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THE KARUSH–KUHN–TUCKER CONDITIONS FOR MULTIPLE OBJECTIVE FRACTIONAL INTERVAL VALUED OPTIMIZATION PROBLEMS

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Abstract. In this article, we focus on a class of a fractional interval multivalued programming problem. For the solution concept, LU-Pareto optimality and LS-Pareto, optimality are discussed, and some nontrivial concepts are also illustrated with small examples. The ideas of LU-V-invex and LS-V-invex for a fractional interval problem are introduced. Using these invexity suppositions, we establish the Karush–Kuhn–Tucker optimality conditions for the problem assuming the functions involved to be gH-differentiable. Non-trivial examples are discussed throughout the manuscript to make a clear understanding of the results established. Results obtained in this paper unify and extend some previously known results appeared in the literature.

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1. Introduction

Interval optimization problems are closely related to the inexact data in real-life engineering and economic problems. These types of problems involve bounds on the values of the coefficient in the objective function. Since the real-world problems involve unsure or imprecise data, therefore, it is not necessary that the coefficients involved in the objective functions are real numbers. Not only interval optimization problems, but also robust optimization problems and fuzzy optimization problems also deal with uncertainty in the data of the optimization problems. However, it is easy to deal with the interval optimization as the hypothesis of probabilistic distributions is not required like we assume in stochastic programming or the hypothesis of possibilistic distributions as in fuzzy programming.

Two types of interval programming problems are available, viz., the interval linear and the interval nonlinear programming. Linear interval problems involving an interval in the coefficients of the objective functions are studied in [13, 17, 22]. An enhanced linear interval problem and its solution algorithm are given in Zhou and Gordan [26]. A study in multiobjective linear interval valued problem is developed in Oliveira [16]. Based on real-life problems, an overall sketch on the utilization of interval arithmetic is studied in Alefeld and Mayer [2].

Keywords. Fractional programming, multiobjective programming, interval valued problem, LU-V/LS-V-invex, gH-differentiable.

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Interval quadratic problems are explored in Ding and Huang [10]. Several algorithms in interval nonlinear programming problems are discussed in Jiang *et al.* [15].

It is essentially important to study the concept of derivative for the interval optimization problems, where the objective functions are set valued function. The notion of H-derivative, given in Hukuhara [12], has developed several mathematical analyses in the set differential equations in Bede and Gal [4] and in the fuzzy differential equations in Chalco-Cano and Roman-Flores [7]. The same concept of derivative is applied to study nonlinear interval optimization problems. However, the H-differentiability concept suffers some disadvantages, which are explained in Singh *et al.* [18]. To overcome the difficulties and to deal with the set valued functions, new concepts of derivative are presented in [4, 8, 20, 21].

The optimality conditions for the interval optimization problems, known as Karush–Kuhn–Tucker, have become an interesting topic of research during recent years. Extending the idea of convexity to LU-convexity/CW-convexity, Wu [24] has given the KKT conditions for the interval-valued optimization problems under LU-convexity/CW-convexity assumptions. The conditions for optimality for the non-convex interval-valued problems under LU-convexity or LU-invexity conditions are discussed in Zhang *et al.* [25]. Recently, Tung [23] considered nonsmooth multiobjective semi-infinite problems and discussed the strong KKT conditions by using tangential subdifferential and suitable regularity conditions. With nondifferentiable multiobjective optimizaton problems, Gupta and Srivastava [11] developed weak and strong KKT conditions and studied duality relations for such problems.

The paper aims to study the KKT conditions for a class of a multivalued fractional interval optimization problem. In many recent publications, like, Ahmad *et al.* [1], Bhurjee and Panda [5], Jayswal *et al.* [14] and Wu [24] the authors have developed sufficient optimality conditions for the interval-valued optimization problems. Motivated by the idea of gH-derivative for the optimization problems having interval-valued objectives, the KKT conditions for optimality are established in Chalco-Cano *et al.* [6]. The current work mainly focuses on non-linear optimization problems in which the objective functions are non-linear fractional interval-valued functions. The main motivation of considering the objective functions as the fraction of intervals is, the uncertainty involved in many practical problems, which may be in the form as the ratio of two intervals of functions.

The paper is organized as follows. In the Section 2, a class of multivalued fractional interval problem is introduced. Two different concepts of partial ordering of fractions of two intervals are introduced-LU ordering and LS ordering. With these concepts of ordering, some definitions of LU/LS-Pareto optimality are given for the fractional multivalued interval problem. In order to justify these definitions, several non-trivial examples are also illustrated. In the same section, the idea of gH-derivative for interval-valued function is further extended to the fractional interval-valued functions. In Section 3, we introduce the definitions of LU-V-invex and LS-V-invex for the multivalued fractional interval problem. Assuming the functions to be gH-differentiable and using the convexity notion, we derive the KKT optimality conditions for the problem. Moreover, considering the concept of LU/LS-V-invexity, the KKT optimality conditions for the multivalued fractional interval problem are developed. In Section 4, using the gH-derivative concept for the fractional interval multivalued problem, the KKT sufficient conditions are derived. To verify the results established, various suitable examples are constructed and validated at suitable places. Eventually, the last section gives a concluding note and some future research directions.

2. Notations and preliminaries

Consider the following fractional interval multivalued programming problem:

$$\text{(FIVP)}\quad \text{Minimize } \frac{F(x)}{G(x)} = \left\{\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_l(x)}{g_l(x)}\right\}$$
$$\text{subject to } x \in X = \{x \in R^n : h_j(x) \leq 0,\ j = 1, 2, \ldots, m\} \subseteq R^n,$$

where $f_k(x) = [f_k^L(x), f_k^U(x)]$ and $g_k(x) = [g_k^L(x), g_k^U(x)],\ k = 1, 2, \ldots, l$ are interval valued functions and X is a convex subset of R^n. It is assumed that $g_k^L > 0,\ \forall\ x \in X$ and $k = 1, 2, \ldots, l$.

Remark 2.1. An interval valued function of the form $\frac{f_k(x)}{g_k(x)} = \frac{[f_k^L(x), f_k^U(x)]}{[g_k^L(x), g_k^U(x)]}$ can be written as follows:

$$\frac{f_k(x)}{g_k(x)} = [f_k^L(x), f_k^U(x)] \times \left[\frac{1}{g_k^U(x)}, \frac{1}{g_k^L(x)}\right] = \left[\frac{f_k^L(x)}{g_k^U(x)}, \frac{f_k^U(x)}{g_k^L(x)}\right],$$

provided $g_k^L > 0$, for all $x \in X$.

2.1. Operations on intervals

Let the set of all closed and bounded intervals in R be represented as I. Let $T = [t^L, t^U] \in I$, $P = [p^L, p^U] \in I$ and $U = [u^L, u^U] \in I$, then

(i) $T + P = \{t + p : \ t \in T \text{ and } p \in P\} = [t^L + p^L, t^U + p^U]$,
(ii) $-T = \{-t : \ t \in T\} = [-t^U, -t^L]$,
(iii) $T - P = T + (-P) = [t^L - p^U, t^U - p^L]$,
(iv) $\beta T = \begin{cases} [\beta t^L, \beta t^U] & \text{if } \beta \geq 0, \\ [\beta t^U, \beta t^L] & \text{if } \beta < 0. \end{cases}$

The generalized Hukuhara difference or gH-difference for the intervals is defined as:

$$T -_g P = U \Longleftrightarrow \begin{cases} T = P + U \text{ or}, \\ P = T + (-1)U \end{cases}$$

where $T -_g P$ always exists and is calculated as (Stefanini and Bede [21]):

$$T -_g P = \Big[\min\{t^L - p^L, t^U - p^U\}, \max\{t^L - p^L, t^U - p^U\}\Big].$$

2.2. Partial ordering and solution concepts

Let $T = [t^L, t^U] \in I$ and $P = [p^L, p^U] \in I$. Then

(i) $T \preceq_{LU} P$ iff $t^L \leq p^L$ and $t^U \leq p^U$.
(ii) $T \prec_{LU} P$ iff $T \preceq_{LU} P$ and $T \neq P$, equivalently,
$\begin{cases} t^L < p^L \\ t^U \leq p^U, \end{cases}$ or $\begin{cases} t^L \leq p^L \\ t^U < p^U, \end{cases}$ or $\begin{cases} t^L < p^L \\ t^U < p^U. \end{cases}$

Let $\frac{A}{B} = \left(\frac{A_1}{B_1}, \ldots, \frac{A_l}{B_l}\right)$ and $\frac{C}{D} = \left(\frac{C_1}{D_1}, \ldots, \frac{C_l}{D_l}\right)$ be two fractional interval valued vectors, where $A_k, \ B_k, \ C_k, \ D_k \in I, \ B_k, \ D_k \in I^+, \ k = 1, 2, \ldots, l$, where I^+ denotes the positive interval. Then

(i) $\frac{A}{B} \preceq_{LU} \frac{C}{D}$ iff $\frac{A_k}{B_k} \preceq_{LU} \frac{C_k}{D_k}$, for all $k = 1, 2, \ldots, l$.
(ii) $\frac{A}{B} \prec_{LU} \frac{C}{D}$ iff $\frac{A_k}{B_k} \preceq_{LU} \frac{C_k}{D_k}$, $k = 1, 2, \ldots, l$ and $\frac{A_j}{B_j} \prec_{LU} \frac{C_j}{D_j}$ for at least one $j \in \{1, 2, \ldots, l\}$.

We recall some definitions introduced by Singh *et al.* [18].

Definition 2.2. A feasible solution $x^0 \in X$ is said to be a LU-Pareto optimal solution of (FIVP) if there does not exist any $x \in X$, such that

$$\frac{F(x)}{G(x)} \prec_{LU} \frac{F(x^0)}{G(x^0)}.$$

Definition 2.3. A feasible solution $x^0 \in X$ is said to be a strongly LU-Pareto optimal solution of (FIVP) if there does not exist any $x \in X$, such that

$$\frac{F(x)}{G(x)} \preceq_{LU} \frac{F(x^0)}{G(x^0)}.$$

Definition 2.4. A feasible solution $x^0 \in X$ is said to be a weakly LU-Pareto optimal solution of (FIVP) if there does not exist any $x \in X$, such that

$$\frac{f_k(x)}{g_k(x)} \prec_{LU} \frac{f_k(x^0)}{g_k(x^0)}, \text{ for all } k = 1, 2, \ldots, l.$$

Remark 2.5. [24] Let X, Y and Z denotes the set of strongly LU-Pareto optimal solutions, LU-Pareto optimal solutions and weakly LU-Pareto optimal solutions, respectively. Then,

$$X \subseteq Y \subseteq Z.$$

Let $T = [t^L, t^U]$ and $P = [p^L, p^U]$ be two intervals. Suppose $t^S = t^U - t^L$ and $p^S = p^U - p^L$ denotes the width of the interval's T and P, respectively. Then another way of defining partial-ordering between T and P (called LS ordering) is as follows:

(*i*) $T \preceq_{\mathrm{LS}} P$ iff $t^L \le p^L$ and $t^S \le p^S$.

(*ii*) $T \prec_{\mathrm{LS}} P$ iff $T \preceq_{\mathrm{LS}} P$ and $T \neq P$, equivalently, $\begin{cases} t^L < p^L \\ t^S \le p^S, \end{cases}$ or $\begin{cases} t^L \le p^L \\ t^S < p^S, \end{cases}$ or $\begin{cases} t^L < p^L \\ t^S < p^S. \end{cases}$

Definition 2.6. A feasible solution $x^0 \in X$ is said to be an LS-Pareto optimal solution of (FIVP) if there does not exist any other $x \in X$, such that

$$\frac{F(x)}{G(x)} \prec_{\mathrm{LS}} \frac{F(x^0)}{G(x^0)}.$$

Definition 2.7. A feasible solution $x^0 \in X$ is said to be a strongly LS-Pareto optimal solution of (FIVP) if there does not exist any other $x \in X$, such that

$$\frac{F(x)}{G(x)} \preceq_{\mathrm{LS}} \frac{F(x^0)}{G(x^0)}.$$

Definition 2.8. A feasible solution $x^0 \in X$ is said to be a weakly LS-Pareto optimal solution of (FIVP) if there does not exist any other $x \in X$, such that

$$\frac{f_k(x)}{g_k(x)} \prec_{\mathrm{LS}} \frac{f_k(x^0)}{g_k(x^0)}, \; k = 1, 2, \ldots, l.$$

Remark 2.9. [18] Let L, M and N denotes the set of strongly LS-Pareto optimal solutions, LS-Pareto optimal solutions and weakly LS-Pareto optimal solutions, respectively. Then,

$$L \subseteq M \subseteq N.$$

Theorem 2.10. *For a feasible solution $\hat{x} \in X_0$ of (FIVP), the following conditions hold:*

(i) If $\hat{x}$ is a strongly LU-Pareto optimal of (FIVP), then $\hat{x}$ is a strongly LS-Pareto optimal of (FIVP).
(ii) If $\hat{x}$ is an LU-Pareto optimal of (FIVP), then $\hat{x}$ is a an LS-Pareto optimal of (FIVP).
(iii) If $\hat{x}$ is a weakly LU-Pareto optimal of (FIVP), then $\hat{x}$ is a weakly LS-Pareto optimal of (FIVP).

Proof. The proof follows the argument on Theorem 3.1 in [18]. □

The following Examples 2.11 and 2.12 illustrates whether or not a feasible solution of type (FIVP) is strongly LU-Pareto optimal, LU-Pareto optimal or weakly LU-Pareto optimal.

Example 2.11. Consider the following problem:

$$\text{(FP1)} \;\; \min \;\; \frac{f(x)}{g(x)} = \left\{ \frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)} \right\},$$

where

$$f_1(x) = [x, x+1], \;\; f_2 = [e^{x+1} - x - 2, 2e^{x+1} - x - 3],$$
$$g_1 = [x+2, x+3] \text{ and } g_2 = [x+2, x+3],$$

subject to

$$x \in X = [-1, 1].$$

Then,

$$\frac{f_1(x)}{g_1(x)} = \Big[\frac{x}{x+3}, \frac{x+1}{x+2}\Big],$$

and

$$\frac{f_2(x)}{g_2(x)} = \Big[\frac{e^{x+1} - x - 2}{x+3}, \frac{2e^{x+1} - x - 3}{x+2}\Big].$$

$\frac{f_1(x)}{g_1(x)}$ and $\frac{f_2(x)}{g_2(x)}$ defined above are the intervals since

$$\phi_{11} = \frac{x}{x+3} - \frac{x+1}{x+2} = \frac{-(2x+3)}{(x+2)(x+3)} \leq 0, \;\; \forall \; x \in X,$$

and from Figure 1

$$\phi_{12} = \frac{e^{x+1} - x - 2}{x+3} - \frac{2e^{x+1} - x - 3}{x+2} = \frac{e^{(x+1)}(-x-4) + 2x + 5}{(x+2)(x+3)} \leq 0, \;\; \forall \; x \in X.$$

The feasible point $\hat{x} = -1$ is strongly LU-Pareto optimal, since from the Figure 2, we can easily obtain that there does not exist any $x \in X$ $x \neq 1$, such that

$$\frac{f_1}{g_1}(x) \preceq_{LU} \frac{f_1}{g_1}(\hat{x} = -1) = \Big[\frac{-1}{2}, 0\Big]$$
$$\text{or, } \; \ell_{11} = \frac{x}{x+3} \leq \frac{-1}{2} \text{ and } \ell_{12} = \frac{x+1}{x+2} \leq 0.$$

Also, from Figure 3, it is clear that there does not exist any $x \in X, \; x \neq 1$, such that

$$\frac{f_2}{g_2}(x) \preceq_{LU} \frac{f_2}{g_2}(\hat{x} = -1) = [0, 0]$$
$$\text{or, } \; \ell_{21} = \frac{e^{x+1} - x - 2}{x+3} \leq 0 \text{ and } \ell_{22} = \frac{2e^{x+1} - x - 3}{x+2} \leq 0.$$

Therefore, $\hat{x} = -1$ is a strongly LU-Pareto optimal point. Hence, $\hat{x} = 1$ is LU-Pareto optimal solution and consequently, a weakly LU-Pareto optimal for the problem (FP1).

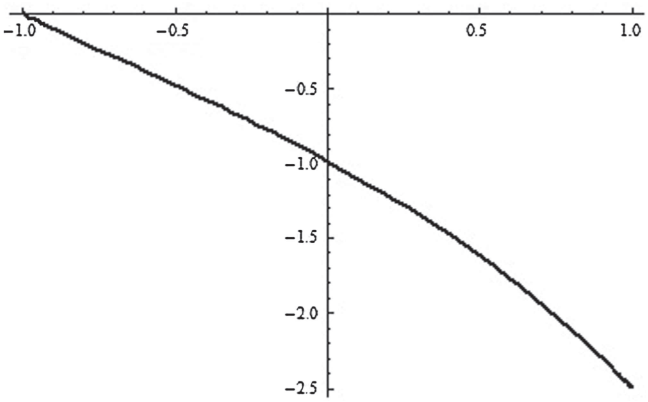

FIGURE 1. The graph of ϕ_{12} subject to $x \in X$.

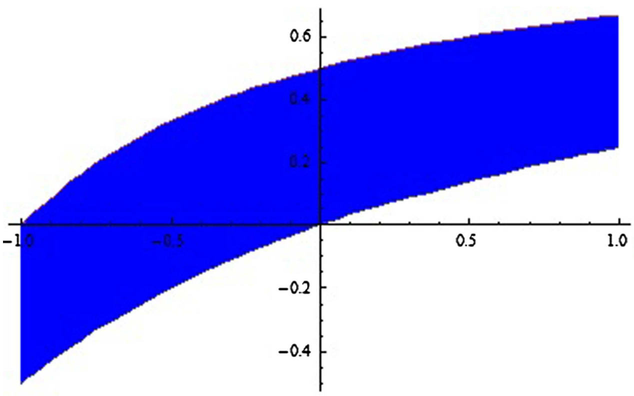

FIGURE 2. The lower graph is of function ℓ_{11} and upper is of ℓ_{12} with respect to $x \in X$.

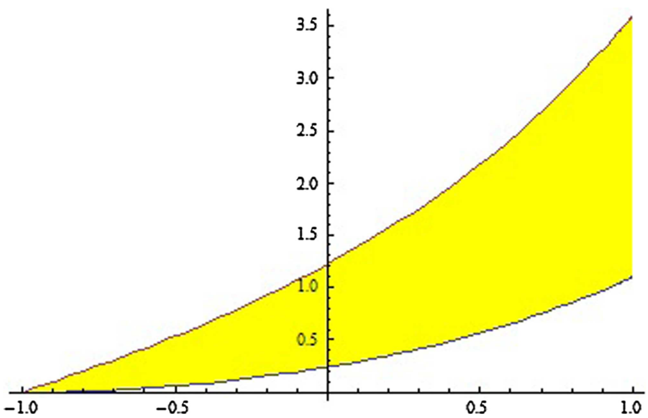

FIGURE 3. The lower graph is of function ℓ_{21} and upper is of ℓ_{22} with respect to $x \in [-1, 1]$.

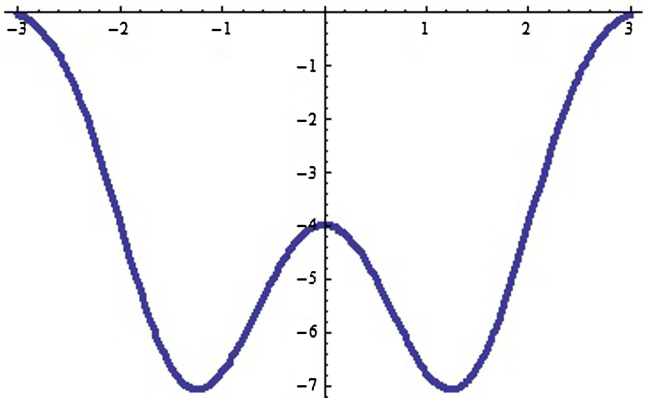

FIGURE 4. The graph of ξ_{11} subject to the constraints.

Example 2.12. Consider the following problem:

$$\text{(FP2)}\quad \text{Minimize}\quad \frac{f_1(x)}{g_1(x)},$$

where

$$f_1(x) = [\sin^2 x, 2(\sin^2 x + \cos x + 1)],\ \ g_1(x) = [\sin^2 x + 1, \sin^2 x + 1.5],$$

subject to

$$x \in X = [-3, 3].$$

Then

$$\frac{f_1(x)}{g_1(x)} = \Big[\frac{\sin^2 x}{\sin^2 x + 1.5}, \frac{2(\sin^2 x + \cos x + 1)}{(\sin^2 x + 1)}\Big].$$

$\frac{f_1(x)}{g_1(x)}$ defined above is an interval since

$$\begin{aligned}\xi_{11} &= \frac{\sin^2 x}{\sin^2 x + 1.5} - \frac{2(\sin^2 x + \cos x + 1)}{\sin^2 x + 1}\\ &= \frac{-(\sin^4 x + 4\sin^2 x + 2\sin^2 x\cos x + 3\cos x + 3)}{(\sin^2 x + 1.5)(\sin^2 x + 1)}\\ &\leq 0, (\text{see Fig. 4})\ \forall\ x \in X.\end{aligned}$$

A.1. The point $x = 2$ is not weakly LU-Pareto optimal

On the contrary, let $x = 2$ be an weakly LU-Pareto optimal for the problem (FP2). Then, by definition there does not exist any $x \in X,\ x \neq 2$ such that

$$\frac{f_1}{g_1}(x) \prec_{LU} \frac{f_1}{g_1}(\hat{x} = 2).$$

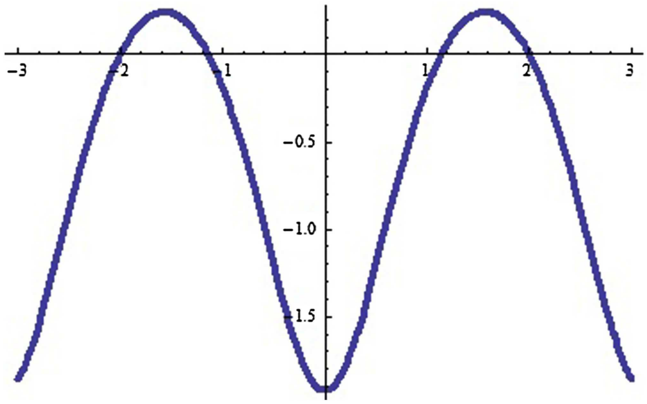

FIGURE 5. The graph of ξ_{21} against $x \in X$.

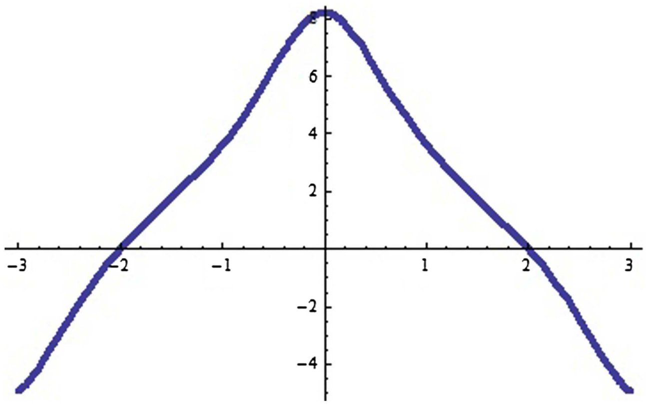

FIGURE 6. The graph of ξ_{22} against $x \in X$.

That is,

$$\left[\frac{\sin^2 x}{\sin^2 x+1.5}, \frac{2(\sin^2 x+\cos x+1)}{(\sin^2 x+1)}\right] \prec_{LU} \left[\frac{\sin^2 2}{\sin^2 2+1.5}, \frac{2(\sin^2 2+\cos 2+1)}{(\sin^2 2+1)}\right],$$

or,

$$\xi_{21} = \frac{3(\sin^2 x - \sin^2 2)}{2(\sin^2 x+1.5)(\sin^2 2+1.5)} < 0 \text{ and}$$

$$\xi_{22} = \frac{2\cos x(\sin^2 2+1) - 2\cos 2(\sin^2 x+1)}{(\sin^2 x+1)(\sin^2 2+1)} < 0.$$

But, from Figures 5 and 6, there are points in $[-3,-2) \cup (2,3]$ such that the above inequalities hold true. Thus, $\hat{x} = 2$ is not a weakly LU-Pareto optimal for (FP2).

Adding another fractional interval function $\frac{f_2(x)}{g_2(x)}$ to problem (FP2) yields:

$$\text{(FP3)}\quad \text{Minimize } \frac{f(x)}{g(x)} = \left\{\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}\right\},$$

where

$$f_2(x) = [x^2, x^2 + 1/2],\ \ g_2(x) = [x^2 + 1, x^2 + 2],$$

subject to

$$x \in X = [-3, 3].$$

Then

$$\frac{f_2(x)}{g_2(x)} = \Big[\frac{x^2}{x^2+2}, \frac{2x^2+1}{2(x^2+1)}\Big].$$

Now, $\frac{f_2(x)}{g_2(x)}$ defined above is again an interval since

$$\frac{x^2}{x^2+2} - \frac{2x^2+1}{2(x^2+1)} = \frac{-(3x^2+2)}{2(x^2+1)(x^2+2)} \leq 0,\ \forall\, x \in X.$$

A.2. The point x=0 is strongly LU-Pareto optimal of (FP3)

Suppose $x = 0$ is not the strongly LU-Pareto optimal solution of (FP3). Then, $\exists$ some $0 \neq x \in X$ such that

$$\frac{f_2(x)}{g_2(x)} \preceq_{LU} \frac{f_2(0)}{g_2(0)}.$$

That is,

$$\Big[\frac{x^2}{x^2+2}, \frac{x^2+1/2}{x^2+1}\Big] \preceq_{LU} \Big[0, \frac{1}{2}\Big]$$
$$\text{or,}\ \ \frac{x^2}{x^2+2} \leq 0 \text{ and } \ \frac{2x^2+1}{2(x^2+1)} \leq \frac{1}{2},$$

which is not possible for all $0 \neq x \in X$. Hence, $\nexists$ any $0 \neq x \in X$ such that

$$\frac{f(x)}{g(x)} \preceq_{LU} \frac{f(0)}{g(0)}.$$

Therefore, $x = 0$ is strongly LU-Pareto optimal solution for (FP3).

A.3. No point in X other than x=0 is strongly LU-Pareto optimal of (FP3)

To prove that $0 \neq \alpha \in X$ is not strongly LU-Pareto optimal, we have to show the existence of a point other than $\hat{x} = \alpha$, such that

$$\frac{f(x)}{g(x)} \preceq_{LU} \frac{f(\hat{x}=\alpha)}{g(\hat{x}=\alpha)}.$$

Since the lower and upper bounds of the intervals of $\frac{f_1(x)}{g_1(x)}$ (see Fig. 7) and $\frac{f_2(x)}{g_2(x)}$ (see Fig. 8) are even, therefore, $x = -\alpha$ is the point such that

$$\frac{f}{g}(x = -\alpha) = \frac{f}{g}(\hat{x} = \alpha).$$

Hence, $\hat{x} = \alpha \neq 0$ is not strongly LU-Pareto optimal for the problem (FP3).

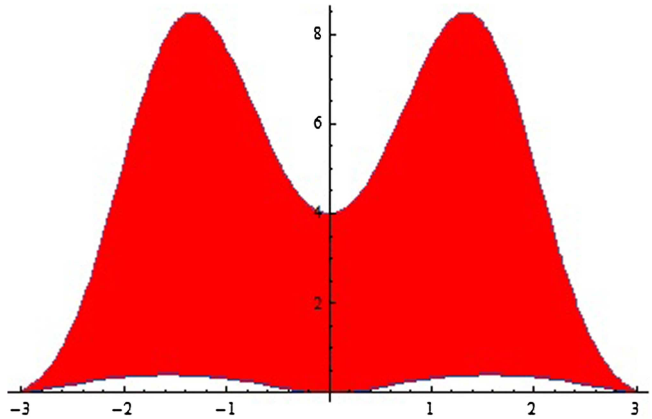

FIGURE 7. The lower and upper graph represents the lower and upper bounds of $\frac{f_1}{g_1}$, respectively.

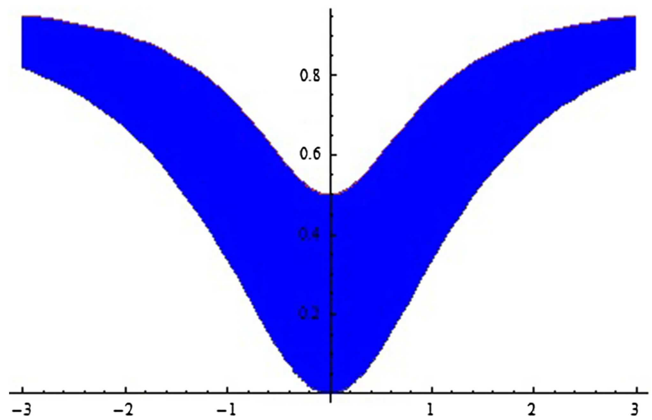

FIGURE 8. The lower and upper graph represents the lower and upper bounds of $\frac{f_2}{g_2}$, respectively.

2.3. *gH*-derivative of fractional interval valued functions

For a nonempty set $X \subseteq R^n$, let $f,\ g:\ X \to I$ and $\frac{f}{g},\ g > 0$ be the fractional interval valued function, where

$$f(x) = [f^L(x), f^U(x)] \text{ and } g(x) = [g^L(x), g^U(x)].$$

Then

$$\frac{f(x)}{g(x)} = \Big[\frac{f^L(x)}{g^U(x)}, \frac{f^U(x)}{g^L(x)}\Big].$$

For further discussion, we consider the class of open intervals (x_1, x_2) and denote it by J.

Definition 2.13. [21] For $f : J \to I$, the gH-derivative is defined as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) -_g f(x_0)}{h}.$$

Further, f is said to be gH-differentiable if $f'(x_0)$ exists at x_0. Moreover, if f is gH-differentiable at each $x_0 \in J$, then f is gH-differentiable on J. Here, in the metric space (I, M) the limits are considered and M is defined as:

$$M(T, P) = \max\Big\{ \max_{t \in T} d(t, P), \max_{p \in P} d(T, p) \Big\},$$

where $d(t, P) = \min_{p \in P} ||t - p||$.

Chalco-Cano *et al.* [8] gave the concept of gH-differentiability for interval valued functions at a point. We extend this concept for fractional interval valued functions.

Theorem 2.14. *If $\frac{f^L}{g^U}$ and $\frac{f^U}{g^L}$ are differentiable at $x_0 \in J$, then $\frac{f}{g}$ is gH-differentiable at x_0 and is defined as*

$$\Big(\frac{f}{g}\Big)'(x_0) = \Big[min\Big\{\Big(\frac{f^L}{g^U}\Big)'(x_0), \Big(\frac{f^U}{g^L}\Big)'(x_0)\Big\}, max\Big\{\Big(\frac{f^L}{g^U}\Big)'(x_0), \Big(\frac{f^U}{g^L}\Big)'(x_0)\Big\}\Big].$$

Theorem 2.15. *The fractional interval function $\frac{f}{g}$ is gH-differentiable at $x_0 \in J$ iff the following conditions hold:*

(i) $\frac{f^L}{g^U}$ *and* $\frac{f^U}{g^L}$ *are differentiable at* x_0.
(ii) $\Big(\frac{f^L}{g^U}\Big)'_-(x_0)$, $\Big(\frac{f^L}{g^U}\Big)'_+(x_0)$, $\Big(\frac{f^U}{g^L}\Big)'_-(x_0)$ *and* $\Big(\frac{f^U}{g^L}\Big)'_+(x_0)$ *exist and* $\Big(\frac{f^L}{g^U}\Big)'_-(x_0) = \Big(\frac{f^U}{g^L}\Big)'_+(x_0)$ *and* $\Big(\frac{f^L}{g^U}\Big)'_+(x_0) = \Big(\frac{f^U}{g^L}\Big)'_-(x_0)$.

Theorem 2.16. [6] *If $\frac{f}{g} : J \to I$ is gH-differentiable at $x_0 \in J$, then $\frac{f^L}{g^U} + \frac{f^U}{g^L}$ is also differentiable at x_0.*

Definition 2.17. [18] Let $X \subseteq R^n$ and $\frac{f}{g}$, be a fractional interval valued function defined on X. For $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in X$. Consider the fractional interval valued function $\frac{P_i(x_i^0)}{Q_i(x_i^0)} = \frac{f(x_1^0,\dots,x_{i-1}^0,x_i^0,x_{i+1}^0,\dots,x_n^0)}{g(x_1^0,\dots,x_{i-1}^0,x_i^0,x_{i+1}^0,\dots,x_n^0)}$. If $\frac{P_i}{Q_i}$ is gH-differentiable at x_i^0, then $\frac{f}{g}$ has the *ith* partial gH-derivative at x^0, also denoted by $\Big(\frac{\partial(f/g)}{\partial x_i}\Big)_g(x^0)$, where

$$\Big(\frac{\partial(f/g)}{\partial x_i}\Big)_g(x^0) = \Big(\frac{P_i}{Q_i}\Big)'(x_i^0).$$

Definition 2.18. A fractional interval valued function $\frac{f}{g}$ is said to be continuously gH-differentiable at x^0 if $\Big(\frac{\partial(f/g)}{\partial x_i}\Big)_g(x^0),\ i = 1, 2, \dots, n$ exist on some neighbourhood of x^0 and are continuous at x^0.

Proposition 2.19. [6] *If $\frac{f}{g}$ is continuously gH-differentiable at x^0, then $\frac{f^L}{g^U} + \frac{f^U}{g^L}$ is also continuously differentiable at x^0.*

Definition 2.20. The function $\frac{F}{G}$ in (FIVP) is continuously gH-differentiable at $x^0 \in X$ if $\frac{f_k}{g_k},\ k = 1, 2, \dots, l$ are continuously gH-differentiable at x^0.

Proposition 2.21. *The function $\frac{F}{G}$ in (FIVP) is continuously gH-differentiable at x^0 if $\frac{f_k^L}{g_k^U} + \frac{f_k^U}{g_k^L},\ k = 1, 2, \dots, l$ are continuously differentiable at x^0.*

3. Optimality conditions

In this section, we study the Karush Kuhn Tucker optimality conditions using the gH-differentiability notion for the fractional interval multivalued problem. Further, the concepts of LU-V-invex and LS-V-invex are introduced, and the KKT conditions are established considering the functions involved as LU-V-invex and LS-V-invex.

Consider the following fractional programming problem:

$$\text{(FP)} \;\; \min \;\; \frac{f(x)}{g(x)} = \frac{f(x_1, x_2, \ldots, x_n)}{g(x_1, x_2, \ldots, x_n)}$$
$$\text{subject to } h_j(x) \leq 0, \;\; j = 1, 2, \ldots, m,$$

where $X_0 = \{x \in R^n : \; h_j(x) \leq 0, \; j = 1, 2, \ldots, m\}$ is a nonempty compact convex feasible set and $f, \; g : R^n \to R$ are continuously differentiable functions on X_0 with $g(x) > 0, \forall \; x \in X_0$.

Theorem 3.1. [19] *Let $\hat{x} \in X_0$ be a feasible solution of (FP) and the functions $f, \; g : R^n \to R$ be continuously differentiable at $\hat{x}$. Let a non-negative function f be convex at $\hat{x}$ and a positive function g be concave at $\hat{x}$. If there exist Lagrange multipliers $\eta_j \geq 0, \; j = 1, 2, \ldots, m$, such that*

(i) $\nabla\Big(\frac{f}{g}\Big)(\hat{x}) + \sum_{j=1}^{m} \eta_j \nabla h_j(\hat{x}) = 0,$
(ii) $\eta_j h_j(\hat{x}) = 0; \;\; j = 1, 2, \ldots, m.$

Then, $\hat{x}$ is optimal for the problem (FP).

In order to derive the KKT conditions for a single objective fractional interval valued problem, the following problem is considered:

$$\text{(SIFP)} \;\; \min \;\; \frac{f(x)}{g(x)} = \Big[\frac{f^L(x)}{g^U(x)}, \frac{f^U(x)}{g^L(x)}\Big]$$
$$\text{subject to } h_j(x) \leq 0, \;\; j = 1, 2, \ldots, m,$$

where $f(x) = [f^L(x), f^U(x)], \;\; g(x) = [g^L(x), g^U(x)], \; k = 1, 2, \ldots, l, \;\; g^L > 0$, are interval valued functions and $h_i, \;\; i = 1, 2, \ldots, m$ are real valued functions.

Theorem 3.2. *Let $\hat{x}$ be a feasible solution of (SIFP). Suppose that*

(i) $\frac{f(x)}{g(x)}$ *is continuously gH-differentiable at $\hat{x} \in X_0$,*
(ii) $(f^L g^L + g^U f^U)$ *is non-negative and a convex function,*
(iii) a positive function $g^L g^U$ is concave.

Further, if there exist Lagrange multipliers $0 \leq \eta_j \in R, \; j = 1, 2, \ldots, m$ such that the following conditions hold:

(a) $\nabla\Big(\Big(\frac{f^L}{g^U}\Big) + \Big(\frac{f^U}{g^L}\Big)\Big)(\hat{x}) + \sum_{j=1}^{m} \eta_j \nabla h_j(\hat{x}) = 0.$
(b) $\eta_j h_j(\hat{x}) = 0, \;\; j = 1, 2, \ldots, m.$

Then $\hat{x}$ is LU-Pareto optimal and hence LS-Pareto optimal for (SIFP).

Proof. Let

$$\frac{\bar{f}(x)}{\bar{g}(x)} = \Big(\frac{f^L}{g^U} + \frac{f^U}{g^L}\Big)(x). \tag{3.1}$$

Now, $\frac{f^L}{g^U}, \; \frac{f^U}{g^L}$ are real valued functions and $\frac{f^L}{g^U} + \frac{f^U}{g^L}$ is continuously differentiable at $\hat{x} \in X_0$, (from Prop. 2.19). Therefore, we have

$$\nabla\Big(\frac{\bar{f}(x)}{\bar{g}(x)}\Big) = \nabla\Big(\frac{f^L}{g^U} + \frac{f^U}{g^L}\Big)(x) = \nabla\Big(\frac{f^L g^L + f^U g^U}{g^L g^U}\Big)(x). \tag{3.2}$$

By the hypotheses (ii) and (iii), there exist Lagrange multipliers $0 \le \eta_j \in R$, $j = 1, 2, \dots, m$, and using (a) and (b), we obtain

$$\nabla\Big(\frac{\bar{f}}{\bar{g}}\Big)(\hat{x}) + \sum_{j=1}^{m} \eta_j \nabla h_j(\hat{x}) = 0 \text{ and}$$
$$\eta_j h_j(\hat{x}) = 0,\ j = 1, 2, \dots, m.$$

Hence, from Theorem 3.1, $\hat{x}$ is optimal for $\Big(\frac{\bar{f}}{\bar{g}}\Big)$ with the constraints of (SIFP).

Thus, for every $\mathbf{x} \in X_0$, we have

$$\Big(\frac{\bar{f}}{\bar{g}}\Big)(\hat{x}) \le \Big(\frac{\bar{f}}{\bar{g}}\Big)(\mathbf{x}). \tag{3.3}$$

Now, we prove that $\hat{x}$ is LU-Pareto optimal for (SIFP). Let $\hat{x}$ be not LU-Pareto optimal.

Then, by definition, $\exists\ \mathbf{x} \in X_0$ such that

$$\frac{f}{g}(\mathbf{x}) \prec_{LU} \frac{f}{g}(\hat{x}),$$

which yields

$$\begin{cases} \frac{f^L}{g^U}(\mathbf{x}) < \frac{f^L}{g^U}(\hat{x}) \\ \frac{f^U}{g^L}(\mathbf{x}) \le \frac{f^U}{g^L}(\hat{x}), \end{cases} \quad \text{or} \quad \begin{cases} \frac{f^L}{g^U}(\mathbf{x}) \le \frac{f^L}{g^U}(\hat{x}) \\ \frac{f^U}{g^L}(\mathbf{x}) < \frac{f^U}{g^L}(\hat{x}), \end{cases} \quad \text{or} \quad \begin{cases} \frac{f^L}{g^U}(\mathbf{x}) < \frac{f^L}{g^U}(\hat{x}) \\ \frac{f^U}{g^L}(\mathbf{x}) < \frac{f^U}{g^L}(\hat{x}), \end{cases}$$

Therefore, (3.1) implies

$$\Big(\frac{\bar{f}}{\bar{g}}\Big)(\mathbf{x}) < \Big(\frac{\bar{f}}{\bar{g}}\Big)(\hat{x}),$$

which is a contradiction to (3.3).

Therefore, $\hat{x}$ is LU-Pareto optimal and thus LS-Pareto optimal (by Thm. 2.10 (ii)). This completes the proof. □

Verification of Theorem 3.2

In order to verify the Theorem 3.2, the following Examples 3.3 and 3.4 are illustrated.

Example 3.3. Suppose in (SIFP), $f(x) = [x, x+1]$, $g(x) = [-x, x+1]$ and $X = [-0.5, -0.1]$.

Then the problem (SIFP) becomes:

$$\text{(E1)}\ \ \text{Min}\ \ \frac{f(x)}{g(x)} = \Big[\frac{x}{x+1}, \frac{x+1}{-x}\Big],$$

subject to

$$-x - 0.5 \le 0,$$
$$x + 0.1 \le 0.$$

Here, the fractional interval $\frac{f}{g}$ is continuously differentiable for every feasible point.

Now, $g^L g^U = -x(x+1)$, is clearly a concave function and positive $\forall\ x \in X$.

Furthermore, $f^L g^L + g^U f^U = 2x + 1$ is a convex function $\forall\ x \in X$ and non-negative for all $x \in X$. Hence, the suppositions given in Theorem 3.2 are satisfied.

Let there exist $\eta_1,\ \eta_2 \ge 0$, such that

(i) $\nabla\Big(\frac{2\hat{x}+1}{-\hat{x}^2-\hat{x}}\Big) + \eta_1\Big(\nabla\Big(-\hat{x}-0.5\Big)\Big) + \eta_2\Big(\nabla\Big(\hat{x}+0.1\Big)\Big) = 0,$
or
$\Big(\frac{2\hat{x}^2+2\hat{x}+1}{\hat{x}^2(\hat{x}+1)^2}\Big) - \eta_1 + \eta_2 = 0.$
(ii) $\eta_1(-\hat{x}-0.5) = 0 = \eta_2(\hat{x}+0.1).$

Now, $\hat{x} = -0.5,\ \eta_1 = 0$ and $\eta_2 = 8$ satisfy the above KKT-conditions.

Next, we will show that $\hat{x} = -0.5$ is an LU-Pareto optimal for the problem (E1).

We see that, there exists no $-0.5 \le x \le -0.1$ such that

$$\Big[\frac{x}{x+1}, \frac{x+1}{-x}\Big] \prec_{LU} \Big[-1, 1\Big],$$

$$\text{which implies } \begin{cases} \frac{x}{x+1} \le -1 \\ \frac{x+1}{-x} < 1, \end{cases} \quad \text{or} \quad \begin{cases} \frac{x}{x+1} < -1 \\ \frac{x+1}{-x} \le 1, \end{cases} \quad \text{or} \quad \begin{cases} \frac{x}{x+1} < -1 \\ \frac{x+1}{-x} < 1, \end{cases}$$

Therefore, $\hat{x} = -0.5$ is LU-Pareto optimal solution and hence LS-Pareto optimal solution to the problem (E1).

Example 3.4. Consider $f(x) = [f^L, f^U]$, $g(x) = [g^L, g^U]$ where

$$f^L(x) = \frac{1}{\log x},\ f^U(x) = e^x,\ g^L(x) = \log x,\ g^U(x) = 1.$$

Then the fractional interval problem (SIFP) becomes:

$$\text{(E2)}\ \ \text{Min}\ \ \frac{f(x)}{g(x)} = \frac{[1/\log x, e^x]}{[\log x, 1]} = \Big[\frac{1/\log x}{1}, \frac{e^x}{\log x}\Big]$$

subject to

$$x - 1.4 \le 0,$$
$$-x + 1.32 \le 0$$

Let $X = \{x \in R:\ 1.32 \le x \le 1.4\}$.

Now, $g^L g^U = \log x$ is a concave function and also $g^L g^U = \log x > 0,\ \forall\ x \in X$.

Also, $f^L g^L + g^U f^U = 1 + e^x > 0$ is a convex function for all x.

Hence, the suppositions given in Theorem 3.2 holds. Suppose there exist multipliers $\eta_1,\ \eta_2 \ge 0$, such that

(i) $\Big(\frac{\hat{x}e^{\hat{x}}\log\hat{x} - 1 - e^{\hat{x}}}{\hat{x}(\log\hat{x})^2}\Big) + \eta_1 - \eta_2 = 0,$
(ii) $\eta_1(\hat{x} - 1.4) = 0 = \eta_2(-\hat{x} + 1.32).$

The above conditions (i) and (ii) are satisfied for $\hat{x} = 1.4,\ \eta_1 = 0.254$ and $\eta_2 = 0$.

Next, we will show that $\hat{x} = 1.4$ is an LU-Pareto optimal for the problem (E2).

We see that, there exists no $1.32 \le x \le 1.4$ such that

$$\Big[\frac{1/\log x}{1}, \frac{e^x}{\log x}\Big] \prec_{LU} \Big[6.85, 27.78\Big],$$

$$\text{which implies} \begin{cases} \frac{1}{\log x} \le 6.85 \\ \frac{e^x}{\log x} < 27.78, \end{cases} \quad \text{or} \quad \begin{cases} \frac{1}{\log x} < 6.85 \\ \frac{e^x}{\log x} \le 27.78, \end{cases} \quad \text{or} \quad \begin{cases} \frac{1}{\log x} < 6.85 \\ \frac{e^x}{\log x} < 27.78, \end{cases}$$

Therefore, $\hat{x} = 1.4$ is LU-Pareto optimal, hence LS-Pareto optimal solution to the problem (E2).

Before deriving the KKT conditions for the problem (FIVP), we introduce the following definitions which will be needed in the sequel.

Definition 3.5. [9] A differentiable function $f : R^n \to R$ is V-invex (strictly V-invex) at the point y with $\beta(.,y) : R^n \times R^n \to R_+$ and $\eta(.,.) : R^n \times R^n \to R^n$, if

$$f(x) - f(y) \geq (>) \beta(x,y)\nabla f(y)\eta(x,y),\ \forall\ x \in R^n.$$

Definition 3.6. The function $\frac{F}{G}$ in (FIVP) is said to be

(a) LU-V-invex iff $\frac{f_k^L}{g_k^U}$ and $\frac{f_k^U}{g_k^L}$ are V-invex for all $k = 1, 2, \ldots, l$

(b) LS-V-invex iff $\frac{f_k^L}{g_k^U}$ and $\left(\frac{f_k^U}{g_k^L} - \frac{f_k^L}{g_k^U}\right)$ are V-invex for all $k = 1, 2, \ldots, l$.

Next, we establish the KKT conditions for the problem (FIVP) using the LU-V-invex and LS-V-invex assumptions.

Theorem 3.7. *Suppose that the fractional interval multivalued function $\frac{F}{G}, F \geq 0,\ G > 0$ is continuously gH-differentiable, also LU-V-invex at $\hat{x} \in X_0$ with respect to $\lambda(x,\hat{x}) > 0$ and $\eta(x,\hat{x})$. Further, assume that $h_j(x),\ j = 1, 2, \ldots, m$, are V-invex at $\hat{x} \in X_0$ with respect to the same $\lambda(x,\hat{x})$ and $\eta(x,\hat{x})$. If there exist Lagrange multipliers $0 < \beta_k \in R$, $k = 1, 2, \ldots, l$ and $0 \leq \xi_j \in R$, $j = 1, 2, \ldots, m$ such that the conditions given below are satisfied:*

(i) $\sum_{k=1}^{l} \beta_k \nabla\Big(\Big(\frac{f_k^L}{g_k^U}\Big) + \Big(\frac{f_k^U}{g_k^L}\Big)\Big)(\hat{x}) + \sum_{j=1}^{m} \xi_j \nabla h_j(\hat{x}) = 0.$

(ii) $\xi_j h_j(\hat{x}) = 0,\ j = 1, 2, \ldots, m.$

Then $\hat{x}$ is LU-Pareto optimal of (FIVP).

Proof. On the contrary, assume that $\hat{x}$ is not an LU-Pareto optimal solution of the problem (FIVP). Then there exists $x \in X_0$, such that

$$\frac{F(x)}{G(x)} \prec_{LU} \frac{F(\hat{x})}{G(\hat{x})}$$

$$\text{which gives } \Big[\frac{f_k^L(x)}{g_k^U(x)}, \frac{f_k^U(x)}{g_k^L(x)}\Big] \prec_{LU} \Big[\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})}, \frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}\Big], \text{ for all } 1 \leq k \leq l.$$

This further implies

$$\begin{cases} \frac{f_k^L}{g_k^U}(x) < \frac{f_k^L}{g_k^U}(\hat{x}) \\ \frac{f_k^U}{g_k^L}(x) \leq \frac{f_k^U}{g_k^L}(\hat{x}), \end{cases} \quad \text{or} \quad \begin{cases} \frac{f_k^L}{g_k^U}(x) \leq \frac{f_k^L}{g_k^U}(\hat{x}) \\ \frac{f_k^U}{g_k^L}(x) < \frac{f_k^U}{g_k^L}(\hat{x}), \end{cases} \quad \text{or} \quad \begin{cases} \frac{f_k^L}{g_k^U}(x) < \frac{f_k^L}{g_k^U}(\hat{x}) \\ \frac{f_k^U}{g_k^L}(x) < \frac{f_k^U}{g_k^L}(\hat{x}). \end{cases}$$

Since $\beta_k > 0$, therefore, we obtain

$$\sum_{k=1}^{l} \beta_k \Big[\Big(\frac{f_k^L(x)}{g_k^U(x)} + \frac{f_k^U(x)}{g_k^L(x)}\Big) - \Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})} + \frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}\Big)\Big] < 0. \tag{3.4}$$

Using the LU-V-invexity of $\frac{F}{G}$ with respect to $\lambda(x,\hat{x}) > 0$ and $\eta(x,\hat{x})$, we have

$$\Big(\frac{f_k^L(x)}{g_k^U(x)} + \frac{f_k^U(x)}{g_k^L(x)}\Big) - \Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})} + \frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}\Big) \geq \lambda(x,\hat{x})\nabla\Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})} + \frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}\Big)\eta(x,\hat{x}),$$

which together with $\beta_k > 0$ yields,

$$\sum_{k=1}^{l} \beta_k \Big[\Big(\frac{f_k^L(x)}{g_k^U(x)} + \frac{f_k^U(x)}{g_k^L(x)}\Big) - \Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})} + \frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}\Big)\Big] \geq \lambda(x,\hat{x})\Big[\sum_{k}^{l} \beta_k \nabla\Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})} + \frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}\Big)\Big]\eta(x,\hat{x}). \tag{3.5}$$

Using the fact that all $h_j,\ j = 1,2,\ldots,m$ are V-invex with respect to $\lambda(x,\hat{x}) > 0$ and $\eta(x,\hat{x})$, we get

$$h_j(x) - h_j(\hat{x}) \geq \lambda(x,\hat{x})\nabla h_j(\hat{x})\eta(x,\hat{x}).$$

Using hypothesis (ii) and $\xi_j \geq 0,\ j = 1,2,\ldots,m$, we obtain

$$\sum_{j=1}^{m} \xi_j h_j(x) \geq \lambda(x,\hat{x})\Big(\sum_{j=1}^{m} \xi_j \nabla h_j(\hat{x})\Big)\eta(x,\hat{x}). \tag{3.6}$$

Now, (3.5) and (3.6), together imply

$$\sum_{k=1}^{l} \beta_k \Big[\Big(\frac{f_k^L(x)}{g_k^U(x)} + \frac{f_k^U(x)}{g_k^L(x)}\Big) - \Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})} + \frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}\Big)\Big] + \sum_{j=1}^{m} \xi_j h_j(x) \geq \lambda(x,\hat{x})\Big[\sum_{k}^{l} \beta_k \nabla\Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})} + \frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}\Big) + \sum_{j=1}^{m} \xi_j \nabla h_j(\hat{x})\Big]\eta(x,\hat{x}).$$

This using (i) yields

$$\begin{aligned}&\sum_{k=1}^{l} \beta_k \Big[\Big(\frac{f_k^L(x)}{g_k^U(x)} + \frac{f_k^U(x)}{g_k^L(x)}\Big) - \Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})} + \frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}\Big)\Big] \\ &\geq -\sum_{j=1}^{m} \xi_j h_j(x) \\ &\geq 0, (\text{since } \xi_j \geq 0 \text{ and } h_j(x) \leq 0),\end{aligned}$$

which contradicts (3.4). Hence, $\hat{x}$ is LU-Pareto optimal for (FIVP). □

Verification of Theorem 3.7

Example 3.8. Consider the fractional interval problem:

$$\text{(E3)}\ \ \text{Min}\ \ \frac{f(x)}{g(x)} = \frac{[x^2+1, x^2+2]}{[x+2, x+3]} = \Big[\frac{x^2+1}{x+3}, \frac{x^2+2}{x+2}\Big],$$

subject to

$$\begin{aligned} x - 2 &\leq 0,\\ -x + 1 &\leq 0.\end{aligned}$$

Let $\lambda(x,y) = 1$ and $\eta(x,y) = x - y$.

Now, we first show that $\frac{f(x)}{g(x)}$ is LU-V-invex at $y = 1$.

$$\begin{aligned}&\Big(\frac{f^L}{g^U}\Big)(x) - \Big(\frac{f^L}{g^U}\Big)(1) - \lambda(x,1)\nabla\Big(\frac{f^L}{g^U}\Big)(1)\eta(x,1)\\&= \frac{x^2+1}{x+3} - \frac{1}{2} - \frac{3}{8}(x-1)\\&= \frac{5(x-1)^2}{8(x+3)}\\&\geq 0, \text{ for all } x \in [1,2].\end{aligned}$$

Also,

$$\begin{aligned}&\Big(\frac{f^U}{g^L}\Big)(x) - \Big(\frac{f^U}{g^L}\Big)(1) - \lambda(x,1)\nabla\Big(\frac{f^U}{g^L}\Big)(1)\eta(x,1)\\&= \frac{x^2+2}{x+2} - 1 - \frac{1}{3}(x-1)\\&= \frac{2(x-1)^2}{3(x+2)}\\&\geq 0, \text{ for all } x \in [1,2].\end{aligned}$$

Next, we shall prove that $h_j,\ j = 1, 2$, are V-invex.
For $h_1(x) = x - 2 \leq 0$,

$$\begin{aligned}&h_1(x) - h_1(1) - \lambda(x,1)\nabla h_1(1)\eta(x,1)\\&= (x-2) - (1-2) - (x-1) = 0.\end{aligned}$$

For $h_2(x) = -x + 1 \leq 0$,

$$\begin{aligned}&h_2(x) - h_2(1) - \lambda(x,1)\nabla h_2(1)\eta(x,1)\\&= (-x+1) - (-1+1) - (-1)(x-1) = 0.\end{aligned}$$

Therefore, the hypotheses of the Theorem 3.7 are satisfied. Further, at $\hat{x} = 1$, there exist $\xi_1 = 0$ and $\xi_2 = \frac{17}{24}$, $\beta = 1$ such that

(i) $\beta\Big(\frac{2\hat{x}^4+20\hat{x}^3+58\hat{x}^2+44\hat{x}-22}{(\hat{x}^2+5\hat{x}+6)^2}\Big) + \xi_1 - \xi_2 = 0$,
(ii) $\xi_1(\hat{x}-2) = 0 = \xi_2(-\hat{x}+1)$.

Next, we will show that $\hat{x} = 1$ is an LU-Pareto optimal for the problem (E3).

We see that, there exists no $1 \leq x \leq 2$ such that

$$\Big[\frac{x^2+1}{x+3}, \frac{x^2+2}{x+2}\Big] \prec_{LU} \Big[\frac{1}{2}, 1\Big],$$

$$\text{which implies } \begin{cases}\frac{x^2+1}{x+3} \leq \frac{1}{2}\\ \frac{x^2+2}{x+2} < 1,\end{cases} \text{ or } \begin{cases}\frac{x^2+1}{x+3} < \frac{1}{2}\\ \frac{x^2+2}{x+2} \leq 1,\end{cases} \text{ or } \begin{cases}\frac{x^2+1}{x+3} < \frac{1}{2}\\ \frac{x^2+2}{x+2} < 1,\end{cases}$$

Therefore, $\hat{x} = 1$ is LU-Pareto optimal solution to the problem (E3).

Theorem 3.9. *Let $\hat{x} \in X_0$ be a feasible solution of (FIVP). Suppose that*

(i) the function $\frac{F}{G}, F \geq 0,\ G > 0$ is continuously gH-differentiable at $\hat{x} \in X_0$,
(ii) $\frac{F}{G}$ is LS-V-invex at $\hat{x} \in X_0$ with respect to $\lambda(x,\hat{x}) > 0$ and $\eta(x,\hat{x})$,
(iii) $h_j(x),\ j = 1, 2, \ldots, m$, are V-invex at $\hat{x} \in X_0$ with respect to the same $\lambda(x,\hat{x}) > 0$ and $\eta(x,\hat{x})$.

Furthermore, let there exist Lagrange multipliers $0 < \beta_k^L, \beta_k^S \in R$, $k = 1, 2, \dots, l$ *and* $0 \le \xi_j \in R$, $j = 1, 2, \dots, m$ *such that*

(a) $\sum_{k=1}^{l} \beta_k^L \nabla\Big(\frac{f_k^L}{g_k^U}\Big)(\hat{x}) + \sum_{k=1}^{l} \beta_k^S \nabla\Big(\frac{f_k^U}{g_k^L} - \frac{f_k^L}{g_k^U}\Big)(\hat{x}) + \sum_{j=1}^{m} \xi_j \nabla h_j(\hat{x}) = 0.$
(b) $\xi_j h_j(\hat{x}) = 0, \ j = 1, 2, \dots, m.$

Then $\hat{x}$ *is LS-Pareto optimal for (FIVP).*

Proof. We shall prove the result by contradiction. Let $\hat{x}$ be not an LS-Pareto optimal solution for (FIVP). Then there exists $x \in X_0$ and $1 \le k \le l$, such that

$$\frac{F(x)}{G(x)} \prec_{\mathrm{LS}} \frac{F(\hat{x})}{G(\hat{x})}$$

which yields $\Big[\frac{f_k^L(x)}{g_k^U(x)}, \frac{f_k^U(x)}{g_k^L(x)}\Big] \prec_{\mathrm{LS}} \Big[\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})}, \frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}\Big]$. Hence,

$$\begin{cases} \frac{f_k^L}{g_k^U}(x) < \frac{f_k^L}{g_k^U}(\hat{x}) \\ \Big(\frac{f_k^U}{g_k^L} - \frac{f_k^L}{g_k^U}\Big)(x) \le \Big(\frac{f_k^U}{g_k^L} - \frac{f_k^L}{g_k^U}\Big)(\hat{x}), \end{cases}$$

$$\text{or } \begin{cases} \frac{f_k^L}{g_k^U}(x) \le \frac{f_k^L}{g_k^U}(\hat{x}) \\ \Big(\frac{f_k^U}{g_k^L} - \frac{f_k^L}{g_k^U}\Big)(x) < \Big(\frac{f_k^U}{g_k^L} - \frac{f_k^L}{g_k^U}\Big)(\hat{x}), \end{cases}$$

$$\text{or } \begin{cases} \frac{f_k^L}{g_k^U}(x) < \frac{f_k^L}{g_k^U}(\hat{x}) \\ \Big(\frac{f_k^U}{g_k^L} - \frac{f_k^L}{g_k^U}\Big)(x) < \Big(\frac{f_k^U}{g_k^L} - \frac{f_k^L}{g_k^U}\Big)(\hat{x}). \end{cases}$$

From $\beta_k^L, \ \beta_k^S > 0$, we obtain

$$\Big[\sum_{k=1}^{l} \beta_k^L \Big(\frac{f_k^L(x)}{g_k^U(x)}\Big) + \sum_{k=1}^{l} \beta_k^S \Big(\frac{f_k^U(x)}{g_k^L(x)} - \frac{f_k^L(x)}{g_k^U(x)}\Big)\Big] -$$
$$\Big[\sum_{k=1}^{l} \beta_k^L \Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})}\Big) + \sum_{k=1}^{l} \beta_k^S \Big(\frac{f_k^U(\hat{x})}{g_k^L(\hat{x})} - \frac{f_k^L(\hat{x})}{g_k^U(\hat{x})}\Big)\Big] < 0. \tag{3.7}$$

Since $\frac{F}{G}$ is LS-V-invex with respect to $\lambda(x, \hat{x}) > 0$ and $\eta(x, \hat{x})$, therefore $\frac{f_k^L}{g_k^U}$ is V-invex and $\frac{f_k^U}{g_k^L} - \frac{f_k^L}{g_k^U}$ are V-invex for same $\eta(x, \hat{x})$ and $\lambda(x, \hat{x})$.

Now, for $\beta_k^L, \ \beta_k^S > 0$, we get
$\sum_{k=1}^{l} \beta_k^L \Big(\frac{f_k^L(x)}{g_k^U(x)}\Big) + \sum_{k=1}^{l} \beta_k^S \Big(\frac{f_k^U(x)}{g_k^L(x)} - \frac{f_k^L(x)}{g_k^U(x)}\Big)$ is V-invex. Therefore, we have

$$\Big[\sum_{k=1}^{l} \beta_k^L \Big(\frac{f_k^L(x)}{g_k^U(x)}\Big) + \sum_{k=1}^{l} \beta_k^S \Big(\frac{f_k^U(x)}{g_k^L(x)} - \frac{f_k^L(x)}{g_k^U(x)}\Big)\Big] - \Big[\sum_{k=1}^{l} \beta_k^L \Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})}\Big)$$
$$+ \sum_{k=1}^{l} \beta_k^S \Big(\frac{f_k^U(\hat{x})}{g_k^L(\hat{x})} - \frac{f_k^L(\hat{x})}{g_k^U(\hat{x})}\Big)\Big] \ge \lambda(x, \hat{x}) \Big[\sum_{k} \beta_k^L \nabla \Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})}\Big) \tag{3.8}$$
$$+ \sum_{k=1}^{l} \beta_k^S \nabla \Big(\frac{f_k^U(\hat{x})}{g_k^L(\hat{x})} - \frac{f_k^L(\hat{x})}{g_k^U(\hat{x})}\Big)\Big] \eta(x, \hat{x}),$$

Again using the fact that all $h_j,\ j=1,2,\dots,m$ are V-invex with respect to $\lambda(x,\hat{x})>0$ and $\eta(x,\hat{x})$, we get

$$h_j(x)-h_j(\hat{x}) \geq \lambda(x,\hat{x})\nabla h_j(\hat{x})\eta(x,\hat{x}).$$

Further, it follows from hypothesis (ii) and $\xi_j \geq 0,\ j=1,2,\dots,m$, that

$$\sum_{j=1}^{m}\xi_j h_j(x) \geq \lambda(x,\hat{x})\Big(\sum_{j=1}^{m}\xi_j\nabla h_j(\hat{x})\Big)\eta(x,\hat{x}). \tag{3.9}$$

Now, using (3.8) and (3.9) and hypothesis (ii), $\xi_j \geq 0,\ h_j(x)\leq 0,\ j=1,2,\dots,m$, we finally obtain

$$\Big[\sum_{k=1}^{l}\beta_k^L\Big(\frac{f_k^L(x)}{g_k^U(x)}\Big)+\sum_{k=1}^{l}\beta_k^S\Big(\frac{f_k^U(x)}{g_k^L(x)}-\frac{f_k^L(x)}{g_k^U(x)}\Big)\Big]-\Big[\sum_{k=1}^{l}\beta_k^L\Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})}\Big)$$
$$+\sum_{k=1}^{l}\beta_k^S\Big(\frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}-\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})}\Big)\Big] \geq 0,$$

which contradicts (3.7). Hence, $\hat{x}$ is an LS-Pareto optimal solution for (FIVP). □

Verification of Theorem 3.9

Example 3.10. Consider the following fractional interval problem:

$$(E4)\quad \text{Min}\quad \frac{f(x)}{g(x)}=\frac{[x^2+x^4,2(x^2+x^4)]}{[x^2+1,x^2+2]}=\Big[\frac{x^2+x^4}{x^2+2},2x^2\Big],$$

subject to

$$1 \leq x \leq 2.$$

Let $\lambda(x,y)=1,\ \eta(x,y)=x-y^2$.

Now, we first prove that $\frac{f(x)}{g(x)}$ is LS-V-invex at $y=1$.

$$\begin{aligned}\psi_{11}&=\Big(\frac{f^L}{g^U}\Big)(x)-\Big(\frac{f^L}{g^U}\Big)(1)-\lambda(x,1)\nabla\Big(\frac{f^L}{g^U}\Big)(1)\eta(x,1)\\&=\frac{x^2+x^4}{x^2+2}-\frac{2}{3}-\frac{14}{9}(x-1)\\&=\frac{9x^4-14x^3+17x^2-28x+16}{9(x^2+2)}\\&\geq 0,\ \text{(see Fig. 9)}.\end{aligned}$$

$$\begin{aligned}\text{Also,}\ \psi_{12}&=\Big(\frac{f^U}{g^L}-\frac{f^L}{g^U}\Big)(x)-\Big(\frac{f^U}{g^L}-\frac{f^L}{g^U}\Big)(1)-\lambda(x,1)\nabla\Big(\frac{f^U}{g^L}-\frac{f^L}{g^U}\Big)(1)\eta(x,1)\\&=\frac{x^6+4x^4+3x^2}{x^4+3x^2+2}-\frac{8}{6}-\frac{22}{9}(x-1)\\&=\frac{18x^6-44x^5+92x^4-132x^3+114x^2-88x+40}{18(x^4+3x^2+2)}\\&\geq 0,\ \text{(see Fig. 10)}.\end{aligned}$$

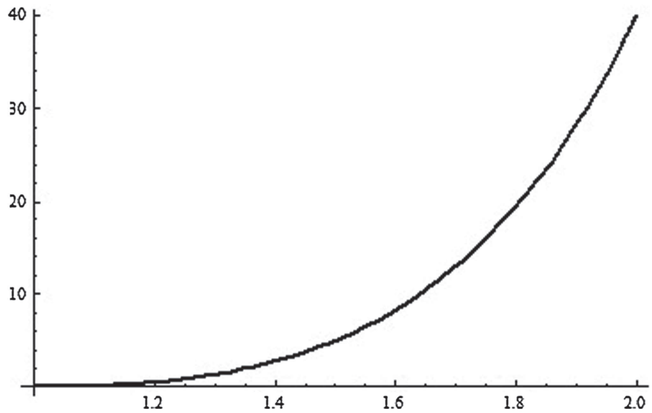

FIGURE 9. The graph of ψ_{11} with respect to $1 \le x \le 2$.

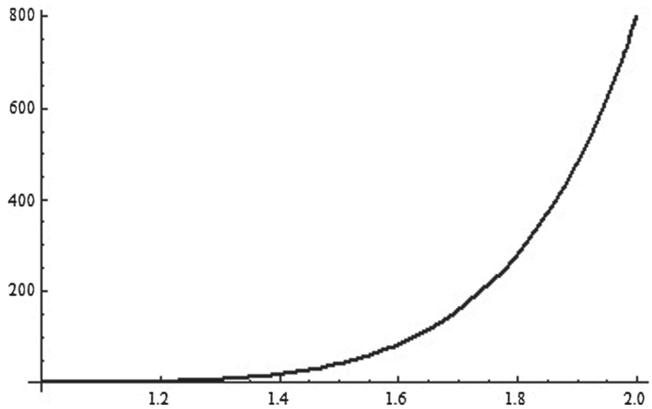

FIGURE 10. The graph of ψ_{12} with respect to $1 \le x \le 2$.

Next, we show that $h_j,\ j = 1, 2$, are V-invex.
For $h_1(x) = x - 2 \le 0$,
$h_1(x) - h_1(1) - \lambda(x, 1)\nabla h_1(1)\eta(x, 1) = (x - 2) - (1 - 2) - (x - 1) = 0.$
For $h_2(x) = -x + 1 \le 0$,
$h_2(x) - h_2(1) - \lambda(x, 1)\nabla h_2(1)\eta(x, 1) = (-x + 1) - (-1 + 1) - (-1)(x - 1) = 0.$
Therefore, all the conditions of Theorem 3.9 are satisfied. Further, at $\hat{x} = 1$, there exist $\xi_1 = 0,\ \xi_2 = 2,\ \lambda_1^L = \frac{9}{14}$ and $\lambda_1^S = \frac{9}{22}$ such that

(i) $\lambda_1^L\left(\frac{2\hat{x}^5+8\hat{x}^3+4\hat{x}}{(\hat{x}^2+2)^2}\right) + \lambda_1^S\left(\frac{2\hat{x}^9+12\hat{x}^7+30\hat{x}^5+32\hat{x}^3+12\hat{x}}{(\hat{x}^4+3\hat{x}^2+2)^2}\right) + \xi_1 - \xi_2 = 0,$
(ii) $\xi_1(\hat{x} - 2) = 0 = \xi_2(-\hat{x} + 1),$

hold.

Now, to show that $\hat{x} = 1$ is an LS-Pareto optimal solution to the problem.

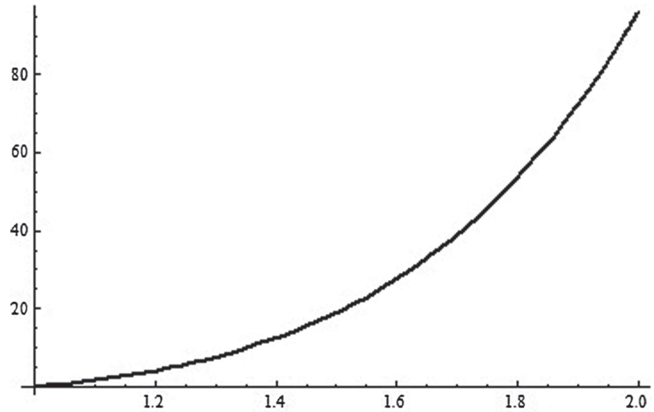

FIGURE 11. The graph of ψ_{21} with respect to $x \in [1, 2]$.

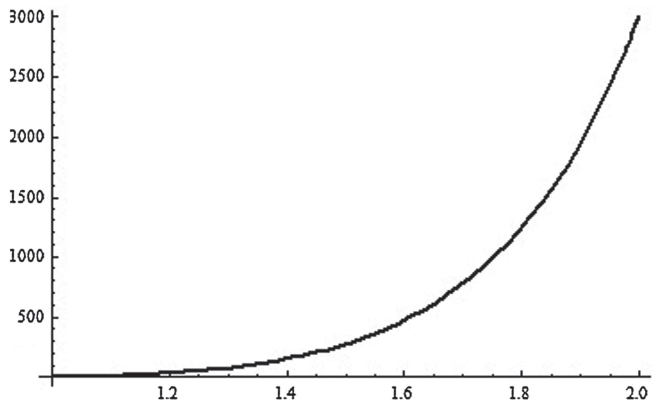

FIGURE 12. The graph of ψ_{22} with respect to $x \in [1, 2]$.

We see (Figs. 11 and 12) that, there exists no $1 \leq x \leq 2$ such that

$$\left[\frac{x^2+x^4}{x^2+2}, \frac{2(x^2+x^4)}{x^2+1}\right] \prec_{\text{LS}} \left[\frac{2}{3}, 2\right],$$

$$\text{which yields} \begin{cases} \frac{x^2+x^4}{x^2+2} \leq \frac{2}{3}, \\ \frac{x^6+4x^4+3x^2}{x^4+3x^2+2} < \frac{4}{3}, \end{cases} \quad \text{or} \ \begin{cases} \frac{x^2+x^4}{x^2+2} < \frac{2}{3}, \\ \frac{x^6+4x^4+3x^2}{x^4+3x^2+2} \leq \frac{4}{3}, \end{cases}$$

$$\text{or} \begin{cases} \frac{x^2+x^4}{x^2+2} < \frac{2}{3}, \\ \frac{x^6+4x^4+3x^2}{x^4+3x^2+2} < \frac{4}{3}, \end{cases}$$

$$\text{Let} \quad \psi_{21} = \frac{x^2+x^4}{x^2+2} - \frac{2}{3} \ \text{and} \ \psi_{22} = \frac{x^6+4x^4+3x^2}{x^4+3x^2+2} - \frac{4}{3}.$$

Therefore, Theorem 3.9 shows that $\hat{x} = 1$ is an LS-Pareto optimal solution to the problem (E4).

4. Optimality conditions with gH-derivative

In this section, the Karush Kuhn Tucker conditions for (FIVP) are obtained using the idea of the gradient of the fractional interval valued function *via* gH-derivative.

The gradient of a fractional interval valued function is defined as:

$$\nabla_g\Big(\frac{f}{g}\Big)(\hat{x}) = \Big(\Big(\frac{\partial(f/g)}{\partial x_1}\Big)_g(\hat{x}), \dots, \Big(\frac{\partial(f/g)}{\partial x_n}\Big)_g(\hat{x})\Big),$$

where $\Big(\frac{\partial(f/g)}{\partial x_j}\Big)_g(\hat{x})$ is the gH-derivative of $\frac{f}{g}$ at $\hat{x}$ and is denoted by

$$\Big(\frac{\partial(f/g)}{\partial x_j}\Big)_g(\hat{x}) = \Big[\min\Big\{\Big(\frac{\partial(f^L/g^U)}{\partial x_j}\Big)(\hat{x}), \Big(\frac{\partial(f^U/g^L)}{\partial x_j}\Big)(\hat{x})\Big\}, \max\Big\{\Big(\frac{\partial(f^L/g^U)}{\partial x_j}\Big)(\hat{x}), \Big(\frac{\partial(f^U/g^L)}{\partial x_j}\Big)(\hat{x})\Big\}\Big],$$

if $\frac{f}{g}$ is gH-differentiable.

The following example illustrates how to find the gH-derivative of a fractional interval valued function.

Example 4.1. Let the fractional interval valued function be given by

$$\frac{f(x)}{g(x)} = \frac{[-x_1^2 - x_2^2, 2(x_1^2 + x_2^2)]}{[x_2^2, x_1^2]} = \Big[\frac{-x_1^2 - x_2^2}{x_1^2}, \frac{2(x_1^2 + x_2^2)}{x_2^2}\Big],\ x = (x_1, x_2),\quad 0 < x_2 \leq x_1.$$

Here,

$$\Big(\frac{\partial(f/g)}{\partial x_1}\Big)(x) = \Big[\min\Big(\frac{2x_2^2}{x_1^3}, \frac{4x_1}{x_2^2}\Big), \max\Big(\frac{2x_2^2}{x_1^3}, \frac{4x_1}{x_2^2}\Big)\Big]$$

and

$$\Big(\frac{\partial(f/g)}{\partial x_2}\Big)(x) = \Big[\min\Big(\frac{-2x_2}{x_1^2}, \frac{-4x_1^2}{x_2^3}\Big), \max\Big(\frac{-2x_2}{x_1^2}, \frac{-4x_1^2}{x_2^3}\Big)\Big]$$

Thus, the gradient of $\frac{f(x)}{g(x)}$ is given as:

$$\nabla_g\Big(\frac{f}{g}\Big)(x) = \Bigg(\Big[\min\Big(\frac{2x_2^2}{x_1^3}, \frac{4x_1}{x_2^2}\Big), \max\Big(\frac{2x_2^2}{x_1^3}, \frac{4x_1}{x_2^2}\Big)\Big], \Big[\min\Big(\frac{-2x_2}{x_1^2}, \frac{-4x_1^2}{x_2^3}\Big), \max\Big(\frac{-2x_2}{x_1^2}, \frac{-4x_1^2}{x_2^3}\Big)\Big]\Bigg).$$

Definition 4.2. [3] The cone of feasible directions of a non-empty feasible set X at $\hat{x}$ is defined as:

$$\tau = \Big\{t \in R^n :\ t \neq 0,\ \exists\ \kappa > 0,\ \text{such that } \hat{x} + \delta t \in X,\ \forall\ \delta \in (0, \kappa)\Big\},$$

where $t \in \tau$ is the feasible direction of X.

Proposition 4.3. [3] *Let $J(\hat{x}) = \{i|\ h_i(\hat{x}) = 0\}$ be the index set of the active constraints for the feasible set $X_0 = \{x \in R^n|\ h_i(x) \leq 0,\ i = 1, 2, \ldots, m\}$ where $\hat{x} \in X_0$. If $h_i(x),\ i = 1, 2, \ldots, m$ are differentiable functions, then*

$$\tau \subseteq \{t \in R^n|\ \nabla h_i(\hat{x})^T t \leq 0,\ i \in I\}.$$

The following discussion is based on the fact that the KKT conditions for the fractional interval valued function (FIVP) can also be obtained using the idea of the gradient of fractional interval valued function via gH-derivative.

For the continuously differentiable functions $\frac{f_k(x)}{g_k(x)},\ k = 1, 2, \ldots, l$ and the real valued functions $h_i,\ i = 1, 2, \ldots, m$ as in (FIVP), $0 < \beta_k \in R,\ k = 1, 2, \ldots, l,\ 0 \leq \eta_j \in R,\ j = 1, 2, \ldots, m$, consider

$$\sum_{k=1}^{l} \beta_k \nabla_g \Big(\frac{f_k}{g_k}\Big)(\hat{x}) + \sum_{j=1}^{m} \eta_j \nabla h_j(\hat{x}) = 0. \tag{4.1}$$

Since $\frac{f_k^L}{g_k^U}$ and $\frac{f_k^U}{g_k^L}$ are continuously differentiable at $\hat{x},\ k = 1, 2, \ldots, l$ (from Thm. 2.15), we have for $j = 1, 2, \ldots, m$,

$$\sum_{k=1}^{l} \beta_k \nabla \Big(\frac{f_k^L}{g_k^U}\Big)(\hat{x}) + \sum_{j=1}^{m} \eta_j \nabla h_j(\hat{x}) = 0 \tag{4.2}$$

$$\text{and } \sum_{k=1}^{l} \beta_k \nabla \Big(\frac{f_k^U}{g_k^L}\Big)(\hat{x}) + \sum_{j=1}^{m} \eta_j \nabla h_j(\hat{x}) = 0. \tag{4.3}$$

Adding (4.2) and (4.3), we obtain

$$\sum_{k=1}^{l} \beta_k \Big[\nabla \Big(\frac{f_k^L}{g_k^U}\Big)(\hat{x}) + \nabla \Big(\frac{f_k^U}{g_k^L}\Big)(\hat{x})\Big] + \sum_{j=1}^{m} \hat{\eta}_j \nabla h_j(\hat{x}) = 0, \tag{4.4}$$

where $\hat{\eta}_j = 2\eta_j$.

Theorem 4.4. *Let $\hat{z} \in X_0$ be a feasible solution for (FIVP). Suppose that the function $\frac{F}{G}, F \geq 0,\ G > 0$ is continuously gH-differentiable, also LU-V-invex at $\hat{z} \in X_0$ with respect to $\lambda(z, \hat{z}) > 0$ and $\eta(z, \hat{z})$. Moreover, assume that $h_j(x),\ j = 1, 2, \ldots, m$ are V-invex at $\hat{z} \in X_0$ with respect to the same $\lambda(z, \hat{z})$ and $\eta(z, \hat{z})$. If there exist Lagrange multipliers $0 < \beta_k \in R,\ k = 1, 2, \ldots, l$ and $0 \leq \xi_j \in R,\ j = 1, 2, \ldots, m$ in such a way that the following conditions are satisfied:*

(i) $\sum_{k=1}^{l} \beta_k \nabla_g \Big(\frac{f_k}{g_k}\Big)(\hat{z}) + \sum_{j=1}^{m} \xi_j \nabla h_j(\hat{z}) = 0,$
(ii) $\xi_j h_j(\hat{z}) = 0,\ j = 1, 2, \ldots, m.$

Then $\hat{x}$ is LU-Pareto optimal for (FIVP).

Proof. Taking $\hat{z} = \hat{x}$ and $\xi_j = \eta_j,\ j = 1, 2, \ldots, m$ in (4.1), we get the condition (i). This is equivalent to (4.4), where $\hat{\eta}_j = 2\eta_j$ and then following the Theorem 3.7, we get the required result. □

In order to establish the KKT optimality conditions for the strongly LU-Pareto optimal solution for the problem (FIVP), the Tucker's theorem of alternative is required. It states that, for two matrices B and D, exactly one of the following has a solution:

(I) $Bx \leq 0,\ Bx \neq 0,\ Dx \leq 0$ for some $x \in R^n$,
(II) $B^T u + D^T v = 0$ for some $u > 0$ and $v \geq 0$,

but never both.

Theorem 4.5. *Let a strictly LU-V-invex fractional interval valued function $\frac{f_k}{g_k}$ be gH-differentiable at $\hat{x}$ with respect to $\beta(x,\hat{x}) > 0$ and $\eta(x,\hat{x})$. If there exist Lagrange multipliers $\xi_j \geq 0 \in R$, $j = 1,2,\ldots,m$ and the following KKT conditions are satisfied:*

(i) $\nabla_g\Big(\frac{f_k^L}{g_k^U} + \frac{f_k^U}{g_k^L}\Big)(\hat{x}) + \sum_{j=1}^m \xi_j \nabla h_j(\hat{x}) = 0.$

(ii) $\xi_j h_j(\hat{x}) = 0,\ j = 1,2,\ldots,m.$

Then $\hat{x}$ is strongly LU-Pareto optimal for the problem (FIVP).

Proof. On the contrary, let $\hat{x}$ be not strongly LU-Pareto optimal for (FIVP). Then there exists $\bar{x} \in X$ such that

$$\Big(\frac{f_k(\bar{x})}{g_k(\bar{x})}\Big) \preceq_{LU} \Big(\frac{f_k(\hat{x})}{g_k(\hat{x})}\Big),$$

which implies

$$\Big(\frac{f_k^L(\bar{x})}{g_k^U(\bar{x})}\Big) \leq \Big(\frac{f_k^L(\hat{x})}{g_k^U(\hat{x})}\Big) \text{ and } \Big(\frac{f_k^U(\bar{x})}{g_k^L(\bar{x})}\Big) \leq \Big(\frac{f_k^U(\hat{x})}{g_k^L(\hat{x})}\Big). \tag{4.5}$$

Since, $\frac{f_k}{g_k}$ is strictly LU-V-invex at $\hat{x}$, we have

$$\frac{f_k^L(\bar{x})}{g_k^U(\bar{x})} - \frac{f_k^L(\hat{x})}{g_k^U(\hat{x})} > \beta(\bar{x},\hat{x})\Big(\nabla\Big(\frac{f_k^L}{g_k^U}\Big)(\hat{x})\Big)^T \eta(\bar{x},\hat{x}). \tag{4.6}$$

and

$$\frac{f_k^U(\bar{x})}{g_k^L(\bar{x})} - \frac{f_k^U(\hat{x})}{g_k^L(\hat{x})} > \beta(\bar{x},\hat{x})\Big(\nabla\Big(\frac{f_k^U}{g_k^L}\Big)(\hat{x})\Big)^T \eta(\bar{x},\hat{x}). \tag{4.7}$$

From (4.5) and (4.6), we obtain

$$\beta(\bar{x},\hat{x})\Big(\nabla\Big(\frac{f_k^L}{g_k^U}\Big)(\hat{x})\Big)^T \eta(\bar{x},\hat{x}) < 0.$$

Again, (4.5) and (4.7) together give

$$\beta(\bar{x},\hat{x})\Big(\nabla\Big(\frac{f_k^U}{g_k^L}\Big)(\hat{x})\Big)^T \eta(\bar{x},\hat{x}) < 0.$$

From the fact that $\beta(\bar{x},\hat{x}) > 0$, we get

$$\Big(\nabla\Big[\Big(\frac{f_k^L}{g_k^U}\Big) + \Big(\frac{f_k^U}{g_k^L}\Big)\Big](\hat{x})\Big)^T \eta(\bar{x},\hat{x}) < 0. \tag{4.8}$$

Let $d = \eta(\bar{x},\hat{x})$. Then, for sufficiently small $\delta > 0$, $\hat{x} + t\eta(\bar{x},\hat{x}) \in X$ for all $t \in (0,\delta)$. Hence $\eta(\bar{x},\hat{x}) \in \tau$. Therefore, from Proposition 4.3, for each $i \in I$ (index set of active constraints of (FIVP)),

$$\nabla h_j(\hat{x})^T \eta(\bar{x},\hat{x}) \leq 0. \tag{4.9}$$

$$\text{Let } B = \Big(\nabla\Big(\frac{f_k^L}{g_k^U} + \frac{f_k^U}{g_k^L}\Big)(\hat{x})\Big)^T \text{ and } D = \Big(\nabla h_j(\hat{x})\Big)^T,\ j \in I$$

are two matrices. Then from (4.8) and (4.9), $\eta(\bar{x},\hat{x})$ solves the system I (of Tucker's condition), which in turn imply that system II has no solution. That is, there does not exist any multiplier $\rho \in R,\ \rho \neq 0$ and $0 \leq \varphi_j \in R,\ j \in I$ such that

$$\rho\nabla\Big(\frac{f_k^L}{g_k^U} + \frac{f_k^U}{g_k^L}\Big)(\hat{x}) + \sum_{j\in I} \varphi_j \nabla h_j(\hat{x}) = 0$$

or, equivalently, there does not exist any multiplier $0 \leq \xi_j \in R,\ j \in I$ such that

$$\nabla\Big(\frac{f_k^L}{g_k^U} + \frac{f_k^U}{g_k^L}\Big)(\hat{x}) + \sum_{j\in I} \xi_j \nabla h_j(\hat{x}) = 0, \tag{4.10}$$

letting $\xi_j = \varphi_j/\rho$.

For the index set I of active constraints, we have $h_j(\hat{x}) < 0,\ j \notin I$. Thus, if $j \notin I$, (ii) yields $\xi_j = 0$.

As a result, (4.10) gives a contradiction to the conditions (i) of the theorem. This ends the proof of the theorem. □

Verification of Theorems 4.4 and 4.5

Example 4.6. Consider the fractional interval problem:

$$(E5)\quad \text{Min}\quad \frac{f(x)}{g(x)} = \frac{[x^2, x^2+1]}{[x+1, x+2]} = \Big[\frac{x^2}{x+2}, \frac{x^2+1}{x+1}\Big]$$

subject to

$$\begin{aligned} x-2 &\leq 0,\\ -x+1 &\leq 0. \end{aligned}$$

Let $\lambda(x,y) = 1$ and $\eta(x,y) = x - y^2$.

Now, we first show that $\frac{f(x)}{g(x)}$ is LU-V-invex at $y = 1$.

$$\begin{aligned} &\Big(\frac{f^L}{g^U}\Big)(x) - \Big(\frac{f^L}{g^U}\Big)(1) - \lambda(x,1)\nabla\Big(\frac{f^L}{g^U}\Big)(1)\eta(x,1)\\ &= \frac{x^2}{x+2} - \frac{1}{3} - \frac{5}{9}(x-1)\\ &= \frac{4(x-1)^2}{9(x+2)}\\ &\geq 0,\ \text{for all } x \in [1,2]. \end{aligned}$$

Also,

$$\begin{aligned} &\Big(\frac{f^U}{g^L}\Big)(x) - \Big(\frac{f^U}{g^L}\Big)(1) - \lambda(x,1)\nabla\Big(\frac{f^U}{g^L}\Big)(1)\eta(x,1)\\ &= \frac{x^2+1}{x+1} - 1 - \frac{1}{2}(x-1)\\ &= \frac{(x-1)^2}{2(x+1)}\\ &\geq 0,\ \text{for all } x \in [1,2]. \end{aligned}$$

Next, we shall prove that $h_j,\ j = 1,2$, are V-invex.

For $h_1(x) = x - 2 \leq 0$,

$$\begin{aligned} &h_1(x) - h_1(1) - \lambda(x,1)\nabla h_1(1)\eta(x,1)\\ &= (x-2) - (1-2) - (x-1) = 0. \end{aligned}$$

For $h_2(x) = -x + 1 \leq 0$,

$$\begin{aligned} &h_2(x) - h_2(1) - \lambda(x,1)\nabla h_2(1)\eta(x,1) \\ &= (-x+1) - (-1+1) - (-1)(x-1) = 0. \end{aligned}$$

Therefore, the hypotheses of the Theorem 4.4 are satisfied. Further, there exist $\beta = 1,\ \xi_1 = 0$ and $\xi_2 = \frac{19}{36}$ such that the conditions

(i) $\beta\nabla_g\left(\frac{f}{g}\right)(\hat{x}) + \xi_1\nabla h_1(\hat{x}) + \xi_2\nabla h_2(\hat{x}) = 0,$ and

(ii) $\xi_1 h_1(\hat{x}) = 0 = \xi_2 h_2(\hat{x})$.

hold.

Next, we will show that $\hat{x} = 1$ is an LU-Pareto optimal for the problem (E5).

We see that, there exists no $1 \leq x \leq 2$ such that

$$\left[\frac{x^2}{x+2}, \frac{x^2+1}{x+1}\right] \prec_{LU} \left[\frac{1}{3}, 1\right]$$

$$\text{which implies } \begin{cases} \frac{x^2}{x+2} \leq \frac{1}{3} \\ \frac{x^2+1}{x+1} < 1, \end{cases} \text{ or } \begin{cases} \frac{x^2}{x+2} < \frac{1}{3} \\ \frac{x^2+1}{x+1} \leq 1, \end{cases} \text{ or } \begin{cases} \frac{x^2}{x+2} < \frac{1}{3} \\ \frac{x^2+1}{x+1} < 1, \end{cases}$$

Therefore, $\hat{x} = 1$ is LU-Pareto optimal solution to the problem (E5). This verifies Theorem 4.4. □

Further, we shall justify the Theorem 4.5 using (E5).

It follows on the similar steps shown above that $\frac{f(x)}{g(x)}$ is strictly LU-V-invex at $\hat{x} = 1$ and $h_1(x),\ h_2(x)$ are V-invex at $\hat{x} = 1$.

Therefore, the hypotheses of the Theorem 4.5 are satisfied. Moreover, there exist $\xi_1 = 0$ and $\xi_2 = \frac{19}{18}$ such that

(v) $\nabla_g\left(\frac{f^L}{g^U} + \frac{f^U}{g^L}\right)(\hat{x}) + \xi_1\nabla h_1(\hat{x}) + \xi_2\nabla h_2(\hat{x}) = 0,$

(vi) $\xi_1 h_1(\hat{x}) = 0 = \xi_2 h_2(\hat{x})$.

Lastly, it remains to show that $\hat{x} = 1$ is a strongly LU-Pareto optimal solution for the problem (E5). For this, we need to show that there exists no $x \in (1,2]$ such that

$$\left[\frac{x^2}{x+2}, \frac{x^2+1}{x+1}\right] \preceq_{LU} \left[\frac{1}{3}, 1\right].$$

Now,

$$\begin{aligned} \left[\frac{x^2}{x+2}, \frac{x^2+1}{x+1}\right] \preceq_{LU} \left[\frac{1}{3}, 1\right] \text{ imply, } & \frac{x^2}{x+2} \leq \frac{1}{3} \text{ and } \frac{x^2+1}{x+1} \leq 1 \\ & \text{or, } 3x^2 - x - 2 \leq 0 \text{ and } x^2 - x \leq 0, \end{aligned}$$

which is not true simultaneously.

Consequently, $\hat{x} = 1$ is a strongly LU-Pareto optimal solution. This verifies Theorem 4.5. □

5. Conclusions and future directions

To the best of our knowledge, the KKT conditions for multiobjective interval valued fractional optimization problems have not appeared in the literature so far. In this article, we have developed the KKT optimality conditions for the multivalued optimization problem having fractional interval objectives. We have presented the idea of LU-V-invex and LS-V-invex for fractional interval valued functions, which generalize the LU-convexity/LS-convexity for interval valued functions. On the interval space, considering two-order relations, namely, LU and LS-relations and using the gH-differentiability for fractional interval valued functions, we have established the KKT conditions for the multivalued fractional interval problem under the LU-V-invex/LS-V-invex assumptions. Moreover, extending the idea of the gradient of the interval valued functions using gH-derivative to the fractional interval valued functions, we have derived the sufficient KKT conditions.

It is to be noted that the constraint functions considered in this paper are inequality constraints and real-valued functions. Using the approach discussed in this paper, one can develop theoretical results for fractional interval valued constraints with inequality conditions. This may be the forthcoming research direction in this area.

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