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ECONOMIC LOT-SIZE PROBLEM FOR A CLEANER MANUFACTURING SYSTEM WITH WARM-UP PERIOD

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Abstract. There are instances that production machines require a warm-up period to reach their anticipated productivity. This study extends an economic manufacturing quantity (EMQ) problem by considering warm-up issue in the model. Warming up the machine decreases production loss, emission, and machine depreciation. Therefore, this study helps industry to enhance the profitability and also to reduce the environmental impact by decreasing waste generation and improving machine efficiency. In this study, we divide our system into three subsystems based on the relationships between production and consumption. Then we provide a mathematical model for each subsystem (three in total). The first two models are single-item EMQ inventory problems and the third one is a multi-item single-machine EMQ problem. In the third model, a machine/facility manufactures some items under a limited manufacturing volume. The purpose of these proposed models is to find the optimum cycle length to minimize the total system cost that consists of manufacturing, inventory and setup costs. Finally, we propose exact solution procedures after proving the convexity of these mathematical models.

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1. Introduction and literature review

The inventory and manufacturing costs are crucial elements of every manufacturing organization. Taft performed one of the first studies on inventory and production modeling in 1918. Taft [26], proposed the classical EMQ problem for a manufacturing system with a single product. He investigated two essential issues, (1) Number of goods to be produced and (2) The production period. Fifty years later, Eilon [5] and Rogers [24] developed Taft's inventory model for producing multiple items. Some researchers focused on the EMQ problems for imperfect manufacturing systems. For instance, Chiu [2] presented a lot sizing production model with a view to scrap and re-workable goods. Inderfurth *et al.* [10] proposed an EMQ model for a defective inventory problem with deteriorating items and rework process. Chiu *et al.* [3] investigated the EMQ problem for a manufacturing system with scrapped items, repair process, and random machine breakdowns to find the optimum production period. Leung [11] considered an EMQ model for an imperfect production problem with immediate rework.

Keywords. Economic manufacturing quantity (EMQ), single machine production, warm-up period, non-linear programming (NLP), exact solution.

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Some instances of recent studies on imperfect quality items in inventory models are Manna *et al.* [12], Manna *et al.* [13], Manna *et al.* [14], De *et al.* [4] and Manna *et al.* [15].

Another extension of the EMQ model is Economic lot scheduling problem (ELSP). ELSP studies cases wherein multiple items are manufactured by a single facility/machine. In the context of ELSP models, demand and manufacturing rates are known, the sequence of producing them is uncertain, and the focus is on decisions about the cycle scheduling, which is repeated periodically. Due to the complexity and non-linearity of the problem, no specific algorithm is proposed to obtain the optimal solution. In these models, most researchers consider a boundary for system, *i.e.* items face an equal production cycle length. In this method, the optimal common cycle can easily be obtained. The advantage of this approach is to find a feasible solution quickly. Some early instance of these models is Elmaghraby [6], he reviewed the literature on ELSP problems until 1976 and highlighted the different approaches for solving this problem. Quenniche and Boctor [23] minimized system costs by addressing sequencing and ELSP in a multi-item production system for the finite horizon in a job shop environment using a mixed integer non-linear programming (MINLP) model. Ben-Daya and Hariga [1] studied ELSP to optimize the number of inspections in a production period when the system deals with a deteriorating process. Giri *et al.* [7] considered capacity constraint and quality-related costs triggered by potential non-conforming items produced in modeling ELSP. Moon *et al.* [16] and Parveen and Rao [21] employed inspection and restoration approach where a fixed interval for conducting inspections is considered to identify process shifts to "out-of-control". Haji *et al.* [9] investigated the impact of sequence-independent setup times for the rework process in an ELSP for an imperfect production system. Defective products reworked to meet acceptable quality. Then Nobil *et al.* [17] revisited Haji *et al.* [9] to obtain conditions under which, the system would not suffer from shortage during setup time. Haji and Haji [8] employed accumulated rework to model an imperfect production system with both conforming and nonconforming items where rework was conducted after a certain period. Pasandideh *et al.* [22] addressed an ELSP with a shortage and non-conforming items where some items were scrapped, and the rest of non-conformed items were reworked based on the severity of product failure. At the same year, Taleizadeh and Wee [27] investigated a manufacturing system with limited capacity, multiple items, partial backordering, immediate rework, and imperfect process performed by a single machine to minimize the total cost by optimizing variables such as production cycle length and backorder. Shafiee-Gol *et al.* [25] investigated ELSP for a defective production system with the rework that employed pricing decisions to maximize system profits. The study assumed that the price of each item followed a random distribution and customers' demand was a linearly decreasing function of price. Nobil *et al.* [18] studied discrete delivery shipment in a multi-echelon supply chain. Nobil *et al.* [19] proposed a defective manufacturing system with permissible shortages, scrap and rework for an ELSP.

All the aforementioned studies, except for Nobil *et al.* [20] research, assumed that goods are produced at a constant rate until the batch quantity reaches a predefined level. However, in all manufacturing industries, this assumption is not held in industries such as tire companies and thermoplastic injection molding. Considering the warm-up period for the machine decreases production loss, emission, and machine depreciation, and increases the efficiency of it and its useful life span. In addition, during the warm-up period, the errors and defects of the manufactured machine become evident, which can be eliminated before the original production begins. As a result, the number of imperfect quality items reduce and the environment remains healthier.

Moreover, we extend Nobil *et al.* [20] research by considering various manufacturing rates with respect to demand rates during warm-up period for a multi-product single machine EMQ problem. Nobil *et al.* [20] assumed that the manufacturing rate during warm-up is greater than the consumption rate. In addition, they proposed the model for a single item EMQ problem. Thus, in this paper, we address an economic lot-scheduling problem with multiple items and single machine, which requires a period before reaching its normal manufacturing rate. So, the production process consists of four parts: setup time, warm-up time, uptime, and downtime. In addition, we assume that the manufacturing rate of items during warm-up period can be lower or higher than the consumption rate. Accordingly, in this study an EMQ inventory problem is studied by considering warm-up period for machines and three models are proposed. Two models address the cases where a machine produces an item for different relations between manufacturing and consumption rates. The first model investigates the case that the manufacturing rate during warm-up period is less than the consumption rate. The second model investigates the opposite (*i.e.* the manufacturing rate during warm-up period is more than consumption rate). The third model develops two previous models by considering production of several goods on a facility/machine or in a production hall. Therefore, the third model is a combination of aforementioned models, *i.e.* items may have the first or the second model conditions.

The remainder of this paper is arranged as follows: definition of problem and models are described in Section 2. In Section 3, notations and mathematical models are presented. In Section 4, the exact solution methods to optimize these models are developed. Section 5, describes numerical examples and analysis. Finally, in Section 6, the managerial insights and conclusion are expressed.

2. Problem definition and assumptions

In this study, we assume a perfect manufacturing system in which the item i is manufactured at rate P_i and is consumed at rate y_i. In this production system, after setup time, machine requires a period of time to reach its intended production rate (P_i). This time is called warm-up period and is indicated by t_i^R wherein machine produces item i at a lower speed than normal period. Therefore, regular manufacturing rate of item i (P_i) is more than warm-up rate (R_i). Also, shortages are not allowed in this manufacturing system, hence the manufacturing rate of item i (P_i), should be greater than or equal to the consumption rate of product (y_i). So, regular manufacturing rate assumed to be greater than or equal to consumption rate and its warm-up counterpart ($P_i \geq y_i$ and $P_i \geq R_i$). After the warm-up period t_i^R, the process stops to undergo corrective maintenance to remove any faults occurred for machine during warm-up period. In this problem, we assume that the corrective maintenance cost and time are negligible. What follows presents possible cases considering relationships between R_i and y_i:

(1) The manufacturing rate during warm-up period is smaller than or equal to consumption rate ($y_i \geq R_i$), *i.e.* $P_i \geq y_i \geq R_i$.
(2) The manufacturing rate during the warm-up period is greater than or equal to consumption rate ($R_i \geq y_i$), *i.e.* $P_i \geq R_i \geq y_i$.

Based on the above-mentioned we would present three EMQ inventory control models considering warm-up period.

3. Model description

The notations of the proposed model are given as follows:

- m The number of items
- i Index of items ($i = 1, \ldots, m$)
- j Index of models ($j = 1, 2$), if $j = 1$, model 1 suits the item warm-up condition; otherwise, model 2 is the suitable model.
- P_i Manufacturing rate for item i after warm-up period (kg/month)
- R_i Manufacturing rate for item i during warm-up period (kg/month)
- y_i Consumption rate for item i (kg/month)
- I_i Maximum inventory level for item i when normal production process finishes (kg)
- A_i Setup cost for manufacturing item i (\$/setup)
- h_i Inventory cost per product per unit time for item i (\$/kg/month)
- c_i Manufacturing cost per unit for item i (\$/month)

t_i^S	Setup time for producing item i
t_i^R	Warm-up period for producing item i
t_i^P	Machine uptime for producing item i
t_i^D	Machine downtime for manufacturing item i
CP_i^j	Total manufacturing cost for item i for model j (\$)
CH_i^j	Total inventory cost for item i for model j (\$)
CA_i^j	Total setup cost for item i for model j (\$)
TC_i^j	Total cost for item i for model j in a month (\$)
Z	Total cost in a month (\$)
Q_i^P	Batch quantity for item i after warm-up period per cycle
Q_i^R	Batch quantity for item i during warm-up period per cycle
Q_i^j	Total batch quantity for item i for model j per cycle ($Q_i^T = Q_i^P + Q_i^R$)
T_i	Cycle length for item i
T	The common cycle length for all products
N	Number of common cycles in a month

In this study if production rate during warm-up is less than or equal to consumption rate them model 1 is the suitable model (*i.e.* $j = 1$). Otherwise (*i.e.* when production rate during warm-up is greater than consumption rate) model 2 suits the item ($j = 2$). What follows investigates models expression and formulation.

3.1. The first model ($j = 1$) for single item when ($y \geq R$)

In this case, the manufacturing rate during warm-up period is lower than or equal to consumption rate and deficiency is not allowed in the system. The inventory graph of the EMQ problem under this condition is shown in Figure 1. In this system, the production rate during warm-up period is lower than the consumption rate, and in order to avoid deficiency, the warm-up period begins before total consumption of inventory of former period.

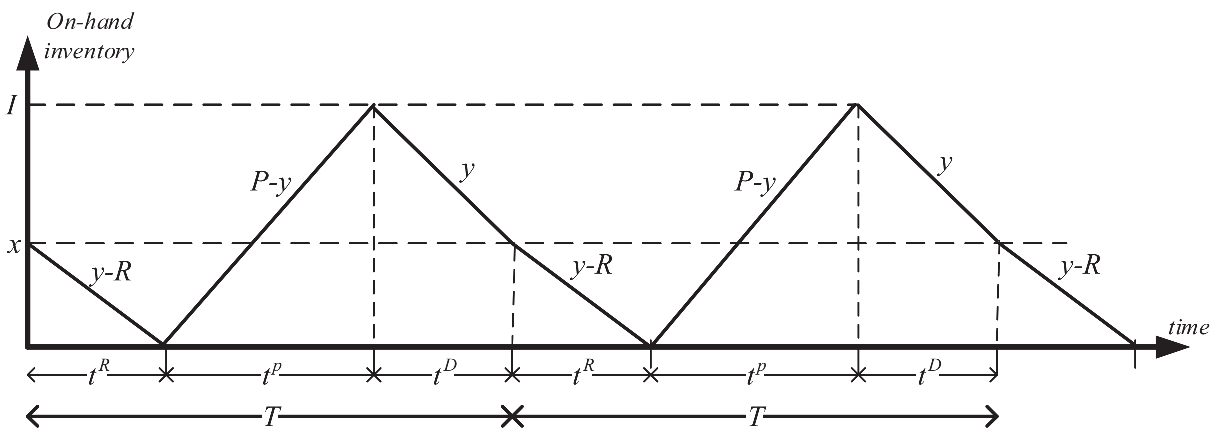

FIGURE 1. The inventory diagram of the single item (first condition).

As can be seen in Figure 1:

$$I = (P - y)\,\frac{Q^P}{P} \tag{3.1}$$

$$x = (y - R)\,t^R. \tag{3.2}$$

For a given t^R we have:

$$Q^R = Rt^R. \tag{3.3}$$

In Figure 1, production period length consists of three parts: uptime t^P, downtime t^D, and warm-up period t^R. Based on Figure 1, these periods are determined as follows:

$$t^P = \frac{Q^P}{P} \tag{3.4}$$

$$t^D = \frac{I - x}{y} = (P - y)\,\frac{Q^P}{Py} - \frac{(y - R)\,t^R}{y}\cdot \tag{3.5}$$

Consequently, thecycle length for the item is:

$$T = t^P + t^D + t^R = \frac{Q^P}{P} + (P - y)\,\frac{Q^P}{Py} - \frac{(y - R)\,t^R}{y} + t^R = \frac{Q^P + Rt^R}{y} = \frac{Q^P + Q^R}{y} = \frac{Q}{y}\cdot \tag{3.6}$$

Hence

$$Q^1 = yT. \tag{3.7}$$

Based on the equations (3.3) and (3.7), we have:

$$Q^P = Q^1 - Q^R = yT - Rt^R. \tag{3.8}$$

Finally, total cost will be:

$$\mathrm{TC}^1 = \mathrm{CA}^1 + \mathrm{CP}^1 + \mathrm{CH}^1. \tag{3.9}$$

3.1.1. The setup cost

Since the setup cost per period is A and N periods exist in a unit time, the total setup cost is obtained by:

$$\mathrm{CA}^1 = NA. \tag{3.10}$$

According to the joint manufacturing policy; $T = 1/N$,

$$\mathrm{CA}^1 = \frac{A}{T}\cdot \tag{3.11}$$

3.1.2. The manufacturing cost

The manufacturing cost per unit time is achieved using the equation (3.12.

$$\mathrm{CP}^1 = NcQ. \tag{3.12}$$

From the equation (3.7), we have:

$$\mathrm{CP}^1 = \frac{1}{T}cyT = yc. \tag{3.13}$$

3.1.3. The inventory cost

In Figure 1, the area under the curve is equal to $\left(x \times t^R/2\right) + \left(I \times t^P/2\right) + \left((I+x)\, t^D/2\right)$. Hence, the total inventory cost of this model based on the holding cost per product per unit time (h) is computed using the equation (3.14) as:

$$\mathrm{CH}^1 = Nh\left[\frac{x \times t^R}{2} + \frac{I \times t^P}{2} + \frac{(I+x)\, t^D}{2}\right]\cdot \tag{3.14}$$

From the equations (3.4) and (3.5), we have:

$$\mathrm{CH}^1 = \frac{h}{T}\left[\frac{xt^R}{2} + \frac{IQ^P}{2P} + \frac{(I+x)\,(I-x)}{2y}\right] = \frac{h}{2T}\left[xt^R + \frac{IQ^P}{P} + \frac{(I)^2 - (x)^2}{y}\right]\cdot \tag{3.15}$$

Substituting I and x from the equations (3.1) and (3.2) respectively, results in:

$$\mathrm{CH}^1 = \frac{h}{2T}\left[(y-R)\left(t^R\right)^2 + \frac{(P-y)\left(Q^P\right)^2}{(P)^2} + \frac{\left((P-y)\,\frac{Q^P}{P}\right)^2 - \left((y-R)\, t^R\right)^2}{y}\right]\cdot \tag{3.16}$$

Finally, substituting Q^P from the equation (3.8), results in:

$$\begin{aligned}
\mathrm{CH}^1 = {} & \frac{h\,(P-y)}{2\,(P)^2\, T}\left[(yT)^2 + \left(Rt^R\right)^2 - 2Rt^R yT\right] + \frac{h\,(P-y)^2}{2\,(P)^2\, yT}\left[(yT)^2 + \left(Rt^R\right)^2 - 2Rt^R yT\right] \\
& - \frac{h\,(y-R)^2\left(t^R\right)^2}{2yT} + \frac{h\,(y-R)\left(t^R\right)^2}{2T} = \frac{h_i\,(P-y)}{2PyT}\left[(yT)^2 + \left(Rt^R\right)^2 - 2Rt^R yT\right] \\
& + \frac{hR\,(y-R)\left(t^R\right)^2}{2yT}
\end{aligned}$$

$$\mathrm{CH}^1 = \left[\frac{hy\,(P-y)\,(T)}{2P} + \frac{h\,(P-y)\left(Rt^R\right)^2}{2Py}\left(\frac{1}{T}\right) - \frac{h\,(P-y)\,Rt^R}{P} + \frac{hR\,(y-R)\left(t^R\right)^2}{2y}\left(\frac{1}{T}\right)\right]\cdot \tag{3.17}$$

Moreover, substituting the equations (3.11), (3.13) and (3.17) into the equation (3.9) yields:

$$\mathrm{TC}^1 = \mathrm{CA}^1 + \mathrm{CP}^1 + \mathrm{CH}^1 = \theta^1 + \alpha^1\left(\frac{1}{T}\right) + \pi^1 T \tag{3.18}$$

where,

$$\theta^1 = yc - \frac{h\,(P-y)\,Rt^R}{P} \tag{3.19}$$

$$\alpha^1 = A + \frac{h\,(P-y)\left(Rt^R\right)^2}{2Py} + \frac{hR\,(y-R)\left(t^R\right)^2}{2y} \tag{3.20}$$

$$\pi^1 = \frac{hy\,(P-y)}{2P}\cdot \tag{3.21}$$

In this inventory system, the cycle length should be greater than total setup, warm-up, and manufacturing times altogether. Therefore,

$$t^S + t^R + t^P \le T. \tag{3.22}$$

This limitation means that the setup must be performed during downtime. Substituting t_i^P shown in the equation (3.4), results in:

$$t^S + t^R + \frac{Q^P}{P} \le T. \tag{3.23}$$

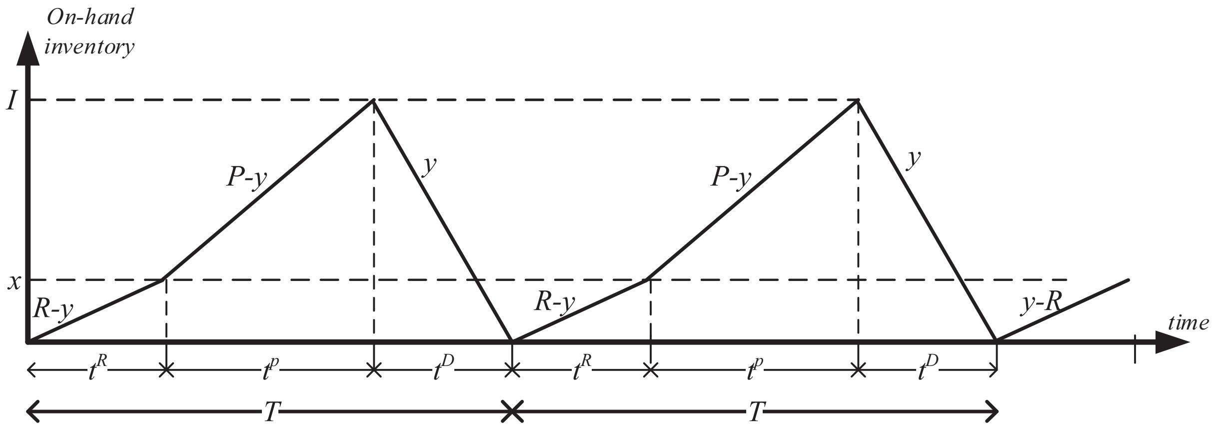

FIGURE 2. The inventory diagram of the single item (second condition).

Substituting Q^P from the equation (3.8), we have:

$$t^S + t^R + \frac{yT - Rt^R}{P} \leq T. \tag{3.24}$$

And after some simple calculations,

$$T^1_{\text{Min}} = \left\{ \frac{t^S + \left(1 - \frac{R}{y}\right) t^R}{\left(1 - \frac{y}{P}\right)} \right\} \leq T. \tag{3.25}$$

Therefore, the final model is as follows:

$$\begin{aligned} &\text{Min TC}^1 = \theta^1 + \alpha^1 \left(\frac{1}{T}\right) + \pi^1 T \\ &\text{s.t. :} \quad T \geq T^1_{\text{Min}} \\ &\qquad\quad T > 0. \end{aligned} \tag{3.26}$$

3.2. The second model ($j = 2$) for single item when ($R \geq y$)

In this case, the manufacturing rate during the warm-up period is equal to or greater than consumption rate. The inventory graph of this EMQ problem is depicted in Figure 2. As can be seen in Figure 2,

$$x = (R - y)\, t^R \tag{3.27}$$

$$I = (P - y)\, \frac{Q^P}{P} + x = (P - y)\, \frac{Q^P}{P} + (R - y)\, t^R \tag{3.28}$$

In Figure 2, the production period length consists of warm-up period t^R, uptime t^P, and downtime t^D. Based on Figure 2, these periods are determined as follows:

$$t^P = \frac{Q^P}{P} \tag{3.29}$$

$$t^D = \frac{I}{y} = \frac{(R - y)\, t^R}{y} + (P - y)\, \frac{Q^P}{yP}. \tag{3.30}$$

Consequently, the production cycle length for item i is:

$$T = t^R + t^P + t^D = t^R + \frac{Q^P}{P} + \frac{(R-y)\, t^R}{y} + (P-y)\, \frac{Q^P}{yP} = \frac{Rt^R + Q^P}{y} = \frac{Q^R + Q^P}{y} = \frac{Q}{y}\cdot \tag{3.31}$$

Hence

$$Q^2 = yT. \tag{3.32}$$

Therefore,

$$Q^P = Q^2 - Q^R = yT - Rt^R. \tag{3.33}$$

So we have:

$$\mathrm{TC}^2 = \mathrm{CA}^2 + \mathrm{CP}^2 + \mathrm{CH}^2. \tag{3.34}$$

3.2.1. The setup cost

The setup cost per unit time associated with this model is computed as the equation (3.35).

$$\mathrm{CA}^2 = NA = \frac{A}{T}\cdot \tag{3.35}$$

3.2.2. The manufacturing cost

Similar to the previous model, the manufacturing cost can be computed as:

$$\mathrm{CP}^2 = \frac{1}{T} cyT = yc. \tag{3.36}$$

3.2.3. The inventory cost

In Figure 2, the area under the curve is equal to $\left(xt^R/2\right) + \left((I+x)\, t^P/2\right) + \left(It^D/2\right)$. Hence, the total inventory cost of this model based on the holding cost per product per unit time (h) is computed by the equation (3.37):

$$\mathrm{CH}^2 = Nh \left[\frac{xt^R}{2} + \frac{(I+x)\, t^P}{2} + \frac{It^D}{2}\right]. \tag{3.37}$$

Substituting t^P and t^D from the equations (3.29) and (3.30), respectively, results in:

$$\mathrm{CH}^2 = \frac{h}{2T} \left[xt^R + \frac{(I+x)\, Q^P}{P} + \frac{(I)^2}{y}\right]. \tag{3.38}$$

Substituting x and I from the equations (3.27) and (3.28), respectively, results in:

$$\mathrm{CH}^2 = \frac{h}{2T} \left[(R-y)\left(t^R\right)^2 + \left(2\,(R-y)\, \frac{t^R Q^P}{P} + (P-y)\left(\frac{Q^P}{P}\right)^2\right) + \left((R-y)\, t^R + (P-y)\, \frac{Q^P}{P}\right)^2 \Big/ y\right]. \tag{3.39}$$

Finally, using Q^P from the equation (3.33), we have:

$$\mathrm{CH}^2 = \left[\frac{h\,(P-y)\left(Rt^R\right)^2}{2Py}\left(\frac{1}{T}\right) - \frac{h\,(R-y)\, R\left(t^R\right)^2}{2y}\left(\frac{1}{T}\right) + \frac{h\,(P-y)\, y}{2P}\,(T) + h\,(R-y)\, t^R - \frac{h\,(P-y)\, Rt^R}{2P}\right]. \tag{3.40}$$

Substituting the equations (3.35), (3.36) and (3.40) into the equation (3.34) we can compute the total cost as follows:

$$\mathrm{TC}^2 = \mathrm{CA}^2 + \mathrm{CP}^2 + \mathrm{CH}^2 = \theta^2 + \alpha^2\left(\frac{1}{T}\right) + \pi^2 T \tag{3.41}$$

where

$$\theta^2 = yc + h\,(R-y)\,t^R - \frac{h\,(P-y)\,Rt^R}{2P} \tag{3.42}$$

$$\alpha^2 = A + \frac{h\,(P-y)\left(Rt^R\right)^2}{2Py} - \frac{h\,(R-y)\,R\left(t^R\right)^2}{2y} \tag{3.43}$$

$$\pi^2 = \frac{hy\,(P-y)}{2P}\cdot \tag{3.44}$$

In this inventory system, the cycle length should be greater than the sum of setup, warm-up, and manufacturing times. Therefore,

$$T \geq t^S + t^R + t^P. \tag{3.45}$$

So we have,

$$T \geq \left\{\frac{t^S + \left(1-\frac{R}{P}\right)t^R}{\left(1-\frac{y}{P}\right)}\right\} = T^2_{\text{Min}}. \tag{3.46}$$

Finally, the model is concluded as:

$$\begin{aligned} &\text{Min TC}^2 = \theta^2 + \alpha^2\left(\frac{1}{T}\right) + \pi^2 T \\ &\text{s.t.}: \quad T \geq T^2_{\text{Min}} \\ &\qquad\quad T > 0. \end{aligned} \tag{3.47}$$

3.3. Multi-item single-machine EMQ problem

Now, we develop a multi-item single-facility/machine EMQ problem based on the 3.1 and 3.2, wherein a facility/machine produces multiple items. A single facility/machine produces m items and each of these products may have one of conditions discussed in Section 3.1 or Section 3.2. Thus, the total cost of this problem consists of setup cost, manufacturing cost and inventory cost of these items. In other words:

$$Z = \sum_{i=1}^{m}\sum_{i\in j}\text{TC}_i^j = \sum_{i=1}^{m}\sum_{i\in j}\left\{\text{CA}_i^j + \text{CP}_i^j + \text{CH}_i^j\right\}. \tag{3.48}$$

So the total cost is a combination of the individual costs of each item. Thus, according to equations (3.18) and (3.41) we have

$$Z = \sum_{i=1}^{m}\sum_{i\in j}\left\{\text{CA}_i^j + \text{CP}_i^j + \text{CH}_i^j\right\} = \sum_{i=1}^{m}\sum_{i\in j}\left\{\theta_i^j + \alpha_i^j\left(\frac{1}{T_i}\right) + \pi_i^j T_i\right\}. \tag{3.49}$$

Besides, all goods are made on one facility/machine with a common cycle length. In other words, $T = T_i = T_1 = T_2 = \cdots = T_m$ (for instance see: [18,22]). Therefore,

$$Z = \sum_{i=1}^{m}\sum_{i\in j}\left\{\theta_i^j + \alpha_i^j\left(\frac{1}{T_i}\right) + \pi_i^j T_i\right\} = \sum_{i=1}^{m}\sum_{i\in j}\left\{\theta_i^j + \alpha_i^j\left(\frac{1}{T}\right) + \pi_i^j T\right\}. \tag{3.50}$$

On the other hand, the cycle length should be greater than the overall time associated with manufacturing, warm-up and setup for all items. Hence,

$$T \geq \sum_{i=1}^{m}\left(t_i^P + t_i^R\right) + \sum_{i=1}^{m} t_i^S. \tag{3.51}$$

So,

$$T \geq \sum_{i=1}^{m}\left(\frac{y_i T - R_i t_i^R}{P_i} + t_i^R\right) + \sum_{i=1}^{m} t_i^S. \tag{3.52}$$

Finally, we have:

$$T \geq \left\{\frac{\sum_{i=1}^{m}\left[t_i^R\left(1-\frac{R_i}{P_i}\right)\right] + \sum_{i=1}^{m}\left(t_i^S\right)}{\left(1-\sum_{i=1}^{m}\frac{y_i}{P_i}\right)}\right\} = T^{\text{Min}}. \tag{3.53}$$

Using the total inventory cost in the equation (3.50) and the constraint in inequality (3.53), the multi-item single-machine EMQ inventory model is obtained as follows:

$$\begin{aligned} &\text{Min } Z = \sum_{i=1}^{m}\sum_{i\in j}\left\{\theta_i^j + \alpha_i^j\left(\frac{1}{T}\right) + \pi_i^j T\right\} \\ &\text{s.t.} : T \geq T^{\text{Min}} \\ &\qquad T > 0. \end{aligned} \tag{3.54}$$

4. The solution procedures

This section focuses on solution procedures for three models proposed in Section 3. These solution procedures are as follows.

4.1. The solution procedure for the first model

The objective function of the model (3.26) is convex. We prove that the second derivative of the objective function with respect to decision variable, shown in the equation (4.1), is positive (see Appendix A). Besides, the constraint in (3.26) is a linear function; hence, it is convex. Thus, the proposed EMQ model in (3.26) is a convex nonlinear mathematical model.

$$\frac{\partial^2 \text{TC}^1}{\partial^2 T} = \frac{-\left(-2T\right)\alpha^1}{\left(T\right)^4} = \frac{2T\alpha^1}{\left(T\right)^4} = \frac{2\alpha^1}{\left(T\right)^3}. \tag{4.1}$$

To find the optimum cycle length, we differentiate the objective function in model (3.26) with respect to T. Hence, we have:

$$T = \sqrt{\frac{\alpha^1}{\pi^1}}. \tag{4.2}$$

Finally, the following algorithm obtains the optimal solution for the aforementioned model:

Step 1. Calculate θ^1, α^1 and π^1 using equations (3.19), (3.20) and (3.21), respectively.
Step 2. Compute T using the equation (4.2).
Step 3. Compute T^1_{Min} using the equation (3.25).
Step 4. If $T < T^1_{\text{Min}}$, then $T^* = T^1_{\text{Min}}$; else, $T^* = T$.
Step 5. According to the value of T^*, obtain Q^{1*} using the equation (3.7) and TC^{1*} using the equation (3.18).

4.2. The solution procedure for the second model

The objective function in model (3.47) is convex (it can be proven as before). Thus, the proposed EMQ model presented in the equation (3.47) is a convex nonlinear mathematical model.

$$\frac{\partial^2 \text{TC}^2}{\partial^2 T} = \frac{2\alpha^2}{\left(T\right)^3}. \tag{4.3}$$

TABLE 1. General data for the first model.

Parameter	A	c	h	t^S	t^R
Value	800	40	10	0.02	0.01

To find the optimum period length, we differentiate the objective function in model (3.26) with respect to T. Hence, we have:

$$T = \sqrt{\frac{\alpha^2}{\pi^2}}. \tag{4.4}$$

Consequently, the following algorithm obtains the optimal solution:

Step 1. Calculate θ^2, α^2 and π^2 using equations (3.42), (3.43) and (3.44), respectively.
Step 2. Compute T using the equation (4.4).
Step 3. Compute T^2_{Min} using the equation (3.46).
Step 4. If $T < T^2_{\text{Min}}$, then $T^* = T^2_{\text{Min}}$; else, $T^* = T$.
Step 5. According to the value of T^*, obtain Q^{2*} using the equation (3.32) and TC^{2*} using the equation (3.41).

4.3. The solution procedure of the developed model

This model is a combination of the aforementioned models such that some items follow conditions of the first model and the rest follows conditions of the second model. The objective function in model (3.54) is convex (see Appendix C). To calculate the optimum cycle length, the first derivative of the objective function (Z) is calculated with respect to T. Hence, the optimal cycle length yields as follows:

$$\frac{\partial Z}{\partial T} = 0 \rightarrow -\frac{\sum_{i=1}^{m}\sum_{i\in j}\alpha_i^j}{(T)^2} + \sum_{i=1}^{m}\sum_{i\in j}\pi_i^j = 0 \rightarrow T = \sqrt{\frac{\sum_{i=1}^{m}\sum_{i\in j}\alpha_i^j}{\sum_{i=1}^{m}\sum_{i\in j}\pi_i^j}}. \tag{4.5}$$

Finally, the following algorithm obtains the optimal solution for the multi-item single-machine EMQ model.

Step 1. If $1 > \sum_{i=1}^{m}\frac{y_i}{P_i}$, go to Step 2. Else, the proposed EMQ problem is not feasible, and go to Step 8.
Step 2. For each item, if $y_i \geq R_i$, then it is of Type I ($j = 1$). Otherwise, it is of Type II ($j = 2$).
Step 3. For Type I items form ($y_i \geq R_i$), calculate θ_i^1, α_i^1 and π_i^1 using equations (3.19), (3.20) and (3.21), respectively. For type II items ($R_i \geq y_i$), calculate $\theta_i^2.\alpha_i^2$ and π_i^2 using equations (3.42), (3.43) and (3.44), respectively.
Step 4. Calculate T using the equation (4.5).
Step 5. Calculate T^{Min} using the equation (3.53).
Step 6. If $T < T^{\text{Min}}$, then $T^* = T^{\text{Min}}$; else, $T^* = T$.
Step 7. According to T^*, obtain Q_i^{j*} by $Q_i^{j*} = y_iT^*$ and Z^* using the equation (3.50).
Step 8. *Terminate the procedure.*

5. Experiments

In this section, we present three numerical examples for the above-mentioned cases. We have employed the proposed algorithms to solve the examples:

5.1. The first model

Consider a single item manufacturing system with $P = 4000$, $y = 2000$ and $R = 500$. The rest of the parameters are given in Table 1. The optimum values are obtained as follows.

Since y is greater than R $(y > R)$, we use the first model's algorithm as follows:

Step 1. *Initial calculations*

We calculate θ^1, α^1 and π^1 using equations (3.19), (3.20) and (3.21), respectively.

$$\theta^1 = cy - \frac{hR\,(P-y)\,t^R}{P} = 40 \times 2000 - \frac{10 \times 500 \times (4000 - 2000) \times 0.01}{4000} = 79975$$

$$\alpha^1 = A + \frac{h\,(P-y)\left(Rt^R\right)^2}{2Py} + \frac{hR\,(y-R)\left(t^R\right)^2}{2y} = 800 + \frac{10 \times (4000-2000) \times (500 \times 0.01)^2}{2 \times 4000 \times 2000}$$
$$+ \frac{10 \times 500 \times (2000-500)\,(0.01)^2}{2 \times 2000} = 800.21875$$

$$\pi^1 = \frac{hy\,(P-y)}{2P} = \frac{10 \times 2000 \times (4000-2000)}{2 \times 4000} = 5000.$$

Step 2. *Finding the cycle length*

Based on the equality (4.2), T is computed as:

$$T = \sqrt{\frac{\alpha^1}{\pi^1}} = \sqrt{\frac{800.21875}{5000}} \cong 0.4.$$

Step 3. *Lower bound*

Using the equation (3.25), T^1_{Min} is obtained as:

$$T^1_{\text{Min}} = \frac{t^S + \left(1-\frac{R}{P}\right)t^R}{\left(1-\frac{y}{P}\right)} = \frac{0.02 + \left(1-\frac{500}{4000}\right) \times 0.01}{\left(1-\frac{2000}{4000}\right)} = 0.0575.$$

Step 4. *Checking the constraint*

As $(T = 0.4) > \left(T^1_{\text{Min}} = 0.0575\right)$, we have $T^* = T = 0.4$.

Step 5. *Finding the optimum values*

According to the value of $T^* = 0.4$, Q^* and TC^{1*} are computed as:

$$Q^{1*} = yT = 2000 \times 0.4 = 800\,\text{kg}$$
$$\text{TC}^{1*} = \theta^1 + \alpha^1\left(\frac{1}{T}\right) + \pi^1 T = 79975 + \frac{800.21875}{0.4} + 5000 \times 0.4 = 83975.5\$.$$

Moreover, according to the above results we can conclude that:

- The manufacturing cost (c) and setup time (t^S) have no effect on the value of T.
- The manufacturing cost (c), setup cost (t^S) and the inventory cost (h) have no effect on the lower bound of the cycle length (T^{Min}).

5.2. The second model

Consider a single-item manufacturing system with $P = 4000$, $R = 3000$, and $y = 2000$. Moreover, the rest of the parameters are proposed in Table 1. The optimal solution is obtained for cases that R is greater than y $(R > y)$, and the exact algorithm of the second model is employed as follows:

Step 1. *Initial calculations*

We calculate θ^2, α^2 and π^2 using equations (3.42), (3.43), and (3.44), respectively as:

$$\theta^2 = cy + h(R-y)t^R - \frac{h(P-y)Rt^R}{2P} = 40 \times 2000 + 10 \times (3000-2000) \times 0.01 - \frac{10 \times (4000-2000) \times 3000 \times 0.01}{4000} = 80025$$

$$\alpha^2 = A + \frac{h(P-y)\left(Rt^R\right)^2}{2Py} - \frac{h(R-y)R\left(t^R\right)^2}{2y} = 800 + \frac{10 \times (4000-2000) \times (3000 \times 0.01)^2}{2 \times 4000 \times 2000} - \frac{10 \times 3000 \times (3000-2000) \times (0.01)^2}{2 \times 2000} = 800.375$$

$$\pi^2 = \frac{hy(P-y)}{2P} = \frac{10 \times 2000 \times (4000-2000)}{2 \times 4000} = 5000.$$

Step 2. *Finding the cycle length*

Based on equality (4.2), T is computed as:

$$T = \sqrt{\frac{\alpha^2}{\pi^2}} = \sqrt{\frac{800.375}{5000}} \cong 0.4.$$

Step 3. *Lower bound*

Using the equation (3.46), T^2_{Min} is obtained as:

$$T^2_{\text{Min}} = \frac{t^S + \left(1 - \frac{R}{P}\right)t^R}{\left(1 - \frac{y}{P}\right)} = \frac{0.02 + \left(1 - \frac{3000}{4000}\right) \times 0.01}{\left(1 - \frac{2000}{4000}\right)} = 0.045.$$

Step 4. *Checking the constraint*

Since, $(T = 0.4) > \left(T^2_{\text{Min}} = 0.045\right)$, we have $T^* = T = 0.4$.

Step 5. *Finding the optimum values*

According to $T^* = 0.4$, Q^* and TC^{2*} are computed as:

$$Q^{2*} = yT = 2000 \times 0.4 = 800\,\text{kg}$$

$$\text{TC}^{2*} = \text{CA}^2 + \text{CP}^2 + \text{CH}^2 = 80025 + \frac{800.375}{0.4} + 5000 \times 0.4 = 84025.9\$.$$

According to these the results:

- The manufacturing cost (c) and setup time (t^S) have no effect on the cycle length (T).
- The manufacturing cost (c), the setup cost (t^S) and the inventory cost (h) have no effect on the lower bound of the cycle length (T^{Min}).

5.3. The extended model

Consider a single-facility/machine manufacturing system with five items. Values for parameters of this problem are given in Table 2 and optimal solution for this model is determined based on the following solution procedure.

Step 1. *Checking the feasibility*

As $\left(1 - \sum_{i=1}^{m} \frac{y_i}{P_i}\right) = 0.0497873471557683$, then go to Step 2.

Step 2. *Specifying items' types*

We specify the items as shown in Table 3.

Step 3. *Initial calculations*

θ_i^j, α_i^j and π_i^j, are computed and shown in Table 3.

TABLE 2. General data for the developed model.

Item	P_i	y_i	R_i	A_i	c_i	h_i	t_i^S	t_i^R
1	8000	2000	500	800	40	10	0.002	0.001
2	9000	2000	3000	1000	50	8	0.003	0.002
3	9500	1000	1500	900	40	12	0.001	0.002
4	10 000	1000	500	1100	45	15	0.002	0.001
5	11 000	3000	4000	1200	50	10	0.003	0.003

TABLE 3. Type of the items and θ_i^j, α_i^j and π_i^j.

Item	y_i	R_i	Situation	Condition	θ_i^j	α_i^j	π_i^j
1	2000	500	$y_1 > R_1$	$j = 1$	79 962.5	800.234375	7500
2	2000	3000	$y_2 < R_2$	$j = 2$	99 973.333333	1003.2	6222.222222
3	1000	1500	$y_3 < R_3$	$j = 2$	39 958.947368	903.031578	5368.421052
4	1000	500	$y_4 > R_4$	$j = 1$	44 932.5	1100.35625	6750
5	3000	4000	$y_5 < R_5$	$j = 2$	149 863.636363	1211.454545	10 909.090909

TABLE 4. Values of Q_i^{j*} and TC_i^{j*}.

	1	2	3	4	5
Q_i^{j*}	736	736	368	368	1104
TC_i^{j*}	84 930.1	105 004.7	44 417.3	50 466.3	15 7262.4

Step 4. *Finding the cycle length*

Based on equality (4.5), T is computed as:

$$T = \sqrt{\frac{\sum_{i=1}^{m} \alpha_i^j}{\sum_{i=1}^{m} \pi_i^j}} = \sqrt{\frac{5018.27674940191}{36749.7341839447}} \cong 0.368.$$

Step 5. *Lower bound*

Using the equation (3.53), T^{Min} is obtained as:

$$T^{\mathrm{Min}} = \frac{\sum_{i=1}^{m} \left[t_i^R \left(1 - \frac{R_i}{P_i} \right) \right] + \sum_{i=1}^{m} \left(t_i^S \right)}{\left(1 - \sum_{i=1}^{m} \frac{y_i}{P_i} \right)} \cong 0.357.$$

Step 6. *Checking the constraint*

Based on the $(T = 0.368) > \left(T^{\mathrm{Min}} = 0.357\right)$, value of T^* is 0.368.

Step 7. *Finding the optimum values*

According to $T^* = 0.368$, values of Q_i^{j*} and TC_i^* for all items are proposed in Table 4. Finally, the minimum annual inventory cost is $Z^* = \sum_{i=1}^{m} \mathrm{TC}_i^* = 442080.8\$$.

Step 8. *Terminate the solution procedure.*

5.4. The sensitivity analysis

In this subsection, the third model, which is a combination of two former models, is chosen for sensitivity analysis. In Table 5, we display the results of the analysis. Table 5 indicates that:

TABLE 5. The sensitivity analysis.

Parameters	% Changes	% Changes in T	T^{Min}	T^*	Z^*
Initial	0	0	0	0	0
P_i	+50	−3.9215	−85.8607	−3.9215	0.2635
	+10	−0.9505	−63.0295	−0.9505	0.0730
	−50	Infeasible			
y_i	+50	Infeasible			
	+10	Infeasible			
	−50	33.2061	−90.5147	33.2061	−48.4407
R_i	+50	0.2435	−6.1351	0.2435	0.0134
	+10	0.2378	−1.2270	0.2378	0.0026
	−50	0.2339	6.1351	3.1946	−0.0094
A_i	+50	22.7628	0	22.7628	1.3782
	+10	5.1287	0	5.1287	0.2993
	−50	−29.1191	0	−2.7705	−1.5776
c_i	+50	0	0	0	46.9371
	+10	0	0	0	9.3874
	−50	0	0	0	−46.9370
h_i	+50	−18.1554	0	−2.7705	1.4866
	+10	−4.4276	0	−2.7705	0.29965
	−50	41.7533	0	41.7533	−1.7928
t_i^S	+50	0	30.8743	0	0
	+10	0	6.1748	0	0
	−50	0	−30.8743	0	0
t_i^R	+50	0.2415	19.1256	15.8252	0.0609
	+10	0.2375	3.8251	0.9486	−0.0004
	−50	0.2338	−19.1256	0.2338	0.0033

– T is a lot sensitive to changes in the amounts of consumption rate, setup cost, and inventory cost, is slightly sensitive to changes in manufacturing rate, warm-up time and manufacturing rate during warm-up period, and setup time and manufacturing cost have no effect on T value.
– T^{Min} is insensitive to changes in manufacturing cost, inventory cost and setup cost, is almost sensitive to changes in manufacturing rate during warm-up period and warm-up time, and is a lot sensitive to changes in consumption rate, manufacturing rate and setup time.
– Z^* is slightly sensitive to the changes in manufacturing rate during warm-up period and warm-up period, is moderately sensitive to changes in inventory and setup costs, and is highly sensitive to changes in consumption rate and manufacturing cost, and setup time has no effect on optimum amount of total inventory cost.

Moreover, effects of changes manufacturing rate during warm-up period (R_i) and warm-up time (t_i^R) on the optimal cycle length and optimal total cost are presented in Figures 3 and 4, respectively.

6. Conclusion and managerial insights

In this study, we have presented three economic manufacturing quantity models by considering warm-up period. In the proposed models a certain time, called warm-up period, has been considered. Warm-up time is the required period for a machine by when it operates efficiently. In the warm-up period, the production machine manufactures the items at a lower rate than the normal production time. After this time, the machine produces the items at normal manufacturing rate. Two models have been proposed considering warm-up period, in one

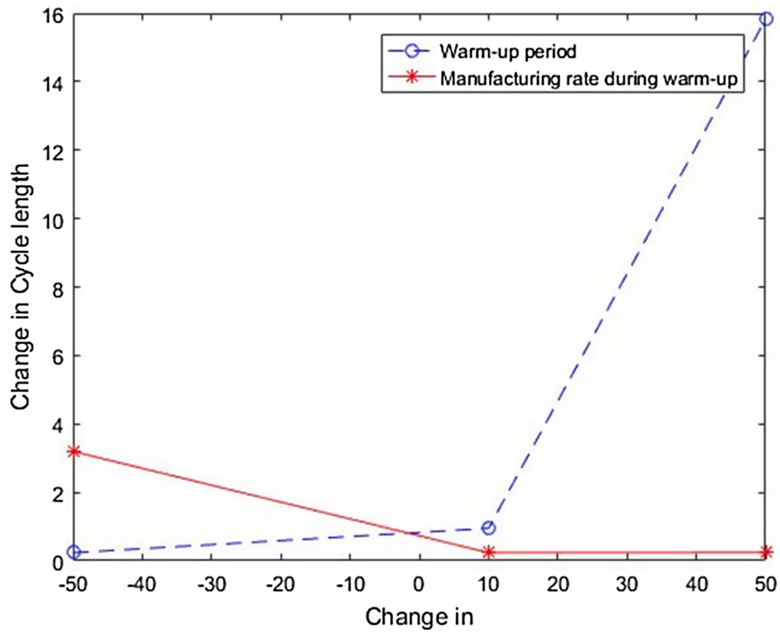

FIGURE 3. Effects of R_i and t_i^R parameters on the optimal cycle length.

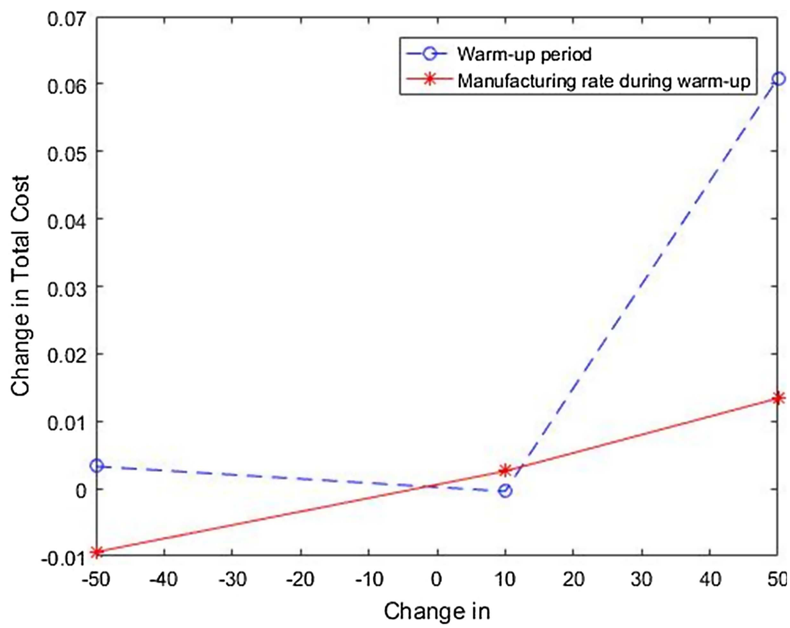

FIGURE 4. Effects of R_i and t_i^R parameters on the optimal total inventory cost.

of them production rate has been greater or equal to consumption rate, in the other production rate has been smaller than the consumption rate. As the nonlinear programming problems of these two models have been proved to be convex, two exact algorithms with five simple steps have been proposed to calculate the optimal solution aiming to minimize the total cost. Finally, based on these two models, a multi-item EMQ model as a combination of the aforementioned models has been developed. In this model, all items are made by a unique facility/machine with a limited manufacturing volume. As the cost function associated with the third model has also been convex, an exact method containing 8 steps has been suggested to obtain the optimal cycle length that minimizes total cost of the inventory system.

Most important managerial insight that is driven from this study is its focus on cleaner production (by taking account of maintenance during warm-up), to provide managers guidelines for choosing the optimal cycle length based on minimizing total cost; including inventory cost, setup cost for normal manufacturing process, and manufacturing cost. Furthermore, considering the warm-up period helps managers to increase the useful life span of the machine and decrease production loss. Moreover, to prevent producing scrapped items, machine needs a warm-up during which the defects of the machine are identified. The defects can be fixed by employing corrective maintenance like astringent, replacement of the parts, and lubrication. The aim of this study is to identify a policy that minimizes the sum of setup, manufacturing, and holding costs.

For future research, some suggestions are as follows:

- Considering approximated and uncertain warm-up time by considering it as a stochastic or fuzzy parameter.
- The models can be extended by considering shortage, in forms of backordered, lost sale, or partial backordering.
- A defective manufacturing system with rework and scraped items can be developed.
- The models can be developed by considering deteriorating process, stock-time dependent demand, sustainable production policy, etc.

Appendix A. Prove convexity of the objective (3.26)

The second order derivative of the objective function shown in the equation (3.26) with respect to the cycle length is:

$$\frac{\partial^2 \mathrm{TC}^1}{\partial^2 T} = \frac{2\alpha^1}{(T)^3}.$$

Based on the equation (3.20), we have:

$$\alpha^1 = A + \frac{h\,(P-y)\left(Rt^R\right)^2}{2Py} + \frac{hR\,(y-R)\left(t^R\right)^2}{2y}.$$

Note that since $P \geq y \geq R$ and all parameters are positive, $\alpha^1 \geq 0$. Also, T is greater than zero, the second derivative of the equation (3.26) is positive.

$$\alpha^1 \geq 0.\mathrm{T} > 0 \rightarrow \frac{\partial^2 \mathrm{TC}^1}{\partial^2 T} = \frac{2\alpha^1}{(T)^3} \geq 0; \qquad (\mathrm{P.S.D})\,.$$

Appendix B. Prove convexity of the objective (3.47)

The second order derivative of the objective function shown in the equation (3.47) with respect to the cycle length is:

$$\frac{\partial^2 \mathrm{TC}^2}{\partial^2 T} = \frac{2\alpha^2}{(T)^3}.$$

Based on the equation (3.43), we have:

$$\alpha^2 = A + \frac{h\left(P-y\right)\left(Rt^R\right)^2}{2Py} - \frac{h\left(R-y\right)R\left(t^R\right)^2}{2y}\cdot$$

The third part of this equation is a negative number. So to prove positivity of this equation, we assume that $A = 0$ (in addition to all parameters of the model are positive). We have:

$$\alpha^2 = \frac{h\left(P-y\right)\left(Rt^R\right)^2}{2Py} - \frac{h\left(R-y\right)R\left(t^R\right)^2}{2y} = \underbrace{\frac{hR\left(t^R\right)^2}{2y}}_{\geq 0}\left[\frac{\left(P-y\right)R}{P} - \left(R-y\right)\right].$$

Since the first part is always positive we have,

$$\frac{\left(P-y\right)R}{P} - \left(R-y\right) = \frac{\left(P-y\right)R}{P} - \frac{\left(R-y\right)P}{P} = \frac{\left(P-y\right)R-\left(R-y\right)P}{\left(P\geq 0\right)}\cdot$$

Because the denominator is always positive, so we have:

$$\left(P-y\right)R - \left(R-y\right)P = \left(PR-yR\right) - \left(PR-yP\right) = yP - yR = y\left(P-R\right).$$

Since $P \geq R$, we can say always

$$\alpha^2 = A + \frac{h\left(P-y\right)\left(Rt^R\right)^2}{2Py} - \frac{h\left(R-y\right)R\left(t^R\right)^2}{2y} \geq 0.$$

Also, T is greater than zero and the second order derivative of the equation (3.47) with respect to period length is positive:

$$\alpha^2 \geq 0.\mathrm{T} > 0 \rightarrow \frac{\partial^2\mathrm{TC}^2}{\partial^2 T} = \frac{2\alpha^2}{\left(T\right)^3} = \frac{2\alpha^2}{\left(T\right)^3} \geq 0; \qquad \left(\mathrm{P.S.D}\right).$$

Appendix C. Prove convexity of the objective (3.54)

Based on the objective function shown in the equation (3.54):

$$Z = \sum_{i=1}^{m}\sum_{i\in j}\left\{\theta_i^j + \alpha_i^j\left(\frac{1}{T}\right) + \pi_i^j T\right\}.$$

Its second order derivative with respect to period length is:

$$\frac{\partial^2 Z}{\partial^2 T} = \frac{2\left(\sum_{i=1}^{m}\sum_{i\in j}\alpha_i^j\right)}{\left(T\right)^3}\cdot$$

In the multi-product single machine EMQ model, the following three conditions exist:

(1) If all items have the first model conditions during the warm-up ime, *i.e.* $y_i \geq R_i$, then for all items, we have $\alpha_i^1 \geq 0$ (see Appendix A). As a result, $\sum_{i=1}^{m}\alpha_i^1 \geq 0$ and the second derivative of objective function shown in the equation (3.54) is positive.

(2) If all items have the first model conditions during the warm-up time, *i.e.* $R_i \geq y_i$, then, for all items, we have $\alpha_i^2 \geq 0$ (see Appendix B). As a result, $\sum_{i=1}^{m}\alpha_i^2 \geq 0$ and the second derivative of objective function shown in the equation (3.54) is positive.

(3) If some items have the first ($y_i \geq R_i$) and remaining items have the second model ($R_i \geq D_i$) conditions during the warm-up time, we prove that $\alpha_i^j; i = 1, 2, \ldots, m$ are positive (see Appendices A and B). Therefore, the sum of positive numbers ($\sum_{i=1}^{m} \alpha_i^j$) is positive. So, the second derivative of objective function in (3.54) is also positive.

Finally, we can say that the second derivative of objective function (3.54) is always positive.

$$T > 0, \alpha_i^j \geq 0; i = 1, 2, \ldots, m \rightarrow \frac{\partial^2 Z}{\partial^2 T} = \frac{2\left(\sum_{i=1}^{m} \alpha_i^j\right)}{(T)^3} \geq 0; \qquad \text{(P.S.D)}.$$

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