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EFFECT OF INSPECTION ERRORS ON IMPERFECT PRODUCTION INVENTORY MODEL WITH WARRANTY AND PRICE DISCOUNT DEPENDENT DEMAND RATE

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Abstract. This paper deals with selling price-discount and warranty period dependent demand in an imperfect production inventory model under the consideration of inspection errors and time dependent development cost. Normally, due to long-run, a production process deteriorates with time and here we assume that the process shifts from "in-control" to "out-of-control" state at any random time. A time dependent development cost has been constructed to increase the reliability of the production system *i.e.*, to decrease the deterioration of the system during the production process. As a result, a few items are rejected. Here, two types of inspection errors such as Type-I error and Type-II error, have been considered during the period of product inspection process. In Type-I error, an inspector may choose falsely a defective item as non-defective and in Type-II error an inspector may choose falsely a non-defective item as defective. Due to these phenomena, the inspection process would consist of the following costs: cost of inspection, cost of inspection errors. The purpose of this paper is to investigate the effects of time dependent development cost on the defective items, selling price-discount and warranty policy on the market demand and finally optimize the expected average profit under consideration of such inspection costs in infinite time horizon. Some numerical examples along with graphical illustrations and sensitivity analysis are provided to test the feasibility of the model.

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1. Introduction

In a real manufacturing system, a long-run production process shifts from "in-control" to "out-of-control" state due to different machinery problems, labor problems, etc. On the other hand, the production of defective units is a natural phenomenon to be occurred due to different difficulties arisen in a long-run production process. Normally, it is seen that a production process is initiated from "in-control" state, because every factors associated with the system are in well condition. Then due to continuous running of system, these factors gradually lose their perfectness. So, after some time, the production process may shift from "in-control" state to "out-of-control" state. For this reason, some imperfect items along with perfect items are produced in every

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manufacturing system. Now, if all product together are sold to the customers, the company must lose his/her good will and henceforth demand of the product will be decreased gradually in the market. So, problem is how to maintain the good will in the market? To search the answer of this question, many researchers investigated EPQ models with imperfect production in which inspection process was considered for screening the defective items. But in these studies, the inspection process has been considered to be error free. Then Jaggi *et al.* [10], Tiwari *et al.* [33], Mallick *et al.* [13], Panja and Mondal [19] proposed that this assumption is not always true in business world. Basically, the inspection process may not be completely error free due to different types of factors such as machinery and human in the system. They considered a possibility of Type I error (falsely rejecting non-defective items) and Type II error (falsely accepting defective items) in their papers to study the EPQ models with imperfect production. But in these studies, if a non-defective item is falsely rejected, there was no scope to get back as a perfect one to be sold directly from the manufacturer. In this paper this lack of analysis has been removed considering Type I error as the possibility of two types such as either (i) when a non-defective item is rejected and it implies the loss of manufacturer or (ii) when a non-defective item is submitted in reworked cell and it implies no loss of manufacturer since it will be detected as a non-defective in the reworked cell. Simultaneously, Type II error is also considered in this paper where the acceptance of the defective item has been considered as a non-defective item and due to that it has a risk to a customer. This is one of the novelties in this paper.

In practice, the manufacturer usually offers a warranty for all selling items for the specific period due to increasing the selling rate and reliability of the product. Warranty period of a product is a duration in which a purchased product provides satisfactory performance to the customer. If any purchased product fails to work within its warranty period, then the servicing center replaces it with a new item or repairs the product by replacing one or more parts of the product. In the literature many researchers (*c.f.* [4, 23, 36]) considered warranty cost as a constant parameter. But there is a connection between warranty period and demand on the product in real business market. So a functional relationship among warranty period, selling price discount and demand should have. Due to this reason, in our model, we consider a new type of demand rate which depends on both selling price discount and warranty period. This is another of the novelty of our paper.

So based on the above facts, we have developed an imperfect production inventory model under considering price discount and warranty period dependent demand with inspection errors.

1.1. Literature survey

In any production system, all produced items are not perfect. So the defective items are inevitable during the production due to many reasons such as defects of machine and other related factors. In 1986, Rosenblatt and Lee [24] considered an EPQ model that deals with imperfect quality. They assumed that at some random point in time the process might shift from an "in-control" to an "out-of-control" state, and a fixed percentage of defective items are produced. At the same year, Porteus [20] was one of the researchers to consider the situation where the production process may shift from an "in-control" state to an "out-of-control" state. Recently, Manna *et al.* [17] studied an imperfect production inventory model considering with the rate of defectiveness which is random and depends on the time length of the "out-of-control" state. They considered the production process shifts from "in-control" state to "out-of control" state, that follows an exponential distribution.

Most of researchers on the EPQ models with imperfect production process considered the inspection process to separate the defective items and they assumed the inspection process perfect *i.e.*, error-free (*c.f.* [10,13,16,26,30]). But in general, this assumption is not always true. Sometimes, the inspection process is not error free due to different types of factors related to machine and human factors in the system. It may be a possibility of Type I error (falsely rejecting non-defective items) or Type II error (falsely accepting defective items) in the inspection process. Raouf *et al.* [21] presented a cost-minimization model for multi characteristic component inspection. Raz and Bricker [22] considered inspection errors during screening in an production process. Rentoul *et al.* [23] studied several ways of inspection errors in manufacturing system which are made by comparing inspection points with a solid model of the desired component. Salameh and Jaber [25] studied a joint lot sizing and inspection policy for an EOQ model when a random proportion of the units in a lot are defective. They assumed a 100%

screening process with no human error. Wang and Sheu [36] proposed an optimal production and maintenance policy under the effect of inspection errors. Darwish and Ben-Daya [4] considered an inventory model in an imperfect production process with the inspection errors. Duffuaa and Khan [5] studied the optimal inspection policy under different kinds of misclassifications. Hsu and Hsu [9] studied the effects of inspection errors of imperfect quality items on an economic order quantity (EOQ) model with shortages and sales returns. Sarkar *et al.* [27] dealt with a problem of optimal production run time and inspection errors in an imperfect production system with warranty.

In an imperfect production process, the rework policy is an important role for eliminating waste and decreasing manufacturing cost. A reworking cost is considered to make the defective items as new as perfect by reworking process. Hayek and Salameh [8] studied the determination of optimal production lot size with reworking of defective items. Flappera and Teunterb [6] showed how reworking plans could both reduce costs and be environment friendly. Chiu *et al.* [2] proposed a more general model that allowed a certain proportion of reworked units to be scrapped. Cárdenas-Barrón [1] proposed an inventory model on optimal batch sizing in a multi-stage production system with rework process. Manna *et al.* [17] examined the effects of rework policy in an imperfect production inventory model with two storage facilities. At the same year, Jain *et al.* [12] proposed a fuzzy imperfect production and repair inventory model with time dependent demand. Recently, Nobil *et al.* [18] consider rework and inspection in an imperfect multi-item single machine production system.

Recently, the supply chain models have generally considered the various demand function such as advertizement (*c.f.* [14]), time dependent (*c.f.* [30]), stock-dependent (*c.f.* [15]), selling prince dependent(*c.f.* [7]) etc. But in real life, the assumption is not always true in general. The demand may vary with selling price discount and warranty period of the product. Due to this reason, a manufacturer considers warranty cost if there exists free-warranty on selling items within the warranty period. Wang and Sheu [35] investigated the imperfect production model with a free warranty for the discrete unit item. Yeh *et al.* [37] developed a production inventory model considering the free warranty and derived the optimal production cycle time. Chung [3] considered an EPQ model with the warranty period-dependent demand, effects on inspection scheduling and supply chain replenishment policy. Jaggi *et al.* [11] introduced price dependent demand in economic ordering policies for non-instantaneous deteriorating items. At the same year, Tiwari *et al.* [29] proposed stock-dependent demand in two-warehouse inventory model for non-instantaneous deteriorating items using particle swarm optimization. Later, Tiwari *et al.* [31] considered price and stock-dependent demand in a supply chain system with deteriorating items under limited storage capacity.

1.2. Contribution of the proposed model

The contributions of our proposed EPQ model with imperfect production are elucidated as follows:

– In this paper, effect of warranty policy has been studied on profit maximization considering market demand to be dependent on both selling price discount and warranty period of selling product.
– Here, a new type inspection error has been proposed for inspection the imperfect items. In this paper, Type I error may be classified as two types such as (i) when a non-defective item is rejected or (ii) when a non-defective item is submitted in reworked cell. Simultaneously, Type II error is also considered where defective item has been considered as a non-defective item.
– Here, warranty period and warranty cost both have been considered separately in such a way that warranty period has been taken as a decision variable for which warranty cost per unit item will be minimized as well as market demand maximized in this profit system.
– Finally, an algorithm has been developed to get the optimal solution of the proposed model.

The remainder of this paper is organized as follows: In Section 2, we define notations and assumptions to be used in this model. The mathematical formulation of our proposed model are described in Section 3. In Section 4, a solution methodology has been developed. A numerical analyses and managerial insights are also presented in Sections 5 and 6 respectively. Finally, practical implication and conclusion with future research are carried out in Sections 7 and 8 respectively.

2. Notation and assumptions of the proposed model

In this section we have described the notation and assumptions of the proposed model which are as follows:

2.1. Notation

To develop the model, following notations have been used.

P	:	Production rate of manufacturer.
D	:	Selling rate of manufacturer/demand rate customer.
η	:	Selling price discount parameter.
ρ	:	Effective parameter of demand on warranty.
τ	:	Random time with mean $\frac{1}{\lambda}$ after which the production system shifts from an "in-control" state to "out-of-control" state.
θ_1	:	Percentage of produced defective items in "in-control'; state.
θ_1	:	Percentage of produced defective items in "out-of-control" state ($\theta_1 < \theta_2$).
δ	:	Probability of rework rate of defective units.
c_p	:	Production cost per item.
c_{sr}	:	Screening cost per item.
h_c	:	Holding cost per item per unit time in production center.
c_w	:	Average warranty cost per item.
s	:	Selling price per item for perfect quality.
c_r	:	Average reworking cost per item of reworkable item.
c_d	:	Disposal cost per item.
A	:	$= (A_0 + \frac{K}{Pt_1})$, set up cost of manufacture.
c_v	:	Development cost during production runtime.
m_1	:	Probability of a Type I error (classifying a non- defective item as defective).
m_2	:	Probability of a Type II error (classifying a defective item as non-defective).
$f(\tau)$	:	Probability density function of τ.
$\phi(\delta)$	:	Probability density function of δ.
$\phi(m_1)$	:	Probability density function of m_1.
$\phi(m_2)$	:	Probability density function of m_2.
c_a	:	The cost of accepting a defective item, where $c_a = c_t + c_l$.
γ	:	Probability of classifying a non-defective item as rework item due to Type I error.
$\phi(\gamma)$	:	Probability density function of γ.
Decision variables:		
t_1	:	Production period.
t_w	:	Warranty period of selling item.
Related to the decision variable:		
T	:	Total business period.

2.2. Assumptions

The mathematical model of the proposed inventory problem is based on the following assumptions:

(i) In this model a manufacturer produces a mixture of defective and non-defective (perfect quality) items. Some portion of defective items are reworked at a cost.

(ii) The manufacturer inspects each produced item to check whether the item is perfect or not. In this inspection process, there may exist some possibility that a non-defective item is treated as defective item and defective item may be considered as a non-defective item which are known as type-I and type-II error respectively. Due to type-I error an item may be truly non-defective or defective. If it is non-defective then it is sent to the inventory of non-defective items after checking from the reworked cell. On the other hand, if it is defective then it is reworked and then sent to the inventory non-defective items, otherwise it is rejected completely from the rework cell. Again in case of type-II error, an defective item is delivered as a non-defective item to the customers. So, after checking it by customer, it is sent back to the manufacturer.

Therefore under type-II error the manufacturer is compelled to bear an extra cost as a miss-classification cost.

(iii) Here we assume that due to continuous long run production process, the components of production system gradually losses their perfectness. So, some time (τ) after production, the production process may shift from the "in-control" state to "out-of-control" state. The time (τ) is an exponential distributed with a finite mean.

(iv) According to assumption (iii), the defective rate (θ_1) in "in-control" state is less than the defective rate (θ_2) in "out-of-control" state and is given by

$$\theta = \begin{cases} \theta_1, 0 \le t \le \tau \\ \theta_2, \tau \le t \le t_1 \end{cases}. \tag{2.1}$$

(v) Here, we consider the warranty cost (c_w) per item is not constant, it depends on the warranty period (t_w) and is given by

$$c_w = a + bt_w. \tag{2.2}$$

(vi) Selling price (s) per item of non-defective item is not fixed always, we consider

$$s = s_0 - \eta s_0, 0 \le t \le T \tag{2.3}$$

where η is the discounts percentage of selling price.

(vii) Here, we consider a new type of selling rate which depends on both selling price discount and warranty period. The selling rate is defined as

$$D = \begin{cases} (D_0 + \rho t_w)e^{k\eta}, 0 \le t \le t_1 \\ D_0 + \rho t_w, \qquad t_1 \le t \le T \end{cases} \tag{2.4}$$

where k and ρ are suitable positive constant.

(viii) Due to long run production process and increasing the duration of "in-control" state, we consider a devolvement cost ($f(\tau)$) per unit time as the form

$$f(\tau) = \begin{cases} B_0, & 0 \le t \le \tau \\ B_0 + B_1(t-\tau)e^{k_1 \frac{v_{\max}-v}{v-v_{\min}}}, & \tau \le t \le t_1 \end{cases}. \tag{2.5}$$

3. The mathematical formulation of the model

This model considers a supply chain system between manufacturer and customer for single type of products such as mobiles, in which the qualities of the production and inspection process are not perfect. In this manufacturing system, it is considered that production, inspections and reworked processes are performed simultaneously. The manufacturer starts production at a rate of P from the beginning and it continues upto the end of the production run, t_1. During the whole production period all produced items are inspected at the rate of P. Initially, the production system starts from "in-control" state and continues to any random time, τ. After this time "in-control" state shifts to "out-of-control" state and it stays until the end of the production-run time, t_1. According to assumption (iv), the probability of the number of defective items in "in-control" state is less than the probability of the number of defective items in "out-of-control" state.

Further, since the inspection process is not perfect, hence it generates both Type-I and Type-II inspection errors. Due to Type-I error there is possibilities of some non-defective items to be considered as defective items of amount $m_1(1-\theta_1)P$ in "in-control" state and $m_1(1-\theta_2)P$ in "out-of-control" state. On the other hand, it classifies some defective items as non-defective items of amount $m_2\theta_1 P$ in "in-control" state and $m_2\theta_2 P$ in "out-of-control" state. Here for sorting the items, an inspection cost per unit (c_{sr}) has been considered. After the inspection of the produced product, some portion (δ) of defective items are send in rework cell to convert it into non-defective item as fixed cost r_c per item. Then all confirming items are sent to the market within

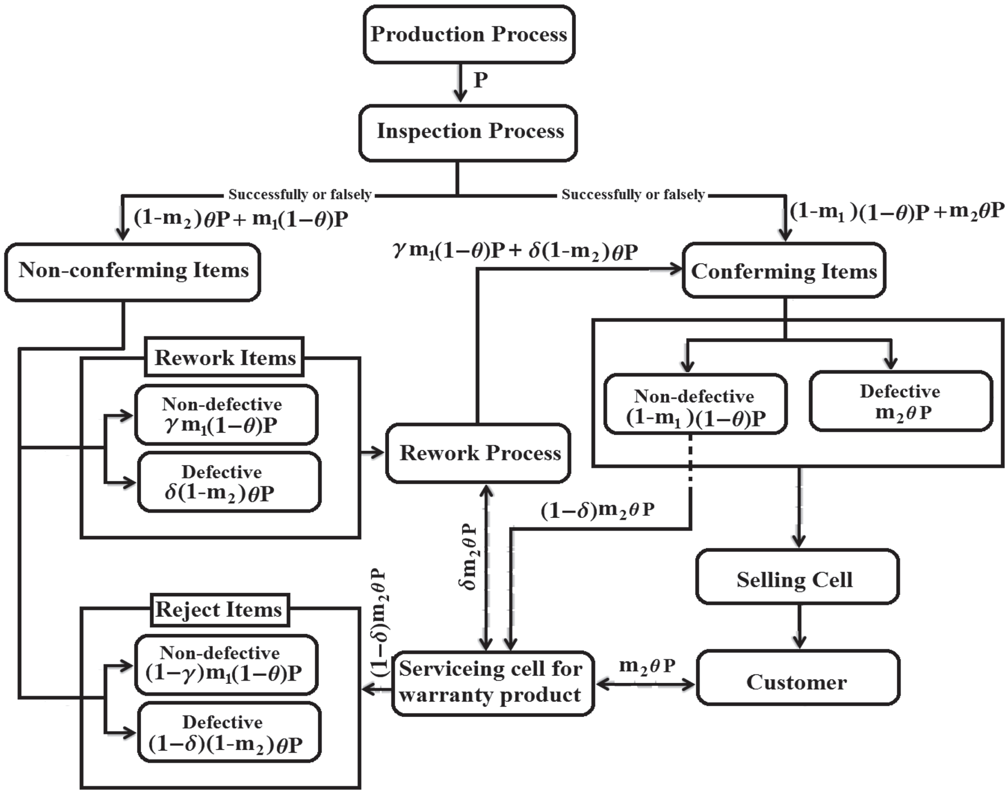

Figure 1. Relationship among production, inspection, selling and rework processes.

warranty period t_w. Figure 1 shows the relationship among production, inspection, rework processes and selling. The manufacturer fulfills the customer demand rate (D) continuously according to the assumption (ix) until the end of business cycle time T. Due to the position of random time τ for which the production system goes from "in-control" state to "out-of-control" state, the model has two different cases such as Case I: $0 < \tau < t_1$; and Case II: $t_1 \leq \tau < \infty$; which are discussed as follows.

Case I: When $0 < \tau < t_1$ *i.e.*, the "out-of-control" state to be occurred during the production-run time

In this case, the production period $[0, t_1]$ can be divided into two sub-intervals such as $[0, \tau]$ and $[\tau, t_1]$. During the time interval $[0, \tau]$, the production process is in "in-control" state and in $[\tau, t_1]$ the process is in "out-of-control" state. Through-out the time interval $[0, \tau]$, the amount of non-defective items, defective items and reworked items are $(1 - \theta_1)P\tau$, $\theta_1 P\tau$ and $\delta\theta_1 P\tau$ respectively. Also on $[\tau, t_1]$, the amount of non-defective items, defective items and reworked items are $(1 - \theta_2)P(t_1 - \tau)$, $\theta_2 P(t_1 - \tau)$ and $\delta\theta_2 P(t_1 - \tau)$ respectively.

During the inspection period $[0, \tau]$, the inspectors accept defective items the amount of $\theta_1 P\tau$ in which the amount of falsely accepted defective items and falsely rejected non-defective items are $m_2\theta_1 P\tau$ and $(1 - \gamma)m_1(1 - \theta_1)P\tau$ respectively. Also, the inspection period $[\tau, t_1]$, inspectors accept the defective items of the amount $\theta_2 P(t_1 - \tau)$ in which the amount of falsely accepted defective items and falsely reject amount of non-defective items are $m_2\theta_2 P(t_1 - \tau)$ and $(1 - \gamma)m_1(1 - \theta_2)P(t_1 - \tau)$ respectively (Fig. 2).

During the period $[0, t_1]$, the inventory level increases due to excess production after fulfill the customer demand upto time $t = t_1$ at which the inventory level reaches at maximum. Therefor the behavior of the

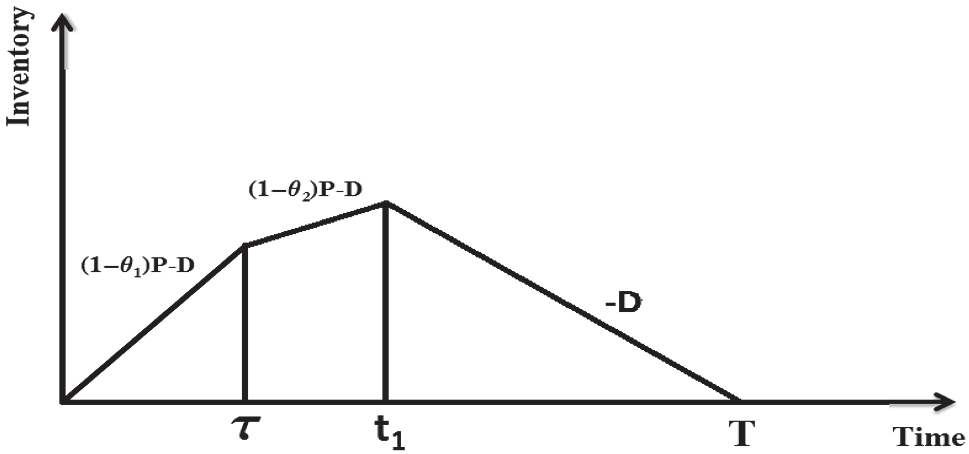

FIGURE 2. Graphical representation of inventory of selling item.

inventory level during $[0,\tau]$ and $[\tau,t_1]$ respectively are given by

$$\begin{aligned}
I_1(t) &= [(1-m_1)(1-\theta_1)P+\gamma m_1(1-\theta_1)P+\delta(1-m_2)\theta_1 P+m_2\theta_1 P-D]t,\ \ 0\le t\le\tau\\
&= [\{1-(1-\gamma)m_1\}(1-\theta_1)P+\{1-(1-\delta)(1-m_2)\}\theta_1 P-(D_0+\rho t_w)e^{k\eta}]t,\ \ 0\le t\le\tau\\
I_2(t) &= [(1-m_1)(1-\theta_2)P+\gamma m_1(1-\theta_2)P+\delta(1-m_2)\theta_2 P+m_2\theta_2 P-D](t-\tau)+I_1(\tau)\tau\le t\le t_1\\
&= [\{1-(1-\gamma)m_1\}(1-\theta_2)P+\{1-(1-\delta)(1-m_2)\}\theta_2 P-(D_0+\rho t_w)e^{k\eta}](t-\tau)\\
&\quad +[\{1-(1-\gamma)m_1\}(1-\theta_1)P+\{1-(1-\delta)(1-m_2)\}\theta_1 P-(D_0+\rho t_w)e^{k\eta}]\tau,\ \ \tau\le t\le t_1.
\end{aligned}$$

Then during the period $[t_1,T]$ the inventory level decline due to meeting customer demand and it reaches zero at T. Therefor the behavior of the inventory level during $[t_1,T]$ is given by

$$\begin{aligned}
I_3(t) &= D(T-t),\ \ t_1\le t\le T\\
&= (D_0+\rho t_w)(T-t),\ \ t_1\le t\le T.
\end{aligned}$$

Lemma 3.1. *When $0<\tau\le t_1$, in a manufacturing system the business period (T) must satisfy the following relation in terms of production rate (P), demand rate (D) warranty period (t_w) and production period (t_1)*

$$\begin{aligned}
T=\frac{1}{(D_0+\rho t_w)}\Big[&\Big\{\{1-(1-\gamma)m_1\}(1-\theta_2)P+\{1-(1-\delta)(1-m_2)\}\theta_2 P+(D_0+\rho t_w)(1-e^{k\eta})\Big\}t_1\\
&+\Big\{\{1-(1-\gamma)m_1\}+\{1-(1-\delta)(1-m_2)\}\Big\}(\theta_2-\theta_1)P\tau\Big].
\end{aligned}$$

Proof. Satisfying the continuity condition of $I_2(t)$ and $I_3(t)$ at $t=t_1$ a relation is obtain following

$$\begin{aligned}
&[\{1-(1-\gamma)m_1\}(1-\theta_2)P+\{1-(1-\delta)(1-m_2)\}\theta_2 P-(D_0+\rho t_w)e^{k\eta}](t_1-\tau)\\
&+[\{1-(1-\gamma)m_1\}(1-\theta_1)P+\{1-(1-\delta)(1-m_2)\}\theta_1 P-(D_0+\rho t_w)e^{k\eta}]\tau=(D_0+\rho t_w)(T-t_1)\\
&\textit{i.e.},\ \ [\{1-(1-\gamma)m_1\}(1-\theta_2)P+\{1-(1-\delta)(1-m_2)\}\theta_2 P-(D_0+\rho t_w)e^{k\eta}]t_1\\
&+[\{1-(1-\gamma)m_1\}+\{1-(1-\delta)(1-m_2)\}](\theta_2-\theta_1)P\tau=(D_0+\rho t_w)(T-t_1)\\
&\textit{i.e.},\ \ (D_0+\rho t_w)T=[\{1-(1-\gamma)m_1\}(1-\theta_2)P+\{1-(1-\delta)(1-m_2)\}\theta_2 P\\
&+(D_0+\rho t_w)(1-e^{k\eta})]t_1+[\{1-(1-\gamma)m_1\}+\{1-(1-\delta)(1-m_2)\}](\theta_2-\theta_1)P\tau
\end{aligned}$$

$$i.e.,\ T=\frac{1}{(D_0+\rho t_w)}\Big[\Big\{\{1-(1-\gamma)m_1\}(1-\theta_2)P+\{1-(1-\delta)(1-m_2)\}\theta_2 P$$
$$+(D_0+\rho t_w)(1-e^{k\eta})\Big\}t_1+\Big\{\{1-(1-\gamma)m_1\}+\{1-(1-\delta)(1-m_2)\}\Big\}(\theta_2-\theta_1)P\tau\Big].$$

Now, the proof is complete. □

Now, we derive the holding cost, manufacturing cost, rework cost, setup cost, inspection cost, return cost, penalty cost, development cost and warranty cost in one cycle as follows.

Holding cost. During the period $[0,T]$, the holding cost is given by

$$\begin{aligned}\mathrm{HC}&=h_c\Big[\int_0^{\tau}I_1(t)\mathrm{d}t+\int_{\tau}^{t_1}I_2(t)\mathrm{d}t+\int_{t_1}^{T}I_3(t)\mathrm{d}t\Big]\\&=\frac{h_c}{2}\Big[\Big\{\{1-(1-\gamma)m_1\}(1-\theta_1)P+\{1-(1-\delta)(1-m_2)\}\theta_1 P\Big\}(2t_1\tau-\tau^2)\\&\quad+\Big\{\{1-(1-\gamma)m_1\}(1-\theta_2)P+\{1-(1-\delta)(1-m_2)\}\theta_2 P\Big\}(t_1-\tau)^2\\&\quad-(D_0+\rho t_w)e^{k\eta}t_1^2+(D_0+\rho t_w)(T-t_1)^2\Big].\end{aligned}$$

Manufacturing, inspection and reworked cost. During the period $[0,t_1]$, total manufacturing, inspection and reworked cost is given by $\mathrm{PC}=(c_p+c_s)Pt_1+c_r\delta(1-m_2)\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}$.

$$\text{Setup cost}=A_0+\frac{K}{Pt_1}\cdot$$

Return and penalty cost. The return cost including communication and reverse logistics per unit (c_t), and penalty cost per unit (c_l), due to inspection errors during the period $[0,T]$ is given by $\mathrm{RC}=(c_t+c_l)m_2\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}$.

Inspection error cost (or misclassification cost). During the period $[0,t_1]$, inspectors accept the amount of $\theta_1 Pt_1$ defective items in which falsely accepted amount of defective items and falsely reject amount of non-defective items are $m_2\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}$ and $m_1(1-\gamma)\{(1-\theta_1)P\tau+(1-\theta_2)P(t_1-\tau)\}$ respectively.

Therefore the inspection error cost is given by, $\mathrm{IEC}=s(1-\gamma)m_1\{(1-\theta_1)P\tau+(1-\theta_2)P(t_1-\tau)\}$.

Development cost. During the period $[0,t_1]$, the development cost is given by

$$c_v=\Big[\int_0^{\tau}B_0\mathrm{d}t+\int_{\tau}^{t_1}\Big\{B_0+B_1(t-\tau)e^{k\frac{v_{\max}-v}{v-v_{\min}}}\Big\}\mathrm{d}t\Big]=B_0t_1+\frac{B_1}{2}(t_1-\tau)^2e^{k_1\frac{v_{\max}-v}{v-v_{\min}}}.$$

Revenue from serviceable items. During the period $[0,T]$, the amount of serviceable items (*i.e.*, falsely accepted defective and successfully accepted non-defective items) is $[(1-m_1)\{(1-\theta_1)P\tau+(1-\theta_2)P(t_1-\tau)\}+m_2\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}]$ at a unit selling price of s, so the sales revenue is $s[(1-m_1)\{(1-\theta_1)P\tau+(1-\theta_2)P(t_1-\tau)\}+m_2\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}]$. Again, the amount of defective items is returned from the customers' due to Type-II inspection errors is $m_2\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}$ for refunds at its full unit price s or replace by non-defective items, it incurs revenue loss which is $sm_2\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}$. Further, the amount of returned non-defective items from rework cell due to inspection errors is $m_1\gamma\{(1-\theta_1)P\tau+(1-\theta_2)P(t_1-\tau)\}$ and the manufacturer obtained sales revenue $sm_1\gamma\{(1-\theta_1)P\tau+(1-\theta_2)P(t_1-\tau)\}$. Also the amount of reworked serviceable items is $\delta(1-m_2)\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}$ at the same unit selling price of s and obtained sales revenue $s\delta(1-m_2)\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}$. Thus, the total sales revenues during the interval $[0,T]$ is given by

$$R=s[\{1-(1-\gamma)m_1\}\{(1-\theta_1)P\tau+(1-\theta_2)P(t_1-\tau)\}+\{\delta+(1-\delta)m_2\}\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}].$$

The warranty cost. During the period $[0, T]$, the warranty cost is given by

$$\mathrm{WC} = c_w[\{1-(1-\gamma)m_1\}\{(1-\theta_1)P\tau + (1-\theta_2)P(t_1-\tau)\} + \{\delta + (1-\delta)m_2\}\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}].$$

Therefore, the profit of the manufacturer in case I is given by

$$\begin{aligned}
\pi_1(t_1, t_w) = {} & (s-c_w)[\{1-(1-\gamma)m_1\}\{(1-\theta_1)P\tau + (1-\theta_2)P(t_1-\tau)\} + \{\delta+(1-\delta)m_2\} \\
& \times \{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}] - [(c_p+c_s)Pt_1 + c_r\delta(1-m_2)\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}] \\
& - \frac{h_c}{2}\Big[\Big\{\{1-(1-\gamma)m_1\}(1-\theta_1)P + \{1-(1-\delta)(1-m_2)\}\theta_1 P\Big\}(2t_1\tau - \tau^2) \\
& + \Big\{\{1-(1-\gamma)m_1\}(1-\theta_2)P + \{1-(1-\delta)(1-m_2)\}\theta_2 P\Big\}(t_1-\tau)^2 \\
& - (D_0+\rho t_w)e^{k\eta}t_1^2 + (D_0+\rho t_w)(T-t_1)^2\Big] - \Big(A_0 + \frac{K}{Pt_1}\Big) \\
& - (c_t+c_l)m_2\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\} - B_0 t_1 - \frac{B_1}{2}(t_1-\tau)^2 e^{k_1\frac{v_{\max}-v}{v-v_{\min}}}.
\end{aligned}$$

The expected profit of the manufacturer in case I is given by

$$\begin{aligned}
E[\pi_1(t_1,t_w)] = {} & (s-c_w)\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)PE[\tau] + (1-\theta_2)PE[(t_1-\tau)]\} \\
& + E[\{m_2+\delta(1-m_2)\}]\{\theta_1 PE[\tau] + \theta_2 PE[(t_1-\tau)]\}\Big] - \Big[(c_p+c_s)Pt_1 \\
& + c_r E[\delta(1-m_2)]\{\theta_1 PE[\tau] + \theta_2 PE[(t_1-\tau)]\}\Big] - \frac{h_c}{2}\Big[\Big\{\{1-E[m_1(1-\gamma)]\} \\
& \times (1-\theta_1)P + \{1-E[(1-\delta)(1-m_2)]\}\theta_1 P\Big\}(2t_1E[\tau] - E[\tau^2]) \\
& + \Big\{\{1-E[(1-\gamma)m_1]\}(1-\theta_2)P + \{1-E[(1-\delta)(1-m_2)]\}\theta_2 P\Big\}E[(t_1-\tau)^2] \\
& + \Big\{(D_0+\rho t_w)(T-t_1)^2 - (D_0+\rho t_w)e^{k\eta}t_1^2\Big\}\int_0^{t_1} f(\tau)\,\mathrm{d}\tau\Big] - B_0 t_1\int_0^{t_1} f(\tau)\,\mathrm{d}\tau \\
& - (c_t+c_l)E[m_2]\{\theta_1 PE[\tau] + \theta_2 PE[(t_1-\tau)]\} - \Big(A_0 + \frac{K}{Pt_1}\Big)\int_0^{t_1} f(\tau)\,\mathrm{d}\tau \\
& - \frac{B_1}{2}e^{k\frac{v_{\max}-v}{v-v_{\min}}}\int_0^{t_1}(t_1-\tau)^2 f(\tau)\,\mathrm{d}\tau \\
= {} & (s-c_w)\Big[\{1-E[m_1(1-\gamma)]\}\Big\{(1-\theta_1)P\lambda t_1^2 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\Big\} + E[\{m_2+\delta(1-m_2)\}] \\
& \times \Big\{\theta_1 P\lambda t_1^2 + \theta_2 P\frac{\lambda t_1^2}{2}\Big\}\Big] - \Big[(c_p+c_s)Pt_1 + c_r E[\delta(1-m_2)]\Big\{\theta_1 P\lambda t_1^2 + \theta_2 P\frac{\lambda t_1^2}{2}\Big\}\Big] \\
& - \frac{h_c}{2}\Big[\Big\{\{1-E[m_1(1-\gamma)]\}(1-\theta_1)P + \{1-E[(1-\delta)(1-m_2)]\}\theta_1 P\Big\}\Big(2\lambda t_1^3 - \frac{1}{2}\lambda^2 t_1^4\Big) \\
& + \Big\{\{1-E[(1-\gamma)m_1]\}(1-\theta_2)P + \{1-E[(1-\delta)(1-m_2)]\}\theta_2 P\Big\}\frac{1}{3}\lambda t_1^3 \\
& + \Big\{(D_0+\rho t_w)(T-t_1)^2 - (D_0+\rho t_w)e^{k\eta}\Big\}\lambda t_1\Big] - \Big(A_0 + \frac{K}{Pt_1}\Big)\lambda t_1 \\
& - (c_t+c_l)E[m_2]\Big\{\theta_1 P\lambda t_1^2 + \theta_2 P\frac{\lambda t_1^2}{2}\Big\} - B_0\lambda t_1^2 - \frac{B_1\lambda t_1^3}{6}e^{k_1\frac{v_{\max}-v}{v-v_{\min}}}.
\end{aligned}$$

See appendix, approximating the function $e^{-\lambda t_1}$ for its expansion.

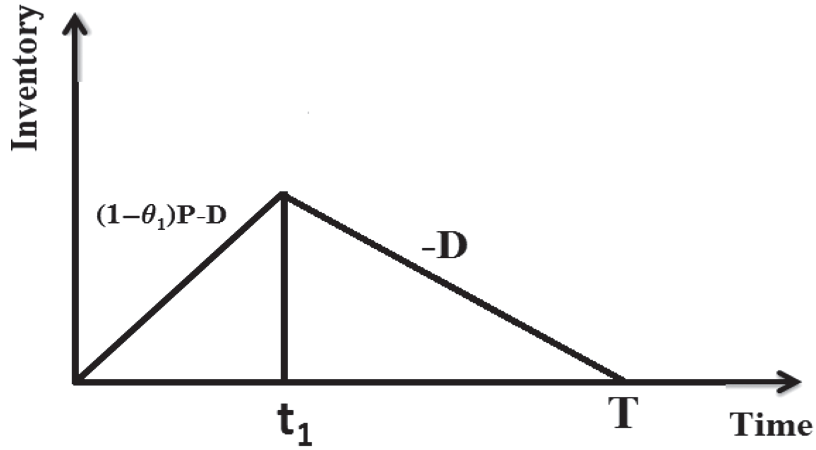

FIGURE 3. Graphical representation of inventory model of perfect quality item.

Case II: When $\tau \geq t_1$, the "out-of-control" state not to be occurred in the production-run time

In this case, the whole production period $[0, t_1]$ is "in-control" state. During the production period $[0, t_1]$, the amount of non-defective items, defective items and reworked items are $(1-\theta_1)Pt_1$, $\theta_1 Pt_1$ and $\delta\theta_1 Pt_1$ respectively. During the inspection period $[0, t_1]$, the inspectors accept defective items of amount $\theta_1 Pt_1$ in which the amount of falsely accepted defective items and falsely rejected non-defective items are $m_2\theta_1 Pt_1$ and $(1-\gamma)m_1(1-\theta_1)Pt_1$ respectively (Fig. 3).

During the period $[0, t_1]$, the inventory level increases due to production after fulfill the customer demand upto time $t = t_1$ at which the inventory level reaches at maximum. Therefore the behavior of the inventory level during the interval $[0, t_1]$ is given by

$$\begin{aligned} I_1(t) &= [(1-m_1)(1-\theta_1)P + \gamma m_1(1-\theta_1)P + \delta(1-m_2)\theta_1 P + m_2\theta_1 P - D]\, t, \ \ 0 \leq t \leq t_1 \\ &= \left[\{1-(1-\gamma)m_1\}(1-\theta_1)P + \{1-(1-\delta)(1-m_2)\}\theta_1 P - (D_0 + \rho t_w)e^{k\eta}\right] t, \ \ 0 \leq t \leq t_1. \end{aligned}$$

Then during the period $[t_1, T]$, the inventory level declines due to meet the customer demand and it reaches zero at T. Therefore the behavior of the inventory level during the interval $[t_1, T]$ is given by

$$\begin{aligned} I_2(t) &= D(T-t), \ \ t_1 \leq t \leq T \\ &= (D_0 + \rho t_w)(T-t), \ \ t_1 \leq t \leq T. \end{aligned}$$

Lemma 3.2. *When $0 < \tau < t_1$, in a manufacturing system the business period (T) must satisfy the following relation in terms of production rate (P), demand rate (D) warranty period (t_w) and production period (t_1) is given by*

$$T = \frac{1}{(D_0 + \rho t_w)}[\{1-(1-\gamma)m_1\}(1-\theta_1)P + \{1-(1-\delta)(1-m_2)\}\theta_1 P + (D_0 + \rho t_w)(1 - e^{k\eta})]t_1.$$

Proof. Satisfying the continuity condition of $I_1(t)$ and $I_2(t)$ at $t = t_1$ a relation is obtain following

$$\begin{aligned} &[\{1-(1-\gamma)m_1\}(1-\theta_1)P + \{1-(1-\delta)(1-m_2)\}\theta_1 P - (D_0 + \rho t_w)e^{k\eta}]t_1 = (D_0 + \rho t_w)(T - t_1) \\ &i.e., \ \ T = \frac{1}{(D_0 + \rho t_w)}\left[\{1-(1-\gamma)m_1\}(1-\theta_1)P + \{1-(1-\delta)(1-m_2)\}\theta_1 P + (D_0 + \rho t_w)(1 - e^{k\eta})\right] t_1. \end{aligned}$$

Now, the proof is complete. □

Now, we derive the holding cost, manufacturing cost, rework cost, inspection cost, setup cost, return cost, penalty cost, development cost and warranty cost in one cycle as follows.

Holding cost. During the period $[0, T]$, the holding cost is given by

$$\begin{aligned}\mathrm{HC} &= h_c\left[\int_0^{t_1} I_1(t)\mathrm{d}t + \int_{t_1}^{T} I_2(t)\mathrm{d}t\right]\\ &= \frac{h_c}{2}\left[\Big\{\{1-(1-\gamma)m_1\}(1-\theta_1)P + \{1-(1-\delta)(1-m_2)\}\theta_1 P\Big\}t_1^2 + (D_0+\rho t_w)(T^2-2t_1T)\right].\end{aligned}$$

Manufacturing, inspection and rework cost. During the period $[0, t_1]$, total manufacturing, inspection and reworked cost is given by $\mathrm{PC} = \{c_p + c_s + c_r\delta(1-m_2)\theta_1\}Pt_1$.

$$\text{Setup cost} = A_0 + \frac{K}{Pt_1}\cdot$$

Return and penalty cost. The return cost including communication and reverse logistics per unit (C_t), and penalty cost per unit (c_l), due to inspection errors during $[0, T]$ is given by $\mathrm{RC} = (c_t + c_l)m_2\theta_1 Pt_1$.

Inspection error cost (or misclassification cost). During the period $[0, t_1]$, inspectors accepts the amount of defective items $\theta_1 Pt_1$ in which falsely accepted amount of defective items and falsely reject amount of non-defective items are $m_2\theta_1 Pt_1$ and $m_1(1-\theta_1)Pt_1$ respectively. Therefore the inspection error cost is given by, $\mathrm{IEC} = s(1-\gamma)m_1(1-\gamma)(1-\theta_1)Pt_1$.

Development cost. During the period $[0, t_1]$, the development cost is given by

$$c_v = \int_0^{t_1} B_0\mathrm{d}t = B_0 t_1.$$

Revenue from serviceable items. During the period $[0, T]$, the amount of serviceable items (*i.e.*, falsely accepted defective and successfully accepted non-defective items) is $[(1-m_1)(1-\theta_1)Pt_1 + m_2\theta_1 Pt_1]$ at a unit selling price of s, so the sales revenue is $s[(1-m_1)(1-\theta_1)Pt_1 + m_2\theta_1 Pt_1]$. Again, amount of defective items to be returned from the customers' due to Type-II inspection errors is $m_2\theta_1 Pt_1$ and refunds at its full unit price s or replace by non-defective items, it incurs revenue loss which is $sm_2\theta_1 Pt_1$. Further, the amount of returned non-defective items from rework cell due to inspection error is $m_1\gamma(1-\theta_1)Pt_1$ and the manufacturer obtains sales revenue $sm_1\gamma(1-\theta_1)Pt_1$. Also the amount of reworked serviceable items is $\delta(1-m_2)\theta_1 Pt_1$ which is sold at the same unit selling price of s and obtains sales revenue $s\delta(1-m_2)\theta_1 Pt_1$. Thus, the total sales revenues during the interval $[0, T]$ is given by

$$R = s[\{1-(1-\gamma)m_1\}(1-\theta_1)Pt_1 + \{m_2 + \delta(1-m_2)\}\theta_1 Pt_1].$$

Warranty cost. During the period $[0, T]$, the warranty cost is given by

$$\mathrm{WC} = c_w[\{1-(1-\gamma)m_1\}(1-\theta_1)Pt_1 + \{m_2 + \delta(1-m_2)\}\theta_1 Pt_1].$$

Therefore, the profit of the manufacturer in case II is given by

$$\begin{aligned}\pi_2(t_1, t_w) = {} & (s-c_w)[\{1-(1-\gamma)m_1\}(1-\theta_1)Pt_1 + \{\delta + (1-\delta)m_2\}\theta_1 Pt_1]\\ & - \{c_p + c_s + c_r\delta(1-m_2)\theta_1\}Pt_1 - \frac{h_c}{2}\Big[\Big\{\{1-(1-\gamma)m_1\}(1-\theta_1)Pt_1\\ & + \{\delta + (1-\delta)m_2\}\theta_1 Pt_1 - (D_0+\rho t_w)e^{k\eta}\Big\}t_1^2 + (D_0+\rho t_w)(T-t_1)^2\Big]\\ & - \left(A_0 + \frac{K}{Pt_1}\right) - \{(c_t + c_l)m_2\theta_1 Pt_1 - s(1-\gamma)m_1(1-\theta_1)Pt_1\} - B_0 t_1.\end{aligned}$$

Therefore, the expected profit of the manufacturer in case II is given by

$$\begin{aligned}E[\pi_2(t_1,t_w)] &= (s-c_w)\Big[\{1-E[m_1(1-\gamma)](1-\theta_1)Pt_1 + E[\{\delta+(1-\delta)m_2\}]\theta_1 Pt_1\Big]\int_{t_1}^{\infty} f(\tau)\,\mathrm{d}\tau \\ &\quad - \{c_p+c_s+c_r E[\delta(1-m_2)]\theta_1\}Pt_1\int_{t_1}^{\infty} f(\tau)\,\mathrm{d}\tau - \frac{h_c}{2}\Big[\Big\{E[\{1-(1-\gamma)m_1\}](1-\theta_1)Pt_1 \\ &\quad + E[\{\delta+(1-\delta)m_2\}]\theta_1 Pt_1 - (D_0+\rho t_w)e^{k\eta}\Big\}t_1^2 + (D_0+\rho t_w)(T-t_1)^2\Big]\int_{t_1}^{\infty} f(\tau)\,\mathrm{d}\tau \\ &\quad - \left[\{(c_t+c_l)E[m_2]\theta_1 Pt_1 + c_r E[(1-\delta)m_1](1-\theta_1)Pt_1\} + \left(A_0+B_0t_1+\frac{K}{Pt_1}\right)\right]\int_{t_1}^{\infty} f(\tau)\,\mathrm{d}\tau \\ &= (s-c_w)\Big[\{1-E[m_1(1-\gamma)](1-\theta_1)Pt_1 + E[\{\delta+(1-\delta)m_2\}]\theta_1 Pt_1\Big](1-\lambda t_1) \\ &\quad - \{c_p+c_s+c_r E[\delta(1-m_2)]\theta_1\}Pt_1(1-\lambda t_1) - \frac{h_c}{2}\Big[\Big\{E[\{1-(1-\gamma)m_1\}](1-\theta_1)Pt_1 \\ &\quad + E[\{\delta+(1-\delta)m_2\}]\theta_1 Pt_1 - (D_0+\rho t_w)e^{k\eta}\Big\}t_1^2 + (D_0+\rho t_w)(T-t_1)^2\Big](1-\lambda t_1) \\ &\quad - \left[\{(c_t+c_l)E[m_2]\theta_1 Pt_1 + c_r E[(1-\delta)m_1](1-\theta_1)Pt_1\} + \left(A_0+B_0t_1+\frac{K}{Pt_1}\right)\right](1-\lambda t_1).\end{aligned}$$

See appendix, approximating the function $e^{-\lambda t_1}$ of its expansion.

3.1. Case I + Case II: Total expected profit during whole business period $[0, T]$

Now combining Case I and Case II, the expected total profit is given by

$$\begin{aligned}E[\pi(t_1,t_w)] &= (s-c_w)\left[\{1-E[m_1(1-\gamma)]\}\left\{(1-\theta_1)Pt_1 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\right\} + E[m_2+\delta(1-m_2)]\right. \\ &\quad \left.\times\left\{\theta_1 Pt_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}\right] - \left[(c_p+c_s)Pt_1 + c_r E[\delta(1-m_2)]\left\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\right\}\right] \\ &\quad - \frac{h_c}{2}\left[\{E[1-(1-\gamma)m_1](1-\theta_1)P + E[\delta+(1-\delta)m_2]\theta_1 P\}\left(t_1^2+\lambda t_1^3 - \frac{\lambda^2 t_1^4}{2}\right)\right. \\ &\quad + \frac{1}{3}\Big\{E[1-(1-\gamma)m_1](1-\theta_2)P + E[\delta+(1-\delta)m_2]\theta_2 P\Big\}\lambda t_1^3 - (D_0+\rho t_w)e^{k\eta}t_1^2 \\ &\quad \left. + (D_0+\rho t_w)(T-t_1)^2\right] - (c_t+c_l)E[m_2]\left\{\theta_1 Pt_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\} - c_r E[(1-\delta)m_1] \\ &\quad \times\left\{(1-\theta_1)Pt_1 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\right\} - B_0 t_1 - \frac{B_1\lambda t_1^3}{6}e^{k_1\frac{\upsilon_{\max}-\upsilon}{\upsilon-\upsilon_{\min}}} - \left(A_0+\frac{K}{Pt_1}\right).\end{aligned}$$

Lemma 3.3. *In the manufacturing system, the business period (T) must satisfy the following relation in terms of production rate (P), demand rate (D), warranty period (t_w) and production period (t_1) as follows*

$$\begin{aligned}T &= \frac{1}{(D_0+\rho t_w)}\Big[\Big\{\{1-E[(1-\gamma)m_1]\}(1-\theta_1)P + \{1-E[(1-\delta)(1-m_2)]\}\{2(\theta_2-\theta_1)\lambda t_1 \\ &\quad + \theta_1\}P + (D_0+\rho t_w)(1-e^{k\eta})\Big\}t_1\Big].\end{aligned}$$

Proof. From Lemma 3.1, the expected value of T in case I ($0 < \tau \le t_1$) is given by

$$\int_0^{t_1} T\, f(\tau)\,\mathrm{d}\tau = \frac{1}{(D_0+\rho t_w)}\Big[\Big\{\{1-E[(1-\gamma)m_1]\}(1-\theta_2)P + \{1-E[(1-\delta)(1-m_2)]\}\theta_2 P + (D_0+\rho t_w)(1-e^{k\eta})\Big\}t_1\int_0^{t_1} f(\tau)\,\mathrm{d}\tau + \Big\{\{1-E[](1-\gamma)m_1]\} + \{1-E[(1-\delta)(1-m_2)]\}\Big\}(\theta_2-\theta_1)P\int_0^{t_1}\tau\, f(\tau)\,\mathrm{d}\tau\Big]. \tag{3.1}$$

Again, from Lemma 3.2, the expected value of T in case I ($t_1 < \tau \le \infty$) is given by

$$\int_{t_1}^{\infty} T\, f(\tau)\,\mathrm{d}\tau = \frac{1}{(D_0+\rho t_w)}[\{1-E[(1-\gamma)m_1]\}(1-\theta_1)P + \{1-E[(1-\delta)(1-m_2)]\}\theta_1 P + (D_0+\rho t_w)(1-e^{k\eta})]t_1\int_{t_1}^{\infty} f(\tau)\,\mathrm{d}\tau. \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$T = \frac{1}{(D_0+\rho t_w)}\Big[\Big\{\{1-E[(1-\gamma)m_1]\}(1-\theta_1)P + \{1-E[(1-\delta)(1-m_2)]\}\{2(\theta_2-\theta_1)\lambda t_1 + \theta_1\}P + (D_0+\rho t_w)(1-e^{k\eta})\Big\}t_1\Big].$$

Now, the proof is complete. □

Hence, the average expected profit is given by

$$\begin{aligned}\mathrm{AEP}(t_1,t_w) &= \frac{E[\pi(t_1,t_w)]}{T}\\ &= \frac{(s-a-bt_w)}{T}\left[\{1-E[m_1(1-\gamma)]\}\left\{(1-\theta_1)Pt_1 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\right\} + E[m_2+\delta(1-m_2)]\right.\\ &\quad\left.\times\left\{\theta_1 Pt_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}\right] - \frac{1}{T}\left[(c_p+c_s)Pt_1 + c_r E[\delta(1-m_2)]\left\{\theta_1 Pt_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}\right]\\ &\quad - \frac{h_c}{2T}\left[\{E[1-(1-\gamma)m_1](1-\theta_1)P + E[\delta+(1-\delta)m_2]\theta_1 P\}\left(t_1^2 + \lambda t_1^3 - \frac{\lambda^2 t_1^4}{2}\right)\right.\\ &\quad + \frac{1}{3}\Big\{E[1-(1-\gamma)m_1](1-\theta_2)P + E[\delta+(1-\delta)m_2]\theta_2 P\Big\}\lambda t_1^3 - (D_0+\rho t_w)e^{k\eta}t_1^2\\ &\quad\left. + (D_0+\rho t_w)(T-t_1)^2\right] - \frac{1}{T}(c_t+c_l)E[m_2]\left\{\theta_1 Pt_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\} - \frac{c_r}{T}E[(1-\delta)m_1]\\ &\quad\times\left\{(1-\theta_1)Pt_1 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\right\} - \frac{1}{T}\left[B_0 t_1 + \frac{B_1\lambda t_1^3}{6}e^{k_1\frac{v_{\max}-v}{v-v_{\min}}} + \left(A_0 + \frac{K}{Pt_1}\right)\right].\end{aligned} \tag{3.3}$$

Now the average expected profit (AEP) is a function of two independent variables t_1 and t_w.

Here it is considered that $\frac{\partial}{\partial t_1}(\mathrm{AEP}(t_1,t_w)) = F(t_1,t_w)$ (see appendix) and $\frac{\partial}{\partial t_w}(\mathrm{AEP}(t_1,t_w)) = G(t_1,t_w)$ (see appendix). Due to complexity of the equations, $F(t_1,t_w)=0$ and $G(t_1,t_w)=0$, it is not possible to show the existence of the solution analytically. Now it is supposed that there exists at least one positive point (t_1^r, t_w^r) for which $F(t_1^r,t_w^r)=0$ and $G(t_1^r,t_w^r)=0$ for some parametric values involved in the system.

Let at (t_1^r,t_w^r), $\frac{\partial F}{\partial t_1}=\Delta_1$, $\frac{\partial G}{\partial t_w}=\Delta_2$ and $\frac{\partial F}{\partial t_w}=\Delta_3$.

Lemma 3.4. *The maximum average profit* $\mathrm{AEP}(t_1^r,t_w^r)$ *exits if* $\Delta_1\Delta_2 > \Delta_3^2$, $\Delta_1 < 0$ *and* $\Delta_2 < 0$.

Proof. Now, from the optimization of calculus, it is known that a function of two variables, $\phi(u,v)$ is maximum at the stationary point (a,b) if $\frac{\partial^2}{\partial u^2}(\phi(a,b))\frac{\partial^2}{\partial v^2}(\phi(a,b))$-$\{\frac{\partial^2}{\partial u\partial v}(\phi(a,b))\}^2>0$, $\frac{\partial^2}{\partial u^2}(\phi(a,b))<0$ and $\frac{\partial^2}{\partial v^2}(\phi(a,b))<0$.

The 1st condition for the existence of maximum value of $\mathrm{AEP}(t_1,t_w)$ at the point (t_1^r,t_w^r) is $\frac{\partial^2}{\partial t_1^2}(\mathrm{AEP}(t_1^r,t_w^r))\frac{\partial^2}{\partial t_w^2}(\mathrm{AEP}(t_1^r,t_w^r))$-$\{\frac{\partial^2}{\partial t_1\partial t_w}(\mathrm{AEP}(t_1^r,t_w^r))\}^2>0$.

i.e., $\frac{\partial}{\partial t_1}(F(t_1^r,t_w^r))\frac{\partial}{\partial t_w}(G(t_1^r,t_w^r))$-$\{\frac{\partial}{\partial t_w}(F(t_1^r,t_w^r))\}^2>0$, since $\frac{\partial}{\partial t_1}(\mathrm{AEP}(t_1,t_w))=F(t_1,t_w)$ and $\frac{\partial}{\partial t_w}(\mathrm{AEP}(t_1,t_w))=G(t_1,t_w)$.

i.e., $\Delta_1\Delta_2>\Delta_3^2$, since $\frac{\partial}{\partial t_1}F(t_1^r,t_w^r)=\Delta_1$, $\frac{\partial}{\partial t_w}G(t_1^r,t_w^r)=\Delta_2$ and $\frac{\partial}{\partial t_w}F(t_1^r,t_w^r)=\Delta_3$.

The 2nd condition for the existence of maximum value of $\mathrm{AEP}(t_1,t_w)$ at the point (t_1^r,t_w^r) is $\frac{\partial^2}{\partial t_1^2}(\mathrm{AEP}(t_1^r,t_w^r))<0$. *i.e.*, $\frac{\partial}{\partial t_1}(F(t_1^r,t_w^r))<0$, since $\frac{\partial}{\partial t_1}(\mathrm{AEP}(t_1,t_w))=F(t_1,t_w)$.
i.e., $\Delta_1<0$, since $\frac{\partial}{\partial t_1}F(t_1^r,t_w^r)=\Delta_1$.

The 3rd condition for the existence of maximum value of $\mathrm{AEP}(t_1,t_w)$ at the point (t_1^r,t_w^r) is $\frac{\partial^2}{\partial t_w^2}(\mathrm{AEP}(t_1^r,t_w^r))<0$. *i.e.*, $\frac{\partial}{\partial t_w}(G(t_1^r,t_w^r))<0$, since $\frac{\partial}{\partial t_w}(\mathrm{AEP}(t_1,t_w))=G(t_1,t_w)$.
i.e., $\Delta_2<0$, since $\frac{\partial}{\partial t_w}G(t_1^r,t_w^r)=\Delta_2$. Now, the proof is complete. □

Lemma 3.5. *There does not exist the maximum average profit* $\mathrm{AEP}(t_1^r,t_w^r)$ *if* $\Delta_1>0$ *and* $\Delta_2>0$.

Lemma 3.6. *There does not exist the maximum average profit* $\mathrm{AEP}(t_1^r,t_w^r)$ *if* $\Delta_1\Delta_2-\Delta_3^2<0$.

4. Solution methodology

From equation (3.3) it is seen that in the proposed model, the objective function $\mathrm{AEP}(t_1,t_w)$ is highly nonlinear. Here t_1 and t_w are two decision variables. Also T is a function of t_1 and t_w obtained according to Lemma 3.3. Since the objective function is highly nonlinear hence to get the optimal solution of the proposed model the following algorithms have been developed.

Algorithm 4.1. For a fixed x, suppose $x=x_0$, the value of y from $\psi(x,y)=0$ can be obtained as follows:

Step 1: For $x=x_0$, compute $\psi(x_0,y)=0$.
Step 2: Select (y_1,y_2) such that $\psi(x_0,y_1)\psi(x_0,y_2)<0$. Then by Roll's theorem there exist a root of $\psi(x_0,y)=0$, between y_1 and y_2.
Step 3: Calculate $m=\frac{(y_1+y_2)}{2}$, be the midpoint of the interval (y_1,y_2).
Step 4: Compute the signs of $\psi(x_0,y_1)$, $\psi(x_0,m)$, and $\psi(x_0,y_2)$.
Step 5: If $\psi(x_0,y_1)\psi(x_0,m)<0$, then a root of $\psi(x_0,y)=0$ lies between y_1 and m. In this case, replace y_2 by m. Otherwise, a root of $\psi(x_0,y)=0$ lies between m and y_2, then replace y_1 by m.
Step 6: Repeat steps 3 through 5 until $|y_1-y_2|<10^{-\varepsilon}$ where ε is a tolerance limit.
Step 7: Then the root of $\psi(x_0,y)=0$ is m such that $m=\frac{(y_1+y_2)}{2}$.

Algorithm 4.2. Since there is no possibility to get the general explicit solution due to absence of linearity of the profit function, to get the maximum profit the following procedure has been devised according to Lemma 3.4 and Algorithm 4.1. Here the optimal values of T, t_1, t_w and $\mathrm{AEP}(t_1,t_w)$ are denoted by T^*, t_1^*, t_w^* and AEP^* respectively.

Step 1: Initialize all parameters associated with the objective function $\mathrm{AEP}(t_1,t_w)$.
Step 2: Set an interval (t_{10},t_{11}) where $t_{10}\in(0,T_0)$ and $t_{11}\in(0,T_0)$. Here $t_w\le T_0$ where T_0 also is initialized.
Step 3: Compute t_{w0F}, t_{w1F}, t_{w0G} and t_{w1G} for t_w from $F(t_{10},t_w)=0$, $F(t_{11},t_w)=0$, $G(t_{10},t_w)=0$ and $G(t_{11},t_w)=0$ respectively by Algorithm 4.1.
Step 4: Compute $\triangle_{t_{10}}=t_{w0F}-t_{w0G}$ and $\triangle_{t_{11}}=t_{w1F}-t_{w1G}$.
Step 5: If $\triangle_{t_{10}}\triangle_{t_{11}}<0$, *i.e.*, the signs of $\triangle_{t_{10}}$ and $\triangle_{t_{11}}$ are opposite, then compute $t_{1m}=\frac{(t_{10}+t_{11})}{2}$.

Step 6: Compute t_{w1mF} and t_{w1mG} for t_w from $F(t_{1m}, t_w) = 0$ and $G(t_{1m}, t_w) = 0$ respectively by Algorithm 4.1.
Step 7: Calculate $\triangle_{t_{1m}} = \triangle_{t_{w1mF}} - \triangle_{t_{w1mG}}$.
Step 8: Compare $\triangle_{t_{1m}}$ with $\triangle_{t_{10}}$. If $\triangle_{t_{10}}\triangle_{t_{1m}} < 0$, *i.e.*, the signs of $\triangle_{t_{10}}$ and $\triangle_{t_{1m}}$ are opposite, then replace t_{11} by t_{1m}. Otherwise replace t_{10} by t_{1m}.
Step 9: Repeat steps 5 through 8 until the absolute values of $(t_{10} - t_{1m})$ or $(\triangle_{t_{10}} - \triangle_{t_{1m}})$ or $(\triangle_{t_{10}} - \triangle_{t_{1m}})$ are within the tolerance limits.
Step 10: The root of $F(t_1, t_w) = 0$ and $G(t_1, t_w) = 0$ is $({t_1}^r, {t_w}^r)$ where ${t_1}^r = t_{1m}$ and $t_{wr} = \frac{t_{w0F}+t_{w1F}}{2}$ or $\frac{t_{w0G}+t_{w1G}}{2}$.
Step 11: Compute $\triangle_1$, $\triangle_2$ and $\triangle_3$ at the point $({t_1}^r, {t_w}^r)$ where $\frac{\partial F}{\partial t_1} = \Delta_1$, $\frac{\partial G}{\partial t_w} = \Delta_2$ and $\frac{\partial F}{\partial t_w} = \Delta_3$.
Step 12: If $\Delta_1 < 0$, $\Delta_2 < 0$ and $\Delta_1\Delta_2 > \Delta_3^2$, then according Lemma 3.4 then $({t_1}^r, {t_w}^r)$ be the optimal solution. So $t_1^* = {t_1}^r$, $t_w^* = {t_w}^r$ and calculate T^* by Lemma 3.3. Also calculate $\text{AEP}^* = \text{AEP}(t_1^*, t_w^*)$.
Step 13: If $\Delta_1 > 0$, $\Delta_2 > 0$ by Lemma 3.5, or $\Delta_1\Delta_2 - \Delta_3^2 < 0$ by Lemma 3.6, then $({t_1}^r, {t_w}^r)$ is not optimal solution. In this case, goto step 1 and change some parametric values.
Step 14: Print the optimal values t_1^*, t_w^*, T^* and AEP^*.

5. Numerical analysis

Here we considered a production inventory system that produces defective and non-defective items as well as continuously fills up the customer demand. The inspection process that screens out the defective items is also imperfect. After a random time at which a the production system goes from "in-control" to "out-of-control" in a cycle has been considered random which is exponentially distributed with mean $\frac{1}{\lambda}$. Similarly, the parameters for inspection errors and rework rate have been considered as uniform distribution. The probability density functions of the inspection errors and rewoke rate are mostly taken from the history of a machine and workers. Using the above mentioned solution procedure (Section 4), the optimum values of t_1, t_w, T and the average expected total profit, $\text{AEP}(t_1, t_w)$ have been calculated for the following values of the parameters of the illustrated model:

$P = 100$ unit/unit time, $D_0 = 45$ unit per unit time, $\theta_1 = 0.05, \theta_2 = 0.12, s_0 = \$95$/unit, $\lambda = 0.01$, $c_p = \$25$/unit, $c_{sr} = \$3$ /unit, $c_r = \$15$ /unit, $(c_t + c_l) = \$5.5$/unit, $h_c = \$1.5$ /unit/unit time, $a = \$15, v_{\max} = 10, v_{\min} = 3, v = 8, B_1 = \$58, B_0 = \$45$/unit time, $A_0 = \$257, k = 1, K = 25$.
The probability density functions of the inspection errors (m_1 and m_2), fraction of rejecting non-defective items (γ) due to type I error and rewoke rate (δ) are considered as follows:

$$\phi(m_1) = \begin{cases} \frac{1}{\alpha}, & 0 \le m_1 \le \alpha \\ 0, & \text{otherwise} \end{cases} \quad \phi(m_2) = \begin{cases} \frac{1}{\beta}, & 0 \le m_2 \le \beta \\ 0, & \text{otherwise} \end{cases}$$
$$\phi(\gamma) = \begin{cases} \frac{1}{\xi}, & 0 \le \gamma \le \xi \\ 0, & \text{otherwise} \end{cases} \quad \phi(\delta) = \begin{cases} \frac{1}{\mu}, & 0 \le \delta \le \mu \\ 0, & \text{otherwise} \end{cases}.$$

Now we calculate $E[m_2]$, $E[\delta]$, $E[(1-\gamma)m_1]$, $E[\delta(1-m_2)]$, and $E[(1-\delta)(1-m_2)]$ and the values are given by

$$E[m_2] = \int_0^\beta m_2\phi(m_2)\,\mathrm{d}\delta = \frac{\beta}{2}, \quad E[\delta] = \int_0^\mu \delta\phi(\delta)\,\mathrm{d}\delta = \frac{\mu}{2}$$
$$E[(1-\gamma)m_1] = \int_0^\alpha m_1\phi(m_1)\,\mathrm{d}m_1 \int_0^\xi (1-\gamma)\phi(\gamma)\,\mathrm{d}\gamma = \frac{\alpha}{2}\left(1 - \frac{\xi}{2}\right)$$
$$E[\delta(1-m_2)] = \int_0^\mu \delta\phi(\delta)\,\mathrm{d}\delta \int_0^\alpha (1-m_2)\phi(m_2)\,\mathrm{d}m_1 = \frac{\mu}{2}\left(1 - \frac{\beta}{2}\right)$$
$$E[(1-\delta)(1-m_2)] = \int_0^\alpha (1-\delta)\phi(\delta)\,\mathrm{d}\delta \int_0^\beta (1-m_2)\phi(m_2)\,\mathrm{d}m_1 = (1 - \frac{\mu}{2})\left(1 - \frac{\beta}{2}\right).$$

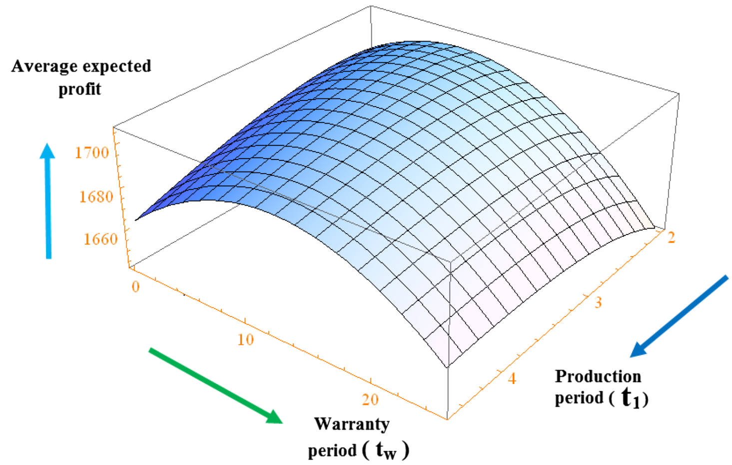

FIGURE 4. Expected average profit $\text{AEP}(t_1, t_w)$ versus production period (t_1) and warranty (t_w).

TABLE 1. Optimal solution of the illustrated model with the effect of discount on selling price and warranty period on demand.

	Production period (t_1^* umit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($\text{AEP}^*(t_1^*, t_w^*)$ \$)
Optimal value	2.59	7.79	4.93	1706.15

Notes. $\eta = 0.10, \rho = 0.50, b_0 = 0.4, k = 0.80.$

TABLE 2. Optimal solution of the illustrated model with the effect of only warranty period on demand.

	Production period (t_1^* umit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($\text{AEP}^*(t_1^*, t_w^*)$ \$)
Optimal value	2.26	5.17	4.61	1624.77

Notes. $\eta = 0.10, k = 0.80.$

Substituting the above expressions in the profit function in equation (3.3), we obtain the optimal values of the expected average profit when $\alpha = 0.04$, $\beta = 0.06$, $\xi = 0.0004$ and $\mu = 0.60$.

For this data set, Figure 4 shows the average expected profit as a function of t_1 and t_w. From this figure it is guarantied that the average expected profit is concave. So there exist unique solution of (t_1, t_w) that maximize the average expected profit $\text{AEP}(t_1, t_w)$. The optimal solutions for the given parametric set with different type of demand rate are represented by following Tables 1–3.

The change in the values of the system parameters can take place an important role in decision-making about the system due to uncertainties and dynamic market conditions. In order to examine the implications of these changes in the values of parameters, the sensitivity analysis will be of great help in a decision-making process. Here, the sensitivity analysis with respect to the parameters such as α, β, μ, η, k, ρ, λ, and b have been carried out. The results of the sensitivity analysis are shown in Tables 4–10.

TABLE 3. Optimal solution of the illustrated model with the effect of only discount on selling price on demand.

	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($\mathrm{AEP}^*(t_1^*, t_w^*)$ \$)
Optimal value	2.28	–	4.73	1694.55

Notes. $b_0 = 0.4, \rho = 0.50$.

TABLE 4. Sensitivity analysis with respect to the probability of Type I error (α).

Parameter (α)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($\mathrm{AEP}^*(t_1^*, t_w^*)$ \$)
0.04	2.59	7.78	4.93	1706.15
0.08	2.66	6.89	5.01	1669.81
0.12	2.72	5.95	5.09	1632.42
0.16	2.80	4.98	5.18	1593.94
0.20	2.87	3.97	5.27	1554.34

TABLE 5. Sensitivity analysis with respect to the probability of Type II error (β).

Parameter (β)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($\mathrm{AEP}^*(t_1^*, t_w^*)$ \$)
0.06	2.59	7.78	4.93	1706.15
0.12	2.56	7.39	4.89	1694.23
0.18	2.53	7.01	4.86	1682.38
0.24	2.50	6.63	4.83	1670.60
0.30	2.48	6.25	4.80	1658.89

– From Table 4, we see that the warranty period (t_w^*), the business cycle period (T^*) and the expected average profit ($\mathrm{AEP}^*(t_1^*, t_w^*)$) decrease with the increase of α *i.e.*, probability of a Type I error. But, the production period (t_1^*) increases when α increases.

– From Table 5, we see that the production period (t_1^*), warranty period (t_w^*), the business cycle period (T^*) and the expected average profit ($\mathrm{AEP}^*(t_1^*, t_w^*)$) decrease with the increase of β *i.e.*, probability of a Type II error.

– From Table 6 it is observed that when μ and c_r increases simultaneously, the production period (t_1^*), warranty period (t_w^*) and the expected average profit ($\mathrm{AEP}^*(t_1^*, t_w^*)$) initially increase, then decrease due to the rapidly increase of average rework cost for defective item.

– Table 7 signifies that when k is fixed, the production period (t_1^*),warranty period (t_w^*), the business cycle period (T^*) and the expected average profit ($\mathrm{AEP}^*(t_1^*, t_w^*)$) increase together due to the increase of discount rate η.

– Table 8 shows that when η increases and k decreases simultaneously, the production period (t_1^*),warranty period (t_w^*), the business cycle period (T^*) and the average expected profit ($\mathrm{AEP}^*(t_1^*, t_w^*)$) initially increase, after that decrease.

Table 6. Sensitivity analysis on the rework rate (μ) and reworked cost (c_r) simultaneously.

Parameter (μ)	Reworked cost (c_r \$)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle time (T^* unit)	Expected average profit ($\text{AEP}^*(t_1^*, t_w^*)$ \$)
0.45	11	2.56	7.81	4.90	1692.84
0.50	12	2.57	7.82	4.91	1697.85
0.55	13	2.58	7.83	4.92	1702.78
0.60	15	2.59	7.79	4.93	1706.15
0.65	20	2.58	7.61	4.92	1704.75
0.70	25	2.58	7.42	4.92	1702.96
0.75	28	2.58	7.22	4.92	1700.77

Table 7. Sensitivity analysis on different values of η with a fixed $k = 0.8$.

Parameter (η)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($\text{AEP}^*(t_1^*, t_w^*)$ \$)
0.06	2.44	6.67	4.78	1671.61
0.08	2.51	7.22	4.85	1688.52
0.10	2.59	7.78	4.92	1706.15
0.12	2.68	8.38	5.01	1724.55
0.14	2.76	9.01	5.11	1743.78

Table 8. Sensitivity analysis with respect to η and k simultaneously.

Parameter (η and k)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($\text{AEP}^*(t_1^*, t_w^*)$ \$)
$\eta = 0.05$, $k = 1.2$	2.50	7.08	4.83	1684.23
$\eta = 0.07$, $k = 1.0$	2.54	7.43	4.88	1695.04
$\eta = 0.10$, $k = 0.8$	2.59	7.78	4.92	1706.15
$\eta = 0.12$, $k = 0.6$	2.55	7.50	4.88	1697.24
$\eta = 0.14$, $k = 0.4$	2.47	6.94	4.81	1679.98

Table 9. Sensitivity analysis for different values of λ.

Parameter (λ)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($\text{AEP}^*(t_1^*, t_w^*)$ \$)
0.005	2.26	7.01	4.32	1690.68
0.010	2.59	7.78	4.92	1706.15
0.015	3.10	8.87	5.84	1724.09
0.020	3.99	10.54	7.41	1745.99
0.025	5.73	13.48	10.40	1774.95

– Table 9 explores that when the value of λ increases, both the production period (t_1^*) and warranty period (t_w^*) increase as well as the corresponding business cycle period T^* and expected average profit ($\text{AEP}^*(t_1^*, t_w^*)$) also increase due to the increase of the "in-control" state.

TABLE 10. Sensitivity analysis for different values of ρ.

Parameter (ρ)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($\text{AEP}^*(t_1^*, t_w^*)$ \$)
0.45	2.36	2.38	4.77	1695.54
0.50	2.59	7.78	4.92	1706.15
0.55	2.87	12.34	5.14	1726.22
0.60	3.22	16.33	5.44	1753.88
0.65	3.71	19.96	5.91	1788.05

– Table 10 shows that when the value of ρ increases, all of the production period (t_1^*), warranty period (t_w^*) and the business cycle period (T^*) increases together. In this case it is also observed that the expected average profit ($\text{AEP}^*(t_1^*, t_w^*)$) increases due to increase of the demand rate.

6. Managerial insights

From numerical analysis of the proposed model, the following managerial insights have been drawn.

(i) From Tables 1–3 it is concluded that the average expected profit is maximum when manufacturer gives both effects such as (a) selling price discount, and (b) warranty period policy on the sale because of attraction of customer. Practically this phenomena is observed in real business system. So this finding supports the real case.

(ii) Again, from Tables 4 and 5 it is inferred that the expected average profit decreases as the probability of a Type I and Type II error increases. This is because of (a) addition to the loss of incorrect rejection of a non-defective item, and (b) return and penalty cost for a defective item to be sold as a non-defective item which is returned from market. In reality, this can be seen in business system. So this finding also supports the real case.

(iii) Table 6 shows that when the probability of reworked rate and average reworked cost simultaneously increase, initially average expected profit increases, after that average expected profit decreases due to a minimum rework cost for reworking some of the defective items and the rest portion of defective items, if reworked, then the rework-cost will be huge. So any manufacturer company can find the optimal reworked rate from this study. Practically, this phenomenon is observed in real business system. So the real situation is supports by finding.

(iv) Table 8 shows that when selling price discount(η) increases and the corresponding effective parameter (k) simultaneously decreases, initially the average expected profit increases, after that the average expected profit decreases. Because at first selling price discount attracts more customers. As a result, the demand rate increases. But later, though discount rate increases, the rate of demand does not increase as much as in the initial stage due to market saturation. So from this study, any manufacturer company can find the optimal selling price discount rate. Practically this phenomenon can be found in real business system. So this finding supports the real example.

7. Practical implication

There are many practical implications of this proposed model. As for example, it is very practicable in the manufacturing system for mobile phones. At the time of production, few defective units (like, scratch, disorder shape, etc) are produced and then some of them get repaired to be sold at the market. Sometimes the company gives a discount on selling price and increases the warranty period to increase selling rate. The decision manager of the company decides the maximization of the profit function, considering the warranty period of each product and the length of the production cycle. For such a real life problem, the present model can be implemented.

From this study, some managerial insights have been drawn which are very useful for the decision maker of any newly established mobile company.

8. Conclusion and future research

In this paper, we have studied the combined effects of inspection errors and warranty policy in an imperfect production system. Here for inspection errors, a newly proposed Type I error has been incorporated to get the effect of profit function. Also warranty period and warranty cost both have been considered simultaneously. Here we have demonstrated how warranty policy and discount on selling price can be effected in the market demand of a manufacturing system. The key decisions are to determine the optimal production period and warranty period to maximize the average expected profit for the manufacturer. To solve the objective function of our proposed model, a computational algorithm has been developed to determine the optimal warranty period and optimal production period as well as average expected profit.

From this study, for the practitioners it is recommended that (i) the discount on selling price at random does not go in favor of profit. Actually, there exists an upper limit of discount for which profit goes to increase, and (ii) products should be carefully inspected to increase the quality of the product in such a way that during warranty period minimum number of sold items are returned from the retailer.

Limitations of this model are that (i) model should be related with imperfect production system, (ii) produced items must be repairable, and (iii) items should be of electronics nature.

There are several interesting future extensions of this research work. First, we may study the effect of trade credit dependent demand [28,32]. Second, we can extend the model to allow machine breakdown with shortages [7,18]. Third, this model can be extended in a supply chain if it contains manufacture with imperfect production [19]. Finally, we can study the effect of carbon emissions for a clean production system [16,34].

Appendix

(i) $$\int_0^{t_1} f(\tau)\,\mathrm{d}\tau = \lambda\int_0^{t_1} e^{-\lambda\tau}\,\mathrm{d}\tau = 1-e^{-\lambda t_1} = \lambda t_1,$$

approximating up to the second term of the expansion of $e^{-\lambda t_1}$.

(ii) $$\int_{t_1}^{\infty} f(\tau)\,\mathrm{d}\tau = \lambda\int_{t_1}^{\infty} e^{-\lambda\tau}\,\mathrm{d}\tau = e^{-\lambda t_1} = 1-\lambda t_1,$$

approximating up to the second term of the expansion of $e^{-\lambda t_1}$.

(iii) $$\int_0^{t_1} \tau f(\tau)\,\mathrm{d}\tau = \lambda\int_0^{t_1} \tau e^{-\lambda\tau}\,\mathrm{d}\tau = \frac{1}{\lambda}\{1-e^{-\lambda t_1}\} - t_1 e^{-\lambda t_1} = \lambda t_1^2,$$

approximating up to the third term of the expansion of $e^{-\lambda t_1}$.

(iv) $$\int_0^{t_1} \Big(t_1-\tau\Big) f(\tau)\,\mathrm{d}\tau = \lambda\int_0^{t_1} \Big(t_1-\tau\Big) e^{-\lambda\tau}\,\mathrm{d}\tau = t_1 - \frac{1}{\lambda}\left\{1-e^{-\lambda t_1}\right\} = \frac{1}{2}\lambda t_1^2,$$

approximating up to the third term of the expansion of $e^{-\lambda t_1}$.

(v) $$\int_0^{t_1} \tau^2 f(\tau)\,\mathrm{d}\tau = \lambda\int_0^{t_1} \tau^2 e^{-\lambda\tau}\,\mathrm{d}\tau = t_1^2 e^{-\lambda t_1} - \frac{2t_1}{\lambda}e^{-\lambda t_1} + \frac{2}{\lambda^2}\{1-e^{-\lambda t_1}\} = \frac{1}{2}\lambda^2 t_1^4,$$

approximating up to the third term of the expansion of $e^{-\lambda t_1}$.

(vi) $$\int_0^{t_1}\Big(t_1-\tau\Big)^2 f(\tau)\,\mathrm{d}\tau=\lambda\int_0^{t_1}\Big(t_1-\tau\Big)^2 e^{-\lambda\tau}\,\mathrm{d}\tau=t_1^2-\frac{2t_1}{\lambda}+\frac{2}{\lambda^2}\{1-e^{-\lambda t_1}\}=\frac{1}{3}\lambda t_1^3,$$

approximating up to the fourth term of the expansion of $e^{-\lambda t_1}$.

(vii) $$\int_0^{t_1}\Big(t_1-\tau\Big)^3 f(\tau)\,\mathrm{d}\tau=\lambda\int_0^{t_1}\Big(t_1-\tau\Big)^3 e^{-\lambda\tau}\,\mathrm{d}\tau=t_1^3-\frac{3}{\lambda}\Big[t_1^2-\frac{2t_1}{\lambda}+\frac{2}{\{\lambda\}^2}\{1-e^{-\lambda t_1}\}\Big],=\frac{1}{4}\lambda t_1^4,$$

using Appendix (iv) & approximating up to the fifth term of the expansion of $e^{-\lambda t_1}$.

By taking the first derivative of AEP$[\pi(t_1,t_w)]$ with respect to t_1 and t_w, we have

$$\begin{aligned}
\frac{\partial}{\partial t_1}\{\mathrm{AEP}(t_1,t_w)\}=&\ \frac{(s-a-bt_w)}{T}\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)P+(1-\theta_2)P\lambda t_1\}\\
&+E[m_2+\delta(1-m_2)]\{\theta_1P+\theta_2P\lambda t_1\}\Big]-\frac{1}{T}\Big[(c_p+c_s)P+c_rE[\delta(1-m_2)]\{\theta_1P\\
&+\theta_2P\lambda t_1\}\Big]-\frac{h_c}{2T}\Big[\{(1-\theta_1)P+E[\delta]\theta_1P-(D_0+\rho t_w)e^{k\eta}\}(2t_1-3\lambda t_1^2+2\lambda^2t_1^3)\\
&+3\{(1-\theta_2)P+E[\delta]\theta_2P-(D_0+\rho t_w)e^{k\eta}\}\lambda t_1^2-2(D_0+\rho t_w)(T-t_1)\Big]\\
&-\frac{1}{T}(c_t+c_l)E[m_2]\{\theta_1P+\theta_2P\lambda t_1\}-\frac{c_r}{T}E[(1-\delta)m_1]\{(1-\theta_1)+(1-\theta_2)\lambda t_1\}P\\
&-\frac{B_0}{T}-\frac{B_1\lambda t_1^2}{2T}e^{k\frac{v_{\max}-v}{v-v_{\min}}}+\frac{K}{PTt_1^2}-\frac{(s-a-bt_w)}{T^2}\Bigg[\{1-E[m_1(1-\gamma)]\}\\
&\times\left\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\right\}+E[m_2+\delta(1-m_2)]\left\{\theta_1Pt_1+\theta_2P\frac{\lambda t_1^2}{2}\right\}\Bigg]\frac{\partial T}{\partial t_1}\\
&+\frac{1}{T^2}\Bigg[(c_p+c_s)Pt_1+c_rE[\delta(1-m_2)]\left\{\theta_1Pt_1+\theta_2P\frac{\lambda t_1^2}{2}\right\}\Bigg]\frac{\partial T}{\partial t_1}\\
&+\frac{h_c}{2T^2}\Big[\{(1-\theta_1)P+E[\delta]\theta_1P-(D_0+\rho t_w)e^{k\eta}\}\left(t_1^2-\lambda t_1^3+\frac{\lambda^2t_1^4}{2}\right)\\
&+\{(1-\theta_2)P+E[\delta]\theta_2P-(D_0+\rho t_w)e^{k\eta}\}\lambda t_1^3+(D_0+\rho t_w)(T-t_1)^2\Big]\frac{\partial T}{\partial t_1}\\
&+\frac{1}{T^2}(c_t+c_l)E[m_2]\left\{\theta_1Pt_1+\theta_2P\frac{\lambda t_1^2}{2}\right\}\frac{\partial T}{\partial t_1}+\frac{c_r}{T^2}E[(1-\delta)m_1]\Bigg\{(1-\theta_1)Pt_1\\
&+(1-\theta_2)P\frac{\lambda t_1^2}{2}\Bigg\}\frac{\partial T}{\partial t_1}+\frac{B_1\lambda t_1^3}{6T^2}\frac{\partial T}{\partial t_1}e^{k_1\frac{v_{\max}-v}{v-v_{\min}}}+\frac{1}{T^2}\left(A_0+B_0t_1+\frac{K}{Pt_1}\right)\frac{\partial T}{\partial t_1}\\
=&\ F(t_1,t_w),\text{say}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t_w}\{\mathrm{AEP}(t_1,t_w)\}=&\ -\frac{b}{T}\Bigg[\{1-E[m_1(1-\gamma)]\}\left\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\right\}+E[m_2+\delta(1-m_2)]\\
&\times\left\{\theta_1Pt_1+\theta_2P\frac{\lambda t_1^2}{2}\right\}\Bigg]+\frac{h_c}{2T}\left[\rho e^{k\eta}\left(t_1^2-\lambda t_1^3+\frac{\lambda^2t_1^4}{2}\right)+\rho e^{k\eta}\lambda t_1^3-\rho(T-t_1)^2\right]\\
&-\frac{(s-a-bt_w)}{T^2}\Bigg[\{1-E[m_1(1-\gamma)]\}\left\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\right\}+E[m_2
\end{aligned}$$

$$+ \delta(1-m_2)]\left\{\theta_1 P t_1 + \theta_2 P \frac{\lambda t_1^2}{2}\right\}\Bigg]\frac{\partial T}{\partial t_w} + \frac{1}{T^2}\Bigg[(c_p+c_s)Pt_1 + c_r E[\delta(1-m_2)]\Bigg\{\theta_1 P t_1$$

$$+ \theta_2 P\frac{\lambda t_1^2}{2}\Bigg\}\Bigg]\frac{\partial T}{\partial t_w} + \frac{h_c}{2T^2}\Bigg[\{(1-\theta_1)P + E[\delta]\theta_1 P - (D_0+\rho t_w)e^{k\eta}\}\left(t_1^2 - \lambda t_1^3 + \frac{\lambda^2 t_1^4}{2}\right)$$

$$+ \{(1-\theta_2)P + E[\delta]\theta_2 P - (D_0+\rho t_w)e^{k\eta}\}\lambda t_1^3 + (D_0+\rho t_w)(T-t_1)^2\Bigg]\frac{\partial T}{\partial t_w}$$

$$+ \frac{1}{T^2}(c_t+c_l)E[m_2]\left\{\theta_1 P t_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}\frac{\partial T}{\partial t_w} + \frac{c_r}{T^2}E[(1-\delta)m_1]\Bigg\{(1-\theta_1)Pt_1$$

$$+ (1-\theta_2)P\frac{\lambda t_1^2}{2}\Bigg\}\frac{\partial T}{\partial t_w} + \frac{B_1\lambda t_1^3}{6T^2}\frac{\partial T}{\partial t_w}e^{k_1\frac{v_{\max}-v}{v-v_{\min}}} + \frac{1}{T^2}\left(A_0 + B_0 t_1 + \frac{K}{Pt_1}\right)\frac{\partial T}{\partial t_w}$$

$$= G(t_1, t_w), \text{ say.}$$

Taking the second derivative of AEP$[\pi(t_1, t_w)]$ with respect to t_w and t_1, we have

$$\frac{\partial^2}{\partial t_w^2}\{\text{AEP}(t_1,t_w)\} = \frac{2b}{T^2}\Bigg[\{1-E[m_1(1-\gamma)]\}\left\{(1-\theta_1)Pt_1 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\right\} + E[m_2+\delta(1-m_2)]$$

$$\times\left\{\theta_1 P t_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}\Bigg]\frac{\partial T}{\partial t_w} + \frac{h_c}{T^2}\left[\rho e^{k\eta}\left(t_1^2 - \lambda t_1^3 + \frac{\lambda^2 t_1^4}{2}\right) + \rho e^{k\eta}\lambda t_1^3 - \rho(T-t_1)^2\right]$$

$$+ (s-a-bt_w)\Bigg[\{1-E[m_1(1-\gamma)]\}\left\{(1-\theta_1)Pt_1 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\right\} + E[m_2+\delta(1$$

$$- m_2)]\left\{\theta_1 P t_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}\Bigg]\left\{\frac{2}{T^3}\left(\frac{\partial T}{\partial t_1}\right)^2 - \frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2}\right\} + \Bigg[(c_p+c_s)Pt_1 + c_r E[\delta(1-m_2)]$$

$$\times\left\{\theta_1 P t_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}\Bigg]\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}\left(\frac{\partial T}{\partial t_1}\right)^2\right\} + \frac{h_c}{2}\Bigg[\{(1-\theta_1)P + E[\delta]\theta_1 P - (D_0$$

$$+ \rho t_w)e^{k\eta}\}\left(t_1^2 - \lambda t_1^3 + \frac{\lambda^2 t_1^4}{2}\right) + \{(1-\theta_2)P + E[\delta]\theta_2 P\}\lambda t_1^3 + (D_0+\rho t_w)\{(T-t_1)^2$$

$$- e^{k\eta}\lambda t_1^3\}\Bigg]\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}\left(\frac{\partial T}{\partial t_1}\right)^2\right\} + \Bigg[(c_t+c_l)E[m_2]\left\{\theta_1 P t_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}$$

$$+ \frac{c_r}{T^2}E[(1-\delta)m_1]\left\{(1-\theta_1)Pt_1 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\right\}\Bigg]\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}\left(\frac{\partial T}{\partial t_1}\right)^2\right\}$$

$$+ \frac{B_1\lambda t_1^3}{6T^2}\Big\{\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}(\frac{\partial T}{\partial t_1})^2\Big\}e^{k_1\frac{v_{\max}-v}{v-v_{\min}}} + \left(A_0 + B_0 t_1 + \frac{K}{Pt_1}\right)\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}\left(\frac{\partial T}{\partial t_1}\right)^2\right\}$$

$$\frac{\partial^2}{\partial t_1^2}\{\text{AEP}(t_1,t_w)\} = \frac{(s-a-bt_w)}{T}\Big[\{1-E[m_1(1-\gamma)]\}(1-\theta_2)P\lambda + E[m_2+\delta(1-m_2)]\theta_2 P\lambda\Big]$$

$$- \frac{c_r}{T}E[\delta(1-m_2)]\theta_2 P\lambda - \frac{h_c}{2T}\Big[\{(1-\theta_1)P + E[\delta]\theta_1 P - (D_0+\rho t_w)e^{k\eta}\}$$

$$\times (2 - 6\lambda t_1 + 6\lambda^2 t_1^2) + 6\{(1-\theta_2)P + E[\delta]\theta_2 P - (D_0+\rho t_w)e^{k\eta}\}\lambda t_1 + 2(D_0+\rho t_w)\Big]$$

$$- \frac{1}{T}(c_t+c_l)E[m_2]\theta_2 P\lambda - \frac{c_r}{T}E[(1-\delta)m_1](1-\theta_2)P\lambda + \frac{B_1\lambda t_1^2}{T}e^{k_1\frac{v_{\max}-v}{v-v_{\min}}} - \frac{2K}{PTt_1^3}$$

$$- \frac{2(s-a-bt_w)}{T^2}\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)P + (1-\theta_2)P\lambda t_1\} + E[m_2+\delta(1-m_2)]$$

$$
\begin{aligned}
&\times \{\theta_1 P + \theta_2 P\lambda t_1\}\Big] \frac{\partial T}{\partial t_1} + \frac{2}{T^2}\Big[(c_p + c_s)P + c_r E[\delta(1-m_2)]\{\theta_1 P + \theta_2 P\lambda t_1\}\Big] \frac{\partial T}{\partial t_1} \\
&+ \frac{h_c}{T^2}\Big[\{(1-\theta_1)P + E[\delta]\theta_1 P - (D_0 + \rho t_w)e^{k\eta}\}(2t_1 - 3\lambda t_1^2 + 2\lambda^2 t_1^3) \\
&+ 3\{(1-\theta_2)P + E[\delta]\theta_2 P - (D_0+\rho t_w)e^{k\eta}\}\lambda t_1^2 - 2(D_0 + \rho t_w)(T - t_1)\Big] \frac{\partial T}{\partial t_1} \\
&+ \frac{2}{T^2}(c_t + c_l)E[m_2]\{\theta_1 P + \theta_2 P\lambda t_1\}\frac{\partial T}{\partial t_1} + \frac{2c_r}{T^2}E[(1-\delta)m_1]\{(1-\theta_1)P \\
&+ (1-\theta_2)P\lambda t_1\}\frac{\partial T}{\partial t_1} + \frac{2B_0}{T^2}\frac{\partial T}{\partial t_1} + \frac{B_1\lambda t_1^2}{2T^2}\frac{\partial T}{\partial t_1}e^{k\frac{\upsilon_{\max}-\upsilon}{\upsilon-\upsilon_{\min}}} + \frac{2K}{PT^2t_1^2}\frac{\partial T}{\partial t_1} \\
&+ (s - a - bt_w)\left[\{1 - E[m_1(1-\gamma)]\}\left\{(1-\theta_1)Pt_1 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\right\}\right. \\
&\left.+ E[m_2 + \delta(1-m_2)]\left\{\theta_1 P t_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}\right]\left\{\frac{2}{T^3}\left(\frac{\partial T}{\partial t_1}\right)^2 - \frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2}\right\} \\
&+ \left[(c_p + c_s)Pt_1 + c_r E[\delta(1-m_2)]\left\{\theta_1 P t_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}\right]\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}\frac{\partial T}{\partial t_1}\right\} \\
&+ \frac{h_c}{2}\left[\left\{(1-\theta_1)P + E[\delta]\theta_1 P - (D_0+\rho t_w)e^{k\eta}\right\}\left(t_1^2 - \lambda t_1^3 + \frac{\lambda^2 t_1^4}{2}\right)\right. \\
&\left.+ \{(1-\theta_2)P + E[\delta]\theta_2 P - (D_0 + \rho t_w)e^{k\eta}\}\lambda t_1^3 + (D_0+\rho t_w)(T-t_1)^2\right] \\
&\times \left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}\frac{\partial T}{\partial t_1}\right\} + (c_t + c_l)E[m_2]\left\{\theta_1 P t_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}\frac{\partial T}{\partial t_1}\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}\frac{\partial T}{\partial t_1}\right\} \\
&+ c_r E[(1-\delta)m_1]\left\{(1-\theta_1)Pt_1 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\right\}\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}\frac{\partial T}{\partial t_1}\right\} \\
&+ \frac{B_1\lambda t_1^3}{6}\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}\frac{\partial T}{\partial t_1}\right\}e^{k_1\frac{\upsilon_{\max}-\upsilon}{\upsilon-\upsilon_{\min}}} + \left(A_0 + B_0 t_1 + \frac{K}{Pt_1}\right)\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}\frac{\partial T}{\partial t_1}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial t_1 \partial t_w}\{\text{AEP}(t_1, t_w)\} = &-\frac{b}{T}\Big[\{1 - E[m_1(1-\gamma)]\}\{(1-\theta_1)P + (1-\theta_2)P\lambda t_1\} + E[m_2 + \delta(1-m_2)] \\
&\times \{\theta_1 P + \theta_2 P\lambda t_1\}\Big] + \frac{h_c}{2T}\Big[\rho e^{k\eta}(2t_1 + 2\lambda^2 t_1^3) + 2\rho(T - t_1)\Big] \\
&+ \frac{b}{T^2}\left[\{1 - E[m_1(1-\gamma)]\}\left\{(1-\theta_1)Pt_1 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\right\} + E[m_2 + \delta(1-m_2)]\right. \\
&\left.\times \left\{\theta_1 P t_1 + \theta_2 P\frac{\lambda t_1^2}{2}\right\}\right]\frac{\partial T}{\partial t_1} - \frac{h_c}{2T^2}\left[\rho e^{k\eta}\left(t_1^2 + \frac{\lambda^2 t_1^4}{2}\right) - \rho(T-t_1)^2\right]\frac{\partial T}{\partial t_1} - \frac{(s - a - bt_w)}{T^2} \\
&\times \Big[\{1 - E[m_1(1-\gamma)]\}\{(1-\theta_1)P + (1-\theta_2)P\lambda t_1\} + E[m_2 + \delta(1-m_2)] \\
&\times \{\theta_1 P + \theta_2 P\lambda t_1\}\Big]\frac{\partial T}{\partial t_w} + \frac{1}{T^2}\Big[(c_p + c_s)P + c_r E[\delta(1-m_2)]\{\theta_1 P + \theta_2 P\lambda t_1\}\Big]\frac{\partial T}{\partial t_w} \\
&+ \frac{h_c}{2T^2}\Big[\{(1-\theta_1)P + E[\delta]\theta_1 P - (D_0+\rho t_w)e^{k\eta}\}(2t_1 - 3\lambda t_1 + 2\lambda^2 t_1^3) \\
&+ \{(1-\theta_2)P + E[\delta]\theta_2 P - (D_0+\rho t_w)e^{k\eta}\}3\lambda t_1^2 - 2(D_0+\rho t_w)(T-t_1)\Big]\frac{\partial T}{\partial t_w} \\
&+ \frac{1}{T^2}(c_t + c_l)E[m_2]\{\theta_1 P + \theta_2 P\lambda t_1\}\frac{\partial T}{\partial t_w} + \frac{c_r}{T^2}E[(1-\delta)m_1]\{(1-\theta_1)P \\
&+ (1-\theta_2)P\lambda t_1\}\frac{\partial T}{\partial t_w} + \frac{B_1\lambda t_1^2}{2T^2}\frac{\partial T}{\partial t_w}e^{k_1\frac{\upsilon_{\max}-\upsilon}{\upsilon-\upsilon_{\min}}} + \frac{1}{T^2}\left(B_0 - \frac{K}{Pt_1^2}\right)\frac{\partial T}{\partial t_w}
\end{aligned}
$$

$$
\begin{aligned}
&+(s-a-bt_w)\Bigg[\{1-E[m_1(1-\gamma)]\}\left\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\right\}\\
&+E[m_2+\delta(1-m_2)]\left\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\right\}\Bigg]\left\{\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}-\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}\right\}\\
&+\left[(c_p+c_s)Pt_1+c_rE[\delta(1-m_2)]\left\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\right\}\right]\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\right\}\\
&+\frac{h_c}{2}\Bigg[\left\{(1-\theta_1)P+E[\delta]\theta_1 P-(D_0+\rho t_w)e^{k\eta}\right\}\left(t_1^2-\lambda t_1^3+\frac{\lambda^2 t_1^4}{2}\right)\\
&+\{(1-\theta_2)P+E[\delta]\theta_2 P-(D_0+\rho t_w)e^{k\eta}\}\lambda t_1^3+(D_0+\rho t_w)(T-t_1)^2\Bigg]\\
&\times\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\right\}+(c_t+c_l)E[m_2]\left\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\right\}\\
&\times\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\right\}+\frac{B_1\lambda t_1^3}{6}\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\right\}e^{k\frac{\upsilon_{\max}-\upsilon}{\upsilon-\upsilon_{\min}}}\\
&+c_rE[(1-\delta)m_1]\left\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\right\}\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\right\}\\
&+\left(A_0+B_0t_1+\frac{K}{Pt_1}\right)\left\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\right\}.
\end{aligned}
$$

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