the input x_o ($x^*_{io} \leq x_{io}$), and thus we have the following relation:

$$C_o \geq C^*_{o_{\bar{T}'}}. \tag{3.10}$$

According to Tone and Tsutsui [30], the difference between these two costs is considered as the cost loss due to the technical inefficiency (in the PPS $\bar{T}'$) of equations:

$$L^*_{o_{\bar{T}}} = L^*_{o_{\bar{T}'}} = C_o - C^*_{o_{\bar{T}'}} \,(\geq 0)\,. \tag{3.11}$$

In the Production Possibility Set of $\bar{T}'$ (3.2), the new technical efficiency is evaluated by the following model (this model is equivalent to model (3.3)):

$$\begin{aligned} &\bar{\theta}'^* = \min \bar{\theta}' \\ \text{s.y. } &\sum_{j=1}^{n^2} \lambda_j \bar{x}'_j + t^- = \bar{\theta}'\bar{x}'_o \quad o = 1, \ldots, n^2 \\ &\sum_{j=1}^{n^2} \lambda_j y_j - t^+ = y_o \qquad o = 1, \ldots, n^2 \\ &\lambda_j \geq 0, t^+ \geq 0, t^- \geq 0. \end{aligned} \tag{3.12}$$

If $\left(\bar{\theta}'^*, \lambda^*, t^{-*}, t^{+*}\right)$ is the optimal solution of model (3.12), then $\bar{\theta}'^*\bar{x}'_o = \left(\bar{\theta}'^* c_{1o}x^*_{1o}, \ldots, \bar{\theta}'^* c_{mo}x^*_{mo}\right)$ $\left(o = 1, \ldots, n^2\right)$ is a radial reduced input vector on the weak efficient frontier of the PPS $\bar{T}'$ (3.2).

The reduced cost vector for observed DMUs in the PPS $\bar{T}'$ is as follows:

$$c^*_{o_{\bar{T}'}} = \bar{\theta}'^* c_o = \left(\bar{\theta}'^* c_{1o}, \ldots, \bar{\theta}'^* c_{mo}\right). \tag{3.13}$$

Here, $c^*_{o_{\bar{T}'}}$ is the radial and reduced input cost for technical efficiency input of x^*_o and it can produce y_o. $\bar{x}'^*_{io}$ is also the projection point of $\bar{x}'_{io}$ in the PPS $\bar{T}'$, and it can be measured by the model (3.12). Model (3.12) finds a reduced cost vector for any DMU such as DMU$_o$ and it can be shown as follows:

$$\bar{x}'^*_{io} = \bar{\theta}'^* \bar{x}'_{io} - t^{-^*} \,, y^*_{ro} = y_{ro} + t^{+^*}. \tag{3.14}$$

Now, strong efficient cost can be introduced as ($C^{**}_{o_{\bar{T}'}}$), which is the technical and price efficient cost (in the PPS $\bar{T}'$) as follows:

Definition 3.5. The strong efficient cost for actual DMUs (in the PPS $\bar{T}'$).

Since $\bar{T} \subseteq \bar{T}'$, then $\bar{\theta}'^* \leq \bar{\theta}^*$ (in which $\bar{\theta}^*$ is the optimal solution of model (3), and $\bar{\theta}'^*$ is the optimal solution of model (3.12)). $C^{**}_{o_{\bar{T}'}} = \sum_{i=1}^{m} \bar{x}'^*_{io} = \sum_{i=1}^{m} \left(\bar{\theta}'^* \bar{x}'_{io} - t^{-*}_{io}\right) \leq \bar{\theta}'^* \sum_{i=‘}^{m} \bar{x}'_{io} \leq \sum_{i=‘}^{m} \bar{x}'_{io} = C^*_{o_{\bar{T}'}} \quad o \in P.$

And

Definition 3.6. Corresponding loss cost of the input price inefficiency (in the PPS $\bar{T}'$) is as follows:

$$L^{**}_{o_{\bar{T}'}} = C^*_{o_{\bar{T}'}} - C^{**}_{o_{\bar{T}'}} \quad o \in P. \tag{3.15}$$

Now, we're going to examine the corresponding loss cost of the allocative inefficiency in the PPS $\bar{T}'$.

In the Production Possibility Set of $\bar{T}'$, the new cost efficiency [*NCost*] can be evaluated by the help of model (3.4) as follows:

$$\begin{aligned} C^{***}_{o_{\bar{T}'}} &= \min e\bar{x}' \\ \text{s.t.} \quad \bar{x}' &\geq \bar{X}'\lambda \\ y_o &\leq Y\lambda \\ \lambda &\geq 0. \end{aligned} \tag{3.16}$$

The optimal solution of model (3.16) is based on the cost of $\left(\bar{x}'^*_o, y_o\right)$. In other words, this model considers the minimum cost in the Production Possibility Set of $\bar{T}'$. Therefore, to evaluate the performance of allocative, we provide the Definition 3.7.

Definition 3.7. The allocative efficiency, which is shown by α^*, can be evaluated for actual DMUs by the following equations:

$$0 \leq \alpha^* = \frac{C^{***}_{o_{\bar{T}'}}}{C^{**}_{o_{\bar{T}'}}} \leq 1 \quad o \in P. \tag{3.17}$$

Theorem 3.8.

$$C_o \geq C^*_{o_{\bar{T}'}} \geq C^{**}_{o_{\bar{T}'}} \geq C^{***}_{o_{\bar{T}'}} \quad o \in P. \tag{3.18}$$

The proof is obvious.

Definition 3.9. The corresponding loss cost of the allocative inefficiency (in the PPS $\bar{T}'$) is as follows:

$$L^{***}_{o_{\bar{T}'}} = C^{**}_{o_{\bar{T}'}} - C^{***}_{o_{\bar{T}'}} \quad o \in P. \tag{3.19}$$

Given the fact that the observed actual cost was decomposed into three correspond losses namely the technical, input price, and allocative inefficiency, a decomposition can be provided for observed actual cost ($o \in P$) of DMU_o in the Production Possibility Set of $\bar{T}$. For actual DMUs, we can evaluate the cost and technical efficiency once with the frontier of $\bar{T}$ and again with frontier of $\bar{T}'$. However, the evaluated technical and cost efficiency with frontier of $\bar{T}'$ are not higher than the evaluated technical and cost efficiency with border of $\bar{T}$. Therefore, the differences of these two groups of efficiency are the new sources of inefficiency called the new technical and cost inefficiency for actual DMUs. In other words, the new sources of inefficiency can be found for DMU_o, $o \in p$ according to the differences between the technical and cost efficiency.

Definition 3.10. The new price inefficiency for actual DMUs can be calculated by the following equation:

$$\text{New price inefficiency of in } \bar{T}' = \frac{\text{Technical efficiency of } \mathrm{DMU}_o \text{ in } \bar{T}'}{\text{Technical efficiency of } \mathrm{DMU}_o \text{ in } \bar{T}} = \frac{\bar{\theta}'^*}{\bar{\theta}^*} \qquad o \in P \; \mathrm{DMU}_o. \tag{3.20}$$

Definition 3.11. New cost inefficiency for actual DMUs can be evaluated by the following equation:

$$\text{New cost inefficiency of } \mathrm{DMU}_o \text{ in } \bar{T}' = \frac{\text{Cost efficiency of } \mathrm{DMU}_o \text{ in } \bar{T}'}{\text{Cost efficiency of } \mathrm{DMU}_o \text{ in } \bar{T}} = \frac{\bar{\gamma}'^*}{\bar{\gamma}^*} \qquad o \in P. \tag{3.21}$$

Theorem 3.12.

$$C^{**}_{o_{\bar{T}}} \geq C^{**}_{o_{\bar{T}'}}. \tag{3.22}$$

Proof. $C^{**}_{o_{\bar{T}'}} = \sum_{i=1}^{m} \bar{x}'^*_{io} = \sum_{i=1}^{m} \bar{\theta}'^* \bar{x}'_{io}$, Since $\bar{T} \subseteq \bar{T}'$, then $\bar{\theta}'^* \leq \bar{\theta}^*$ and $\bar{x}'_{io} = \bar{x}_{io}$, therefore

$$C^{**}_{o_{\bar{T}'}} = \sum_{i=1}^{m} \bar{x}'^*_{io} = \sum_{i=1}^{m} \bar{\theta}'^* \bar{x}'_{io} \leq \sum_{i=1}^{m} \bar{\theta}^* \bar{x}_{io} = \sum_{i=1}^{m} \bar{x}^*_{io} = C^{**}_{o_{\bar{T}}}.$$

□

Definition 3.13. The loss corresponding to the new price inefficiency (in the PPS $\bar{T}'$) is as follows:

$$L^{4*}_{o_{\bar{T}'}} = C^{**}_{o_{\bar{T}}} - C^{**}_{o_{\bar{T}'}}. \tag{3.23}$$

In order to address $L^{****}_{o_{\bar{T}'}}$ conveniently, $L^{4*}_{o_{\bar{T}'}}$ is used as a substitute.

Theorem 3.14.

$$C^{***}_{o_{\bar{T}'}} \leq C^{***}_{o_{\bar{T}}}. \tag{3.24}$$

Proof.

$$\bar{\alpha}^* = \frac{C^{***}_{o_{\bar{T}}}}{C^{**}_{o_{\bar{T}}}} \Rightarrow C^{***}_{o_{\bar{T}}} = \bar{\alpha}^* C^{**}_{o_{\bar{T}}} \text{ based on Theorem 3.12 } C^{**}_{o_{\bar{T}}} \geq C^{**}_{o_{\bar{T}'}}, \tag{3.25}$$

then

$$\bar{\alpha}^* = \frac{C^{***}_{o_{\bar{T}}}}{C^{**}_{o_{\bar{T}}}} \Rightarrow C^{***}_{o_{\bar{T}}} = \bar{\alpha}^* C^{**}_{o_{\bar{T}}} \geq \bar{\alpha}^* C^{**}_{o_{\bar{T}'}} \geq \bar{\alpha}'^* C^{**}_{o_{\bar{T}'}} = C^{***}_{o_{\bar{T}'}}.$$

□

Definition 3.15. The loss corresponding to the new cost inefficiency (in the PPS $\bar{T}'$) is as follows:

$$L^{5*}_{o_{\bar{T}'}} = C^{***}_{o_{\bar{T}}} - C^{***}_{o_{\bar{T}'}}. \tag{3.26}$$

In order to address $L^{*****}_{o_{\bar{T}'}}$ conveniently, $L^{5*}_{o_{\bar{T}'}}$ is used as a substitute.

Accordingly, a new decomposition can be obtained for the observed actual cost for DMUs in the new Production Possibility Set of $\bar{T}'$ which is different from its corresponding decomposition in $\bar{T}$.

New decomposition of observed cost for actual DMUs ($\mathrm{DMU}_o, o \in P$) in $\bar{T}'$

$$\text{Observed cost for } \mathrm{DMU}_o\text{:} \quad C_o = L^{*}_{o_{\bar{T}'}} + L^{**}_{o_{\bar{T}'}} + L^{***}_{o_{\bar{T}'}} + C^{***}_{o_{\bar{T}'}} \quad o \in P. \tag{3.27}$$

Now, The costs and loss of both Production Possibility Sets of $\bar{T}$ and $\bar{T}'$.

New decomposition of observed actual cost in $\bar{T}$:

The actual observed cost can be shown in two (A and B) different methods.

New decomposition A:

$$C_o = L^*_{o_{\bar{T}'}} + C^*_{o_{\bar{T}'}}. \tag{3.28}$$

From the equation (3.15) this formula, $C^*_{o_{\bar{T}'}} = L^{**}_{o_{\bar{T}'}} + C^{**}_{o_{\bar{T}'}}$, is achieved: hence using it in equation (3.28), equation (3.28) can be revised as below:

$$C_o = L^*_{o_{\bar{T}}} + L^{**}_{o_{\bar{T}'}} + C^{**}_{o_{\bar{T}'}}. \tag{3.29}$$

Hence $C^{**}_{o_{\bar{T}'}} = L^{4*}_{o_{\bar{T}'}} + C^{**}_{o_{\bar{T}}}$ therefore, equation (3.29) is reconstructed like;

$$C_o = L^*_{o_{\bar{T}}} + L^{**}_{o_{\bar{T}'}} + L^{4*}_{o_{\bar{T}'}} + C^{**}_{o_{\bar{T}}}. \tag{3.30}$$

Since $L^{***}_{o_{\bar{T}}} = C^{**}_{o_{\bar{T}}} - C^{***}_{o_{\bar{T}}}$, then $C^{**}_{o_{\bar{T}}} = L^{***}_{o_{\bar{T}}} + C^{***}_{o_{\bar{T}}}$ therefore, new equation (3.30) is:

$$C_o = L^*_{o_{\bar{T}}} + L^{**}_{o_{\bar{T}'}} + L^{4*}_{o_{\bar{T}'}} + L^{***}_{o_{\bar{T}}} + C^{***}_{o_{\bar{T}}}. \tag{3.31}$$

Finally revising equation (3.26), $C^{***}_{o_{\bar{T}}} = L^{5*}_{o_{\bar{T}'}} + C^{***}_{o_{\bar{T}'}}$ the below equation (3.32) is resulted:

$$C_o = L^*_{o_{\bar{T}}} + L^{**}_{o_{\bar{T}'}} + L^{4*}_{o_{\bar{T}'}} + L^{***}_{o_{\bar{T}}} + L^{5*}_{o_{\bar{T}'}} + C^{***}_{o_{\bar{T}'}}. \tag{3.32}$$

New decomposition B:

$$C_o = L^*_{o_{\bar{T}'}} + C^*_{o_{\bar{T}'}}. \tag{3.33}$$

Because $C^*_{o_{\bar{T}'}} = C^*_{o_{\bar{T}}}$ and $L^*_{o_{\bar{T}'}} = L^*_{o_{\bar{T}}}$ therefore;

$$C_o = L^*_{o_{\bar{T}}} + C^*_{o_{\bar{T}}}. \tag{3.34}$$

Since $L^{**}_{o_{\bar{T}}} = C^*_{o_{\bar{T}}} - C^{**}_{o_{\bar{T}}}$, then $C^*_{o_{\bar{T}}} = L^{**}_{o_{\bar{T}}} + C^{**}_{o_{\bar{T}}}$. So equation (3.34) can be defined as:

$$C_o = L^*_{o_{\bar{T}}} + L^{**}_{o_{\bar{T}}} + C^{**}_{o_{\bar{T}}}. \tag{3.35}$$

Considering $L^{***}_{o_{\bar{T}}} = C^{**}_{o_{\bar{T}}} - C^{***}_{o_{\bar{T}}}$, then $C^{**}_{o_{\bar{T}}} = L^{***}_{o_{\bar{T}}} + C^{***}_{o_{\bar{T}}}$. Therefore, equation (3.35) can be revised as:

$$C_o = L^*_{o_{\bar{T}}} + L^{**}_{o_{\bar{T}}} + L^{***}_{o_{\bar{T}}} + C^{***}_{o_{\bar{T}}}. \tag{3.36}$$

Using equation (3.26), then $C^{***}_{o_{\bar{T}}} = L^{5*}_{o_{\bar{T}'}} + C^{***}_{o_{\bar{T}'}}$. The below final equation can be concluded:

$$C_o = L^*_{o_{\bar{T}}} + L^{**}_{o_{\bar{T}}} + L^{***}_{o_{\bar{T}}} + L^{5*}_{o_{\bar{T}'}} + C^{***}_{o_{\bar{T}'}}. \tag{3.37}$$

The equation above is almost similar to the decomposition by Tone and Tsutsui [30] who found that they were different in terms of $L^{5*}_{o_{\bar{T}'}}$ and $C^{***}_{o_{\bar{T}'}}$. Theorem 3.14 proves $C^{***}_{o_{\bar{T}'}} \leq C^{***}_{o_{\bar{T}}}$, and thus the Decomposition (3.37) includes higher loss than the decomposition by Tone and Tsutsui [30]. Therefore, this decomposition of observed cost is more accurate than what was provided by Tone and Tsutsui [30].

TABLE 1. Price data and information of DMUs.

DMU	(x_1, x_2)	(y_1, y_2)	(c_1, c_2)
1	$(20, 151)$	$(100, 90)$	$(500, 100)$
2	$(19, 131)$	$(150, 50)$	$(350, 80)$
3	$(25, 160)$	$(160, 55)$	$(450, 90)$
4	$(27, 168)$	$(180, 72)$	$(600, 120)$
5	$(22, 158)$	$(94, 66)$	$(300, 70)$
6	$(55, 255)$	$(230, 90)$	$(450, 80)$
7	$(33, 235)$	$(220, 88)$	$(500, 100)$
8	$(31, 206)$	$(152, 80)$	$(450, 85)$
9	$(30, 244)$	$(190, 100)$	$(380, 76)$
10	$(50, 268)$	$(250, 100)$	$(410, 75)$
11	$(53, 306)$	$(260, 147)$	$(440, 80)$
12	$(38, 284)$	$(250, 120)$	$(400, 70)$

TABLE 2. Data and efficiency of DMUs in $\bar{T}$.

DMU		$(\bar{x}_1, \bar{x}_2)$	(y_1, y_2)	$\bar{\theta}^*$	$\bar{\gamma}^*$	$\bar{\alpha}^*$
$*_1$	1	$(10\,000, 15\,100)$	$(100, 90)$	0.99	0.96	0.96
$*_2$	2	$(66\,500, 10\,480)$	$(150, 50)$	1	1	1
$*_3$	3	$(11\,250, 14\,400)$	$(160, 55)$	0.78	0.72	0.92
$*_4$	4	$(16\,200, 20\,160)$	$(180, 72)$	0.66	0.62	0.94
$*_5$	5	$(6600, 11\,060)$	$(94, 66)$	1	1	1
$*_6$	6	$(24\,750, 20\,400)$	$(230, 90)$	0.83	0.63	0.76
$*_7$	7	$(16\,500, 23\,500)$	$(220, 88)$	0.69	0.69	1
$*_8$	8	$(13\,950, 17\,510)$	$(152, 80)$	0.75	0.73	0.96
$*_9$	9	$(11\,400, 18\,544)$	$(190, 100)$	0.97	0.95	0.98
$*_{10}$	10	$(20\,500, 20\,100)$	$(250, 100)$	0.92	0.78	0.84
$*_{11}$	11	$(23\,320, 24\,480)$	$(260, 147)$	0.99	0.86	0.87
$*_{12}$	12	$(15\,200, 19\,880)$	$(250, 120)$	1	1	1

Notes. The symbol * represents the actual DMUs in Table 2 and their corresponding DMUs in Table 3.

4. Numerical example

In this section, in order to investigate the accuracy of provided method in the previous sections, observe new defined inefficiency, and compare different types of efficiency in $\bar{T}$ and $\bar{T}'$ numerical example have been brought.

The data of this example is the taken from [29]. In [29] studied an example in which the situations of 12 hospitals each of which had two inputs (numbers of physicians and nurses) and two outputs (number of ambulatory and hospitalized patients in terms of 100 people per month) were studied.

The data of these DMUs is available with two inputs and two outputs and their price information. Table 1 includes this information. In Table 2, the inputs are shown in the second column, and the outputs are shown in the third column, and the input prices are shown in the last column.

Now, the inputs is multiplied in their prices (according to (2.2)) and show the new data set as $(\bar{x}_1, \bar{x}_2, y_1, y_2)$ and scores of technical efficiency $(\bar{\theta}^*)$ and cost efficiency $(\bar{\gamma}^*)$ and allocative efficiency $(\bar{\alpha}^*)$ in Table 2.

According to the Table 2, the efficiency scores of DMU_2, DMU_5 and DMU_{12} are equal to 1 (these DMUs are efficient). Furthermore, the Table 2 indicates that the technical and cost efficiency scores of DMU_4 are lower than other units, and thus this DMU has the highest technical and cost inefficiency.

Table 3. The coordinates of the inputs and outputs and the actual and virtual DMUs efficiencies in $\bar{T}'$.

	DMU		$(\bar{x}'_1, \bar{x}'_2)$	(y_1, y_2)	$\bar{\theta}'^*$	$\bar{\gamma}'^*$	$\bar{\alpha}'^*$
1	$*_1$	1	(10 000, 15 100)	(100, 90)	0.70	0.66	0.94
		2	(7000, 12 080)	(100, 90)	0.88	0.87	0.99
		3	(9000, 13 590)	(100, 90)	0.78	0.73	0.94
		4	(12 000, 18 120)	(100, 90)	0.58	0.55	0.95
		5	(6000, 10 570)	(100, 90)	1	1	1
		6	(9000, 12 080)	(100, 90)	0.88	0.79	0.90
		7	(10 000, 15 100)	(100, 90)	0.70	0.66	0.94
		8	(9000, 12 835)	(100, 90)	0.82	0.76	0.93
		9	(7600, 11 476)	(100, 90)	0.92	0.87	0.95
		10	(8200, 11 325)	(100, 90)	0.93	0.85	0.91
		11	(8800, 12 080)	(100, 90)	0.88	0.79	0.90
		12	(8000, 10 570)	(100, 90)	1	0.89	0.89
2		13	(9500, 13 100)	(150, 50)	0.70	0.66	0.94
	$*_2$	14	(6650, 10 480)	(150, 50)	0.88	0.87	0.99
		15	(8550, 11 790)	(150, 50)	0.78	0.73	0.94
		16	(11 400, 15 720)	(150, 50)	0.58	0.55	0.95
		17	(5700, 9170)	(150, 50)	1	1	1
		18	(85 500, 10 480)	(150, 50)	0.88	0.78	0.99
		19	(95 00, 13 100)	(150, 50)	0.70	0.66	0.94
		20	(8550, 11 135)	(150, 50)	0.82	0.76	0.93
		21	(7220, 9956)	(150, 50)	0.92	0.87	0.95
		22	(7790, 9825)	(150, 50)	0.93	0.84	0.93
		23	(8360, 10 480)	(150, 50)	0.88	0.79	0.90
		24	(7600, 9170)	(150, 50)	1	0.89	0.89
3		25	(12 500, 1600)	(160, 55)	0.49	0.09	0.13
		26	(8750, 12 800)	(160, 55)	0.70	0.12	0.17
	$*_3$	27	(11 250, 14 400)	(160, 55)	0.55	0.10	0.18
		28	(15 000, 19 200)	(160, 55)	0.41	0.08	0.19
		29	(7500, 112 000)	(160, 55)	0.82	0.13	0.16
		30	(11 250, 128 000)	(160, 55)	0.55	0.12	0.22
		31	(12 500, 160 000)	(160, 55)	0.49	0.09	0.18
		32	(11 250, 136 000)	(160, 55)	0.55	0.11	0.2
		33	(9500, 121 600)	(160, 55)	0.65	0.12	0.18
		34	(102 50, 120 000)	(160, 55)	0.60	0.12	0.2
		35	(11 000, 128 000)	(160, 55)	0.56	0.12	0.21
		36	(10 000, 112 000)	(160, 55)	0.61	0.13	0.21
4		37	(13 500, 16 800)	(180, 72)	0.70	0.64	0.91
		38	(9450, 13 440)	(180, 72)	0.88	0.84	0.95
		39	(12 150, 15 120)	(180, 72)	0.78	0.71	0.91
	$*_4$	40	(16 200, 20 160)	(180, 72)	0.58	0.53	0.91
		41	(8100, 11 760)	(180, 72)	1	0.97	0.97
		42	(12 150, 13 440)	(180, 72)	0.88	0.75	0.85
		43	(13 500, 16 800)	(180, 72)	0.70	0.64	0.91
		44	(12 150, 14 280)	(180, 72)	0.82	0.73	0.89
		45	(10 260, 12 768)	(180, 72)	0.92	0.84	0.91
		46	(11 070, 12 600)	(180, 72)	0.93	0.81	0.87
		47	(11 880, 13 440)	(180, 72)	0.88	0.76	0.86
		48	(10 800, 11 760)	(180, 72)	1	0.85	0.85

Notes. The symbol * represents the actual DMUs (observed DMUs) in Table 3.

Table 3. Continued.

	DMU		$(\bar{x}'_1, \bar{x}'_2)$	(y_1, y_2)	$\bar{\theta'}^*$	$\bar{\gamma'}^*$	$\bar{\alpha'}^*$
5		49	(11 000, 15 800)	(94, 66)	0.53	0.50	0.94
		50	(7700, 12 640)	(94, 66)	0.67	0.66	0.98
		51	(9900, 14 220)	(94, 66)	0.59	0.56	0.95
		52	(13 200, 18 960)	(94, 66)	0.44	0.42	0.95
	$*_5$	53	(6600, 11 060)	(94, 66)	0.76	0.76	1
		54	(9900, 12 640)	(94, 66)	0.67	0.59	0.88
		55	(11 000, 15 800)	(94, 66)	0.53	0.50	0.94
		56	(9900, 13 430)	(94, 66)	0.63	0.57	0.90
		57	(8360, 12 008)	(94, 66)	0.70	0.66	0.94
		58	(9020, 11 850)	(94, 66)	0.71	0.64	0.90
		59	(9680, 12 640)	(94, 66)	0.67	0.60	0.89
		60	(8800, 11 060)	(94, 66)	0.76	0.67	0.88
6		61	(27 500, 25 500)	(230, 90)	0.54	0.46	0.85
		62	(19 250, 20 400)	(230, 90)	0.77	0.61	0.79
		63	(24 750, 22 950)	(230, 90)	0.60	0.51	0.85
		64	(33 000, 30 600)	(230, 90)	0.45	0.38	0.84
		65	(16 500, 17 850)	(230, 90)	0.90	0.71	0.79
	$*_6$	66	(24 750, 20 400)	(230, 90)	0.60	0.54	0.9
		67	(27 500, 25 500)	(230, 90)	0.54	0.46	0.85
		68	(24 750, 1675)	(230, 90)	0.60	0.52	0.87
		69	(20 900, 19 380)	(230, 90)	0.71	0.60	0.84
		70	(22 550, 19 125)	(230, 90)	0.66	0.58	0.88
		71	(24 200, 20 400)	(230, 90)	0.62	0.55	0.89
		72	(22 000, 17 850)	(230, 90)	0.68	0.61	0.90
7		73	(16 500, 23 500)	(220, 88)	0.61	0.59	0.97
		74	(11 550, 18 800)	(220, 88)	0.78	0.78	1
		75	(14 850, 21 150)	(220, 88)	0.68	0.65	0.95
		76	(19 800, 28 200)	(220, 88)	0.51	0.49	0.96
		77	(9900, 16 450)	(220, 88)	0.90	0.89	0.99
		78	(14 850, 18 800)	(220, 88)	0.76	0.70	0.92
	$*_7$	79	(16 500, 23 500)	(220, 88)	0.61	0.59	0.97
		80	(14 850, 19 975)	(220, 88)	0.72	0.68	0.94
		81	(12 540, 17 860)	(220, 88)	0.80	0.77	0.96
		82	(13 530, 17 625)	(220, 88)	0.82	0.76	0.93
		83	(14 520, 18 800)	(220, 88)	0.76	0.71	0.93
		84	(13 200, 16 450)	(220, 88)	0.87	0.79	0.90
8		85	(15 500, 20 600)	(152, 80)	0.56	0.51	0.91
		86	(10 850, 16 480)	(152, 80)	0.70	0.68	0.97
		87	(13 950, 18 540)	(152, 80)	0.62	0.57	0.91
		88	(18 600, 24 720)	(152, 80)	0.46	0.43	0.93
		89	(9300, 14 420)	(152, 80)	0.80	0.78	0.97
		90	(13 950, 16 480)	(152, 80)	0.70	0.61	0.87
		91	(15 500, 20 600)	(152, 80)	0.56	0.51	0.91
	$*_8$	92	(13 950, 17 510)	(152, 80)	0.66	0.59	0.89
		93	(11 780, 15 656)	(152, 80)	0.73	0.67	0.92
		94	(12 710, 15 450)	(152, 80)	0.74	0.66	0.89
		95	(13 640, 16 480)	(152, 80)	0.70	0.61	0.87
		96	(12 400, 14 420)	(152, 80)	0.80	0.69	0.86

TABLE 3. Continued.

	DMU		$(\bar{x}'_1, \bar{x}'_2)$	(y_1, y_2)	$\bar{\theta'}^*$	$\bar{\gamma'}^*$	$\bar{\alpha'}^*$
9		97	$(15\,000, 24\,400)$	$(190, 100)$	0.59	0.59	1
		98	$(10\,500, 19\,520)$	$(190, 100)$	0.82	0.77	0.94
		99	$(13\,500, 21\,960)$	$(190, 100)$	0.66	0.65	0.98
		100	$(18\,000, 29\,280)$	$(190, 100)$	0.49	0.49	1
		101	$(9000, 17\,080)$	$(190, 100)$	0.96	0.89	0.93
		102	$(13\,500, 19\,520)$	$(190, 100)$	0.74	0.70	0.95
		103	$(15\,000, 24\,400)$	$(190, 100)$	0.59	0.59	1
		104	$(13\,500, 20\,740)$	$(190, 100)$	0.69	0.68	0.98
	$*_9$	105	$(11\,400, 18\,544)$	$(190, 100)$	0.78	0.77	0.99
		106	$(12\,300, 18\,300)$	$(190, 100)$	0.78	0.76	0.94
		107	$(13\,200, 19\,520)$	$(190, 100)$	0.74	0.71	0.96
		108	$(12\,000, 17\,080)$	$(190, 100)$	0.84	0.80	0.95
10		109	$(25\,000, 26\,800)$	$(250, 100)$	0.61	0.52	0.85
		110	$(17\,500, 21\,440)$	$(250, 100)$	0.76	0.69	0.91
		111	$(22\,500, 24\,120)$	$(250, 100)$	0.68	0.57	0.84
		112	$(30\,000, 32\,160)$	$(250, 100)$	0.51	0.43	0.78
		113	$(15\,000, 18\,760)$	$(250, 100)$	0.87	0.79	0.91
		114	$(22\,500, 21\,440)$	$(250, 100)$	0.76	0.61	0.80
		115	$(25\,000, 26\,800)$	$(250, 100)$	0.61	0.52	0.85
		116	$(22\,500, 22\,780)$	$(250, 100)$	0.72	0.59	0.82
		117	$(19\,000, 20\,368)$	$(250, 100)$	0.80	0.68	0.85
	$*_{10}$	118	$(20\,500, 20\,100)$	$(250, 100)$	0.81	0.66	0.82
		119	$(22\,000, 21\,440)$	$(250, 100)$	0.76	0.62	0.81
		120	$(20\,000, 18\,760)$	$(250, 100)$	0.87	0.69	0.79
11		121	$(26\,500, 30\,600)$	$(260, 147)$	0.67	0.58	0.87
		122	$(18\,550, 24\,480)$	$(260, 147)$	0.84	0.76	0.90
		123	$(23\,850, 27\,540)$	$(260, 147)$	0.74	0.64	0.86
		124	$(31\,800, 36\,720)$	$(260, 147)$	0.56	0.48	0.86
		125	$(15\,900, 21\,420)$	$(260, 147)$	0.96	0.88	0.92
		126	$(23\,850, 24\,480)$	$(260, 147)$	0.84	0.68	0.81
		127	$(26\,500, 30\,600)$	$(260, 147)$	0.67	0.58	0.86
		128	$(23\,850, 26\,010)$	$(260, 147)$	0.79	0.66	0.83
		129	$(20\,140, 23\,256)$	$(260, 147)$	0.88	0.76	0.86
		130	$(21\,730, 22\,950)$	$(260, 147)$	0.89	0.74	0.83
		$*_{11}$131	$(23\,320, 24\,480)$	$(260, 147)$	0.84	0.69	0.88
		132	$(21\,200, 21\,420)$	$(260, 147)$	0.96	0.77	0.80
12		133	$(19\,000, 28\,400)$	$(250, 120)$	0.63	0.61	0.97
		134	$(13\,300, 22\,720)$	$(250, 120)$	0.82	0.81	0.98
		135	$(17\,100, 25\,560)$	$(250, 120)$	0.70	0.68	0.97
		136	$(22\,800, 34\,080)$	$(250, 120)$	0.53	0.51	0.96
		137	$(11\,400, 19\,880)$	$(250, 120)$	0.96	0.93	0.97
		138	$(17\,100, 22\,720)$	$(250, 120)$	0.79	0.73	0.92
		139	$(19\,000, 28\,400)$	$(250, 120)$	0.63	0.61	0.97
		140	$(17\,100, 24\,140)$	$(250, 120)$	0.74	0.71	0.96
		141	$(14\,440, 21\,584)$	$(250, 120)$	0.83	0.81	0.96
		142	$(15\,580, 21\,300)$	$(250, 120)$	0.84	0.70	0.83
		143	$(16\,720, 22\,720)$	$(250, 120)$	0.79	0.74	0.94
	$*_{12}$	144	$(15\,200, 19\,880)$	$(250, 120)$	0.90	0.83	0.92

TABLE 4. Information about the loss of observed actual cost in $\bar{T}$.

DMU	C	C^{*}	$C^{**}_{\bar{T}}$	$C^{***}_{\bar{T}}$	L^{*}	$L^{**}_{\bar{T}}$	$L^{***}_{\bar{T}}$
1	25 100	25 100	19 140	18 386	0	5960	754
2	17 130	17 130	17 130	17 130	0	0	0
3	25 650	22 093	19 435	18 404	3557	2658	1031
4	36 360	36 360	22 890	21 507	0	13 470	1383
5	17 660	13 483	13 483	13 483	4177	0	0
6	45 150	32 239	30 472	27 323	12 911	1767	3149
7	40 000	35 744	26 813	26 287	4256	8931	526
8	31 460	25 053	20 823	19 684	6407	4230	1139
9	29 944	26 690	25 073	24 605	3254	1617	468
10	40 600	32 875	32 875	29 871	7725	0	3004
11	47 800	42 433	37 403	34 476	5367	5030	2927
12	35 080	32 732	32 732	31 457	2348	0	1275

Now, DMU_1 from Table 1 is multiplied to the cost vectors of $(500, 100), \ldots, (400, 70)$, respectively with (3.1). Therefore, equivalent DMUs to DMU_1 is achieved. Similarly, the same action for all 12 DMUs of Table 1 will be done; hence, the actual and virtual DMUs are built as shown in Table 3.

Table 3 shows the technical efficiency ($\bar{\theta'}^{*}$), cost efficiency ($\bar{\gamma'}^{*}$), and allocative efficiency ($\bar{\alpha'}^{*}$) for all DMUs (including actual and virtual) by the help of equations (3.3), (3.5) and (3.7), respectively in the PPS $\bar{T}'$ (3.2) in the Columns 5–7.

Table 3 also separately shows all equivalent DMUs of each DMU in Table 2.

The following DMUs are equivalent in both PPSs of $\bar{T}'$ and $\bar{T}$:

$$(1\&1), (14\&2), (27\&3), (40\&4), (53\&5), (66\&6), (79\&7), (92\&8), (105\&9), (118\&10), (113\&11), (114\&12)$$

where the first components are related to $\bar{T}'$ and the second components are related to $\bar{T}$.

According to Tables 2 and 3, DMU_2 and DMU_5 and DMU_{12} from Table 2 correspond DMU_{14}, DMU_{53} and DMU_{144} from Table 3, and have technical and cost efficiency in PPS $\bar{T}$, but they are inefficient in evaluation with PPS $\bar{T}'_{\text{frontier}}$. Accordingly, there all types of efficiency are reduced according to the compared all DMUs of Table 2 with their equivalent DMUs in Table 3. This difference between scores of efficiency in $\bar{T}$ and $\bar{T}'$ indicates the existence of more inefficiency sources as the new cost and price inefficiency.

Each inefficiency source creates a loss of actual cost for each DMU. The created loss which corresponds the technical, cost and allocative inefficiency respectively in the PPSs of $\bar{T}'$ and $\bar{T}$, is shown in Tables 4 and 5.

According to the compared DMUs in $\bar{T}'$ and $\bar{T}$, the technical, cost and allocative efficiency in $\bar{T}'$ is not higher than the technical, cost and allocative efficiency in $\bar{T}$. In other words, the amount of efficiency in $\bar{T}'$ is less than or equal to the amount of efficiency in $\bar{T}$. These different amounts of efficiency in these two PPSs confirm the existence of more inefficient sources in the PPS $\bar{T}'$. Utilizing these inefficiency sources and loss, the weaknesses and thus increase the amounts of efficiency can be detected.

Finally, we compare the results obtained for the cost efficiency of observed DMU using our method with the results obtained for the cost effectiveness of the same DMU with the method Sahoo *et al.* [25] and show the results in Table 6.

It is observed that the cost efficiency of method [25] is larger than the results obtained for the cost efficiency of observed DMU using our method in $\bar{T}'$ this indicates the existence of more loss resources in $\bar{T}'$ and the superiority of our proposed method towards that paper.

TABLE 5. Information about the cost and loss of observed actual cost in $\bar{T}'$.

DMU	C	C^*	$C^{**}_{\bar{T}'}$	$C^{***}_{\bar{T}'}$	L^*	$L^{**}_{\bar{T}'}$	$L^{***}_{\bar{T}'}$	$L^{4*}_{\bar{T}'}$	$L^{5*}_{\bar{T}'}$
1	25 100	25 100	17 627.66	16 570	0	7472.34	1057.66	1512.34	1816
14	17 130	17 130	15 020.20	14 870	0	2109.8	150.2	2109.8	2260
27	25 650	22 093	19 345.93	16 057.12	3557	2747.07	3288.81	89.07	2346.88
40	36 360	36 360	21 157.86	19 253.65	0	15 202.14	1904.21	1732.14	3636.35
53	17 660	13 483	13 390.85	13 390.85	4177	92.15	0	92.15	92.15
66	45 150	32 239	27 074.38	24 366.94	12 911	5164.62	2707.44	3397.62	2956.06
79	40 000	35 744	24 260.04	23 532.24	4256	11 483.96	727.8	2552.96	2754.76
92	31 460	25 053	20 802.32	18 514.07	6407	4250.68	2288.25	20.68	1169.93
105	29 944	26 690	23 376.35	23 142.59	3254	3313.65	233.76	1696.65	1462.41
118	40 600	32 875	32 611.20	26 741.18	7725	263.8	5870.02	263.8	3129.82
131	47 800	42 433	37 343.25	32 862.06	5367	5089.75	4481.19	59.75	1613.94
144	35 080	32 732	31 620.21	29 090.59	2348	1111.79	2529.62	1111.79	2366.41

TABLE 6. Results obtained for the cost efficiency of observed DMU in $\bar{T}'$ and [25].

DMU	$(\bar{x}_1, \bar{x}_2)$	(y_1, y_2)	$\bar{\gamma}^*_{\text{sahoo}}$	$\bar{\gamma}'^*$
1	$(10\,000, 15\,100)$	$(100, 90)$	0.96	0.66
2	$(66\,500, 10\,480)$	$(150, 50)$	1	0.87
3	$(11\,250, 14\,400)$	$(160, 55)$	0.88	0.10
4	$(16\,200, 20\,160)$	$(180, 72)$	0.90	0.53
5	$(6600, 11\,060)$	$(94, 66)$	1	0.76
6	$(24\,750, 20\,400)$	$(230, 90)$	0.80	0.54
7	$(16\,500, 23\,500)$	$(220, 88)$	0.95	0.59
8	$(13\,950, 17\,510)$	$(152, 80)$	0.90	0.59
9	$(11\,400, 18\,544)$	$(190, 100)$	0.87	0.77
10	$(20\,500, 20\,100)$	$(250, 100)$	0.86	0.66
11	$(23\,320, 24\,480)$	$(260, 147)$	0.86	0.69
12	$(15\,200, 19\,880)$	$(250, 120)$	1	0.83

5. Conclusion

In previous studies, cost efficiency had been evaluated in cost Production Possibility Set $\bar{T}$. In those studies, the actual costs observed for DMU_o are decomposed into several types of inefficiency. Those studies decomposed the observed actual cost for DMU_o for several types of inefficiency. It seems that this decomposition is not complete; hence, we sought to provide a new decomposition of observed actual cost for DMU_o in the new cost Production Possibility Set $\bar{T}'$. In this decomposition, the amounts of inefficiency and thus the amount of loss would be higher than the previous states. Therefore, the new Production Possibility Set was built with larger number of DMUs than the primary Production Possibility Set by application of price data for each under evaluation unit and multiplying the input and output of each unit by the price information of other DMUs and by the help of (3.1). Some of these DMUs, which were multiplied by their own price vectors (actual DMUs), were also available in the PPS $\bar{T}$, but others which were obtained from multiplying the input and output of a DMU by the price vectors of other DMUs (virtual DMUs), were not available in the PPS $\bar{T}$. This changed the number of DMUs modified the frontier of efficiency leading to the different scores of evaluated efficiency in the new set for each actual DMU with the same actual DMU in the PPS $\bar{T}$. This different efficiency of a DMU in both Production Possibility Sets indicated the existence of more inefficiency sources in the PPS $\bar{T}'$ leading the lower efficiency score for DMU. The model proposed in this research provides this capability to decompose the observed actual cost of each DMU by the loss of an efficiency source through identification of these new inefficiency sources. Being aware of these inefficiency sources, enables us to overcome or minimize these sources and the created loss, and ultimately improve the efficiency scores.

For future researches, it is suggested to implemen and investigate the Malmquist Productivity Index on the defined production possibility set in the present research. A new Production Possibility Sets for fuzzy data in the future studies also can be defined. DMUs, which have created from a price vector, in the Production Possibility Set of $\bar{T}'$ can be compared, and it let us investigate whether those DMUs are better than other similar units that are created with different input and output and equal price. At the end this paper also suggests investigating states with some negative price data in the future studies.

Acknowledgements. Thank the three reviewers for their influential and very helpful suggestions. Their instructive implications have improved the earlier version of the present manuscript. Thank the Department of Mathematics, Khorramabad Branch, Islamic Azad University, Khorramabad, Iran.

References

[1] A. Ashrafi and M.M. Kaleibar, Cost, revenue and profit efficiency models in generalized fuzzy data envelopment analysis. *Fuzzy Inf. Eng.* **9** (2017) 237–246.

[2] K. Ayouba, J.P. Boussemart, H.B. Lefer, H. Leleu and R. Parvulescu, A measure of price advantage and its decomposition into output-and input-specific effects. *Eur. J. Oper. Res.* **276** (2019) 688–698.

[3] V. Bilotkach, S. Gitto, R. Jovanović, J. Mueller and E. Pels, Cost-efficiency benchmarking of European air navigation service providers. *Transp. Res. Part A: Policy Pract.* **77** (2015) 50–60.

[4] A.S. Camanho and R.G. Dyson, A generalisation of the Farrell cost efficiency measure applicable to non-fully competitive settings. *Omega* **36** (2008) 147–162.

[5] A.S. Camanho and R.G. Dyson, Cost efficiency measurement with price uncertainty: a DEA application to bank branch assessments. *Eur. J. Oper. Res.* **161** (2005) 432–446.

[6] G. Cesaroni, Industry cost efficiency in data envelopment analysis. *Soc.-Econ. Planning Sci.* **61** (2018) 37–43.

[7] A. Charnes, W.W. Cooper and E. Rhodes, Measuring the efficiency of decision-making units. *Eur. J. Oper. Res.* **3** (1979) 339.

[8] G. Chortareas, G. Kapetanios and A. Ventouri, Credit market freedom and cost efficiency in US state banking. *J. Empirical Finance* **37** (2016) 173–185.

[9] H.A. Cinemre, V. Ceyhan, M. Bozoğlu, K. Demiryürek and O. Kılıç, The cost efficiency of trout farms in the Black Sea Region, Turkey. *Aquaculture* **251** (2006) 324–332.

[10] W.W. Cooper, L.M. Seiford and K. Tone, Data envelopment analysis: a comprehensive text with models, applications, references and DEA-solver software. Springer Science & Business Media. New York, USA (2007).

[11] S. De Jaeger and N. Rogge, Cost-efficiency in packaging waste management: the case of Belgium. *Resour. Conserv. Recycl.* **85** (2014) 106–115.

[12] N. Ederer, Evaluating capital and operating cost efficiency of offshore wind farms: a DEA approach. *Renew. Sustainable Energy Rev.* **42** (2015) 1034–1046.

[13] L. Fang and H. Li, Lower bound of cost efficiency measure in DEA with incomplete price information. *J. Product. Anal.* **40** (2013) 219–226.

[14] L. Fang and H. Li, Cost efficiency in data envelopment analysis under the law of one price. *Eur. J. Oper. Res.* **240** (2015) 488–492.

[15] R. Färe and S. Grosskopf, CAK Lovell The Measurement of Efficiency of Production Kluwer. Nijhoff Publishing (1985).

[16] M.J. Farrell, The measurement of productive efficiency. *J. R. Stat. Soc. Ser. A (General)* **120** (1957) 253–290.

[17] M. Ghiyasi, Inverse DEA based on cost and revenue efficiency. *Comput. Ind. Eng.* **114** (2017) 258–263.

[18] B. Hampf, Cost and environmental efficiency of US electricity generation: accounting for heterogeneous inputs and transportation costs. *Energy* **163** (2018) 932–941.

[19] G.R. Jahanshahloo, M. Soleimani-Damaneh and A. Mostafaee, Cost efficiency analysis with ordinal data: a theoretical and computational view. *Int. J. Comput. Math.* **84** (2007) 553–562.

[20] G.R. Jahanshahloo, M. Soleimani-Damaneh and A. Mostafaee, A simplified version of the DEA cost efficiency model. *Eur. J. Oper. Res.* **184** (2008) 814–815.

[21] H.A. Kader, M. Adams, P. Hardwick and W.J. Kwon, Cost efficiency and board composition under different takaful insurance business models. *Int. Rev. Financial Anal.* **32** (2014) 60–70.

[22] A. Mostafaee and F.H. Saljooghi, Cost efficiency measures in data envelopment analysis with data uncertainty. *Eur. J. Oper. Res.* **202** (2010) 595–603.

[23] M.R. Mozaffari, P. Kamyab, J. Jablonsky and J. Gerami, Cost and revenue efficiency in DEA-R models. *Comput. Ind. Eng.* **78** (2014) 188–194.

[24] M.C.A.S. Portela and E. Thanassoulis, Economic efficiency when prices are not fixed: disentangling quantity and price efficiency. *Omega* **47** (2014) 36–44.

[25] B.K. Sahoo, M. Mehdiloozad and K. Tone, Cost, revenue and profit efficiency measurement in DEA: a directional distance function approach. *Eur. J. Oper. Res.* **237** (2014) 921–931.

[26] R.K. Shiraz, H. Fukuyama, M. Tavana and D. Di Caprio, An integrated data envelopment analysis and free disposal hull framework for cost-efficiency measurement using rough sets. *Appl. Soft Comput.* **46** (2016) 204–219.

[27] E. Thanassoulis, D. Sotiros, G. Koronakos and D. Despotis, Assessing the cost-effectiveness of university academic recruitment and promotion policies. *Eur. J. Oper. Res.* **264** (2018) 742–755.

[28] M. Toloo, A cost efficiency approach for strategic vendor selection problem under certain input prices assumption. *Measurement* **85** (2016) 175–183.

[29] K. Tone, A strange case of the cost and allocative efficiencies in DEA. *J. Oper. Res. Soc.* **53** (2002) 1225–1231.

[30] K. Tone and M. Tsutsui, Decomposition of cost efficiency and its application to Japanese-US electric utility comparisons. *Soc.-Econ. Planning Sci.* **41** (2007) 91–106.

[31] A. Venkatesh and S. Kushwaha, Short and long-run cost efficiency in Indian public bus companies using Data Envelopment Analysis. *Soc.-Econ. Planning Sci.* **61** (2018) 29–36.