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SIMULTANEOUS CONTROL ON LEAD TIME ELEMENTS AND ORDERING COST FOR AN INFLATIONARY INVENTORY-PRODUCTION MODEL WITH MIXTURE OF NORMAL DISTRIBUTIONS LTD UNDER FINITE CAPACITY

Hesamoddin Tahami[1,*], Abolfazl Mirzazadeh[2] and Aref Gholami-Qadikolaei[2]

Abstract. The significance of inflation and time value of money in inventory/production systems is indisputable for modern decision makers. Consequently, the paper aims to study the influence of inflationary condition on a stochastic continuous review integrated vendor–buyer inventory system in the presence of a multilevel reorder strategy for the system. It is considered that lead time components and ordering cost are controllable. Lead time is decomposed into its components: set-up time, production time and transportation time. Based on lead time components, demand during the lead time for different batches is assumed to be a mixture of normal distributions. The objective is to minimize joint inventory expected cost by simultaneously optimizing ordering quantity, reorder points of different batches, ordering cost, setup time, transportation time, production time and a number of deliveries under space constraint while the lead time demand follows a normal distribution. To minimize the expected inventory cost, a Lagrange multiplier method is applied in order to solve the problem, and an iterative algorithm is designed to find optimal values. The behavior of the model is illustrated in numerical examples. It was found that for a fixed value of transportation time, setup time and number of batches, with an augment in inflation rate, the two optimal reorder points for different batches were increased. Also, optimum joint expected annual cost with inflation for two kinds of customers' demand is larger than one kind of customers' demand. Furthermore, sensitivity analysis and managerial insights are given to show the applicability of the model.

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1. Introduction

In many deterministic or stochastic inventory research papers, one facility (*e.g.* a buyer or a vendor) is assumed to minimize its own cost or maximize its own profit. This one-sided optimal strategy is not appropriate for the global market. However, according to the just-in-time (JIT) philosophy, many researchers concentrate on integration between buyer and vendor. Once they form a long-term strategic alliance, both facilities can cooperate and share information to achieve improved benefits. Nowadays, inflation is an observable phenomenon

Keywords. Controllable lead time, integrated vendor–buyer inventory model, nonlinear constrained stochastic optimization, mixture of normal distributions, inflation.

[1] Department of Engineering Management & Systems Engineering, Old Dominion University, Norfolk, VA, USA.

[2] Department of Industrial Engineering, Kharazmi University, Tehran, Iran.

[*]Corresponding author: htahami@odu.edu, hesamtahami@gmail.com

in most societies, and its impact on determining optimal policies of inventory/production systems is a momentous concern for inventory managers. In economics, inflation is a sustained increase in the general price level of goods and services in an economy over a period of time. When the general price level rises, with each unit of currency, we can buy fewer goods and services. Consequently, inflation reflects a reduction in the purchasing power per unit of money – a loss real value in the medium of exchange and unit of account within the economy. A chief measure of price inflation is the inflation rate, the annualized percentage change in a general price index. In addition, lead time plays an important role in the logistics management. In the inventory management literature, lead time is treated as predetermined constant or stochastic parameters. If it is assumed that lead time can be decomposed into several components, such as setup time, process time, and transportation time, it can be assumed that each component may be reduced at a crashing cost. Also, as a consequence of high cost of land acquisition in the most societies, most of inventory systems have limited storage space to stock goods. Moreover, as can be seen in various industries, bi-modal or multi-modal distributions arise frequently in the cases where demand is observed from multiple, distinct sources (customers). So, in order to model the real environment, it is better not to use only a single distribution. Consequently, the present paper tries to capture all of the mentioned factors and fill the gap in the literature by developing an inflationary integrated vendor–buyer inventory model where the lead time components and ordering cost are controllable and demand during the lead time follows a mixture of normal distributions.

2. Literature review

The inventory problem regarding integration between buyer and vendor has received a lot of attention in recent years. Firms are appreciating that a more appropriate result is achieved over better coordination and cooperation of all parties involved in a supply chain. Starting with Goyal [12], much research has been conducted that aims to obtain coordinated inventory replenishment decisions among individual companies to benefit the entire supply network rather than a single company. Banerjee [1] extended the model of Goyal [12], assumed that the manufacturing rate was finite, and considered a lot-for-lot model where the vendor produced each buyer shipment as a separate batch. Goyal [13] extended Banerjee's work by relaxing the lot-for-lot production policy for the vendor and assumed that the vendor's lot size was an integral multiple of the buyer's order quantity. An extension of Banerjee [1] and Goyal [13] was proposed by Landeros and Lyth [20], which assumed that inventory carrying charges of both buyer and vendor included different cost components. Lu [23] presented a model for integrated vendor–buyer problems and developed a heuristic approach for the one-vendor multi-buyer case. The model was an improvement over the models of Banerjee [1] and Goyal [13]. Regarding previously mentioned studies, demand and lead time were assumed to be deterministic. However, demand or lead time in various industries was stochastically distributed; hence, considering uncertainty in the integrated inventory models is appropriate and meaningful. Also, through the Japanese successful experience of using Just-In-Time (JIT) production, the advantages and benefits associated with efforts to control the lead time can be clearly perceived. Regarding the mentioned issues, Pan and Yang [33] extended Goyal [13] by considering lead time demand as a probabilistic variable and considered lead time as a controllable variable.

Ouyang *et al.* [32] extended Pan and Yang's [33] model by further considering the reorder point as a decision variable and shortage occurrence, simultaneously optimizing ordering quantity, reorder point, lead time and the number of lots in an integrated single-vendor single-buyer inventory model. Ben-Daya and Hariga [3] considered a single-vendor single-buyer integrated inventory problem and assumed that the lead time was composed of lot-size dependent run time and constant delay times, such as moving, waiting, and setup times. Chang *et al.* [6] proposed an integrated vendor–buyer cooperative inventory model in which lead time and ordering cost were controllable. They considered that the buyer's lead time can be reduced at an extra crashing cost which depends on the lead time length and ordering lot size. They also investigated a case when the ordering cost and lead time reduction acted dependently. Hsiao [18] modified the model developed by Ben-Daya and Hariga [3] by adding the assumption that production time was the same as the lead time of the first batch and developed an integrated inventory model with two reorder points and service levels. Glock [11] extended Hsiao's [18] model considering that lead time could be decomposed to production, setup, and transportation time, and therefore, lead time could be reduced by shortening setup and transportation time with the same crashing cost or by increasing the production rate, which resulted in reduced production time.

Moon *et al.* [31] developed the distribution-free continuous-review inventory model to minimize the total cost by using a negative exponential lead time crashing cost function, and derived the closed-form expressions for the optimal order quantity, reorder point, and lead time. Sarkar *et al.* [39] examined the effects of setup cost reduction and quality improvement in a two-echelon supply chain model with deterioration. Their objective was to minimize the total cost of the entire supply chain model by simultaneously optimizing setup cost, process quality, number of deliveries and lot size. Sarkar *et al.* [40] developed a sustainable integrated inventory model by considering Stackelberg's game policy, where a discrete investment was used to reduce the setup cost, as well as introducing fixed and variable transportation and carbon emission cost.

Majumder *et al.* [24] investigated a two-echelon supply chain model considering the quality improvement of products and setup cost reduction under controllable lead time. They considered two cases where in the first one, lead time demand followed a normal distribution and in the second one, no specific distribution was assumed except a mean and standard deviation. Kim and Sarkar [19] developed a model to minimize the total cost throughout the supply chain network under single-supplier and single-buyer for a single type of product and single-setup-multi delivery (SSMD) policy. They assumed that the supplier offered trade-credit-period to the buyer and the buyer used the delay time to increase his/her profit. Also, a continuous review inventory model was considered for both supplier and buyer, investment was used to reduce setup cost, and another investment was used to improve the quality of products. Soni *et al.* [43] presented a continuous review inventory model with backorders and lost sales with fuzzy demand and learning considerations. They showed that the learning effect in fuzziness reduced the ambiguity associated with the decision making progress. In another paper of Soni *et al.* [44], they investigated the effects of lost sales reduction and quality improvement in an imperfect production process under an imprecise environment while simultaneously optimizing reorder point, order quantity, and lead time. They also assumed that lead time demand followed a mixture of normal distributions.

Majumder *et al.* [25] developed a continuous review single-vendor multi-buyer inventory model with partially back ordered demand, where the vendor produced products in a batch production process under a variable production rate. Also, it was assumed that a crashing cost was incurred by all buyers to reduce their lead time. Sarkar *et al.* [41] extended Glock's [11] paper by adding the concept of quality improvement and setup cost reduction in a two-echelon supply chain. They considered the combination of setup time reduction, quality improvement of products, and setup time as well as transportation time crashing cost together to reduce the whole supply chain cost. Dey *et al.* [8] studied a sustainable inventory model with a controllable lead time, discrete setup cost reduction and consideration of environmental issues. They assumed that the customer demand was dependent on selling-price and the lead time demand followed a Poisson distribution. Sarkar and Mahapatra [34] investigated a periodic review fuzzy inventory model with lead time, reorder point, and cycle length as decision variables, where the main goal was to minimize the expected total annual cost by simultaneously optimizing cycle length, reorder point, and lead time for the whole system based on fuzzy demand. They assumed that lead time demand could follow either normal distribution or distribution-free and then solved their model considering both conditions. Sarkar *et al.* [37] studied a continuous review inventory model considering setup cost reduction and quality improvement by using logarithmic expression, and solved their model with a distribution-free approach. Sarkar and Majumder [35] developed two integrated vendor–buyer supply chain models where in the first one, the lead time demand followed a normal distribution and in the second one the distribution-free approach was applied for the lead time demand.

Regarding the inventory management literature about the space constrained problem, very few papers have worked on stochastic demand models. In fact, previous research on inventory problems with space restriction focused on the case of multiple items with deterministic demand [15,17]. Veinott [46] was one of the first authors to propose storage space constrained inventory problems in the stochastic environment. Hariga [16] presented a stochastic space constrained continuous review inventory system for a single item and random demand, wherein the order quantity and reorder point were decision variables. Xu and Leung [48] proposed an analytical model in a two-party vendor managed system where the retailer restricted the maximum space allocated to the vendor. Moon and Ha [29] proposed three extended models with variable capacity. First, they presented an EOQ model with random yields. Second, they developed a multi-item EOQ model with storage space and solved the model with the Lagrange multiplier method. Third, they applied a distribution-free approach to the (Q, r) with variable capacity.

The analysis of inventory systems under an inflationary condition in the literature can be carried out using two procedures. The first one determines the optimal values of the control variables by minimizing the average annual cost, and the second one determines the optimal ordering policy by minimizing the discounted value of all future costs. Hadley [14] showed with the detailed computations in the simplest way that the ordering quantities computed by minimizing the average annual cost and also by minimizing the discounted cost did not differ significantly. Mirzazadeh [27] extended Hadley's work by minimizing the inflation and time value of money under uncertain conditions, shortages and the effect of deterioration. The above-mentioned system was formulated with two methods, which were derived under some assumptions, and the objective was to minimize the average annual cost and the discounted cost. These methods were compared to each other carefully, and the results revealed that the mentioned methods (the average annual cost and the discounted cost) had a negligible difference to each other.

A number of papers have considered the effect of inflation on the inventory system since 1975. Buzacott [5] developed an economic order quantity model with a fixed inflation rate for all related costs. Bierman and Thomas [4] proposed an inventory model under an inflationary condition that also incorporated the discount rate. Then, Misra [26] extended an inventory model with different inflation rates for variously related costs. Yang *et al.* [49] developed different inventory models with time-varying demand patterns under inflation. Moon *et al.* [30] developed an inflation EOQ model for both ameliorating and deteriorating items, assuming not only a constant length of each replenishment, but also a constant fraction of shortage length with respect to the cycle length. Mirzazadeh *et al.* [28] proposed an inventory model with probabilistic inflationary conditions. The developed model also implicated to a finite replenishment rate and finite time horizon with a shortage. The objective was to minimize the expected present value of costs over the time horizon. Hence, in this study, a stochastic (Q, r) integrated inventory system under the inflationary condition is proposed.

In the case of probabilistic demand, as can be seen in various industries, bi-modal or multi-modal distributions arise frequently in the cases where demand is observed from multiple, distinct sources (customers). So, we cannot use only a single distribution [3, 6, 10, 11, 29, 32, 36, 38, 42]. Wu and Tsai [47] considered the mixture of normal distributions for lead time demand to find optimal ordering quantity and lead time for the buyer based on the work by Everitt and Hand [9]. Lee *et al.* [21] proposed a one-sided inventory system with defective goods wherein the lead time demand followed a mixture of normal distributions to find buyer's optimal inventory strategy when reorder point, lead time and ordering quantity were the decision variables. Lin [22] extended the research by Lee *et al.* [21] by assuming lost sale rate as a controllable variable. However, in the previously mentioned research, one facility, *e.g.* a buyer, was assumed to minimize its own cost. This one-sided optimal strategy is not appropriate for the global market. Therefore, in this study, we consider a mixture of normally distributed lead time demands for an integrated single-vendor single-buyer inventory model rather than considering only one facility.

The present paper extends the mentioned works considering multi-reorder level inventory systems by adding lead time components. In Glock's [11] study, it was assumed that transportation time was a fraction of setup time with the same crashing cost. But, in practice, setup and transportation time and their crashing costs are different from each other, and mostly the mentioned assumption cannot be used in a real environment. Therefore, this paper assumes transportation and setup time and their crashing costs act independently. Also, in order to fit some real environment, transportation time crashing cost is presented as a function of reduced transportation time and the quantities in the orders. This paper also considers a random space constraint for random demand and positive lead time when maximum permissible storage space is restricted. A few papers have considered space constraint for stochastic demand. However, to our knowledge, no paper has assumed a mixture of distributions for demand while considering space constraint to the model. In addition, as mentioned in the previous paragraph, inflationary condition for the proposed stochastic demand and deterministic variable lead time is considered, helping the model to be more appropriate for the real environment. The paper aims to minimize joint inventory expected cost by simultaneously optimizing ordering quantity, reorder points of different batches, ordering cost, setup time, transportation time, production time, and a number of deliveries under space constraint, while the lead time demand follows a normal distribution. Hence, a Lagrangian method is applied to solve the problem, and a solution procedure is proposed to find optimal values. The behavior of the model is also illustrated in numerical examples.

The rest of the paper is organized as follows. In Section 3, the notations and assumptions are given. In Section 4, we present the mathematical model. In Section 5, a numerical example and sensitivity analysis are given to illustrate the model and its solution procedure. Finally, we conclude this paper.

3. Notations and assumptions

Notations

Following notations have been used through the paper:

Decision variables

Q	Buyer's order quantity (units).
r	Buyer's reorder point (units).
A	Buyer's ordering cost at the time zero ($/order).
t	Transportation time (days).
s	Setup time (days).
m	The number of lots in which the product is delivered from the vendor to the buyer in one Production cycle, a positive integer (units).

Parameters

D	Annual demand for buyer (units/year).
P	Production rate in units per unit time (units/unit time).
p	$1/P$.
a	Vendor's setup cost per set up at the time zero ($/setup).
π	Buyer's stock out cost per unit at the time zero ($/unit).
h_v	Vendor's holding cost per unit per year at the time zero ($/unit/year).
h_b	Buyer's holding cost per unit per year at the time zero ($/unit/year).
I	Inflation rate.
n	Number of cycle.
n_s	Number of shipment cycle.
f	Space used per unit (m^2/unit).
F	Maximum permissible storage space (m^2).
$I(A)$	Buyer's capital investment required to achieve ordering cost A, $0 < A \leq A_0$.
b	Percentage decrease in ordering cost A per dollar increase in investment $I(A)$.
θ	Fractional opportunity cost of capital investment per year ($/year).
C_{pu}	Buyer's purchasing cost per unit at the time zero ($/unit).
c_s	Vendor's Setup cost per setup at the time zero ($/setup).
c_{pr}	Vendor's Production cost per unit at the time zero ($/unit).
A_0	Buyer's original ordering cost per order ($/order).
X	Demand during lead time, as a random variable.
X^+	Maximum value of x and 0.
$E(\cdot)$	Mathematical expectation.

Assumptions

1. There are single-vendor and single-buyer for a single product in this paper.
2. The vendor's production rate is finite and greater than the buyer's demand rate, *i.e.*, $P > D$, where P and D are given.
3. The buyer orders a lot, of size mQ and the vendor manufactures a lot, of size mQ, but transfer a shipment of size Q to the buyer. Once a vendor produces the first Q units, he will deliver them to the buyer. The vendor will schedule successive deliveries every Q/D units of time.
4. We assume that the capital investment, $I(A)$ in reducing buyer's ordering cost is a logarithmic function of the ordering cost$A$. That is,
$$I\left(A\right)=\frac{1}{\delta}\ln\left(\frac{A_0}{A}\right)\text{ for }0<A\leq A_0$$
where δ is the fraction of the reduction in A per dollar increase in investment.
5. Setup time s consists of n^s mutually independent components. The ith component has a normal duration NS_i and minimum duration MS_i, $i=1,2,\ldots,n^s$. If we let $s_0=\sum_{j=1}^{n^s}\mathrm{NS}_j$ and s_i be the length of setup time with components $1,2,\ldots,i$, crashed to their minimum duration, then s_i can be expressed as $s_i = s_0-\sum_{j=1}^{i}(\mathrm{NS}_j-\mathrm{MS}_j)$, $i=1,2,\ldots,n^s$ and the setup time crashing cost is given by
$$\mathrm{CS}\left(s\right)=c_{si}\left(s_{i-1}-s\right)+\sum_{j=1}^{i-1}c_{sj}\left(\mathrm{NS}_j-\mathrm{MS}_j\right).$$
6. The transportation time t consists of n^t mutually independent components. The ith component has a normal duration NT_i and minimum duration $\mathrm{MT}_i, i=1,2,\ldots,n^t$.
7. For the ith component of transportation time, the crashing cost per unit time c_{ti}, depends on the ordering lot size Q and is described by $c_{ti}=a_i+b_iQ$, where $a_i>0$ is the fixed cost, and $b_i>0$ is the unit variable cost, for $i=1,2,\ldots,n^t$.
8. For any two crash cost lines $c_{ti}=a_i+b_iQ$ and $c_{tj}=a_j+b_jQ$, where $a_i>a_j$, $b_i<b_j$, for $i\neq j$ and $i,j=1,2,\ldots,n^t$, there is an intersection point Q^S such that $c_{ti}=c_{tj}$. These intersection points are arranged in ascending order so that $Q_0^S<Q_1^S<\ldots<Q_w^S<Q_{w+1}^S$, where $Q_0^S=0$, $Q_{w+1}^S=\infty$ and $w\leq n^t(n^t-1)/2$. For any order quantity range (Q_i^S,Q_{i+1}^S), $c_i s$ are arranged such that $c_1\leq c_2\leq\ldots\leq c_{n^t}$, and the lead time components are crashed one at a time starting with the component of least c_i, and so on.
9. Let $t_0\equiv\sum_{j=1}^{n^t}\mathrm{NT}_j$ and t_i be the length of transportation time with components $1,2,\ldots,i$ crashed to their minimum duration, then t_i can be expressed as $t_i=t_0-\sum_{j=1}^{i}(\mathrm{NT}_j-\mathrm{MT}_j), i=1,2,\ldots,n^t$ and the transportation time crashing cost per cycle $\mathrm{CT}(t)$ is given by $\mathrm{CT}(t)=c_{ti}(t_{i-1}-t)+\sum_{j=1}^{i-1}c_j(\mathrm{NT}_j-\mathrm{MT}_j)$, where $t\epsilon[t_i,t_{i-1}]$, and $c_j=a_j+b_jQ$ for $j=1,2,\ldots,i$.
10. We consider the deterministic variable lead time L and assume that the demand of the lead time X follows the mixture of normal distributions, $F_*=\alpha F_1+(1-\alpha)F_2$, where F_1 has a normal distribution with finite mean μ_1 and standard deviation $\sigma\sqrt{L}$ and F_2 has a normal distribution with finite mean μ_2 and standard deviation $\sigma\sqrt{L}$. Therefore, the lead time demand, X has a mixture of probability density function (PDF) which is given by
$$f\left(x\right)=\alpha\frac{1}{\sqrt{2\pi}\sigma\sqrt{L}}\times\exp\left[-\frac{1}{2}\left(\frac{x-\mu_1L}{\sigma\sqrt{L}}\right)^2\right]+\left(1-\alpha\right)\frac{1}{\sqrt{2\pi}\sigma\sqrt{L}}\times\exp\left[-\frac{1}{2}\left(\frac{x-\mu_2L}{\sigma\sqrt{L}}\right)^2\right]$$
where $\mu_1-\mu_2=k_1\sigma/\sqrt{L}$ or $\mu_1L-\mu_2L=k_1\sigma\sqrt{L}$, $k_1>0$, $-\infty<x<\infty$, $0\leq\alpha\leq1$, $\sigma>0$. Moreover, the mixture of normal distributions is unimodal for all α if $(\mu_1-\mu_2)^2<27\sigma^2/8L$ or $k_1<\sqrt{\frac{27}{8}}$. Also, when $(\mu_1-\mu_2)^2>4\sigma^2/L$ or $k_1>2$, at least we can find a value of $\alpha(0\leq\alpha\leq1)$ which makes the mixture of normal distributions to be a bimodal distribution.

11. The reorder point r = expected demand during lead time + safety stock (ss), and $ss = k\times$(standard deviation of lead time demand), that is $r = \mu_* L + k\sigma_*\sqrt{L}$, where $\mu_* = \alpha\mu_1 + (1-\alpha)\mu_2$, $\sigma_* = \sqrt{1+\alpha(1-\alpha)k_1^2}\sigma$, $\mu_1 = \mu_* + (1-\alpha)k_1\sigma/\sqrt{L}$, $\mu_2 = \mu_* - \alpha k_1\sigma/\sqrt{L}$, and k is the safety factor which satisfies $P(X > r) = 1 - p\Phi(r_1) - (1-p)\Phi(r_2) = q$, where Φ represents the cumulative distribution function of the standard normal random variable, q represents the allowable stock-out probability during L, $r_1 = (r - \mu_1 L/\sigma\sqrt{L}) = (r - \mu_* L/\sigma\sqrt{L}) - (1-\alpha)k_1$, and $r_2 = (r - \mu_2 L/\sigma\sqrt{L}) = (r - \mu_* L/\sigma\sqrt{L}) + k_1\alpha$.
12. Lead time for the first shipment is proportional to the lot size produced by the vendor and consists of the sum of setup time, transportation time and production time, *i.e.* $L(Q,s,t) = s + pQ + t$. For shipments $2,\ldots,m$ only transportation time has to be considered for calculating lead time, *i.e.* $L(t) = t$. Since, due to $P > D$, shipments $2,\ldots,m$ have been completed when the order of buyers arrives. Hence, considering the mixture of normal distributions, the lead time demand for the first batch, X^1, has a probability density function $f(x^1, \mu_1 L(Q,s,t), \mu_2 L(Q,s,t), \sigma\sqrt{L(Q,s,t)}, \alpha)$ with the mean $\mu_1 L(Q,s,t), \mu_2 L(Q,s,t)$ and standard deviation $\sigma\sqrt{L(Q,s,t)}$ and for the other batches, the lead time demand, X^2, has a probability density function $f(x^2, \mu_1 L(t), \mu_2 L(t), \sigma\sqrt{L(t)}, \alpha)$ with the mean $\mu_1 L(t), \mu_2 L(t)$ and standard deviation $\sigma\sqrt{L(t)}$.

4. Model formulation

In this section, we establish a continuous review integrated inventory model involving backorders, variable lead time elements, mixture of normal distributions for lead time demand, bi-level reorder strategy and storage space constraint. We assume that buyer order mQ units and vendor produces mQ units, but transfer a shipment of size Q to the buyer. Therefore the length of buyer's and vendor's cycle is mQ/D, but the shipment cycle is Q/D.

4.1. Buyer's total expected cost per unit time

Due to random demand, shortage may occur at the buyer side. The expected shortage for the first batch is equal to $E(X^1 - r^1)^+ = \int_{r^1}^{\infty}(x^1 - r^1)f(x^1)\mathrm{d}x^1$; And for the other batches $E(X^2 - r^2)^+ = \int_{r^2}^{\infty}(x^2 - r^2)f(x^2)\mathrm{d}x^2$. For bi-level reorder point system, the expected net inventory level for the first batch just before an order arrival is equal to $E[(X^1 - r^1)^- I_{0<X^1<r^1}] - E(X^1 - r^1)^+$ and the expected net inventory level at the beginning of the cycle equals $Q + E[(X^1 - r^1)^- I_{0<X^1<r^1}] - E(X^1 - r^1)^+$. For the other batches, expected net inventory level for the first batch just before an order arrival is equal to $E[(X^2 - r^2)^- I_{0<X^2<r^2}] - E(X^2 - r^2)^+$ and the expected net inventory level at the beginning of the cycle equals $Q + E[(X^2 - r^2)^- I_{0<X^2<r^2}] - E(X^2 - r^2)^+$. Hence, average inventory can be expressed by

$$\frac{Q}{2} + \frac{1}{m}\left\{E\left[\left(X^1 - r^1\right)^- I_{0<X^1<r^1}\right] - E(X^1 - r^1)^+\right\} + \frac{m-1}{m}\left\{E\left[\left(X^2 - r^2\right)^- I_{0<X^2<r^2}\right] - E(X^2 - r^2)^+\right\}. \tag{4.1}$$

As mentioned in assumption 12, the demand during the lead time is a mixture of normal distributions. For the first batch, means and standard deviation are $\mu_1 L(pQ,s,t)$, $\mu_2 L(pQ,s,t)$, $\sigma\sqrt{L(pQ,s,t)}$ respectively, and for the jth batch, $j = 2,\ldots,m$. They are equal to $\mu_1 L(t)$, $\mu_2 L(t)$ and $\sigma\sqrt{L(t)}$ respectively. Therefore, the safety stock (SS), can be expressed as follows:

$$\mathrm{SS} = \sigma\sqrt{s+t+pQ}\left\{\frac{r_1^1 + k_1(1-\alpha)}{\sqrt{1+k_1^2\alpha(1-\alpha)}}\right\}. \tag{4.2}$$

The safety stock also can be expressed as follows:

$$\mathrm{SS} = \sigma\sqrt{t}\left\{\frac{r_1^2 + k_1(1-\alpha)}{\sqrt{1+k_1^2\alpha(1-\alpha)}}\right\}. \tag{4.3}$$

According to [18], from equations (4.2) and (4.3), we have

$$\sigma\sqrt{s+t+pQ}\left\{\frac{r_1^1+k_1\left(1-\alpha\right)}{\sqrt{1+k_1^2\alpha\left(1-\alpha\right)}}\right\}=\sigma\sqrt{t}\left\{\frac{r_1^2+k_1\left(1-\alpha\right)}{\sqrt{1+k_1^2\alpha\left(1-\alpha\right)}}\right\}. \tag{4.4}$$

The expected shortage of the first batch shipment is given as (see Appendix A)

$$E(X^1-r^1)^+=\int_{r^1}^{\infty}\left(x^1-r^1\right)f\left(x^1\right)\mathrm{d}x^1=\sigma\sqrt{t+s+pQ}\,\psi\left(r_1^1,r_2^1,\alpha\right). \tag{4.5}$$

For batches $2,\ldots,m$, the expected shortage amount is

$$E(X^2-r^2)^+=\int_{r^2}^{\infty}\left(x^2-r^2\right)f\left(x^2\right)\mathrm{d}x^2=\sigma\sqrt{t}\,\psi\left(r_1^2,r_2^2,\alpha\right). \tag{4.6}$$

Hence, considering the inflationary condition for the inventory costs, (see Appendix B), the buyer's expected annual cost under inflationary condition can be obtained as follows

$$\begin{aligned}
&\mathrm{MIN\,EAC}\left(Q,A,r^1,r^2,t\right)\\
&=\frac{\theta}{\delta}\ln\left(\frac{A_0}{A}\right)+A\left[\frac{D}{mQ}\left(1+\frac{I}{2}\right)-\frac{I}{2}\right]\\
&\quad+\frac{h_b\left(1+\frac{I}{2}\right)}{m}\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]\right.\\
&\quad\left.+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}\\
&\quad+\frac{(m-1)\,h_b\left(1+\frac{I}{2}\right)}{m}\sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)\right]\right.\\
&\quad\left.+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}+\frac{h_bQ}{2}\left(1+\frac{I}{2}\right)\\
&\quad+\frac{D\pi}{mQ}\left[1+(s+pQ+t)\,I+\frac{I}{2}\left(1-\frac{Q}{D}\right)\right]\left[\sigma\sqrt{t+s+pQ}\,\psi\left(r_1^1,r_2^1,\alpha\right)+(m-1)\,\sigma\sqrt{t}\,\psi\left(r_1^2,r_2^2,\alpha\right)\right]\\
&\quad+\frac{D}{Q}\left[ut+(a_i+b_iQ)\,(t_{i-1}-t)+\sum_{j=1}^{i-1}(a_j+b_jQ)\,(\mathrm{NT}_j-\mathrm{MT}_j)\right]\left[1+(s+pQ)\,I+\frac{I}{2}\left(1-\frac{Q}{D}\right)\right]\\
&\quad+Dc_{pu}\left[1+\frac{I}{2}\left(1-\frac{mQ}{D}\right)\right].
\end{aligned} \tag{4.7}$$

With today's high cost of land acquisition in most societies, most of the inventory systems have limited storage space to stock goods. Therefore, for the proposed inventory system, it is assumed that maximum permissible storage space is limited. The proposed constraint is probabilistic since buyer's maximum inventory level is a random variable. The mentioned probabilistic constraint can be expressed by

$$P\left\{f\left[Q+r-X\right]\leq F\right\}\geq\gamma. \tag{4.8}$$

The above constraint forces the probability that the total used space is within maximum permissible storage space to be no smaller than γ. It is problematic to solve the constrained inventory system when the space constraint is written as (4.8). Hence, by using the chance-constrained programming technique which is proposed by Charnes and Cooper [7] and considering Markov inequality, the random constraint for a mixture of normal distributions is converted to the crisp one which is given by (see Appendix C)

$$\gamma Q+\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma}+(1-\alpha)\,k_1\right)-\phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma+(1-\alpha)\,k_1}\right)\right]+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}-\frac{F}{f}\le 0 \quad (4.9)$$

and

$$\gamma Q+\sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)\,k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)\,k_1\right)\right]+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}-\frac{F}{f}\le 0. \quad (4.10)$$

4.2. Vendor's total expected cost per unit time

The initial stock in the system, QD/P, is the amount of inventory required by the buyer during the protection period of the first shipment Q. As soon as the production run is started, the total stock increases at a rate of $(P-D)$ until the complete batch quantity, mQ, has been manufactured. Hence, the average inventory level per unit time for the vendor can be calculated as follows:

$$\left\{mQ\left[\frac{Q}{P}+(m-1)\frac{Q}{D}-\frac{m^2Q^2}{2P}\right]-\left[\frac{Q^2}{D}\left(1+2+\cdots+(m-1)\right)\right]\right\}\Big/\left(\frac{mQ}{D}\right)=\left[m\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right]. \quad (4.11)$$

Considering the inflationary condition, the vendor's expected total cost per year is computed as given below

$$\mathrm{EAC}_v=\left[as+c_{si}\left(s_{i-1}-s\right)+\sum_{j=1}^{i}c_{sj}\left(\mathrm{NS}_j-\mathrm{MS}_j\right)\right]\left[\frac{D}{mQ}\left(1+\frac{I}{2}\right)-\frac{I}{2}\right]+Dc_{Pr}\left[1+sI+\frac{I}{2}\left(1-\frac{mQ}{D}\right)\right]+\frac{h_vQ}{2}\left(1+\frac{I}{2}\right)\left[m\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right]. \quad (4.12)$$

4.3. Joint total expected cost per unit time

Once the buyer and vendor have built up a long-term strategic partnership, they can jointly determine the best policy for both parties. Accordingly, the joint total expected cost per unit time can be obtained as the sum of the buyer's and the vendor' total expected costs per unit time. That is,

$$\begin{aligned}
&\mathrm{MIN\ JEAC}(Q,A,r^1,r^2,s,t,m)\\
&=\frac{\theta}{\gamma}\ln\left(\frac{A_0}{A}\right)+A\left[\frac{D}{mQ}\left(1+\frac{I}{2}\right)-\frac{I}{2}\right]\\
&-\frac{h_b\left(1+\frac{I}{2}\right)}{m}\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(a-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(a-\alpha)k_1\right)\right]\right.\\
&\left.+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-ak_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-ak_1\right)\right]\right\}\\
&+\frac{(m-1)h_b\left(1+\frac{I}{2}\right)}{m}\sigma\sqrt{L}\left\{\alpha\left[r-1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+1-\alpha K_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+1-\alpha K_1\right)\right]\right.\\
&\left.+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}+\frac{h_bQ}{2}\left(1+\frac{I}{2}\right)
\end{aligned}$$

$$+\frac{D\pi}{mQ}\left[1+(s+pQ+t)I+\frac{I}{2}\left(1-\frac{Q}{D}\right)\right][\sigma\sqrt{t+S+pQ}\psi(r_1^1,r_2^1,\alpha)+(m-1)\sigma\sqrt{t}\psi(r_1^2,r_2^2,\alpha)]$$
$$+\frac{D}{Q}\left[ut+(a_i+b_iQ)(t_{i-1}-t)+\sum_{j=1}^{i-1}(a-j+b_jQ)(\mathrm{NT}_j-\mathrm{MT}_j)\right]\left[1+(s+pQ)I+\frac{I}{2}\left(1-\frac{Q}{D}\right)\right]$$
$$+\left[as+c_{si}(S_{i-1}-S)+\sum_{j=1}^{i-1}s_{si}(\mathrm{NS}_j-\mathrm{MS}_j)\right]\left[\frac{D}{mQ}\left(1+\frac{I}{2}\right)-\frac{I}{2}\right]+D\,c_{Pr}\left[1+sI+\frac{I}{2}\left(1-\frac{mQ}{D}\right)\right]$$
$$+Dc_{pu}\left[1+\frac{I}{2}\left(1-\frac{mQ}{D}\right)\right]+\frac{h_vQ}{2}\left(1+\frac{I}{2}\right)\left[m\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right].$$

Subject to

$$\sigma\sqrt{s+t+pQ}\left[r_1^1+k_1(1-\alpha)\right]-\sigma\sqrt{t}\left[r_1^2+k_1(1-\alpha)\right]=0$$
$$\gamma Q+\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]\right.$$
$$\left.+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}-\frac{F}{f}\le 0$$
$$\gamma Q+\sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)\right]\right.$$
$$\left.+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}-\frac{F}{f}\le 0$$
$$\text{Over } Q,r^1,r^2\ge 0, A\epsilon\,(\,0,A_0\,], t\epsilon\,[t_i,\,t_{i-1}], s\epsilon\,[s_i,s_{i-1}], m>0\,\text{integer.} \tag{4.13}$$

The above model (4.13) can be solved with the Lagrange multiplier method as given below:

$$\mathrm{JEAC}\left(Q,A,r^1,r^2,s,t,m,\lambda_1,\lambda_2,\lambda_3\right)$$
$$=\frac{\theta}{\delta}\ln\left(\frac{A_0}{A}\right)+A\left[\frac{D}{mQ}\left(1+\frac{I}{2}\right)-\frac{I}{2}\right]$$
$$+\frac{h_b\left(1+\frac{I}{2}\right)}{m}\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]\right.$$
$$\left.+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}$$
$$+\frac{(m-1)h_b\left(1+\frac{I}{2}\right)}{m}\sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)\right]\right.$$
$$\left.+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}+\frac{h_bQ}{2}\left(1+\frac{I}{2}\right)$$
$$+\frac{D\pi}{mQ}\left[1+(s+pQ+t)I+\frac{I}{2}\left(1-\frac{Q}{D}\right)\right]\left[\sigma\sqrt{t+s+pQ}\psi\left(r_1^1,r_2^1,\alpha\right)+(m-1)\sigma\sqrt{t}\psi\left(r_1^2,r_2^2,\alpha\right)\right]$$
$$+\frac{D}{Q}\left[ut+(a_i+b_iQ)(t_{i-1}-t)+\sum_{j=1}^{i-1}(a_j+b_jQ)(\mathrm{NT}_j-\mathrm{MT}_j)\right]\left[1+(s+pQ)I+\frac{I}{2}\left(1-\frac{Q}{D}\right)\right]$$
$$+\left[as+c_{si}(s_{i-1}-s)+\sum_{j=1}^{i}c_{sj}(\mathrm{NS}_j-\mathrm{MS}_j)\right]\left[\frac{D}{mQ}\left(1+\frac{I}{2}\right)-\frac{I}{2}\right]+Dc_{Pr}\left[1+sI+\frac{I}{2}\left(1-\frac{mQ}{D}\right)\right]$$

$$
\begin{aligned}
&+ Dc_{pu}\left[1+\frac{I}{2}\left(1-\frac{mQ}{D}\right)\right]+\frac{h_vQ}{2}\left(1+\frac{I}{2}\right)\left[m\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right] \\
&+ \lambda_1\left\{\sigma\sqrt{s+t+pQ}\left[r_1^1+k_1(1-\alpha)\right]-\sigma\sqrt{t}\left[r_1^2+k_1(1-\alpha)\right]\right\}+\lambda_2 Q\varphi_1 \\
&+ \lambda_2\sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]\right. \\
&\left.+ (1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}-\lambda_2\frac{F}{f} \\
&+ \lambda_3\sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)\right]\right. \\
&\left.+ (1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}+\lambda_3 Q\varphi_2-\lambda_3\frac{F}{f}
\end{aligned}
\tag{4.14}
$$

where λ_1 is free in sign and λ_2 and λ_3 are nonnegative variables. To solve the above nonlinear programming problem, this study temporarily ignores the constraint $0 \leq A \leq A_0$ and relaxes the integer requirement on m (the number of shipments from the vendor to the buyer during a cycle). It can be shown that for fixed $Q, A, r^1, r^2, s, t, m, \lambda_1, \lambda_2, \lambda_3$, the optimal setup and transportation time occur at the end of points of interval $s\epsilon[s_i, s_{i-1}]$ and $t\epsilon[t_i, t_{i-1}]$ respectively (see Pan and Yang [33], Chang *et al.* [6], Glock [11], Ben-Daya and Hariga [2]). This result simplifies considerably the search for the optimal solution to this inventory problem. Therefore, the Kuhn-Tucker necessary conditions for minimization of the function (4.14) are as follows:

$$
\begin{aligned}
\frac{\partial \mathrm{JEAC}}{\partial Q} =& -\frac{D\left(1+\frac{I}{2}\right)}{mQ^2}\left[A+as+c_{si}(s_{i-1}-s)+\sum_{j=1}^{i}c_{sj}(\mathrm{NS}_j-\mathrm{MS}_j)\right] \\
&-\frac{D\pi}{mQ^2}\left(1+sI+tI+\frac{I}{2}\right)\left[\sigma\sqrt{t+s+pQ}\psi\left(r_1^1,r_2^1,\alpha\right)+(m-1)\sigma\sqrt{t}\psi\left(r_1^2,r_2^2,\alpha\right)\right] \\
&-\frac{D}{Q^2}\left[ut+a_i(t_{i-1}-t)+\sum_{j=1}^{i-1}a_j(\mathrm{NT}_j-\mathrm{MT}_j)\right]\left(1+sI+\frac{I}{2}\right) \\
&+\left[b_i(t_{i-1}-t)+\sum_{j=1}^{i-1}b_j(\mathrm{NT}_j-\mathrm{MT}_j)\right]\left(DpI-\frac{I}{2}\right)+\frac{\left(\frac{h_b}{m}\left(1+\frac{I}{2}\right)+\lambda_2\right)p\sigma}{2\sqrt{s+t+pQ}} \\
&\times\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]\right. \\
&\left.+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\right\} \\
&+\frac{\left(\frac{h_b}{m}\left(1+\frac{I}{2}\right)+\lambda_2\right)\mu_* p}{2}\left[\alpha\left(r_1^1+\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right. \\
&\left.+(1-\alpha)\left(r_2^1+\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]+\frac{h_b}{2}\left(1+\frac{I}{2}\right) \\
&+\frac{Dp\pi\sigma\psi\left(r_1^1,r_2^1,p\right)\left(1+sI+tI+\frac{I}{2}\right)}{2mQ\sqrt{s+t+pQ}}+\frac{\lambda_1\sigma p\left[r_1^1+k_1(1-\alpha)\right]}{2\sqrt{s+t+pQ}}+\frac{Dp\pi\sigma\psi\left(r_1^1,r_2^1,p\right)\left(pI+\frac{I}{2D}\right)}{2m\sqrt{s+t+pQ}} \\
&-\frac{mI}{2}(c_{pr}+c_{pu})+\frac{h_v}{2}\left(1+\frac{I}{2}\right)\left[m\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right]+\lambda_2\varphi_1 \\
&+\lambda_3\varphi_2=0
\end{aligned}
\tag{4.15}
$$

$$\frac{\partial \mathrm{JEAC}}{\partial r^1} = \left[\frac{h_b\left(1+\frac{I}{2}\right)}{m} + \lambda_2\right] \sigma\sqrt{s+t+pQ}\left[\alpha\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} + (1-\alpha)\,k_1\right)\right.$$
$$\left. + (1-\alpha)\,\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} - \alpha k_1\right)\right] + \lambda_1\sigma\sqrt{t+s+pQ} + \frac{D\pi}{mQ}\left[1+(s+pQ+t)\,I + \frac{I}{2}\left(1-\frac{Q}{D}\right)\right]$$
$$\times\left[\sigma\sqrt{t+s+pQ}\left(\alpha\Phi\left(r_1^1\right) + (1-\alpha)\,\Phi\left(r_2^1\right) - 1\right)\right] = 0 \tag{4.16}$$

$$\frac{\partial \mathrm{JEAC}}{\partial r^2} = \frac{D\pi\,(m-1)}{mQ}\left[1+(s+pQ+t)\,I + \frac{I}{2}\left(1-\frac{Q}{D}\right)\right]\left[\sigma\sqrt{t}\left(\alpha\Phi\left(r_1^2\right) + (1-\alpha)\,\Phi\left(r_2^2\right) - 1\right)\right]$$
$$+\left(\frac{(m-1)\,h_b}{m} + \lambda_3\right)\sigma\sqrt{t}\left[\alpha\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma} + (1-\alpha)\,k_1\right) + (1-\alpha)\,\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma} - \alpha k_1\right)\right] - \lambda_1\sigma\sqrt{t} = 0 \tag{4.17}$$

$$\frac{\partial \mathrm{JEAC}\,(Q,A,k,\lambda_1,\lambda_2)}{\partial A} = -\frac{\theta}{\delta A} + \frac{DA\left(1+\frac{I}{2}\right)}{mQ} = 0 \tag{4.18}$$
$$\sigma\sqrt{s+t+pQ}\left[r_1^1 + k_1\,(1-\alpha)\right] - \sigma\sqrt{t}\left[r_1^2 + k_1\,(1-\alpha)\right] = 0 \tag{4.19}$$

$$\lambda_1\left\{\gamma Q + \sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} + (1-\alpha)\,k_1\right) - \phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} + (1-\alpha)\,k_1\right)\right]\right.\right.$$
$$\left.\left. +\ (1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} - \alpha k_1\right) - \phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} - \alpha k_1\right)\right]\right\} - \frac{F}{f}\right\} = 0 \tag{4.20}$$
$$\lambda_2\left\{\gamma Q + \sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma} + (1-\alpha)\,k_1\right) - \phi\left(\frac{\mu_*\sqrt{t}}{\sigma} + (1-\alpha)\,k_1\right)\right]\right.\right.$$
$$\left.\left. +\ (1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma} - \alpha k_1\right) - \phi\left(\frac{\mu_*\sqrt{t}}{\sigma} - \alpha k_1\right)\right]\right\} - \frac{F}{f}\right\} = 0. \tag{4.21}$$

Solving equations (4.15)–(4.18) respectively produce

$$Q = \sqrt{\frac{\frac{D}{m}\left\{\left(1+\frac{I}{2}\right)\left[A+as+\mathrm{CS}\,(s)\right] + \left(1+sI+tI+\frac{I}{2}\right)Y\left(r^1,r^2\right) + m\left[ut+U\,(t)\right]\left(1+sI+\frac{I}{2}\right)\right\}}{H_v + H_{b,\lambda_1,\lambda_2,\lambda_3} + \frac{Dp\pi\sigma\psi\left(r_1^1,r_2^1,p\right)}{2m\sqrt{s+t+pQ}}\left[\frac{\left(1+sI+tI+\frac{I}{2}\right)}{Q} + \left(pI+\frac{I}{2D}\right)\right] - \frac{mI}{2}\left(c_{pr}+c_{pu}\right) + \left(\lambda_2\varphi_1 + \lambda_3\varphi_2\right)}} \tag{4.22}$$

where

$$U\,(t) = a_i\,(t_{i-1}-t) + \sum_{j=1}^{i-1} a_j\left(\mathrm{NT}_j - \mathrm{MT}_j\right) \tag{4.23}$$
$$\mathrm{CS}\,(s) = c_{si}\,(s_{i-1}-s) + \sum_{j=1}^{i-1} c_{sj}\left(\mathrm{NS}_j - \mathrm{MS}_j\right) \tag{4.24}$$
$$Y\left(r^1,r^2\right) = \pi\left[\sigma\sqrt{t+s+pQ}\,\psi\left(r_1^1,r_2^1,\alpha\right) + (m-1)\,\sigma\sqrt{t}\,\psi\left(r_1^2,r_2^2,\alpha\right)\right] \tag{4.25}$$
$$H_v = \frac{h_v}{2}\left(1+\frac{I}{2}\right)\left[m\left(1-\frac{D}{P}\right) - 1 + \frac{2D}{P}\right] \tag{4.26}$$
$$H_{b,\lambda_1,\lambda_2} = \frac{\left(\frac{h_b}{m}\left(1+\frac{I}{2}\right)+\lambda_2\right)p\sigma}{2\sqrt{s+t+pQ}}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} + (1-\alpha)\,k_1\right) - \phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma} + (1-\alpha)\,k_1\right)\right]\right.$$

$$+ (1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]\Bigg\}$$
$$+\frac{\left(h_b\left(1+\frac{I}{2}\right)+\lambda_2\right)\mu_* p}{2}\left[\alpha\left(r_1^1+\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)\right.$$
$$\left.+(1-\alpha)\left(r_2^1+\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]+\frac{h_b}{2}\left(1+\frac{I}{2}\right) \tag{4.27}$$

and

$$\left[\frac{h_b\left(1+\frac{I}{2}\right)}{m}+\lambda_1+\lambda_2\right]\sigma\sqrt{s+t+pQ}\left[\alpha\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)k_1\right)+(1-\alpha)\right.$$
$$\left.\times\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\right]+\lambda_1\sigma\sqrt{t+s+pQ}=\frac{D\pi}{mQ}\left[1+(s+pQ+t)I+\frac{I}{2}\left(1-\frac{Q}{D}\right)\right]\left(1-F_*\left(r^1\right)\right) \tag{4.28}$$

$$\left(\frac{(m-1)h_b}{m}+\lambda_3\right)\sigma\sqrt{t}\left[\alpha\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)+(1-\alpha)\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]-\lambda_1\sigma\sqrt{t}$$
$$=\frac{D\pi(m-1)}{mQ}\left[1+(s+pQ+t)I+\frac{I}{2}\left(1-\frac{Q}{D}\right)\right]\left(1-F_*\left(r^2\right)\right) \tag{4.29}$$

$$A=\frac{\theta mQ}{D\delta\left(1+\frac{I}{2}\right)}\cdot \tag{4.30}$$

Where $F_*(r^1)=\alpha\Phi(r_1^1)+(1-\alpha)\Phi(r_2^1)$ and $F_*(r^2)=\alpha\Phi(r_1^2)+(1-\alpha)\Phi(r_2^2)$. On the other hand, for fixed s, t and m, it can be shown that $\text{JEAC}(Q, A, r^1, r^2, s, t, m, \lambda_1, \lambda_2, \lambda_3)$ is convex in (A, r^1, r^2) since the objective function, $\text{JEAC}(Q, A, r^1, r^2, s, t, m)$, is convex in (A, r^1, r^2) by examining second order sufficient condition and also the constraints are linear in (A, r^1, r^2); however, may not be convex in (Q, A, r^1, r^2). Therefore, the following algorithm can be used to find an approximate solution to the above problem.

Algorithm

Step 1. Set $m = 1$.
Step 2. Compute the intersection points Q^s of the crash cost lines $c_i = a_i + b_iQ$ and $c_j = a_j + b_jQ$, for all i and j, where $a_i > a_j$, $b_i < b_j$, $i \neq j$ and $i, j = 1, 2, \ldots, n^t$. Arrange these intersection points such that $Q_1^s < Q_2^s < \ldots < Q_w^s$ and let $Q_0^s = 0$, $Q_{w+1}^s = \infty$.
Step 3. Rearrange c_i such that $c_1 \le c_2 \le \ldots \le c_{n^t}, j = 1, 2, \ldots, w$, for the order quantity range (Q_{j-1}^s, Q_j^s).
Step 4. For each t_i and s_z, $i = 0, 1, \ldots, n^t$, $z = 0, 1, \ldots, n^s$, perform Step 4-1 to Step 4-10.
Step 4-1. Set $\lambda_2 = 0$ and $\lambda_3 = 0$ and solve the problem without space constraint.
Step 4-2. Compute $Q_{iz}^1 = \sqrt{\frac{D}{m}\{\left(1+\frac{I}{2}\right)[A+as+\text{CS}(s)]+m[ut+U(t)]\left(1+sI+\frac{I}{2}\right)\}/H_v}$.
Step 4-3. Find A_{iz}^1 from equation (4.30).
Step 4-4. Find $r_{iz}^{1^1}$, $r_{iz}^{2^1}$ in terms of λ_1 from equations (4.28) and (4.29).
Setp 4-5. Setting the values $Q_{iz}^1, r_{iz}^{1^1}$ and $r_{iz}^{2^1}$ in equation (4.19) and find $\lambda_{1_{iz}}^1$.
Step 4-6. Compute Q_{iz}^2 from (4.22) using A_{iz}^1, $r_{iz}^{1^1}$, $r_{iz}^{2^1}$ and $\lambda_{1_{iz}}^1$.
Step 4-7. Repeat Step 4-2 to Step 4-6 until no changes occur in the values of Q_{iz}, A_{iz}, r_{iz}^1 and r_{iz}^2.
Step 4-8. Check whether $A_{iz} < A_0$ and $Q_i \in [Q_{j-1}^s, Q_j^s]$:
Step 4-8-1. If $A_{iz} < A_0$ and $Q_{iz} \in [Q_{j-1}^s, Q_j^s]$, then the solution found in Step 4-2 to Step 4-7 is optimal for given t_i and s_z go to Step 4.5.
Step 4-8-2. If $A_{iz} \ge A_0$, for given t_i and s_z, set $A_{iz} = A_0$ and obtain Q_{iz}, r_{iz}^1, r_{iz}^2, $\lambda_{1_{iz}}$ by solving equations (4.22), (4.28), (4.29) and (4.19) iteratively until convergence.

Step 4-8-3. If $Q_{iz} \leq Q^s_{j-1}$, let $Q_{iz} = Q^s_{j-1}$ and if $Q^s_j \leq Q_{iz}$ let $Q^s_j = Q_{iz}$. Using Q_{iz} as a constant, obtain A_{iz}, r^1_{iz}, r^2_{iz} and $\lambda_{1_{iz}}$ by solving equations (4.28)–(4.30) and (4.19) iteratively until convergence.
Step 4-9. If the solution for Q_{iz}, A_{iz}, r^1_{iz}, r^2_{iz} and $\lambda_{1_{iz}}$ satisfies the space constraint from a model (4.13), then go to step 4.5 otherwise go to step (4-10).
Step 4-10. If the solution for Q_{iz}, A_{iz}, r^1_{iz}, r^2_{iz} and $\lambda_{1_{iz}}$ don't satisfy the space constraint, determine the new Q_{iz}, A_{iz}, r^1_{iz}, r^2_{iz}, $\lambda_{1_{iz}}$, $\lambda_{2_{iz}}$ and $\lambda_{3_{iz}}$ by a procedure similar to given In Step 4 then go to Step 5.
Step 5. Find $\min \text{JTEC}(Q_{iz}, A_{iz}, r^1_{iz}, r^2_{iz}, t_i, s_z) = \text{JTEC}(Q^m, A^m, r^{1^m}, r^{2^m}, t^m, s^m)$ for $i = 0, 1, \ldots, n^t, z = 0, 1, \ldots, n^s$.
Step 6. Set $m = m + 1$, and repeat Steps 2 to 5 to get $\text{JTEC}(Q^m, A^m, r^{1^m}, r^{2^m}, t^m, s^m)$.
Step 7. If $\text{JTEC}(Q^m, A^m, r^{1^m}, r^{2^m}, t^m, s^m, m) \leq \text{JTEC}(Q^{m-1}, A^{m-1}, r^{1^{m-1}}, r^{2^{m-1}}, t^{m-1}, s^{m-1}, m-1)$, then go to step 6, otherwise go to step 8.
Step 8. Set $(Q^*, A^*, r^{1^*}, r^{2^*}, t^*, s^*, m^*) = (Q^m, A^m, r^{1^m}, r^{2^m}, t^m, s^m, m)$, then $(Q^*, A^*, k_1^*, t^*, s^*, m^*)$ is the optimal solution and $\text{JTEC}(Q^*, A^*, r^{1^*}, r^{2^*}, t^*, s^*, m^*)$ is the minimum joint expected annual cost.

5. Numerical example

To illustrate the behavior of the model developed in this paper, let us consider an inventory problem with the following data: $D = 624$ units per year, $h_b = 10\$$ per unit per year, $h_v = 3\$$ per unit per year, $A_0 = 50\$$ per order, $a = 1000\$$ per week, $u = 7\$$ per week, $p = 1/125$ week per unit, $\sigma = 15$ units per week, $\pi = 70\$$ per unit per year, $f = 3\,\text{M}^2$ per unit, $F = 400\,\text{M}^2$, $\phi_1 = 0.99$, $\phi_2 = 0.99$, $\theta = 0.1$ and $\delta = 1/700$. Moreover, we consider 1 yr = 48 weeks. The lead time has three components with data shown in Table 1. The Table 2's data are first used to evaluate the intersection points, order quantity rage interval and component crash priorities in each interval. Table 2 shows the crash sequence corresponding to each order quantity range. Setup times and their respective crashing costs are tabulated in Table 3. To show the performance of the proposed inventory system under the inflationary condition, we first assume the model without space constraint and solve the case when $\alpha = 0.0, 0.3, 0.8,$ 1.0 and $I = 0.00, 0.01, 0.02$ and $k_1 = 0.7$. Applying the proposed algorithm yields the optimal solutions as tabulated in Tables 4–7. Results of optimal decisions show that for a fixed value of s, t and m, with an augment in the expected inflation rate, the optimal ordering quantity increases. Also, results reveal that with an increase in inflation rate the optimal number of shipment increases consequently. We also observe that when $\alpha = 0$ or 1, the model considers only one kind of customers' demand and when $0 \leq \alpha \leq 1$, the model considers two kinds of customers' demand. It implies that the minimum joint expected annual cost with two kinds of customers' demand is larger than the minimum expected annual cost with of one kind of customers' demand. Thus, the minimum joint expected annual cost with inflation increases as the distance between α and 0 (or 1) increased for the fixed inflation rate. Also, the summary of results for the model without space constraint are listed in Table 8.

Then, we consider space constrained model and solve the case where, $\alpha = 0.0, 0.3, 0.8,$ 1.0 and $I = 0.00, 0.01, 0.02$ and $k_1 = 0.7$. Utilizing the presented algorithm, optimal decisions are obtained which are tabulated in Tables 9–12. Similar to the unconstrained model, for a fixed value of m, t and s, with an augment in

Table 1. Transportation time data.

Transportation time component i	1	2	3
Normal duration NT_i (days)	20	20	16
Minimum duration MT_i (days)	6	6	9
Unit fixed crash cost a_i (\$/day)	0.5	1.3	5.1
Unit variable crash cost b_i (\$/unit/day)	0.012	0.004	0.0012

TABLE 2. The values of Q^s, order quantity ranges and crash sequence.

Inspection points (Q^s)	Order quantity range	Crash sequence of components
100	(0, 100]	1, 2, 3
426	(100, 426]	2, 1, 3
1357	(426, 1357]	2, 3, 1
–	(1357, ∞)	3, 2, 1

TABLE 3. Setup time data.

Setup time component i	1	2	3
Normal duration NS_i (days)	0.14	0.14	0.07
Minimum duration MT_i (days)	0.105	0.105	0.049
Unit fixed crash cost c_{si} ($/day)	2000	3000	5000

α, the two optimal reorder points for different batches are increased. Also, optimum joint expected annual cost with inflation for two kind of customers' demand is larger than one kind of customers' demand. The summary of results for the model under space constraint are listed in Table 13.

5.1. Sensitivity analysis

Change in the value of system parameters can take place due to uncertainties and dynamic market conditions in any DM situation. Therefore, the sensitivity analysis will be of great help to study these changes in the value

TABLE 4. Results of solution procedure for the proposed model without space constraint for $\alpha = 0$.

m	Q	A	r^1	r^2	s(weeks)	t(weeks)	λ_1	JEACWI
$I = 0.0$								
1	122	13.63	148	123	0.05	4	0	64 754.08
2	102	22.83	140	118	0.05	4	1.89	64 626.06*
3	90	30.42	137	118	0.05	4	1.74	64 631.69
$I = 0.01$								
1	127	14.14	150	123	0.05	4	0	65 044.67
2	111	24.63	142	118	0.05	4	2.01	64 870.96
3	101	33.80	139	117	0.05	4	1.89	64 838.68*
4	94	41.95	137	117	0.05	4	1.61	64 845.42
$I = 0.02$								
1	133	14.72	152	124	0.05	4	0	65 232.31
2	122	26.95	144	118	0.05	4	2.14	65 106.15
3	117	38.64	142	117	0.05	4	2.06	65 026.80
4	113	50.00	140	116	0.05	4	1.80	64 990.58
5	109	50.00	139	116	0.05	4	1.55	64 974.32
6	105	50.00	138	115	0.05	4	1.35	64 969.36*
7	102	50.00	137	115	0.05	4	1.19	64 790.83

Notes. (*) shows the optimal decision.

Table 5. Results of solution procedure for the proposed model without space constraint for $\alpha = 0.3$.

m	Q	A	r^1	r^2	s(weeks)	t(weeks)	λ_1	JEACWI
$I = 0.0$								
1	121	13.60	152	127	0.05	4	0	64 794.67
2	102	22.85	144	102	0.05	4	1.87	64 661.37*
3	91	30.48	141	121	0.05	4	1.72	64 665.52
$I = 0.01$								
1	126	14.10	154	121	0.05	4	0	65 086.06
2	111	24.64	145	127	0.05	4	1.98	64 906.75
3	101	33.87	143	122	0.05	4	1.86	64 872.74*
4	94	42.05	141	121	0.05	4	1.59	64 878.75
$I = 0.02$								
1	132	14.67	156	128	0.05	4	0	65 374.50
2	122	26.96	148	122	0.05	4	2.11	65 142.43
3	117	38.71	145	120	0.05	4	2.04	65 061.08
4	113	50.00	143	119	0.05	4	1.78	65 023.87
5	109	50.00	142	119	0.05	4	1.54	65 007.10
6	105	50.00	142	119	0.05	4	1.34	65 001.83*
7	102	50.00	141	119	0.05	4	1.18	65 003.12

Notes. (*) shows the optimal decision.

Table 6. Results of solution procedure for the proposed model without space constraint for $\alpha = 0.8$.

m	Q	A	r^1	r^2	s(weeks)	t(weeks)	λ_1	JEACWI
$I = 0.0$								
1	121	13.58	151	125	0.05	4	0	64 770.67
2	101	22.79	142	120	0.05	4	1.90	64 639.48*
3	90	30.40	139	120	0.05	4	1.75	64 644.05
$I = 0.01$								
1	126	14.08	153	126	0.05	4	0	65 061.65
2	110	24.58	144	121	0.05	4	2.01	64 884.69
3	101	33.77	141	119	0.05	4	1.89	64 851.28*
4	94	41.92	139	119	0.05	4	1.62	64 857.46
$I = 0.02$								
1	132	14.66	154	126	0.05	4	0	65 349.70
2	122	26.89	147	121	0.05	4	2.14	65 120.25
3	116	38.59	144	119	0.05	4	2.07	65 039.72
4	113	50.00	142	118	0.05	4	1.81	65 002.88
5	108	50.00	141	118	0.05	4	1.56	64 986.24
6	105	50.00	140	117	0.05	4	1.36	64 981.03*
7	102	50.00	139	117	0.05	4	1.20	64 982.34

Notes. (*) shows the optimal decision.

TABLE 7. Results of solution procedure for the proposed model without space constraint for $\alpha = 1.0$.

m	Q	A	r^1	r^2	s(weeks)	t(weeks)	λ_1	JEACWI
$I = 0.0$								
1	122	13.63	148	123	0.05	4	0	64 754.08
2	102	22.83	140	118	0.05	4	1.89	64 626.06*
3	90	30.42	137	118	0.05	4	1.74	64 631.69
$I = 0.01$								
1	127	14.14	150	123	0.05	4	0	65 044.67
2	111	24.63	142	118	0.05	4	2.01	64 870.96
3	101	33.80	139	117	0.05	4	1.89	64 838.68*
4	94	41.95	137	117	0.05	4	1.61	64 845.42
$I = 0.02$								
1	133	14.72	152	124	0.05	4	0	65 232.31
2	122	26.95	144	118	0.05	4	2.14	65 106.15
3	117	38.64	142	117	0.05	4	2.06	65 026.80
4	113	50.00	140	116	0.05	4	1.80	64 990.58
5	109	50.00	139	116	0.05	4	1.55	64 974.32
6	105	50.00	138	115	0.05	4	1.35	64 969.36*
7	102	50.00	137	115	0.05	4	1.19	64 790.83

Notes. (*) shows the optimal decision.

TABLE 8. Summary of results for the model without space constraint.

I	Q	A	r^1	r^2	m	s(weeks)	t(weeks)	$E(X^1 - r^1)^+$	$E(X^2 - r^2)^+$	JEACWI
$\alpha = 0.0$										
0.00	102	22.83	140	118	2	0.05	4	0.4420	0.1416	64 626.06
0.01	101	33.80	139	117	3	0.05	4	0.4770	0.1577	64 838.68
0.02	105	50.00	138	115	6	0.05	4	0.5567	0.1863	64 969.36
$\alpha = 0.3$										
0.00	102	22.85	144	122	2	0.05	4	0.4654	0.1493	64 661.37
0.01	101	33.87	143	121	3	0.05	4	0.5031	0.1662	64 872.74
0.02	105	50.00	141	119	6	0.05	4	0.5880	0.1966	65 001.83
$\alpha = 0.8$										
0.00	102	22.79	142	120	2	0.05	4	0.4478	0.1415	64 639.48
0.01	101	33.77	141	119	3	0.05	4	0.4844	0.1579	64 851.28
0.02	105	50.00	140	117	6	0.05	4	0.5671	0.1872	64 981.03
$\alpha = 1.0$										
0.00	102	22.83	140	118	2	0.05	4	0.4420	0.1416	64 626.06
0.01	101	33.80	139	117	3	0.05	4	0.4770	0.1577	64 838.68
0.02	105	50.00	138	115	61	0.05	4	0.5567	0.1863	64 969.36

TABLE 9. Results of solution procedure for the proposed model under space constraint $\alpha = 0.0$.

m	Q	A	r^1	r^2	s(weeks)	t(weeks)	$\lambda_1, \lambda_2, \lambda_3$	JEACWI
$I = 0.0$								
1	74	8.32	131	115	0.05	4	0, 9.64, 0	64 955.22
2	75	16.78	131	115	0.05	4	−0.56, 5.77, 0	64 701.50
3	74	25.05	131	115	0.05	4	0.52, 3.82, 0	64 663.75*
4	73	32.97	132	116	0.05	4	0.13, 2.34, 0	64 682.66
$I = 0.01$								
1	74	8.24	132	116	0.05	4	0, 10.27, 0	65 275.91
2	75	16.71	131	114	0.05	4	−0.91, 6.78, 0	64 980.25
3	75	25.09	131	114	0.05	4	−1.27, 5.24, 0	64 903.87
4	75	33.31	131	115	0.05	4	−1.27, 4.74, 0	64 885.76*
5	74	41.37	131	115	0.05	4	−1.07, 3.28, 0	64 892.01
$I = 0.02$								
1	74	8.15	132	116	0.05	4	0, 10.90, 0	65 596.44
2	75.	16.64	131	114	0.05	4	−1.26, 7.80, 0	65 258.98
3	76	25.13	130	114	0.05	4	−2.03, 6.69, 0	65 143.46
4	76	35.57	130	113	0.05	4	−2.45, 6.06, 0	65 087.22
5	76	41.97	130	113	0.05	4	−2.68, 5.60, 0	65 056.01
6	76	50.00	130	113	0.05	4	−2.78, 5.24, 0	65 037.95
7	76	50.00	130	114	0.05	4	−2.73, 4.81, 0	65 028.56
8	76	50.00	130	114	0.05	4	−2.65, 4.45, 0	65 025.28*
9	76	50.00	130	114	0.05	4	−2.55, 4.12, 0	65 026.07

Notes. (*) shows the optimal decision.

TABLE 10. Results of solution procedure for the proposed model under space constraint $\alpha = 0.3$.

m	Q	A	r^1	r^2	s(weeks)	t(weeks)	$\lambda_1, \lambda_2, \lambda_3$	JEACWI
$I = 0.0$								
1	72	8.07	134	118	0.05	4	0, 10.66, 0	65 025.37
2	72	16.27	134	118	0.05	4	−0.95, 6.65, 0	64 755.43
3	72	24.25	134	118	0.05	4	−1.04, 4.66, 0	64 710.49*
4	71	31.95	135	119	0.05	4	−0.70, 3.18, 0	64724.44
$I = 0.01$								
1	72	7.98	134	119	0.05	4	0, 11.32, 0	65 348.81
2	73	16.20	133	118	0.05	4	−1.30, 7.68, 0	65 037.73
3	73	24.31	133	118	0.05	4	−1.78, 6.09, 0	64 955.17
4	72	32.28	134	118	0.05	4	−1.84, −5.00, 0	64 933.16*
5	72	40.08	134	118	0.05	4	−1.69, −4.11, 0	64 936.42
$I = 0.02$								
1	71	7.09	135	119	0.05	4	0, 11.98, 0	65 672.05
2	73	16.31	133	118	0.05	4	−1.66, 8.71, 0	65 320.00
3	74	24.35	133	117	0.05	4	−2.54, 7.54, 0	65 199.36
4	74	32.54	133	117	0.05	4	−3.02, 6.88, 0	65 140.33
5	74	40.67	133	117	0.05	4	−3.28, 6.24, 0	65 107.26
6	74	48.75	133	117	0.05	4	−3.41, 6.05, 0	65 087.81
7	74	50.00	133	117	0.05	4	−3.39, 5.63, 0	65 077.00
8	73	50.00	133	117	0.05	4	−3.32, 5.26, 0	65 072.51
9	73	50.00	133	117	0.05	4	−3.23, 4.93, 0	65 072.25*
10	73	50.00	133	117	0.05	4	−3.22, 4.63	65 074.93

Notes. (*) shows the optimal decision.

TABLE 11. Results of solution procedure for the proposed model under space constraint $\alpha = 0.8$.

m	Q	A	r^1	r^2	s(weeks)	t(weeks)	$\lambda_1, \lambda_2, \lambda_3$	JEACWI
$I = 0.0$								
1	73	8.20	133	117	0.05	4	0, 10.03, 0	64 982.63
2	74	16.55	133	117	0.05	4	−0.68, 6.10, 0	64 721.69
3	73	24.68	133	117	0.05	4	−0.69, 4.13, 0	64 680.73*
4	72	32.53	134	118	0.05	4	−0.32, 2.65, 0	64 697.46
$I = 0.01$								
1	73	8.12	133	118	0.05	4	0, 10.67, 0	65 304.32
2	74	16.48	132	116	0.05	4	−1.02, 7.12, 0	65 001.83
3	74	24.74	132	116	0.05	4	−1.43, 5.56, 0	64 922.68
4	74	32.86	133	117	0.05	4	−1.45, 4.48, 0	64 902.83*
5	73	40.82	133	117	0.05	4	−1.27, 3.59, 0	64 907.76
$I = 0.02$								
1	72	8.03	134	118	0.05	4	0, 11.31, 0	65 625.83
2	74	16.41	132	116	0.05	4	−1.37, 8.13, 0	65 281.95
3	75	24.78	132	116	0.05	4	−2.18, 7.00, 0	65 164.13
4	75	33.12	132	115	0.05	4	−2.62, 6.36, 0	65 106.62
5	75	41.40	132	115	0.05	4	−2.87, 5.90, 0	65 074.58
6	75	49.63	132	115	0.05	4	−2.99, 5.54, 0	65 055.89
7	75	50.00	132	115	0.05	4	−2.95, 5.11, 0	65 045.86
8	75	50.00	132	115	0.05	4	−2.87, 4.74, 0	65 042.05*
9	74	50.00	132	116	0.05	4	−2.77, 4.42, 0	65 043.38

Notes. (*) shows the optimal decision.

TABLE 12. Results of solution procedure for the proposed model under space constraint $\alpha = 1.0$.

m	Q	A	r^1	r^2	s(weeks)	t(weeks)	λ_1	JEACWI
$I = 0.0$								
1	74	8.32	131	115	0.05	4	0, 9.64, 0	64 955.22
2	75	16.78	131	115	0.05	4	−0.56, 5.77, 0	64 701.50
3	74	25.05	131	115	0.05	4	0.52, 3.82, 0	64 663.75*
4	73	32.97	132	116	0.05	4	0.13, 2.34, 0	64 682.66
$I = 0.01$								
1	74	8.24	132	116	0.05	4	0, 10.27, 0	65 275.91
2	75	16.71	131	114	0.05	4	−0.91, 6.78, 0	64 980.25
3	75	25.09	131	114	0.05	4	−1.27, 5.24, 0	64 903.87
4	75	33.31	131	115	0.05	4	−1.27, 4.74, 0	64 885.76*
5	74	41.37	131	115	0.05	4	−1.07, 3.28, 0	64 892.01
$I = 0.02$								
1	74	8.15	132	116	0.05	4	0, 10.90, 0	65 596.44
2	75.	16.64	131	114	0.05	4	−1.26, 7.80, 0	65 258.98
3	76	25.13	130	114	0.05	4	−2.03, 6.69, 0	65 143.46
4	76	35.57	130	113	0.05	4	−2.45, 6.06, 0	65 087.22
5	76	41.97	130	113	0.05	4	−2.68, 5.60, 0	65 056.01
6	76	50.00	130	113	0.05	4	−2.78, 5.24, 0	65 037.95
7	76	50.00	130	114	0.05	4	−2.73, 4.81, 0	65 028.56
8	76	50.00	130	114	0.05	4	−2.65, 4.45, 0	65 025.28*
9	76	50.00	130	114	0.05	4	−2.55, 4.12, 0	65 026.07

Notes. (*) shows the optimal decision.

TABLE 13. Summary of results for the model under space constraint.

I	Q	A	r^1	r^2	m	s(weeks)	t(weeks)	$E(X^1-r^1)^+$	$E(X^2-r^2)^+$	JEACWI
$\alpha=0.0$										
0.00	74	25.05	131	115	3	0.05	4	0.4441	0.1916	64 663.75
0.01	75	33.31	131	115	4	0.05	4	0.4616	0.2000	64 885.76
0.02	76	50.00	130	114	8	0.05	4	0.4977	0.2176	65 025.28
$\alpha=0.4$										
0.00	72	24.25	134	118	3	0.05	4	0.4747	0.2108	64 710.49
0.01	72	32.28	134	118	4	0.05	4	0.4908	0.2187	64 933.16
0.02	73	50.00	133	117	9	0.05	4	0.5183	0.2325	65 071.25
$\alpha=0.8$										
0.00	73	24.68	133	118	3	0.05	4	0.4528	0.1955	64 680.73
0.01	74	32.86	133	117	4	0.05	4	0.4700	0.2038	64 902.83
0.02	75	50.00	132	116	8	0.05	4	0.5064	0.2215	65 042.05
$\alpha=1.0$										
0.00	74	25.05	131	115	3	0.05	4	0.4441	0.1916	64 663.75
0.01	75	25.05	131	115	4	0.05	4	0.4616	0.2000	64 885.76
0.02	76	50.00	130	114	8	0.05	4	0.4977	0.2176	65 025.28

of parameters. Following the previous example and considering a constrained model with $I=0.01$ and $\alpha=0.8$, the sensitivity analysis with respect to various system parameters has been done. The results of the sensitivity analysis are listed in Table 14. Keeping all parameters fixed, with an increase in maximum allowable space, F, the optimal ordering quantity, Q, is increased which indicates that if allowable space increases, ordering quantity should be increased to diminish total expected annual cost, JEAC. Also, with an augment in buyer's shortage cost, π, the optimal reorder point for the first and other batches, r^1, r^2, will be increased consequently. From an economic viewpoint, this implies that once shortage cost in the system increases, reorder point for all of the batches should be increased to avoid large shortage in the system which results in higher joint expected annual cost. On the other hand, an augment in buyer's demand standard deviation, σ, results in increasing reorder point for all of the batches, r^1, r^2, and also reducing transportation time, t. A simple economic interpretation says if the variation of demand increases, then reorder point for all of the batches should be increased and transportation time should be decreased simultaneously to reduce inventory costs. In addition, with an increase in vendor's holding cost and keeping the remaining parameters unchanged, the number of shipment, m, decreases. It shows that the vendor should reduce his stored items when his holding cost is increased to reduce the total expected annual cost. As can be seen in Table 14, the optimal setup time for the system is always 0.35 days or 0.05 weeks. It means that the crashing cost of setup time is so high that it doesn't allow the system to crash setup time. Table 15 displays expected annual cost of the system for $I=0.01$ and $\alpha=0.8$ with their respective crashing cost wherein optimal joint expected annual cost is obtained 64 902.83$ with $s^*=0.05$. The crashing costs for the setup components must be highly reduced to change optimal setup time. For instance, if c_{s1}, setup crashing cost of first components, is reduced from 2000 $ to 150$, the total expected annual cost is reduced to 64 901$ which gives optimal setup time as 0.045 weeks.

6. Conclusion

The purpose of this paper was to propose a multilevel reorder inventory-production model in which buyer's LTD followed the mixture of distributions. Lead time components and ordering cost were considered to be controllable. In practice, setup and transportation time and their crashing costs are different from each. Therefore, the paper assumed that transportation and setup time and their crashing costs acted independently. Also, in order to fit some real environment, transportation time crashing cost was presented as a function of reduced

TABLE 14. Effect of changes in various parameters.

	Q	A	r^1	r^2	m	s(weeks)	t(weeks)	JEACWI
F								
+50%	101.10	33.77	141	119	3	0.05	4	64 851.28
+30%	101.10	33.77	141	119	3	0.05	4	64 851.28
+10%	85.48	38.06	137	118	4	0.05	4	64 864.36
−10%	62.16	34.60	128	115	5	0.05	4	64 984.29
−30%	41.51	27.74	118	108	6	0.05	4	65 430.44
−50%	25.56	34.16	104	98	10	0.05	4	67 106.84
π								
+50%	68.87	38.33	137	122	5	0.05	4	64 972.39
+30%	70.76	31.52	135	120	4	0.05	4	64 947.82
+10%	72.67	32.37	133	117	4	0.05	4	64 919.12
−10%	75.04	33.42	131	115	4	0.05	4	64 885.10
−30%	78.13	34.80	129	112	4	0.05	4	64 843.51
−50%	82.54	36.75	125	107	4	0.05	4	64 789.14
σ								
+50%	71.76	39.93	137	122	5	0.05	3	65 236.19
+30%	85.01	33.73	127	111	4	0.05	3	65 131.59
+10%	70.19	31.27	138	122	4	0.05	4	64 979.62
−10%	70.19	34.72	127	111	4	0.05	4	64 825.71
−30%	88.47	35.75	125	108	4	0.05	4	64 787.08
−50%	95.10		138	117	3	0.05	6	64 506.98
A_0								
+50%	74	25.00	133	117	4	0.05	4	64 905.21
+30%	74	32.86	133	117	4	0.05	4	64 902.83
+10%	74	32.86	133	117	4	0.05	4	64 902.83
−10%	74	32.86	133	117	4	0.05	4	64 902.83
−30%	74	32.86	133	117	4	0.05	4	64 902.83
−50%	74	32.86	133	117	4	0.05	4	64 902.83
h_b								
+50%	75.96	33.83	131	114	4	0.05	4	65 380.77
+30%	75.11	33.45	131	115	4	0.05	4	65 190.24
+10%	74.23	33.06	132	116	4	0.05	4	64 998.85
−10%	73.33	32.66	133	117	4	0.05	4	64 806.56
−30%	72.40	32.25	134	118	4	0.05	4	64 613.43
−50%	71.45	31.83	134	119	4	0.05	4	64 419.34
h_v								
+50%	72.85	24.35	133	118	3	0.05	4	65 021.89
+30%	72.70	32.38	133	118	4	0.05	4	64 988.43
+10%	73.42	32.70	133	117	4	0.05	4	64 931.51
−10%	73.83	41.08	132	116	5	0.05	4	64 870.31
−30%	74.31	50.00	132	116	7	0.05	4	64 786.81
−50%	75.27	50.00	131	115	12	0.05	4	64 649.54

TABLE 15. Setup times with their expected annual costs.

s_i (weeks)	Q	A	r^1	r^2	m	t(weeks)	JEACWI
0.05	74	32.86	133	117	4	4	64 902.83*
0.045	175	33.58	131	115	4	4	65 037.55
0.04	78	34.52	129	113	4	4	65 240.44
0.037	79	35.35	127	111	4	4	65 442.74

Notes. (*) shows the optimal decision.

transportation time and the quantities in the orders. Also, in this paper, it was assumed that buyer's maximum permissible storage space was limited and therefore a random space constraint was added to the respective inventory system. In addition, inflationary condition for the proposed stochastic demand and deterministic variable lead time was considered, helping the model to be more appropriate for the real environment. The paper intended to minimize joint inventory expected cost by simultaneously optimizing ordering quantity, reorder points of different batches, ordering cost, setup time, transportation time, production time and a number of deliveries under space constraint while the lead time demand followed a normal distribution. A Lagrangian method was utilized in order to solve the model and a solution procedure was proposed to find optimal values. Then, the behavior of the model was illustrated in numerical examples. In the numerical experiment, two cases were considered. First, the inflationary inventory system without space constraint was considered. The results showed that with an augment in the expected inflation rate, the optimal ordering quantity was increased. Also, results revealed that with an increase in inflation rate, the optimal number of shipment was increased consequently. We also observed that the minimum joint expected annual cost with two kinds of customers' demand was larger than the minimum expected annual cost with of one kind of customers' demand. In the second case, the inflationary inventory system with space constraint was considered. Similar to the unconstrained model, for a fixed value of transportation time, setup time and number of batches, with an augment in inflation rate, the two optimal reorder points for different batches were increased. Also, optimum joint expected annual cost with inflation for two kind of customers' demand was larger than one kind of customers' demand. To increase the scope of our analysis, the model presented in this article could be extended in several ways. For example, shortage cost can be calculated as a mixture of backorder and lost sales. Thus, with an increasing or a decreasing in a backorder rate, the optimal order quantity and reorder level may be higher or lower. Another important aspect that has not been addressed in this paper is that, relaxing the assumption of the form of a mixture of normal distributions function and assuming buyer's lead time demand follows a mixture of free distribution and then applies minimax distribution-free procedure to find the most unfavorable expected cost. In this way, comparing the expected cost with a mixture of normal and free distributions for buyer's LTD and obtaining expected additional cost when using minimax distribution-free procedure. Also, investigating some other LTD approach such as gamma and lognormal distribution could be considered. Other kinds of constraints such as budget constraint could be added in order to make the system more closely to the real environment.

Appendix A.

The expected shortage, $E(X-r)^+$, is computed as follows.

$$
\begin{aligned}
E(X-r)^+ = & \int_r^{\infty}(x-r)\left\{\alpha\frac{1}{\sqrt{2\pi}\sigma\sqrt{L}}\times\exp\left[-\frac{1}{2}\left(\frac{x-\mu_1 L}{\sigma\sqrt{L}}\right)^2\right]+(1-\alpha)\frac{1}{\sqrt{2\pi}\sigma\sqrt{L}}\right.\\
& \left.\times\exp\left[-\frac{1}{2}\left(\frac{x-\mu_2 L}{\sigma\sqrt{L}}\right)^2\right]\right\}\mathrm{d}x\\
= & \left[\alpha\sigma\sqrt{L}\int_{r_1}^{\infty}(z-r_1)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)\mathrm{d}z\right]\\
& +\left[(1-\alpha)\sigma\sqrt{L}\int_{r_2}^{\infty}(z-r_2)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)\mathrm{d}z\right]\\
= & \,\sigma\sqrt{L}\left\{\alpha\left[\phi(r_1)-r_1\overline{\Phi(r_1)}\right]+(1-\alpha)\left[\phi(r_2)-r_2\overline{\Phi(r_2)}\right]\right\}.
\end{aligned}
\tag{A.1}
$$

Hence, for bi-level reorder point system, the expected shortage for the first batch is given by

$$\begin{aligned} E(X^1 - r^1)^+ &= \sigma\sqrt{t+s+pq}\left\{\alpha\left[\phi\left(r_1^1\right) - r_1^1\overline{\Phi\left(r_1^1\right)}\right] + (1-\alpha)\left[\phi\left(r_2^1\right) - r_2^1\overline{\Phi\left(r_2^1\right)}\right]\right\} \\ &= \sigma\sqrt{t+s+pq}\psi\left(r_1^1, r_2^1, \alpha\right). \end{aligned} \tag{A.2}$$

The expected shortage of the other batches is

$$E\left(X^2 - r^2\right)^+ = \sigma\sqrt{t}\left\{\alpha\left[\phi\left(r_1^2\right) - r_1^2\overline{\Phi\left(r_1^2\right)}\right] + (1-\alpha)\left[\phi\left(r_2^2\right) - r_2^2\overline{\Phi\left(r_2^2\right)}\right]\right\} = \sigma\sqrt{t}\psi\left(r_1^2, r_2^2, \alpha\right). \tag{A.3}$$

The expected net inventory level just before an order arrival is $E[(X-r)^- I_{0<X<r}] - E(X-r)^+$, which is computed as follows:

$$\begin{aligned} &E\left[(X-r)^- I_{0<X<r}\right] - E(X-r)^+ \\ &= \alpha\sigma\sqrt{L}\left\{r_1\left[\Phi(r_1) - \Phi\left(\frac{-\mu_1 L}{\sigma\sqrt{L}}\right)\right] + \left[\phi(r_1) - \phi\left(\frac{-\mu_1 L}{\sigma\sqrt{L}}\right)\right]\right\} \\ &\quad + (1-\alpha)\sigma\sqrt{L}\left\{r_2\left[\Phi(r_2) - \Phi\left(\frac{-\mu_2 L}{\sigma\sqrt{L}}\right)\right] + \left[\phi(r_2) - \phi\left(\frac{-\mu_2 L}{\sigma\sqrt{L}}\right)\right]\right\} \\ &\quad - \sigma\sqrt{L}\left\{\alpha\left[\phi(r_1) - r_1\overline{\Phi(r_1)}\right] + (1-\alpha)\left[\phi(r_2) - r_2\overline{\Phi(r_2)}\right]\right\} \\ &= \sigma\sqrt{L}\left\{\alpha\left[r_1\Phi\left(\frac{\mu_1\sqrt{L}}{\sigma}\right) - \phi\left(\frac{\mu_1\sqrt{L}}{\sigma}\right)\right] + (1-\alpha)\left[r_2\Phi\left(\frac{\mu_2\sqrt{L}}{\sigma}\right) - \phi\left(\frac{\mu_2\sqrt{L}}{\sigma}\right)\right]\right\} \\ &= \sigma\sqrt{L}\left\{\alpha\left[r_1\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-\alpha)k_1\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} + (1-\alpha)k_1\right)\right]\right. \\ &\quad \left. + (1-\alpha)\left[r_2\Phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - \alpha k_1\right) - \phi\left(\frac{\mu_*\sqrt{L}}{\sigma} - \alpha k_1\right)\right]\right\} \end{aligned} \tag{A.4}$$

where

$$\begin{aligned} E[(X-r)^- I_{0<X<r}] &= \int_0^r -(x-r)\left\{\alpha\frac{1}{\sqrt{2\pi}\sigma\sqrt{L}} \times \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1 L}{\sigma\sqrt{L}}\right)^2\right] + (1-\alpha)\frac{1}{\sqrt{2\pi\sigma\sqrt{L}}}\right. \\ &\quad \left.\times \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1 L}{\sigma\sqrt{L}}\right)^2\right]\right\}\mathrm{d}x \\ &= \left[\alpha\sigma\sqrt{L}\int_{\frac{=\mu_1 L}{\sigma\sqrt{L}}}^{r_1}(r_1 - Z)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}Z^2\right)\mathrm{d}Z\right] \\ &= \alpha\sigma\sqrt{L}\left\{r_1\left[\Phi(r_1) - \phi\left(\frac{-\mu_1 L}{\sigma\sqrt{L}}\right)\right] + \left[\Phi(r_1) - \phi\left(\frac{-\mu_1 L}{\sigma\sqrt{L}}\right)\right]\right\} \\ &\quad + (1-\alpha)\sigma\sqrt{L}\left\{r_2\left[\Phi(r_2) - \phi\left(\frac{-\mu_2 L}{\sigma\sqrt{L}}\right)\right] + \left[\Phi(r_2) - \phi\left(\frac{-\mu_2 L}{\sigma\sqrt{L}}\right)\right]\right\}. \end{aligned} \tag{A.5}$$

The following functions are used

$$Y^- = \begin{cases} -Y, & Y < 0 \\ 0, & Y > 0 \end{cases} \tag{A.6}$$

and

$$I_{(0<X<r)} = \begin{cases} 1, & 0 < X < r \\ 0, & \text{otherwise} \end{cases}. \tag{A.7}$$

For bi-level reorder point system, the expected net inventory level for the first batch just before an order arrival is

$$\begin{aligned}E[(X^1-r^1)^- I_{0<X^1<r^1}] - E(X^1-r^1)^+ = \sigma\sqrt{t+s+pQ}\Bigg\{\alpha\Bigg[r_1^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)\,k_1\right) \\ -\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}+(1-\alpha)\,k_1\right)\Bigg]+(1-\alpha) \\ \times\Bigg[r_2^1\Phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right) \\ -\phi\left(\frac{\mu_*\sqrt{t+s+pQ}}{\sigma}-\alpha k_1\right)\Bigg]\Bigg\}\cdot \qquad (A.8)\end{aligned}$$

For bi-level reorder point system, the expected net inventory level of other batches just before an order arrival is

$$\begin{aligned}&E[(X^2-r^2)^- I_{0<X^2<r^2}] - E(X^2-r^2)^+ \\ &= \sigma\sqrt{t}\Bigg\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)\,k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)\,k_1\right)\right] \\ &\quad+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\Bigg\}\cdot \qquad (A.9)\end{aligned}$$

Appendix B.

B.1. Buyer's inventory costs with inflation

Stock out cost is assumed to be paid at the end of the shipment cycle. Stock out cost during an infinite planning horizon considering inflationary condition is shown in the following matrix

$$\begin{bmatrix} \pi(1+(s+pQ+t)I) & \pi\left(1+(s+pQ+t)I+\frac{Q}{D}I\right) & & \pi\left(1+(s+pQ+t)I+\frac{(m-1)Q}{D}I\right) \\ \pi\left(1+(s+pQ+t)I+\frac{mQ}{D}I\right) & \pi\left(1+(s+pQ+t)I+\frac{(m+1)Q}{D}I\right) & \cdots & \pi\left(1+(s+pQ+t)I+\frac{(2m-1)Q}{D}I\right) \\ & \vdots & \ddots & \vdots \\ \pi\left(1+(s+pQ+t)I+\frac{(n-1)mQ}{D}I\right) & \pi\left(1+(s+pQ+t)I+\frac{[(n-1)m+1]Q}{D}I\right) & \cdots & \pi\left(1+(s+pQ+t)I+\frac{[nm-1]Q}{D}I\right) \end{bmatrix}_{n\times m}.$$

Considering the above matrix, the average stock out cost per unit can be computed as follows:

$$\frac{1}{n_s}\sum_{j=0}^{n_s-1}\pi\left(1+(s+pQ+t)\,I+\frac{Q}{D}Ij\right)=\pi\left[1+(s+pQ+t)\,I+\frac{I}{2}\left(1-\frac{Q}{D}\right)\right]\cdot \qquad (B.1)$$

Buyer's transportation cost is paid once the vendor sends a lot size to the buyer. The period of paying is repeated in every shipment cycle. Hence, Buyer's transportation cost during an infinite planning horizon with inflation is presented in the following matrix

$$\begin{bmatrix} c_t(1+(s+pQ)I) & c_t\left(1+(s+pQ)I+\frac{Q}{D}I\right) & & c_t\left(1+(s+pQ)I+\frac{(m-1)Q}{D}I\right) \\ c_t\left(1+(s+pQ)I+\frac{mQ}{D}I\right) & c_t\left(1+(s+pQ)I+\frac{(m+1)Q}{D}I\right) & \cdots & c_t\left(1+(s+pQ)I+\frac{(2m-1)Q}{D}I\right) \\ & \vdots & \ddots & \vdots \\ c_t\left(1+(s+pQ)I+\frac{(n-1)mQ}{D}I\right) & c_t\left(1+(s+pQ)I+\frac{[(n-1)m+1]Q}{D}I\right) & \cdots & c_t\left(1+(s+pQ)I+\frac{[nm-1]Q}{D}I\right) \end{bmatrix}_{n\times m}.$$

With the summation of above components based on a number of shipment cycle, the total transportation cost is obtained by

$$\sum_{j=0}^{n_s-1} c_t\left(1+(s+pQ)\,I+\frac{Q}{D}Ij\right)=\frac{c_tD}{Q}\left[1+(s+pQ)\,I+\frac{I}{2}\left(1-\frac{Q}{D}\right)\right] \tag{B.2}$$

where

$$c_t=ut+\left[(a_i+b_iQ)\,(t_{i-1}-t)+\sum_{j=1}^{i-1}(a_j+b_jQ)\,(\mathrm{NT}_j-\mathrm{MT}_j)\right].$$

Buyer's ordering cot is assumed to be paid at the beginning of buyer's and vendor's cycle. Buyer's ordering cost will be paid at the following matrix elements

$$\left[AA\left(1+\frac{mQ}{D}I\right)A\left(1+\frac{2mQ}{D}I\right)\quad\ldots\quad A\left(1+\frac{(n-1)mQ}{D}I\right)\right].$$

The summation of the above elements is given the buyer's total ordering cost which is given below

$$\sum_{j=0}^{n-1} A\left(1+\frac{mQ}{D}Ij\right)=A\left[\frac{D}{mQ}\left(1+\frac{I}{2}\right)-\frac{I}{2}\right]. \tag{B.3}$$

Buyer's purchasing cost is also paid at the beginning of the cycle which is shown in the following matrix elements.

$$\left[c_{pu}\,c_{pu}\left(1+\frac{mQ}{D}I\right)c_{pu}\left(1+\frac{2mQ}{D}I\right)\ldots c_{pu}\left(1+\frac{(n-1)mQ}{D}I\right)\right].$$

The average purchasing cost per unit is obtained by

$$\frac{1}{n}\sum_{j=0}^{n-1} c_{pu}\left(1+\frac{mQ}{D}Ij\right)=c_{pu}\left[1+\frac{I}{2}\left(1-\frac{mQ}{D}\right)\right]. \tag{B.4}$$

Hence, the total annual buyer's purchasing cost is

$$Dc_{pu}\left[1+\frac{I}{2}\left(1-\frac{mQ}{D}\right)\right]. \tag{B.5}$$

In the average annual method, buyer's holding cost is dependent on the buyer's average inventory level. Hence, buyer's annual holding cost can be computed as follows

$$\begin{aligned}&\left[\int_0^{nT} h_b\left(\frac{Q}{2}+\frac{1}{m}\left\{E\left[\left(X^1-r^1\right)^- I_{0<X^1<r^1}\right]-E\left(X^1-r^1\right)^+\right\}\right.\right.\\&\quad\left.\left.+\ \frac{m-1}{m}\left\{E\left[\left(X^2-r^2\right)^- I_{0<X^2<r^2}\right]-E\left(X^2-r^2\right)^+\right\}\right)(1+It)\,\mathrm{d}t\right]\\&=h_b\left(1+\frac{I}{2}\right)\left(\frac{Q}{2}+\frac{1}{m}\left\{E\left[\left(X^1-r^1\right)^- I_{0<X^1<r^1}\right]-E\left(X^1-r^1\right)^+\right\}\right.\\&\quad\left.+\frac{m-1}{m}\left\{E\left[\left(X^2-r^2\right)^- I_{0<X^2<r^2}\right]-E\left(X^2-r^2\right)^+\right\}\right).\end{aligned} \tag{B.6}$$

B.2. Vendor's inventory costs with inflation

Vendor's Setup cost is obtained at the beginning of the cycle. Setup cost payment's time is as follows

$$\left[c_s\, c_s\left(1+\frac{mQ}{D}I\right)\; c_s\left(1+\frac{2mQ}{D}I\right)\;\cdots\; c_s\left(1+\frac{(n-1)mQ}{D}I\right)\right].$$

Considering the above matrix, the total setup cost is obtained by

$$\sum_{j=0}^{n-1} c_s\left(1+\frac{mQ}{D}Ij\right)=c_s\left[\frac{D}{mQ}\left(1+\frac{I}{2}\right)-\frac{I}{2}\right] \tag{B.7}$$

where

$$c_s = as + c_{si}\left(s_{i-1}-s\right)+\sum_{j=1}^{i} c_{sj}\left(\mathrm{NS}_j-\mathrm{MS}_j\right).$$

Vendor's Production cost will be paid after setup which is shown in the following matrix

$$\left[c_{pr}(1+sI)c_{pr}\left(1+sI+\frac{mQ}{D}I\right)\; c_{pr}\left(1+sI+\frac{2mQ}{D}I\right)\;\cdots\; c_{pr}\left(1+\frac{(n-1)mQ}{D}I\right)\right].$$

The average production cost per unit is obtained by

$$\frac{1}{n}\sum_{j=0}^{n-1} c_{Pr}\left(1+sI+\frac{mQ}{D}Ij\right)=c_{Pr}\left[1+sI+\frac{I}{2}\left(1-\frac{mQ}{D}\right)\right]\cdot \tag{B.8}$$

Hence, the total annual vendor's production cost is

$$Dc_{Pr}\left[1+sI+\frac{I}{2}\left(1-\frac{mQ}{D}\right)\right]\cdot \tag{B.9}$$

Also, a vendor's holding cost is computed as follows

$$\int_0^{nT}\frac{h_vQ}{2}\left[m\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right](1+It)\,\mathrm{d}t=\frac{h_vQ}{2}\left(1+\frac{I}{2}\right)\left[m\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right]\cdot \tag{B.10}$$

Appendix C.

The space constraint can be expressed as follows:

$$P\left\{f\left[Q+r-X\right]\le F\right\}\ge\gamma. \tag{C.1}$$

The above constraint can be written as follows:

$$P\left\{(X-r)+\frac{F}{f}\ge Q\right\}\ge\gamma. \tag{C.2}$$

Thus, using Markov inequality, we have

$$\gamma\le P\left\{(X-r)^+-(X-r)^-+Q\gamma+\frac{F}{f}\ge Q\right\}\le\frac{E\left[(X-r)^+I_{X>r}\right]-E\left[(X-r)^-I_{0<X<r}\right]+\frac{F}{f}}{Q}\cdot \tag{C.3}$$

Hence, the space constraint is obtained as follows

$$\gamma Q + E\left[(X-r)^{-} I_{0<X<r}\right] - E\left[(X-r)^{+} I_{X>r}\right] - \frac{F}{f} \leq 0. \tag{C.4}$$

Note that the constraint in (C.4) is a relaxation of (C.3) obtained by applying the Markov Inequality. Therefore, substituting (C.3) by (C.4) gives a lower bound to the minimization problem. For bi-level reorder point system, the space constraint can be written as follows

$$\begin{aligned}\gamma Q + \sigma\sqrt{s+t+pQ}\left\{\alpha\left[r_1^1\Phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma}+(1-\alpha)k_1\right)\right]\right.\\ \left.+(1-\alpha)\left[r_2^1\Phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{s+t+pQ}}{\sigma}-\alpha k_1\right)\right]\right\}-\frac{F}{f}\leq 0\end{aligned} \tag{C.5}$$

and

$$\begin{aligned}\gamma Q + \sigma\sqrt{t}\left\{\alpha\left[r_1^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}+(1-\alpha)k_1\right)\right]\right.\\ \left.+(1-\alpha)\left[r_2^2\Phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)-\phi\left(\frac{\mu_*\sqrt{t}}{\sigma}-\alpha k_1\right)\right]\right\}-\frac{F}{f}\leq 0.\end{aligned} \tag{C.6}$$

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