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OPTIMAL PRODUCTION LOT SIZE UNDER BACKUP AGREEMENT ALLOWING SUBSTITUTE PRODUCTS

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Abstract. In the fashion industry, a flexible backup agreement contract allows the retailer to order a partial amount from the backup quantity to allay the risk of uncertain market demand. However, under such a contract, the manufacturer faces the risk of bearing a huge leftover if the quantity realized by the retailer in backup is small. Accordingly, the present study considers a modified backup agreement in which the manufacturer is permitted to urgently purchase substitute products to satisfy the backup order from a third-party supplier, but at a unit purchase cost greater than the original unit manufacturing cost. The corresponding expected total profit function for the manufacturer is established and shown to be concave. The profit function is used to explore various useful properties for determining the optimal production lot size. In addition, an illustrated numerical example is provided to analyze the impact of the backup contract terms on the optimal production lot size and manufacturer's profit.

Mathematics Subject Classification. 90B05.

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1. Introduction

Fashion products such as clothing, shoes, egg-shaped virtual pet games [12], video game software [7], and some mail-ordered fashion merchandise have, by definition, a fairly short life cycle, and hence pose a challenging order decision-making problem to the retailer [2]. In fashion buying, the decision maker (*i.e.*, the retailer) must decide on the order quantity required to satisfy the (uncertain) demand for the coming sales season. Once the sales season is over, any products left over are simply discarded or sold at their salvage values. Since the fashion industry is characterized by both short product life cycles and erratic consumer demand, retailers generally prefer more flexible procurement agreements with the manufacturer in order to guard against the risk of dull sales. These agreements may take various forms, as described in the following.

When sales are dull, suppliers can allow buyers to sell the unsold items back to them at the end of the selling season under a buy-back contract [1, 14]. Alternatively, the supplier may offer a contract that combines order quantity flexibility (QB) with price discount schemes [9] and [18]. For example, when buyers purchase above the forecast quantity, the supplier may offer a discount for the units sold in excess of the QB-contracted quantity in order to induce the buyers to hold a higher inventory in exchange for a lower unit purchase cost. Choi *et al.* [6]

Keywords. Fashion supply chain, backup contract, economic production quantity.

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reported that a quick response of the supplier to the retailer's demands brings significant benefit to the retailer in the fashion supply chain. Thus, Donohue [10] suggested the addition of a quick response option to a traditional flexible-price contract, wherein the buyer is offered two prices, namely a lower price for a standard turnaround time and a higher price for a quicker turnaround time in response to emergency demands. However, Yuan *et al.* [21] cautioned the need for care in establishing the emergency purchase price since an overly low price may damage the supplier's profit.

A backup agreement contract, in which the retailer is permitted to order up to a certain pre-agreed backup quantity from the manufacturer at some point after sales begin, provides an attractive arrangement for both the retailer and the manufacturer in mitigating the effects of dull sales. For example, Eppen and Iyer [11] considered a two-stage order decision-making problem, referred to as an EI problem (EIP), in which the retailer places an order on two separate occasions, namely at the beginning of the sales season (with the time being set to zero) and at a certain point during the season, designated as time t. The second-order decision uses the known demand in the first period $(0, t)$ to forecast the unknown demand in the second period (t, T), where T $(t < T)$ represents the end of the sales season. Thus, the EIP consists of forecasting the demand in period 2 based on the first-period demand for the same product and determining, at the beginning of the second period, the quantity of products to be fulfilled from the backup quantity. However, in the EIP model, the demand forecasts for the second sales period preclude the following: price changes in one or both periods (*e.g.*, [8,15–17]); the appearance of other competitive products in the market (*e.g.*, [20]); and exogenous information [3,19], such as information collected from the distribution of sales vouchers or quotations [4].

Although a backup contract helps to stimulate the buyer's order quantity, the supplier bears a high risk of dull sales; particularly when the backup ratio is large. Accordingly, the present study establishes a production model, where if the pre-determined production lot size cannot satisfy the backup order, the manufacturer is permitted to urgently purchase substitute products from other suppliers to make up the shortfall in the backup commitment. In particular, the study considers the problem of maximizing the manufacturer's expected profit under such a production model subject to the constraint that the unit purchase cost of the substitute products is greater than the original unit manufacturing cost. It is also noted that the cost incurred by the retailer in processing each returned unit (*e.g.*, the mailing cost, repackaging cost, and so on) is ignored in Eppen and Iyer's [11] model, but is incorporated within the present model in the form of a return cost parameter, s_r.

The remainder of this paper is organized as follows. Section 2 extends the EIP model to include the cost incurred by the retailer in processing customer returns during or after each sales period. Based on the optimal retailer order policy obtained from the EIP model, Section 3 establishes a profit function for the manufacturer from which to derive the optimal production lot size. A numerical example is provided to illustrate the benefits of the proposed production strategy. Finally, Section 4 summarizes the major findings and contributions of the study and indicates the intended direction of future research.

2. Problem description and optimal buyer order policy

In developing the proposed manufacturer's production model, the following notations (mainly adopted from Eppen and Iyer [11]) are used:

y: retailer's commitment quantity for total sales season (a decision variable).
y_2: pre-determined inventory level at beginning of period 2 for retailer (a decision variable).
c: retailer's purchase cost per unit (\$/item).
r: retailer's sales price per unit (\$/item), where $0 < c < r$.
π: retailer's stockout cost (\$/item).
$h_1(h_2)$: retailer's holding cost per unit for each unit on hand at end of first (second) period (\$/item).
s_2: salvage value per unit at end of period 2 (\$/item), where $s_2 < c$.

ρ:	flexible backup ratio, *i.e.*, percentage of y units for which manufacturer allows retailer to determine order quantity at beginning of period 2.
b:	penalty cost per unit payable to manufacturer for each unit not taken from backup (\$/item).
$X_1(X_2)$:	amount of sales in period 1 (period 2) for retailer.
$\Phi_i(\xi_i) = \Pr(X_i \le \xi_i)$:	distribution function of X_i, where ξ_i represents the random demand in period i, $i = 1, 2$.
$\phi_i(\xi_i)$:	probability density function of X_i, $i = 1, 2$.
$\mathrm{BU}(\xi_1)$:	number of units taken from backup, ρy^*, based on optimal order decision (y^* and $y_2^*(\xi_1)$), at beginning of period 2 given known demand in period 1 ξ_1, where $0 \le \mathrm{BU}(\xi_1) \le \rho y^*$.
s_r:	processing cost to retailer for each returned unit, including repacking and other costs incurred for resale (\$/item).

Eppen and Iyer [11] stated that fashion products are characterized by a highly uncertain demand. Thus, in the EIP, the retailer agrees to spend a cost of c per unit (\$/item) to order from the manufacturer, while the manufacturer agrees to hold a certain percentage (say, ρ) of the commitment y in reserve as a "backup quantity" (equal to ρy) for the retailer. The retailer then decides how many units to take from this backup based on the observed demand in period 1 (typically around two weeks after the start of sales) to satisfy the forecast market demand in the second period. For each unit not taken from the backup, the retailer agrees to pay the manufacturer a penalty cost of b (\$/item).

The EIP model takes explicit account of the customer return rate on the retailer's order decision-making process. In particular, the model assumes that a constant percentage, $v\%$, of the sales in period 1 are returned. Furthermore, it is assumed that of these returned units, a constant percentage, $u\%$, of those arriving at the beginning of period 2 are repacked by the retailer for resale to new customers. The remaining percentage $(1-u\%)$ are assumed to arrive at the end of the second period and are offered for resale at a certain salvage value. Similarly, $v\%$ of the units sold in period 2 are also returned, where $u\%$ of these items arrive in time for resale during the same period, while the remaining fraction $(1-u\%)$ arrive at the end of the second period and are offered for resale at their salvage value.

In the EIP model, $v\% \approx 30\%$ and $u\% \approx 36\%$. However, the cost incurred by the retailer in processing the returns is not considered. Thus, to obtain a more realistic retailer ordering model, the present study extends the original EIP model proposed by Eppen and Iyer [11] to include a return cost parameter, denoted as s_r (\$/item).

Consider that the retailer commits y units for the season and the manufacturer holds back a certain percentage ρ of this commitment in reserve. The retailer's inventory at the beginning of the first period is thus equal to $y(1-\rho)$ units. Assuming that the demand in period 1 is equal to ξ_1, and y_2 items are to be held in inventory at the beginning of period 2, the expected total profit over the entire sales season (denoted as $f_2(y, \xi_1)$) can be obtained as described in the following. (Note that as in Eppen and Iyer [11], $f_2(y, \xi_1)$ is derived subject to the demand ranges $\xi_1 < y(1-\rho)$ and $\xi_1 \ge y(1-\rho)$.)

The sales revenue in period 1 is equal to $r\xi_1(1-v)$ given ξ_1 since $\xi_1(1-v)$ units are not returned. Of those units which are returned in the first period, $\xi_1 vu$ units are available in time for resale in period 2; albeit with an associated processing cost of $s_r vu\xi_1$ and holding cost of $h_1 vu\xi_1$. A further $\xi_1 v(1-u)$ units are returned at the end of period 2. The resulting holding cost and salvage value at the end of period 2 are thus given by $(h_1+h_2)\xi_1 v(1-u)$ and $s_2 v(1-u)\xi_1$, respectively. Assume that the retailer has $y(1-\rho) - \xi_1 + \xi_1 uv$ units on hand at the beginning of period 2; incurring a holding cost of $h_1[y(1-\rho) - \xi_1 + \xi_1 uv)]$, where $y(1-\rho) - \xi_1$ represents the number of unsold units at the end of period 1 and $\xi_1 uv$ represents the number of returned units received at the end of period 1, respectively. (Note that both quantities are available for resale during period 2.) Consequently, the order quantity from the backup at the beginning of period 2 is equal to $y_2 - [y(1-\rho) - \xi_1 + \xi_1 uv]$ with a purchase cost of $c\{y_2(\xi_1) - [y(1-\rho) - \xi_1 + \xi_1 uv]\}$, while the quantity not taken from backup is equal to $\rho y - \{y_2(\xi_1) - [y(1-\rho) - \xi_1 + \xi_1 uv]\}$ with a unit penalty cost b. Assuming a demand in period 2 of ξ_2, then $\xi_2 vu$ units are returned prior to the end of period 2 and are available for resale in period 2 after repacking with a total processing cost of $s_r vu\xi_2$. Thus, when $y(1-\rho)-\xi_1+\xi_1 uv \le y_2(\xi_1) \le y-\xi_1+\xi_1 uv$, $f_2(y,\xi_1)$ can be computed by the following equation:

$$\begin{aligned} f_2(y,\xi_1) = \max_{y(1-\rho)-\xi_1+\xi_1 uv \le y_2(\xi_1) \le y-\xi_1+\xi_1 uv} &\{[r(1-v)-s_r vu]\,\xi_1 - c\{y_2 \\ &- [y(1-\rho)-\xi_1+\xi_1 uv]\} \\ &- h_1[y(1-\rho)-\xi_1+\xi_1 uv)] - (h_1+h_2-s_2)\,\xi_1 v(1-u) \\ &- b(\rho y - \{y_2 - [y(1-\rho)-\xi_1+\xi_1 uv]\}) + G_2(y_2,\xi_1)\}, \end{aligned} \tag{2.1}$$

where $G_2(y_2,\xi_1)$ represents the expected profit of period 2 assuming that the inventory level at the beginning of period 2 is set equal to y_2 and the demand in period 1 is ξ_1. As described in Eppen and Iyer [11], $G_2(y_2,\xi_1)$ can be formulated as

$$\begin{aligned} G_2(y_2,\xi_1) = &\int_0^{y_2/(1-uv)} r(1-v)\,\xi_2\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2 \\ &- \int_0^{y_2/(1-uv)} s_r vu\xi_2\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2 \\ &- (h_2-s_2)\int_0^{y_2/(1-uv)} [y_2-(1-v)\,\xi_2]\,\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2 \\ &+ \int_{y_2/(1-uv)}^{\infty} r(1-v)\,\frac{y_2}{1-uv}\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2 \\ &- \int_{y_2/(1-uv)}^{\infty} s_r vu\frac{y_2}{1-uv}\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2 \\ &- (h_2-s_2)\int_{y_2/(1-uv)}^{\infty} \frac{y_2 v(1-u)}{1-uv}\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2 \\ &- \pi\int_{y_2/(1-uv)}^{\infty} \left(\xi_2 - \frac{y_2}{1-uv}\right)\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2. \end{aligned} \tag{2.2}$$

Note that in deriving equation (2.2), it is assumed that the returned items can be recycled as many times as necessary without limit. Thus, $y_2/(1-uv)$ units are available for sale in period 2 (see [11]).

In this case, the stockout in period 1 is given by $\xi_1 - y(1-\rho)$ and results in a stockout cost of $\pi[\xi_1 - y(1-\rho)]$. (Note that the other cost and profit terms are as described above for Case 1). Therefore,

$$\begin{aligned} f_2(y,\xi_1) = \max_{y(1-\rho)uv \le y_2(\xi_1) \le y(1-\rho)uv+\rho y} &\{[r(1-v)-s_r vu]\,y(1-\rho) - c\{y_2 - y(1-\rho)\,uv\} \\ &- \pi[\xi_1 - y(1-\rho)] - h_1 y(1-\rho)\,uv) - (h_1+h_2-s_2)\,y(1-\rho)\,v(1-u) \\ &- b(\rho y - \{y_2 - y(1-\rho)\,uv\}) + G_2(y_2,\xi_1)\}. \end{aligned} \tag{2.3}$$

In summary, if y items are committed, the buyer's profit is

$$G_1(y) = -cy(1-\rho) + \int_0^{\infty} f_2(y,\xi_1)\,\phi_1(\xi_1)\,\mathrm{d}\xi_1.$$

Obviously, the guaranteed optimal order quantity $y^* = \underset{y>0}{\operatorname{argmax}} \{G_1(y)\}$. As in Eppen and Iyer [11], it can be shown that (see Appendix A for the proof)

$$\Phi_2\left(\left.\frac{y_2^*(\xi_1)}{1-uv}\right| X_1=\xi_1\right)=A, \tag{2.4}$$

where $A = \frac{r(1-v)-s_r vu+\pi-(h_2-s_2)v(1-u)+(b-c)(1-uv)}{r(1-v)-s_r vu+(h_2-s_2)(1-v)+\pi}$. Note that the results provided by Eppen and Iyer [11] can be obtained by setting $s_r = 0$ in equation (2.4).

Lemma 2.1. *In equation (2.4), assume that* $r(1-v) - s_r vu + \pi - (h_2 - s_2)v(1-u) + (b-c)(1-uv) > 0$ *and* $b < c + h_2 - s_2$. *One then has* $0 < A < 1$, *and the value of* A *decreases as* s_r *increases.*

Note that the optimal policy for the case of $b \ge c + h_2 - s_2$ *is to fulfill all of the backup amount for the retailer, which has been shown by Eppen and Iyer [11]. In this case, the manufacturer's optimal production lot size is equal to the retailer's commitment quantity.*

The solution procedure to obtain y^* *is briefly stated as follows: (1) substitute* $y_2^*(\xi_1)$, *which can be obtained from equation (2.4), into* $f_2(y,\xi_1)$ *as given in equation (2.1) to equation (2.3) to obtain* $G_1(y)$ *for a given* $y > 0$*; and (2) vary* y *to search for the value of* y^* *that maximizes* $G_1(y)$.

Example 2.2. As described in Eppen and Iyer [11], let $u = 0.36$, $v = 0.3$, $r = 2.25$ (\$/item), $c = 1$ (\$/item), $h_1 = 0.004$, $h_2 = 0.036$, $\pi = 0.25$ (\$/item) and $s_2 = 0.1$ (\$/item). Let the following parameter values be additionally considered: $s_r = 0.1 \cdot r$, $b = 0.2$, and $\rho = 0.3$. Assume that the demands in period 1 and period 2 follow a normal distribution (refer to [5], more precisely, $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$. Furthermore, assume that the two demands are related with a correlation coefficient of γ. Let the associated parameter values be set as $\mu_1 = 55$, $\mu_2 = 300$, $\sigma_1 = 12$, $\sigma_2 = 55$ and $\gamma = 0.65$. Since $X_2 | X_1 = \xi_1 \sim N(\mu_2 + \gamma(\sigma_2/\sigma_1)(\xi_1 - \mu_1), \sigma_2^2(1-\gamma^2))$, by equation (2.4), it follows that

$$\Phi_{\text{std}}\left(\left.\frac{\left\{\frac{y_2^*(\xi_1)}{1-uv} - [\mu_2 + \gamma(\sigma_2/\sigma_1)(\xi_1-\mu_1)]\right\}}{\sigma_2\sqrt{1-\gamma^2}}\right| X_1 = \xi_1\right) = A, \tag{2.5}$$

where $\Phi_{\text{std}}(\cdot)$ represents the standardized normal distribution function. From equation (2.5) one then obtains

$$y_2^*(\xi_1) = (1-uv)\left\{\mu_2 + \gamma(\sigma_2/\sigma_1)(\xi_1-\mu_1) + \Phi_{\text{std}}^{-1}(A)\,\sigma_2\sqrt{1-\gamma^2}\right\}. \tag{2.6}$$

Under the above parameter values, when $y = 342$, one has $y_2(\xi_1) < y(1-\rho) - \xi_1 + \xi_1 uv$ for $\xi_1 < 21.87$ and $y_2(\xi_1) > y - \xi_1 + \xi_1 uv$ for $\xi_1 > 24.77$ (see Fig. 1). Since their corresponding profit functions are not addressed in the Eppen and Iyer [11] model, they are provided here as follows.

The four cases $y_2(\xi_1) < y(1-\rho) - \xi_1 + \xi_1 uv$, $y_2(\xi_1) > y - \xi_1 + \xi_1 uv$, $y(1-\rho)uv > y_2(\xi_1)$, and $y_2(\xi_1) > y(1-\rho)uv + \rho y$, ignored by Eppen and Iyer [11], are described as follows:

Case 1: $\xi_1 < y(1-\rho)$

$$\begin{aligned} f_2(y,\xi_1) = \max_{y_2(\xi_1)<y(1-\rho)-\xi_1+\xi_1 uv} &\{[r(1-v) - s_r vu]\,\xi_1 - h_1[y(1-\rho) - \xi_1 + \xi_1 uv] \\ &- (h_1 + h_2 - s_2)\,\xi_1 v(1-u) - b\rho y + G_2(y_2 = y(1-\rho) - \xi_1 + \xi_1 uv, \xi_1)\}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} f_2(y,\xi_1) = \max_{y_2(\xi_1)>y-\xi_1+\xi_1 uv} &\{[r(1-v) - s_r vu]\,\xi_1 - c\rho y - h_1[y(1-\rho) - \xi_1 + \xi_1 uv] \\ &- (h_1 + h_2 - s_2)\,\xi_1 v(1-u) + G_2(y_2 = y - \xi_1 + \xi_1 uv, \xi_1)\}, \end{aligned} \tag{2.8}$$

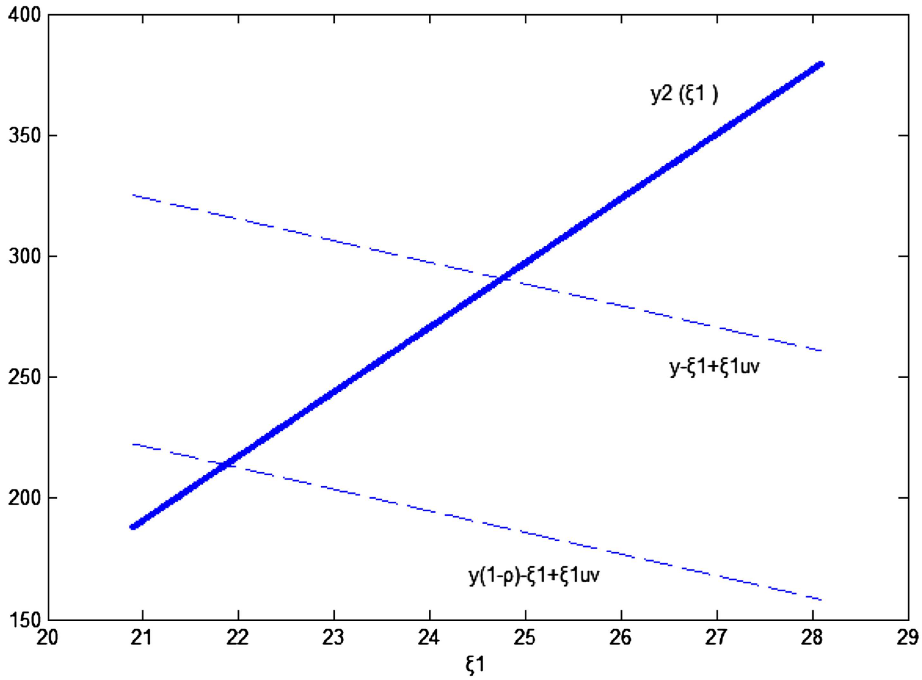

FIGURE 1. Selected y_2 values with different values of ξ_1.

where $G_2(y_2, \xi_1)$ is as shown in equation (2.2).

In equation (2.7), when $y_2(\xi_1) < y(1-\rho) - \xi_1 + \xi_1 uv$, which implies that the order quantity for the second period is zero. Thus, one sets $y_2 = y(1-\rho) - \xi_1 + \xi_1 uv$ to compute $G_2(y_2, \xi_1)$. However, $y_2(\xi_1) > y - \xi_1 + \xi_1 uv$ in equation (2.8), which implies that the order quantity in the second period is ρy and there is no backup left; thus, one sets $y_2 = y - \xi_1 + \xi_1 uv$ to calculate $G_2(y_2, \xi_1)$.

Case 2: $\xi_1 \geq y(1-\rho)$

$$f_2(y, \xi_1) = \max_{y_2(\xi_1) < y(1-\rho)uv} \{[r(1-v) - s_r vu]\, y(1-\rho) - \pi[\xi_1 - y(1-\rho)] - h_1 y(1-\rho)uv) - (h_1 + h_2 - s_2)y(1-\rho)v(1-u) - b\rho y + G_2(y_2 = y(1-\rho)uv, \xi_1)\}, \tag{2.9}$$

and

$$f_2(y, \xi_1) = \max_{y_2(\xi_1) > y(1-\rho)uv + \rho y} \{[r(1-v) - s_r vu]\, y(1-\rho) - c\rho y - \pi[\xi_1 - y(1-\rho)] - h_1 y(1-\rho)uv) - (h_1 + h_2 - s_2)\, y(1-\rho)\, v(1-u) + G_2(y_2 = y(1-\rho)uv + \rho y, \xi_1)\}, \tag{2.10}$$

where $G_2(y_2, \xi_1)$ is as shown in equation (2.2).

In equation (2.9), since $y_2(\xi_1) < y(1-\rho)uv$, the order quantity for the second period is zero, and the backup quantity that will not be fulfilled is ρy; thus, one sets $y_2 = y(1-\rho)uv$ to compute $G_2(y_2, \xi_1)$. In equation (2.10), since $y_2(\xi_1) > y(1-\rho)uv + \rho y$, the order quantity in the second period is ρy; thus, one sets $y_2 = y(1-\rho)uv + \rho y$ to compute $G_2(y_2, \xi_1)$.

Adding the profit functions stated in (2.7)–(2.10) to the model, we use the previous parameter values to analyze the optimal order quantity and profit. The results are summarized in Table 1, where $\Delta_R\% = \frac{G_1(y^*;\rho>0) - G_1(y^*;\rho=0)}{G_1(y^*;\rho=0)} \times 100\%$ represents the improvement percentage in the buyer's expected profit with the

TABLE 1. Order decisions for retailer.

		$s_r = 0$			$s_r = 0.1 \cdot r$		
ρ	b	y^*	$G_1(y^*)$	$\Delta_R\%$	y^*	$G_1(y^*)$	$\Delta_R\%$
0.5	0	396	212.46	8.74%	395	204.17	9.03%
0.5	0.1	363	207.06	5.97%	363	198.79	6.16%
0.5	0.2	350	203.80	4.30%	349	195.57	4.43%
0.5	0.3	343	201.52	3.14%	342	193.31	3.23%
0.5	0.4	338	199.85	2.28%	337	191.66	2.34%
0.5	0.5	334	198.61	1.65%	333	190.44	1.70%
0.3	0	350	205.96	5.41%	349	197.73	5.59%
0.3	0.1	346	203.88	4.34%	346	195.65	4.48%
0.3	0.2	343	202.03	3.40%	342	193.82	3.50%
0.3	0.3	339	200.43	2.58%	338	192.23	2.65%
0.3	0.4	336	199.12	1.91%	335	190.94	1.96%
0.3	0.5	333	198.06	1.37%	332	189.9	1.40%
0.1	0	324	193.91	−0.76%	322	185.78	−0.79%
0.1	0.1	325	194.43	−0.49%	324	186.28	−0.53%
0.1	0.2	326	194.80	−0.30%	326	186.67	−0.32%
0.1	0.3	327	195.09	−0.15%	325	186.97	−0.16%
0.1	0.4	326	195.30	−0.04%	326	187.18	−0.05%
0.1	0.5	326	195.44	0.03%	326	187.32	0.03%
0		327	195.39		327	187.27	

flexible ordering parameter ρ as the other parameter values are the same as before. For example, in Table 1, if $s_r = 0.1 \cdot r$, $\rho = 0.3$, and $b = 0.2$, then $y^* = 342$, the profit is 193.82 (also see Fig. 2) and $\Delta_R\% = 3.5\%$. Figure 2 shows that negative values occur in the retailer's profit due to improper order quantities. That is, $y^* \leq 127$ (with a higher stockout cost) or $y^* \geq 563$ (with a higher unfulfilled-backup penalty cost, returned item processing cost, and holding cost; and a lower salvage value).

An inspection of Table 1 reveals the following:

– When the return cost is ignored (*i.e.*, $s_r = 0$), the buyer overestimates the order quantity and profit (*i.e.*, y^* and $G_1(y^*)$ are overestimated by around $0 \sim 0.6\%$ and $3.9\% \sim 4.2\%$, respectively).
– For a given value of b, $\Delta_R\%$ increases as ρ increases.
– For $\rho \geq 0.3$ (*i.e.*, $\rho = 0.3$ or $\rho = 0.5$), $\Delta_R\%$ decreases as b increases. For $\rho < 0.3$ (say, $\rho = 0.1$), however, $\Delta_R\%$ increases with increasing b, where $\Delta_R\%$ is either a small negative value or a small positive value (close to zero in both cases). In other words, when ρ is small (*i.e.*, $\rho = 0.1$), a backup agreement contract is not attractive to the buyer.

The following section explores the problem of determining the optimal manufacturer's production lot size based on the buyer's optimal order policy (y^* and y_2^*) under a backup agreement, where the manufacturer is permitted to urgently purchase substitute products from a third-party supplier when the available production quantity is insufficient to fulfill the buyer's backup order.

3. Manufacturer's production decision

The following additional notations and assumptions are used to formulate the manufacturer's production lot size decision under the buyer's optimal order policy (y^* and y_2^*).

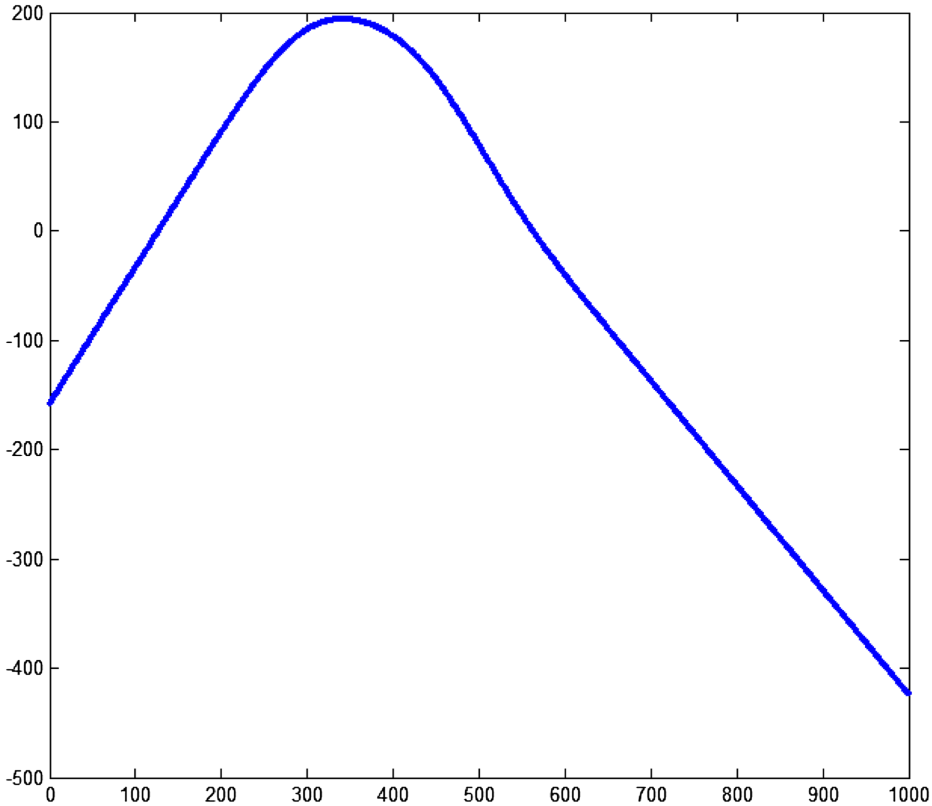

FIGURE 2. Retailers' profit for various values of y (with $s_r = 0.1 \cdot r$, $\rho = 0.3$, and $b = 0.2$).

N: manufacturer's production lot size (a decision variable), where $(1-\rho)\, y^* \leq N \leq y^*$.
N^*: manufacturer's optimal production lot size.
K: manufacturer's setup cost for production ($/operation).
w: manufacturing cost per unit ($/item).
s_m: manufacturer's revenue per unit sold for each unit left over in backup ($/item).

Assumptions:

- When the manufacturer's production lot size cannot satisfy the retailer's backup order, the manufacturer is permitted to supply the retailer with substitute products, where these products can be accepted by the customer [13], *e.g.*, they have a similar style and color, and do not affect the market demand.
- The substitute products can be urgently purchased from others suppliers. However, the unit purchased cost, denoted by c_r ($/item), is larger than the unit manufacture cost, w. In addition, $c_r > c - b$ is assumed; hence, the retailer's optimal order policy ($y^*$and y_2^*) is unchanged.
- Quantity discounts for the emergency purchase are not available.

Given a commitment quantity $y^* > 0$, the manufacturer is assumed to produce a single lot of size N to fulfill the order. At the beginning of the first period, $(1-\rho)\, y^*$ units are delivered to the retailer, and the remaining $N - (1-\rho)\, y^*$ units are held in reserve to satisfy the backup order quantity $\mathrm{BU}\,(\xi_1)$ placed by the retailer at the beginning of period 2 given a demand of ξ_1 in period 1. It is assumed that if $N - (1-\rho)\, y^*$ is greater than $\mathrm{BU}\,(\xi_1)$, the products left over at the end of period 2 will be sold, for example, at the manufacturer's outlet stores, at a salvage value of s_m ($/item). On the other hand, if $N - (1-\rho)\, y^*$ is less than $\mathrm{BU}\,(\xi_1)$, the manufacturer spends a unit cost of c_r (\$/item) to purchase substitute products for each unsatisfied unit of the retailer's backup order.

Based on the optimal order policy ($y^*$and y_2^*) of the retailer under the backup agreement, the manufacturer's expected total profit $\mathrm{TR}(N)$ for a production lot size N can be obtained as follows (see also Eppen and Iyer [11]):

$$\begin{aligned}\mathrm{TR}(N) = {} & cy^*(1-\rho) - K - wN + c\int_0^\infty \mathrm{BU}(\xi_1)\,\phi_1(\xi_1)\,\mathrm{d}\xi_1 \\ & - c_r\int_0^\infty \max\{\mathrm{BU}(\xi_1) - [N - y^*(1-\rho)], 0\}\,\phi_1(\xi_1)\,\mathrm{d}\xi_1 \\ & + b\int_0^\infty [\rho y^* - \mathrm{BU}(\xi_1)]\,\phi_1(\xi_1)\,\mathrm{d}\xi_1 \\ & + s_m\int_0^\infty \max\{N - y^*(1-\rho) - \mathrm{BU}(\xi_1), 0\}\,\phi_1(\xi_1)\,\mathrm{d}\xi_1, \end{aligned} \tag{3.1}$$

where $c_r > \max\{w, c-b\}$, $0 \le \mathrm{BU}(\xi_1) \le \rho y^*$ and $y^*(1-\rho) \le N \le y^*$.

When the demand in the first period, ξ_1, is high, the second-period demand is also likely to be high. To satisfy the probable high demand in period 2, the inventory on hand $y_2^*(\xi_1)$ at the retailer at the beginning of the second period needs to be high. The optimal production lot size is thus determined by the assumption that $y_2^*(\xi_1)$ is a continuous function increasing with ξ_1.

The relationship between the demand in the first period (ξ_1), the associated backup order quantity ($\mathrm{BU}(\xi_1)$) and the manufacturer's production lot size (N) is analyzed as follows:

Case 1: $0 \le \xi_1 \le y^*(1-\rho)$

Let

$$\xi_{L_1} = \min\{\max\{\sup\{\xi_1 \in \mathbf{R} | \mathrm{BU}(\xi_1) = 0 \Leftrightarrow y_2^*(\xi_1) \le y^*(1-\rho) - \xi_1 + \xi_1 uv\}, 0\}, y^*(1-\rho)\}, \tag{3.2}$$

and

$$\xi_{U_1} = \min\{\max\{\inf\{\xi_1 \in \mathbf{R} | \mathrm{BU}(\xi_1) = \rho y^* \Leftrightarrow y_2^*(\xi_1) \ge y^* - \xi_1 + \xi_1 uv\}, \xi_{L_1}\}, y^*(1-\rho)\}. \tag{3.3}$$

Expression (3.2) represents the buyer's backup order quantity at the beginning of period 2 as zero, and expression (3.3) as ρy^*, when the first-period demand ξ_1 is less than ξ_{L_1} and greater than ξ_{U_1}, respectively. Note that $\mathrm{BU}(\xi_1) = y_2^*(\xi_1) - [y^*(1-\rho) - \xi_1 + \xi_1 uv]$ for $\xi_{L_1} \le \xi_1 \le \xi_{U_1}$, which implies that $\mathrm{BU}(\xi_1)$ is a continuous function, increasing from zero at $\xi_1 = \xi_{L_1}$ to ρy^* as $\xi_1 = \xi_{U_1}$.

Let $\xi_{N_1}(N)$ be the first-period demand such that the buyer's backup order quantity at the beginning of the second period is equal to $N - y^*(1-\rho)$, *i.e.*, the production lot with size of N is exactly equal to the buyer's total order quantity. Since $0 \le N - y^*(1-\rho) \le \rho y^*$ and $\mathrm{BU}(\xi_1)$ increases with ξ_1, we have

$$\begin{aligned}\xi_{N_1}(N) = \min\{\max\{\{\xi_1 \in \mathbf{R} | \mathrm{BU}(\xi_1) = N - y^*(1-\rho) \Leftrightarrow y_2^*(\xi_1) - [y^*(1-\rho) - \xi_1 + \xi_1 uv] \\ = N - y^*(1-\rho)\}, \xi_{L_1}\}, \xi_{U_1}\}.\end{aligned} \tag{3.4}$$

Note that equations (3.2)–(3.4) indicate that $\xi_{L_1} \le \xi_{N_1}(N) \le \xi_{U_1} \le y^*(1-\rho)$. Therefore, if $\xi_{L_1} = y^*(1-\rho)$, then $\xi_{L_1} = \xi_{N_1}(N) = \xi_{U_1} = y^*(1-\rho)$. Likewise, if $\xi_{U_1} = \xi_{L_1}$, then $\xi_{U_1} = \xi_{N_1}(N) = \xi_{L_1}$.

Case 2: $\xi_1 > y^*(1-\rho)$

As with the analysis in Case 1, we discuss the case $\xi_1 > y^*(1-\rho)$ as follows. Let

$$\xi_{L_2} = \max\{\sup\{\xi_1 \in \mathbf{R} | \mathrm{BU}(\xi_1) = 0 \Leftrightarrow y_2^*(\xi_1) \le y^*(1-\rho)uv\}, y^*(1-\rho)\}. \tag{3.5}$$

and

$$\xi_{U_2} = \max\left\{\inf\left\{\xi_1 \in \mathbf{R} | \mathrm{BU}\left(\xi_1\right) = \rho y^* \Leftrightarrow y_2^*\left(\xi_1\right) \geq y^*\left(1-\rho\right) uv + \rho y^*\right\}, \xi_{L_2}\right\}. \tag{3.6}$$

Equation (3.5) indicates the buyer's backup order quantity at the beginning of period 2 as zero, and equation (3.6) as ρy^*, when the first-period demand ξ_1 is less or equal to ξ_{L_2} and greater or equal to ξ_{U_2}, respectively.

Let $\xi_{N_2}(N)$ be the first-period demand such that the buyer's backup order quantity is equal to $N - y^*(1-\rho)$, *i.e.*, the production lot size with of N is exactly equal to the buyer's total order quantity. Thus,

$$\begin{aligned}\xi_{N_2}(N) = \min\{&\max\{\{\xi_1 \in \mathbf{R} | \mathrm{BU}\left(\xi_1\right) = N - y^*\left(1-\rho\right) \Leftrightarrow y_2\left(\xi_1\right) - y^*(1-\rho)uv \\ &= N - y^*(1-\rho)\}, \xi_{L_2}\}, \xi_{U_2}\}.\end{aligned} \tag{3.7}$$

Equations (3.5)–(3.7) show that $y^*\left(1-\rho\right) \leq \xi_{L_2} \leq \xi_{N_2}(N) \leq \xi_{U_2}$.

Summarizing the results in Cases 1 and 2, one obtains $0 \leq \xi_{L_1} \leq \xi_{N_1}(N) \leq \xi_{U_1} \leq y^*\left(1-\rho\right) \leq \xi_{L_2} \leq \xi_{N_2}(N) \leq \xi_{U_2} < \infty$. Thus, equation (3.1) can be rewritten as

$$\begin{aligned}\mathrm{TR}(N) = {} & cy^*\left(1-\rho\right) - K - wN + b\int_0^{\xi_{L_1}} \rho y^* \phi_1\left(\xi_1\right) \mathrm{d}\xi_1 \\ & + \int_{\xi_{L_1}}^{\xi_{U_1}} \left\{(c-b)\left\{y_2^*\left(\xi_1\right) - \left[y^*\left(1-\rho\right) - \xi_1 + \xi_1 uv\right]\right\} + b\rho y^*\right\} \phi_1\left(\xi_1\right) \mathrm{d}\xi_1 \\ & + c\int_{\xi_{U_1}}^{y^*(1-\rho)} \rho y^* \phi_1\left(\xi_1\right) \mathrm{d}\xi_1 + b\int_{y^*(1-\rho)}^{\xi_{L_2}} \rho y^* \phi_1\left(\xi_1\right) \mathrm{d}\xi_1 \\ & + \int_{\xi_{L_2}}^{\xi_{U_2}} \left\{(c-b)\left\{y_2^*\left(\xi_1\right) - y^*\left(1-\rho\right) uv\right\} + b\rho y^*\right\} \phi_1\left(\xi_1\right) \mathrm{d}\xi_1 + c\int_{\xi_{U_2}}^{\infty} \rho y^* \phi_1\left(\xi_1\right) \mathrm{d}\xi_1 \\ & + s_m \int_0^{\xi_{L_1}} \left[N - y^*\left(1-\rho\right)\right] \phi_1\left(\xi_1\right) \mathrm{d}\xi_1 \\ & + s_m \int_{\xi_{L_1}}^{\xi_{N_1}(N)} \left\{N - y^*\left(1-\rho\right) - \left\{y_2^*\left(\xi_1\right) - \left[y^*\left(1-\rho\right) - \xi_1 + \xi_1 uv\right]\right\}\right\} \phi_1\left(\xi_1\right) \mathrm{d}\xi_1 \\ & - c_r \int_{\xi_{N_1}(N)}^{\xi_{U_1}} \left\{\left\{y_2^*\left(\xi_1\right) - \left[y^*\left(1-\rho\right) - \xi_1 + \xi_1 uv\right]\right\} - \left[N - y^*\left(1-\rho\right)\right]\right\} \phi_1\left(\xi_1\right) \mathrm{d}\xi_1 \\ & - c_r \int_{\xi_{U_1}}^{y^*(1-\rho)} \left\{\rho y^* - \left[N - y^*\left(1-\rho\right)\right]\right\} \phi_1\left(\xi_1\right) \mathrm{d}\xi_1 \\ & + s_m \int_{y^*(1-\rho)}^{\xi_{L_2}} \left[N - y^*\left(1-\rho\right)\right] \phi_1\left(\xi_1\right) \mathrm{d}\xi_1\end{aligned}$$

$$+ s_m \int_{\xi_{L_2}}^{\xi_{N_2}(N)} \left\{N - y^* (1-\rho) - [y_2^* (\xi_1) - y^* (1-\rho) uv]\right\} \phi_1 (\xi_1) \, \mathrm{d}\xi_1$$
$$- c_r \int_{\xi_{N_2}(N)}^{\xi_{U_2}} \left\{[y_2^* (\xi_1) - y^* (1-\rho) uv] - [N - y^* (1-\rho)]\right\} \phi_1 (\xi_1) \, \mathrm{d}\xi_1$$
$$- c_r \int_{\xi_{U_2}}^{\infty} \left\{\rho y^* - [N - y^* (1-\rho)]\right\} \phi_1 (\xi_1) \, \mathrm{d}\xi_1. \tag{3.8}$$

The optimal production lot size N^* can be obtained by maximizing the manufacturer's expected profit function given in equation (3.8). The properties of N^* are explored in the following lemma.

Lemma 3.1. *Consider $c_r > max\{w, c-b\} \geq w > s_m$ and let $\frac{c_r - w}{c_r - s_m} = \alpha$.*

i. If $\Phi_1 (\xi_{U_1}) - \Phi_1 (y^* (1-\rho)) + \Phi_1 (\xi_{U_2}) < \alpha$, then $N^* = y^*$;
ii. If $\Phi_1 (\xi_{L_1}) - \Phi_1 (y^* (1-\rho)) + \Phi_1 (\xi_{L_2}) \leq \alpha$ and $\Phi_1 (\xi_{U_1}) - \Phi_1 (y^* (1-\rho)) + \Phi_1 (\xi_{U_2}) \geq \alpha$, then N^* can be obtained by solving $\Phi_1 (\xi_{N_1} (N^*)) - \Phi_1 (y^* (1-\rho)) + \Phi_1 (\xi_{N_2} (N^*)) = \alpha$, where $y^* (1-\rho) \leq N^* \leq y^*$;
iii. If $\Phi_1 (\xi_{L_1}) - \Phi_1 (y^* (1-\rho)) + \Phi_1 (\xi_{L_2}) > \alpha$, then $N^* = y^* (1-\rho)$.

When c_r is significantly larger than w, the value of α is also larger. Thus, the optimal production lot size is more likely to be equal to the commitment y^*, as shown in Lemma 3.1-i. On the other hand, if c_r is close to w, then the manufacturer prefers to minimize the production quantity, $y^* (1-\rho)$ (see Lem. 3.1-iii), and use an emergency purchase strategy to satisfy the backup order instead. The following corollary is provided to examine the impact of c_r on the optimal production lot size.

Corollary 3.2. *Assume that $1 - [\Phi_1 (\xi_{U_1}) - \Phi_1 (y^* (1-\rho)) + \Phi_1 (\xi_{U_2})] < 1$. Let* $\frac{w - [\Phi_1(\xi_{L_1}) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{L_2})] s_m}{1 - [\Phi_1(\xi_{L_1}) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{L_2})]} = c_r^{\mathrm{L}}$ *and* $\frac{w - [\Phi_1(\xi_{U_1}) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{U_2})] s_m}{1 - [\Phi_1(\xi_{U_1}) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{U_2})]} = c_r^{\mathrm{U}}$.

Case I. $\max\{w, c-b\} < c_r^{\mathrm{L}}$: If $c_r > c_r^{\mathrm{U}}$, then $N^* = y^*$, else if $c_r^{\mathrm{L}} \leq c_r \leq c_r^{\mathrm{U}}$, then $y^* (1-\rho) \leq N^* \leq y^*$; otherwise, if $c_r < c_r^{\mathrm{L}}$, then $N^* = y^* (1-\rho)$.
Case II. $c_r^{\mathrm{L}} \leq \max\{w, c-b\} \leq c_r^{\mathrm{U}}$: If $c_r > c_r^{\mathrm{U}}$, then $N^* = y^*$; otherwise, if $\max\{w, c-b\} \leq c_r \leq c_r^{\mathrm{U}}$, then $y^* (1-\rho) \leq N^* \leq y^*$.
Case III. $c_r^{\mathrm{U}} < \max\{w, c-b\}$: $N^* = y^*$.

Corollary 3.2 is a consequence of Lemma 3.1.

For illustration purposes, consider the demands in periods 1 and 2 to have a bivariate normal distribution (as described previously in Sect. 2). The manufacturer's optimal production lot size is then determined as follows.

First, notice that, in equation (3.2), $\sup\{\xi_1 \in \mathbf{R} | y_2^* (\xi_1) \leq y^* (1-\rho) - \xi_1 + \xi_1 uv\}$ is equivalent to the root of the following equation:

$$y^* (1-\rho) - \xi_1 + \xi_1 uv = (1-uv) \left\{\mu_2 + \gamma (\sigma_2/\sigma_1) (\xi_1 - \mu_1) + \Phi_{\text{std}}^{-1} (A) \sigma_2 \sqrt{1-\gamma^2}\right\},$$

where $y_2^* (\xi_1)$ is as in equation (2.6). Thus,

$$\xi_{L_1} = \min\left\{\max\left\{\frac{y^* (1-\rho) / (1-uv) - \mu_2 + \gamma (\sigma_2/\sigma_1) \mu_1 - \Phi_{\text{std}}^{-1} (A) \sigma_2 \sqrt{1-\gamma^2}}{\gamma (\sigma_2/\sigma_1) + 1}, 0\right\}, y^* (1-\rho)\right\}\cdot \tag{3.9}$$

In equation (3.3), $\inf\{\xi_1 \in \mathbf{R} | y_2^* (\xi_1) \geq y^* - \xi_1 + \xi_1 uv\}$ can be obtained by solving the following equation:

$$y^* - \xi_1 + \xi_1 uv = (1-uv) \left\{\mu_2 + \gamma (\sigma_2/\sigma_1) (\xi_{U_1} - \mu_1) + \Phi_{\text{std}}^{-1} (A) \sigma_2 \sqrt{1-\gamma^2}\right\}.$$

Therefore, we have

$$\xi_{U_1} = \min\left\{\max\left\{\frac{y^*/(1-uv) - \mu_2 + \gamma(\sigma_2/\sigma_1)\mu_1 - \Phi_{\text{std}}^{-1}(A)\sigma_2\sqrt{1-\gamma^2}}{\gamma(\sigma_2/\sigma_1)+1}, \xi_{L_1}\right\}, y^*(1-\rho)\right\}, \tag{3.10}$$

where ξ_{L_1} can be obtained from equation (3.9).

Then, substituting $y_2^*(\xi_1)$ as in equation (2.6) into equation (3.4) gives

$$\xi_{N_1}(N) = \min\left\{\max\left\{\frac{N/(1-uv) - \mu_2 + \gamma(\sigma_2/\sigma_1)\mu_1 - \Phi_{\text{std}}^{-1}(A)\sigma_2\sqrt{1-\gamma^2}}{\gamma(\sigma_2/\sigma_1)+1}, \xi_{L_1}\right\}\xi_{U_1}\right\},$$

where ξ_{L_1} and ξ_{U_1} are given by equations (3.9) and (3.10), respectively.

Remark 3.3. Since $A = 1$, it follows that $\Phi_{\text{std}}^{-1}(A) \to \infty$, which implies that $\xi_{L_1} = \xi_{U_1} = 0$ (see Eqs. (3.9) and (3.10)). Thus, if the first-period demand ξ_1 is greater than $\xi_{U_1} = 0$, the backup order quantity for the second period is always ρy.

An explicit form of ξ_{L_2}, ξ_{U_2}, and $\xi_{N_2}(N)$ can be obtained as follows. From equation (3.5), we have

$$\xi_{L_2} = \max\left\{\frac{y^*(1-\rho)uv/(1-uv) - \mu_2 - \Phi_{\text{std}}^{-1}(A)\sigma_2\sqrt{1-\gamma^2}}{\gamma(\sigma_2/\sigma_1)} + \mu_1, y^*(1-\rho)\right\}\cdot \tag{3.11}$$

Furthermore, from equation (3.6),

$$\xi_{U_2} = \max\left\{\frac{[\rho y^* + y^*(1-\rho)uv]/(1-uv) - \mu_2 - \Phi_{\text{std}}^{-1}(A)\sigma_2\sqrt{1-\gamma^2}}{\gamma(\sigma_2/\sigma_1)} + \mu_1, \xi_{L_2}\right\}, \tag{3.12}$$

where ξ_{L_2} is obtained from equation (3.11). Then, from equation (3.7), we have $\xi_{N_2}(N) = \min\left\{\max\left\{\frac{N/(1-uv)-\mu_2-\Phi_{\text{std}}^{-1}(A)\sigma_2\sqrt{1-\gamma^2}-y^*(1-\rho)}{\gamma(\sigma_2/\sigma_1)} + \mu_1, \xi_{L_2}\right\}, \xi_{U_2}\right\}$, where ξ_{L_2} and ξ_{U_2} are obtained from equations (3.11) and (3.12), respectively.

The following numerical example illustrates the use of Lemma 3.1 in determining the optimal production lot size, *i.e.*, the lot size that maximizes the manufacturer's expected total profit given in equation (3.8). Note that the parameter values used in Example 2.2 are once again adopted, and the following additional parameter values are also introduced: $K = 10$, $w = 0.75$ and $s_m = 0.1$. The benefit of the proposed optimal production lot size formulation is evaluated by means of the percentage profit increase (denoted by $\Delta_M\%$) obtained from producing N^* instead of the commitment quantity, y^*.

Let the values of ρ and b be varied in order to assess their impacts on c_r^{L} and c_r^{U}. Furthermore, let the value of c_r be set as either 1.1 or 1.2 in order to investigate the impact of the substitute product purchase cost on the production policy. The results are summarized in Table 2 and Figure 3. An observation of the results reveals the following:

- When a backup contract is offered, the manufacturer always has a lower total profit than when no backup contract is available (71.75) since the backup contract increases the risk of leftover stock at the manufacturer. However, if the retailer agrees to accept substitute products for the backup order, the resulting total profit of the manufacturer can be improved (see Tab. 2); particularly for the case of a more flexible backup contract (*i.e.*, a larger ρ and/or smaller b).
- From Table 2, for given values of ρ and c_r, both the optimal production lot size and the manufacturer's profit increase with increasing b. Conversely, for constant values of ρ and b, the optimal production lot size increases with increasing c_r, while the manufacturer's profit decreases.

TABLE 2. Impact of c_r on manufacturer's decisions.

ρ	b	c_r^{L}	c_r^{U}	N^*	$\mathrm{TR}\,(N^*)$	$\mathrm{TR}\,(y^*)$	$\Delta_M\%$	c_r
0.5	0	0.75101	14.02368	307	55.66	47.61	16.90%	1.1
				314	53.95	47.61	13.31%	1.2
0.5	0.1	0.75019	3.26892	311	59.83	53.36	12.12%	1.1
				318	58.25	53.36	9.16%	1.2
0.5	0.2	0.75007	1.96211	316	63.46	58.46	8.56%	1.1
				323	62.03	58.46	6.11%	1.2
0.5	0.3	0.75003	1.52564	321	66.58	62.93	5.80%	1.1
				327	65.32	62.93	3.81%	1.2
0.5	0.4	0.75001	1.27963	326	69.22	66.79	3.64%	1.1
				333	68.17	66.79	2.08%	1.2
0.5	0.5	0.75001	1.11668	332	71.42	70.06	1.95%	1.1
				338	70.65	70.06	0.84%	1.2
0.3	0	0.77109	2.46407	307	56.30	50.69	11.07%	1.1
				314	54.80	50.69	8.10%	1.2
0.3	0.1	0.76467	2.03652	311	59.93	55.64	7.71%	1.1
				318	58.58	55.64	5.29%	1.2
0.3	0.2	0.75952	1.67930	316	63.10	59.99	5.17%	1.1
				323	61.93	59.99	3.22%	1.2
0.3	0.3	0.75593	1.41763	321	65.84	63.79	3.23%	1.1
				327	64.88	63.79	1.72%	1.2
0.3	0.4	0.75368	1.24128	326	68.21	67.04	1.75%	1.1
				333	67.49	67.04	0.68%	1.2
0.3	0.5	0.75214	1.10295	332	70.24	69.76	0.69%	1.1
				338	69.83	69.76	0.10%	1.2
0.1	0	0.92717	1.36372	307	61.79	60.48	2.17%	1.1
				314	61.03	60.48	0.91%	1.2
0.1	0.1	0.90934	1.31107	311	63.32	62.60	1.15%	1.1
				318	62.79	62.60	0.29%	1.2
0.1	0.2	0.89213	1.25975	316	64.77	64.48	0.45%	1.1
				323	64.49	64.48	0.02%	1.2
0.1	0.3	0.86159	1.15950	321	66.15	66.11	0.07%	1.1
				324	66.11	66.11	0.00%	1.2
0.1	0.4	0.84259	1.09933	324	67.50	67.50	0.00%	1.1
				324	67.50	67.50	0.00%	1.2
0.1	0.5	0.82154	1.03013	324	68.66	68.66	0.00%	1.1
				324	68.66	68.66	0.00%	1.2

– Figure 3 shows that when ρ is large, b is small and/or c_r small, the profit increase percentage, $\Delta_M\%$, increases. In other words, when the manufacturer provides the retailer with a more flexible contract, the use of substitute products can effectively reduce the risk of leftover, and hence improve the manufacturer's profit (particularly when c_r is small). However, if the retailer is offered a less flexible contract (*i.e.*, a smaller ρ and/or a larger b), the value of $\Delta_M\%$ reduces to around 0–2%.

4. Conclusions and discussions

This study has considered a two-period ordering model for fashion products, in which the manufacturer agrees to hold a certain percentage of the total commitment for the coming sales season in reserve for the retailer. After observing the first-period demand, the retailer determines how many units to purchase from this backup

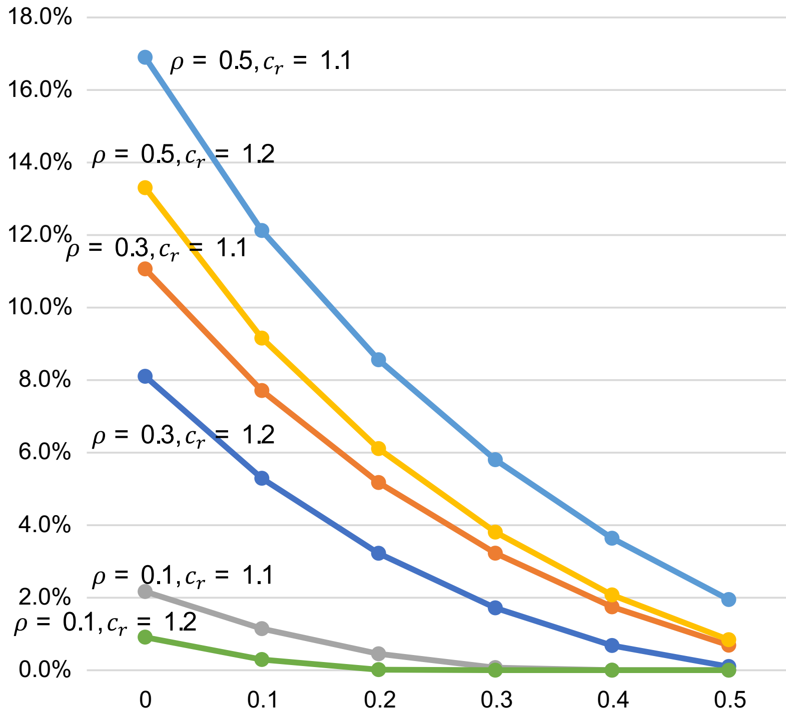

FIGURE 3. Impact of ρ, b and c_r on $\Delta_M\%$.

quantity at the beginning of the second period. Under the terms of the backup contract, the retailer is permitted to order fewer units than the backup quantity when the demand in period 1 is low. However, this may result in a significant reduction in the manufacturer's profit. Therefore, this paper has proposed a model for determining the optimal production lot size which maximizes the manufacturer's profit without affecting the retailer's profit under a modified backup agreement in which the manufacturer is permitted to supply the retailer with substitute products to satisfy the backup order in the event that the production quantity is insufficient. The study has additionally extended the retailer's profit function to take account of a return cost not considered in previous studies. The numerical results have shown that the profit accruing to the manufacturer under the proposed modified backup contract increases as the flexibility of the contract offered to the retailer increases.

This study has considered that the manufacturer is permitted to place an emergency order on a third-party supplier to provide substitute products in order to make up the shortfall in the backup quantity. However, in practice, the emergency purchase could in fact be replaced by an emergency production run by the manufacturer himself. Assume that the production cost comprises a fixed setup cost ($K = 10$) and a variable cost ($w = 0.75$). Assume further that the unit acquisition cost for the insufficient quantity (supposed to be λ) is given as $c_r = (K + \lambda w''))$, where w'' is the unit variable cost for the emergency production (supposed to be 1.1 times that of the unit manufacturing cost w). It can be shown that $c_r \leq 1.119$ when the emergency production quantity is larger than 34. When the offered contract terms are set as $\rho = 0.5$ and $b \leq 0.2$, the percentage increase in the manufacturer's profit, $\Delta_M\%$, is around $6.11\% \sim 8.56\%$. When w'' is increased from 1.1 times to 1.2 times the value of w, the unit acquisition cost is $c_r \leq 1.194$ (1.488) for an emergency production quantity of more than 34 (17) units, respectively. Note that for the case of $c_r \approx 1.488$, $\Delta_M\%$ is positive only when a very flexible contract is offered (*i.e.*, $\rho = 0.5$, $b \leq 0.3$, or $\rho = 0.3$, $b \leq 0.2$ (refer to Tab. 2).

A delayed production strategy can also provide a potential means of reducing the cost incurred by the manufacturer in making up the backup quantity shortfall. In particular, the manufacturer may consider completing the second production run and delivering to the retailer at a point during period 2 before the quantity $y^* (1-\rho)$ delivered at the beginning of the first period is sold out. Such an approach cannot only reduce the retailer's inventory cost, but can also give the manufacturer sufficient time to prepare for production, thereby eliminating the need for larger unit variable costs during emergency production. This topic will be explored further in future studies.

Appendix A. (see [11], p. 1481, Eppen and Iyer for a similar derivation)

Filtering the terms of y_2 from the buyer's expected profit function, $G_1(y) = -cy(1-\rho) + \int_0^\infty f_2(y_2,\xi_1)\,\phi_1(\xi_1)\,\mathrm{d}\xi_1$ gives

$$\begin{aligned}
-cy_2 + by_2 + G_2(y_2,\xi_1) = &-cy_2 + by_2 + \int_0^{y_2/(1-uv)} r(1-v)\,\xi_2\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2\\
&- \int_0^{y_2/(1-uv)} s_r vu\xi_2\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2 - (h_2 - s_2)\\
&\times \int_0^{y_2/(1-uv)} [y_2 - (1-v)\xi_2]\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2\\
&+ \int_{y_2/(1-uv)}^{\infty} r(1-v)\,\frac{y_2}{1-uv}\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2\\
&- \int_{y_2/(1-uv)}^{\infty} s_r vu\frac{y_2}{1-uv}\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2 - (h_2 - s_2)\\
&\times \int_{y_2/(1-uv)}^{\infty} \frac{y_2 v(1-u)}{1-uv}\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2\\
&- \pi\int_{y_2/(1-uv)}^{\infty} \left(\xi_2 - \frac{y_2}{1-uv}\right)\phi_2(\xi_2|\,\xi_1)\,\mathrm{d}\xi_2.
\end{aligned}$$

Differentiating $G_1(y)$ with respect to y_2 yields

$$\begin{aligned}
-c + b + &\frac{1}{1-uv} r(1-v)\,[y_2/(1-uv)]\,\phi_2(y_2/(1-uv)|\,\xi_1)\\
&- \frac{1}{1-uv} s_r vu\,[y_2/(1-uv)]\,\phi_2(y_2/(1-uv)|\,\xi_1)\\
&- (h_2 - s_2)\left\{\frac{1}{1-uv}\left[y_2 - (1-v)\,\frac{y_2}{1-uv}\right]\phi_2(y_2/(1-uv)\,|\xi_1)\right.\\
&\left.+\Phi_2(y_2/(1-uv)\,|\xi_1)\right\}\\
&- \frac{1}{1-uv} r(1-v)\,\frac{y_2}{1-uv}\phi_2(y_2/(1-uv)\,|\xi_1)\\
&+ r(1-v)\,\frac{1}{1-uv}\,[1 - \Phi_2(y_2/(1-uv)|\,\xi_1)]\\
&+ \frac{1}{1-uv} s_r vu\frac{y_2}{1-uv}\phi_2(y_2/(1-uv)\,|\xi_1)\\
&- s_r vu\frac{1}{1-uv}\,[1 - \Phi_2(y_2/(1-uv)|\,\xi_1)]\\
&+ (h_2 - s_2)\left\{\frac{y_2 v(1-u)}{1-uv} \times \frac{1}{1-uv}\phi_2(y_2/(1-uv)|\,\xi_1)\right.
\end{aligned}$$

$$-\frac{v\,(1-u)}{1-uv}\left[1-\Phi_2\left(y_2/\left(1-uv\right)\middle|\,\xi_1\right)\right]\Bigg\}$$
$$+\,\pi\frac{1}{1-uv}\left(y_2/\left(1-uv\right)-\frac{y_2}{1-uv}\right)\phi_2\left(y_2/\left(1-uv\right)\middle|\,\xi_1\right)$$
$$+\,\pi\frac{1}{1-uv}\left[1-\Phi_2\left(y_2/\left(1-uv\right)\middle|\,\xi_1\right)\right].$$

Note that the coefficient of $\phi_2\left(y_2/\left(1-uv\right)|\xi_1\right)$ is zero; therefore, we obtain equation (2.4).

Appendix B.

Proof of Lemma 2.1

Let $A\left(s_r\right)=\frac{n(s_r)}{m(s_r)}$, where $m\left(s_r\right)=r\left(1-v\right)-s_rvu+\left(h_2-s_2\right)\left(1-v\right)+\pi$ and $n\left(s_r\right)=r\left(1-v\right)-s_rvu+\pi-\left(h_2-s_2\right)v\left(1-u\right)+\left(b-c\right)\left(1-uv\right)>0$. Note that when $b<c+h_2-s_2$, one has $m\left(s_r\right)>n\left(s_r\right)>0$, which implies that $0<A<1$. It is then easily shown that $\frac{\mathrm{d}A(s_r)}{\mathrm{d}s_r}=\frac{-uv[m(s_r)-n(s_r)]}{[m(s_r)]^2}<0$ since $m\left(s_r\right)>n\left(s_r\right)$. □

Appendix C.

Proof for Lemma 3.1

First, note that the first derivative of $\mathrm{TR}\left(N\right)$, as in equation (3.8), with respect to N is given by

$$\begin{aligned}
\mathrm{TR}'\left(N\right)=&-w+s_m\left[\Phi_1\left(\xi_{L_1}\right)-\Phi_1\left(0\right)\right]\\
&+s_m\frac{\mathrm{d}\xi_{N_1}\left(N\right)}{dN}\left\{N-y^*\left(1-\rho\right)\right.\\
&\left.-\left\{y_2^*\left(\xi_{N_1}\left(N\right)\right)-\left[y^*\left(1-\rho\right)-\xi_{N_1}\left(N\right)+\xi_{N_1}\left(N\right)uv\right]\right\}\right\}\phi_1\left(\xi_{N_1}\left(N\right)\right)\\
&+s_m\left[\Phi_1\left(\xi_{N_1}\left(N\right)\right)-\Phi_1\left(\xi_{L_1}\right)\right]\\
&+c_r\frac{\mathrm{d}\xi_{N_1}\left(N\right)}{dN}\left\{\left\{y_2^*\left(\xi_{N_1}\left(N\right)\right)-\left[y^*\left(1-\rho\right)-\xi_{N_1}\left(N\right)+\xi_{N_1}\left(N\right)uv\right]\right\}-\left[N-y^*\left(1-\rho\right)\right]\right\}\\
&\phi_1\left(\xi_{N_1}\left(N\right)\right)+c_r\left[\Phi_1\left(\xi_{U_1}\right)-\Phi_1\left(\xi_{N_1}\left(N\right)\right)\right]\\
&+c_r\left[\Phi_1\left(y^*\left(1-\rho\right)\right)-\Phi_1\left(\xi_{U_1}\right)\right]+s_m\left[\Phi_1\left(\xi_{L_2}\right)-\Phi_1\left(y^*\left(1-\rho\right)\right)\right]\\
&+\frac{\mathrm{d}\xi_{N_2}\left(N\right)}{dN}\left\{N-y^*\left(1-\rho\right)-\left[y_2^*\left(\xi_{N_2}\left(N\right)\right)-y^*\left(1-\rho\right)uv\right]\right\}\phi_1\left(\xi_{N_2}\left(N\right)\right)\\
&+s_m\left[\Phi_1\left(\xi_{N_2}\left(N\right)\right)-\Phi_1\left(\xi_{L_2}\right)\right]\\
&+c_r\frac{\mathrm{d}\xi_{N_2}\left(N\right)}{dN}\left\{\left[y_2^*\left(\xi_{N_2}\left(N\right)\right)-y^*\left(1-\rho\right)uv\right]-\left[N-y^*\left(1-\rho\right)\right]\right\}\phi_1\left(\xi_{N_2}\left(N\right)\right)\\
&+c_r\left[\Phi_1\left(\xi_{U_2}\right)-\Phi_1\left(\xi_{N_2}\left(N\right)\right)\right]+c_r\left[\Phi_1\left(\infty\right)-\Phi_1\left(\xi_{U_2}\right)\right].
\end{aligned}\tag{C.1}$$

In equation (C.1), if $\xi_{N_1}\left(N\right)=\xi_{L_1}$ or ξ_{U_1}, then $\frac{\mathrm{d}\xi_{N_1}(N)}{\mathrm{d}N}=0$; otherwise, $\xi_{N_1}\left(N\right)$ satisfies $N-y^*\left(1-\rho\right)=y_2^*\left(\xi_{N_1}\left(N\right)\right)-\left[y^*\left(1-\rho\right)-\xi_{N_1}\left(N\right)+\xi_{N_1}\left(N\right)uv\right]$ (see Eq. (3.4)).

These imply that

$$\frac{\mathrm{d}\xi_{N_1}\left(N\right)}{\mathrm{d}N}\left\{N-y^*\left(1-\rho\right)-\left\{y_2^*\left(\xi_{N_1}\left(N\right)\right)-\left[y^*\left(1-\rho\right)-\xi_{N_1}\left(N\right)+\xi_{N_1}\left(N\right)uv\right]\right\}\right\}=0.$$

Similarly, we can show that

$$\frac{\mathrm{d}\xi_{N_2}(N)}{\mathrm{d}N}\left\{N - y^*(1-\rho) - \left[y_2^*(\xi_{N_2}(N)) - y^*(1-\rho)uv\right]\right\} = 0.$$

Therefore, equation (C.1) becomes

$$\mathrm{TR}'(N) = -w + c_r + (s_m - c_r)\left[\Phi_1(\xi_{N_1}(N)) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{N_2}(N))\right]. \tag{C.2}$$

By equation (C.2), the following result is easy to verify:

$$\mathrm{TR}''(N) = (s_m - c_r)\left[\phi_1(\xi_{N_1}(N))\frac{\mathrm{d}\xi_{N_1}(N)}{\mathrm{d}N} + \phi_1(\xi_{N_2}(N))\frac{\mathrm{d}\xi_{N_2}(N)}{\mathrm{d}N}\right]. \tag{C.3}$$

If $\xi_{N_1}(N) = \xi_{L_1}$ or ξ_{U_1}, then $\frac{\mathrm{d}\xi_{N_1}(N)}{\mathrm{d}N} = 0$; otherwise, $\xi_{N_1}(N)$ satisfies
$N - y^*(1-\rho) = y_2^*(\xi_{N_1}(N)) - [y^*(1-\rho) - \xi_{N_1}(N) + \xi_{N_1}(N)uv]$ (see Eq. (3.4)).
This shows that

$$N = y_2^*(\xi_{N_1}(N)) + \xi_{N_1}(N)(1-uv). \tag{C.4}$$

In equation (C.4), we notice that $\xi_{N_1}(N)$ increases with N and $y_2^*(\xi_1)$ increases with ξ_1. From the above analysis, we obtain $\frac{\mathrm{d}\xi_{N_1}(N)}{\mathrm{d}N} \geq 0$ for $N > 0$.

Likewise, if $\xi_{N_2}(N) = \xi_{L_2}$ or ξ_{U_2}, then $\frac{\mathrm{d}\xi_{N_2}(N)}{\mathrm{d}N} = 0$; otherwise, $\xi_{N_2}(N)$ satisfies the following equation:

$$N - y_2^*(\xi_{N_2}(N)) = y^*(1-\rho)(1-uv). \tag{C.5}$$

In equation (C.5), we notice that $\xi_{N_2}(N)$ increases with N. Based on the above, we obtain $\frac{\mathrm{d}\xi_{N_2}(N)}{\mathrm{d}N} \geq 0$ for $N > 0$.

Therefore, if $s_m < c_r$, then $\mathrm{TR}''(N) < 0$ (see Eq. (C.3)). This implies that $\mathrm{TR}(N)$ is a concave function in N. In addition, $\xi_{N_i}(N = y^*(1-\rho)) = \xi_{L_i}$, $\xi_{N_i}(N = y^*) = \xi_{U_i}$ for $i = 1, 2$. Therefore,

1. If $\mathrm{TR}'(y^*(1-\rho)) > 0$ and $\mathrm{TR}'(y^*) > 0$ iff $\Phi_1(\xi_{L_1}) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{L_2}) < \alpha$ and $\Phi_1(\xi_{U_1}) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{U_2}) < \alpha$, then $\mathrm{TR}(N)$ increases with N for $y^*(1-\rho) \leq N \leq y^*$, so $N^* = y^*$.
2. If $\mathrm{TR}'(y^*(1-\rho)) > 0$ and $\mathrm{TR}'(y^*) \leq 0$ iff $\Phi_1(\xi_{L_1}) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{L_2}) < \alpha$ and $\Phi_1(\xi_{U_1}) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{U_2}) \geq \alpha$, then $\mathrm{TR}(N)$ increases with N until $N = N^*$, where $\mathrm{TR}'(N^*) = 0$, and $\mathrm{TR}(N)$ decreases with N from $N = y^*(1-\rho)$ until $N = y^*$.
3. Finally, if $\mathrm{TR}'(y^*(1-\rho)) < 0$ and $\mathrm{TR}'(y^*) < 0$ iff $\Phi_1(\xi_{L_1}) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{L_2}) > \alpha$ and $\Phi_1(\xi_{U_1}) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{U_2}) > \alpha$, then $\mathrm{TR}(N)$ decreases with N for $y^*(1-\rho) \leq N \leq y^*$, resulting in $N^* = y^*(1-\rho)$.

Points 1–3 above should be considered together with the following:

(a) $\Phi_1(\xi_{L_1}) + \Phi_1(\xi_{L_2}) \leq \Phi_1(\xi_{U_1}) + \Phi_1(\xi_{U_2})$,
(b) $\mathrm{TR}'(N^*) = 0$ or, equivalently, $\Phi_1(\xi_{N_1}(N^*)) - \Phi_1(y^*(1-\rho)) + \Phi_1(\xi_{N_2}(N^*)) = \alpha$ (see Eq. (C.2)).

Thus, Lemma 3.1 is completely proved. □

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