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SYMMETRIC DUALITY RESULTS FOR SECOND-ORDER NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING PROBLEM

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Abstract. In this article, we study the existence of G_f-bonvex/G_f-pseudo-bonvex functions and construct various nontrivial numerical examples for the existence of such type of functions. Furthermore, we formulate Mond-Weir type second-order nondifferentiable multiobjective programming problem and give a nontrivial concrete example which justify weak duality theorem present in the paper. Next, we prove appropriate duality relations under aforesaid assumptions.

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1. Introduction

The vector optimization problem and its dual are said to be symmetric if the dual of the dual is the original problem. Second-order duality is significant due to its computational importance as it provides more higher bounds whenever approximation is used. Mangasarian [16] was the first one who introduced the concept of second-order duality for non-linear programming. Furthermore, Gulati and Gupta [10] have been introduced the concept of η_1-bonvexity/η_2-boncavity and derived duality results for a Wolfe type model. Several researchers [5, 7] have done their work in the related areas.

Antczak [1] introduced the concept of G-invex function and derived some optimality conditions for constrained optimization problem. Later Antzcak [2] extended his earlier work of G_f-invex function, he proved necessary and sufficient optimality conditions for a multiobjective nonlinear programming problem. Ferrara and Stefaneseu [9] discussed the conditions of optimality and duality for multiobjective programming problem. In last several years, various optimality and duality results have been obtained for multi objective fractional programming problems. In Chen [4], multi objective fractional problem and its duality theorems have been considered under higher-order (F, α, ρ, d)-convexity.

Jayswal *et al.* [12] discussed multiobjective fractionl programming problem involving-invex function. Kang *et al.* [13] defined G-invexity for a locally Lipschitz function and obtained optimality conditions for

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multiobjective programming problems. Recently, generalizing the notion of invexity to $(p, r) - \rho - (\eta, \theta)$-invex function, Mandal and Nahak [15] developed symmetric duality results for a Mond-Weir type model. Stefaneseu and Ferrara [21] studied new invexities for multiobjective programming problem. Bhatia and Garg [3] discussed the concept of (V, p)-invexity has been introduced for non-smooth vector functions and is used to establish duality results for multi objective programs. Later on, Reuda *et al.* [18] generalized the concept of invex function by introducing type-I and type-II functions.

In this paper, we construct several nontrivial numerical examples which illustrate the existence of such type of functions and also formulate a pair of nondifferentiable multiobjective Mond-Weir type symmetric primal-dual problems over arbitrary cones. Next, We construct numerical examples to illustrate the existence of G_f-bonvex/G_f-pseudobonvex functions but it is neither η-bonvex/η-pseudobonvex functions nor η-invex/η-pseudoinvex functions. Further, under the G_f-bonvex/G_f-pseudobonvex assumptions, we prove the weak, strong and strict converse duality theorems. We also formulate an example which justifies the Weak duality theorem presented in the paper.

2. Preliminaries and definitions

Definition 2.1. The positive polar cone S^* of a cone $S \subseteq R^s$ is defined by

$$S^* = \{y \in R^s : x^T y \geq 0\}.$$

Consider the following vector minimization problem:

$$\textbf{(P)} \qquad \text{Minimize } f(x) = \left\{ f_1(x), f_2(x), \ldots, f_k(x) \right\}^T$$
$$\text{Subject to } X^0 = \{x \in X \subset R^n : g_j(x) \leq 0, j = 1, 2, \ldots, m\}$$

where $f = \{f_1, f_2, \ldots, f_k\} : X \to R^k$ and $g = \{g_1, g_2, \ldots, g_m\} : X \to R^m$ are differentiable functions defined on X.

Definition 2.2. A point $\bar{x} \in X^0$ is said to be an efficient solution of (P) if there exists no other $x \in X^0$ such that $f_r(x) < f_r(\bar{x})$, for some $r = 1, 2, \ldots, k$ and $f_i(x) \leq f_i(\bar{x})$, for all $i = 1, 2, \ldots, k$.
Let $f = (f_1, \ldots, f_k) : X \to R^k$ be a vector-valued differentiable function defined on a non-empty open set $X \subseteq R^n$, and $I_{f_i}(X), i = 1, \ldots, k$, be the range of f_i, that is, the image of X under f_i.

Definition 2.3. Let $f : X \to R^k$, $(X \subseteq R^n)$ be a differentiable function. If there exists a vector valued function $\eta : X \times X \to R^n$ such that $\forall x \in X$,

$$f_i(x) - f_i(u) \geq \eta^T(x, u)\nabla_x f_i(u), \quad \text{for all } i = 1, 2, \ldots, k,$$

then f is called invex at $u \in X$ with respect to η.
If the above inequality sign changes to $\leq$, then f is called incave at $u \in X$ with respect to η.

Definition 2.4. Let $f : X \to R^k$, $(X \subseteq R^n)$ be a differentiable function. If there exists a vector valued function $\eta : X \times X \to R^n$ such that $\forall x \in X$,

$$\eta^T(x, u)G'_{f_i}(f_i(u))\nabla_x f_i(u) \geq 0 \Rightarrow G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geq 0, \text{ for all } i = 1, 2, \ldots, k,$$

then f is called G_f-pseudoinvex at $u \in X$ with respect to η.
If the above inequality sign changes to $\leq$, then f is called G_f-pseudoincave at $u \in X$ with respect to η.
Now, we give the definition of a differentiable vector valued G_f-bonvex/G_f-pseudobonvex functions.

Definition 2.5. Let $f : X \to R^k$ be a vector-valued differentiable function. If there exists a differentiable function $G_f = (G_{f_1}, G_{f_2}, \dots, G_{f_k}) : R \to R^k$ such that every component $G_{f_i} : I_{f_i}(X) \to R$ is strictly increasing on the range of I_{f_i} and a vector valued function $\eta : X \times X \to R^n$ such that $\forall\ x \in X$ and $p_i \in R^n$,

$$\begin{aligned}
&G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \\
&\quad \geq \eta^T(x,u)\left[G'_{f_i}(f_i(u))\nabla_x f_i(u) + \left\{G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\right\} p_i\right] \\
&\quad\quad - \frac{1}{2} p_i^T \left[G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\right] p_i, \text{ for all } i = 1, 2, \dots, k,
\end{aligned}$$

then f is called G_f-bonvex at $u \in X$ with respect to η.
If the above inequality sign changes to $\leq$, then f is called G_f-boncave at $u \in X$ with respect to η.

Definition 2.6. Let $f : X \to R^k$ be a vector-valued differentiable function. If there exists a differentiable function $G_f = (G_{f_1}, G_{f_2}, \dots, G_{f_k}) : R \to R^k$ such that every component $G_{f_i} : I_{f_i}(X) \to R$ is strictly increasing on the range of I_{f_i} and a vector valued function $\eta : X \times X \to R^n$ such that $\forall\ x \in X$ and $p_i \in R^n$, for all $i = 1, 2, \dots, k$,

$$\begin{aligned}
&\eta^T(x,u)[G'_{f_i}(f_i(u))\nabla_x f_i(u) + \left\{G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T \right. \\
&\quad \left. + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\right\} p_i] \geq 0 \Rightarrow G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \\
&\quad + \frac{1}{2} p_i^T \left[G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\right] p_i \geq 0,
\end{aligned}$$

then f is called G_f-pseudobonvex at $u \in X$ with respect to η.
If the above inequality sign changes to $\leq$, then f is called G_f-pseudoboncave at $u \in X$ with respect to η.
We now give an example of G_f-bonvexity with respect to η, which is neither η-bonvex nor η-invex.

Example 2.1. Let $f : [0,1] \to R^4$ be defined as

$$f(x) = \{f_1(x), f_2(x), f_3(x), f_4(x)\},$$

where $f_1(x) = x^8$, $f_2(x) = \text{arc (sinx)}$, $f_3(x) = \text{arc (tanx)}$, $f_4(x) = \text{arc (cotx)}$ and $G_f = \{G_{f_1}, G_{f_2}, G_{f_3}, G_{f_4}\} : R \to R^4$ be defined as:

$$G_{f_1}(t) = t^9 + 2,\ G_{f_2}(t) = sint,\ G_{f_3}(t) = tant,\ G_{f_4}(t) = cott.$$

Let $\eta : [0,1] \times [0,1] \to R$ be given as:

$$\eta(x,u) = -\frac{1}{9}x^{14} + x - \frac{1}{12}x^7u^3 - \frac{1}{4}x^2u^2 + u.$$

For showing that f is G_f-bonvex at $u = 0$ with respect to η, for this we have to claim that

$$\begin{aligned}
\pi_i &= G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) - \eta^T(x,u)\left[G'_{f_i}(f_i(u))\nabla_x f_i(u) + \left\{G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T\right]\right. \\
&\quad \left. + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\right\} p_i + \frac{1}{2} p_i^T \left[G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\right] p_i \geq 0, \\
&\quad i = 1, 2, 3, 4.
\end{aligned}$$

Putting the values of $f_1,\ f_2,\ f_3,\ f_4,\ G_{f_1},\ G_{f_2},\ G_{f_3}$ and G_{f_4} in the above expressions, we have

$$\pi_1 = (x^{72}+2) - (u^{72}+2) - 72u^{70}\Big(-\frac{1}{9}x^{14} + x - \frac{1}{12}x^7u^3 - \frac{1}{4}x^2u^2 + u\Big)(u+71p_1) + 2556u^{70}p_1^2,$$

$$\pi_2 = \sin(\text{arc}(\text{sinx})) - \sin(\text{arc}(\text{sinu})) - \Big(-\frac{1}{9}x^{14} + x - \frac{1}{12}x^7u^3 - \frac{1}{4}x^2u^2 + u\Big)\Big(\cos(\text{arc}(\text{sinu})) \times \frac{1}{(1-u^2)^{\frac{1}{2}}} + \Big\{\frac{-u}{1-u^2} + \cos(\text{arc}(\text{sinu})) \times \frac{u}{(1-u^2)^{\frac{3}{2}}}\Big\}p_2\Big) + \frac{p_2^2}{2}\Big\{\frac{-u}{1-u^2} + \cos(\text{arc}(\text{sinu})) \times \frac{u}{(1-u^2)^{\frac{3}{2}}}\Big\},$$

$$\pi_3 = \tan(\text{arc}(\text{tanx})) - \tan(\text{arc}(\text{tanu})) - \Big(-\frac{1}{9}x^{14} + x - \frac{1}{12}x^7u^3 - \frac{1}{4}x^2u^2 + u\Big) \times \Big(sec^2(\text{arc}(\text{tanu})) \times \frac{1}{(1+u^2)} + \Big\{2sec^2(\text{arc}(\text{tanu}))\tan(\text{arc}(\text{tanu})) + \sec(\text{arc}(\text{tanu})) \times \frac{-2u}{(1+u^2)^2}\Big\}p_3\Big) + \frac{p_3^2}{2}\Big\{2sec^2(\text{arc}(\text{tanu}))\tan(\text{arc}(\text{tanu})) + \sec(\text{arc}(\text{tanu})) \times \frac{-2u}{(1+u^2)^2}\Big\}\Big)$$

and

$$\pi_4 = \cot(\text{arc}(\text{cotx})) - \cot(\text{arc}(\text{cotu})) - \Big(-\frac{1}{9}x^{14} + x - \frac{1}{12}x^7u^3 - \frac{1}{4}x^2u^2 + u\Big) \times \Big(\text{cosec}^2(\text{arc}(\text{cotu})) \times \frac{1}{1+u^2} + \Big\{2\text{cosec}^2(\text{arc}(\text{cotu}))\cot(\text{arc}(\text{cotu})) \times \Big(\frac{1}{(1+u^2)^2} - \text{cosec}^2(\text{arc}(\text{cotu})) \times \frac{2u}{1+u^2}\Big)\Big\}p_4\Big) + \frac{p_4^2}{2}\Big\{2\text{cosec}^2(\text{arc}(\text{cotu}))\cot(\text{arc}(\text{cotu})) \times \frac{1}{(1+u^2)^2} - \text{cosec}^2(\text{arc}(\text{cotu})) \times \frac{2u}{1+u^2}\Big\}.$$

At the point $u = 0 \in [0,1]$. It is clear from figures (1) and (2), the above expressions hold the inequalities:

$$\pi_1 \geq 0,\ \pi_2 \geq 0,\ \pi_3 \geq 0 \text{ and } \pi_4 \geq 0,\ \text{for all } x \in [0,1],\ \forall\ p_i,\ i = 1,2,3,4.$$

Therefore, f is G_f-bonvex at $u = 0$ with respect to η and p.
Now, suppose

$$\xi = f_3(x) - f_3(u) - \eta^T(x,u)[\nabla_x f_3(u) - \nabla_{xx} f_3(u)p_3] + \frac{1}{2}p_3^T[\nabla_{xx} f_3(u)]p_3$$

or

$$\xi = \text{arc (tanx)} - \text{arc (tanu)} - \Big(-\frac{1}{9}x^{14} + x - \frac{1}{12}x^7u^3 - \frac{1}{4}x^2u^2 + u\Big)\Big[\frac{1}{1+u^2} - \frac{2up_3}{(1+u^2)^2}\Big] - \frac{up_3^2}{(1+u^2)^2},$$

$$\xi = \text{arc (tanx)} + \frac{1}{9}x^{14} - x \quad \text{at } u = 0.$$

It follows that

$$\xi \ngeq 0,\ \forall\ x \in [0,1], \quad \text{(see the Fig. 3)}.$$

Therefore, f_3 is not η-bonvex at $u = 0$ with respect to p_3. Hence, $f = (f_1,\ f_2,\ f_3,\ f_4)$ is not η-bonvex at $u = 0$ with respect to $p = (p_1, p_2, p_3, p_4)$.
Next,

$$\delta = f_3(x) - f_3(u) - \eta^T(x,u)\nabla_x f_3(u)$$

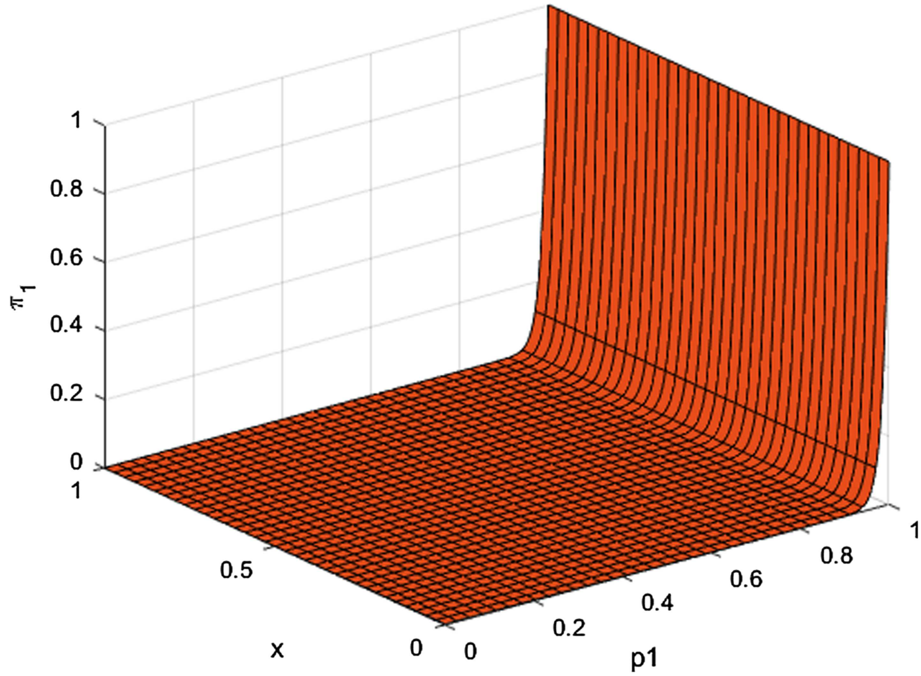

FIGURE 1. $\pi_1 = x^{72}, \forall x \in [0,1]$

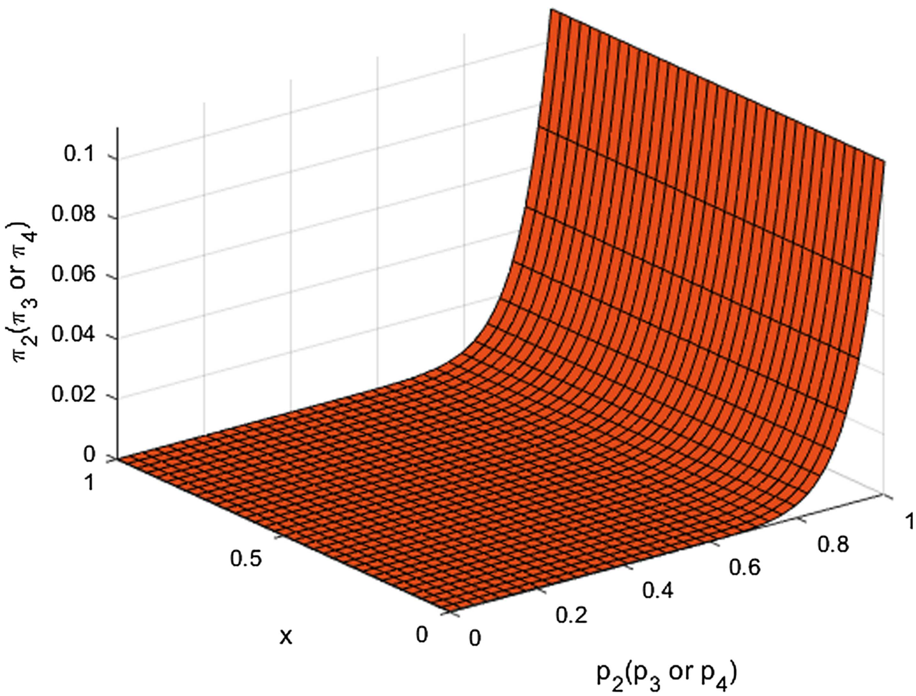

FIGURE 2. $\pi_2 = \pi_3 = \pi_4 = \sin(arc(\sin x)) = \tan(arc(\tan x)) = \cot(arc(\cot x)), \forall x \in [0,1]$

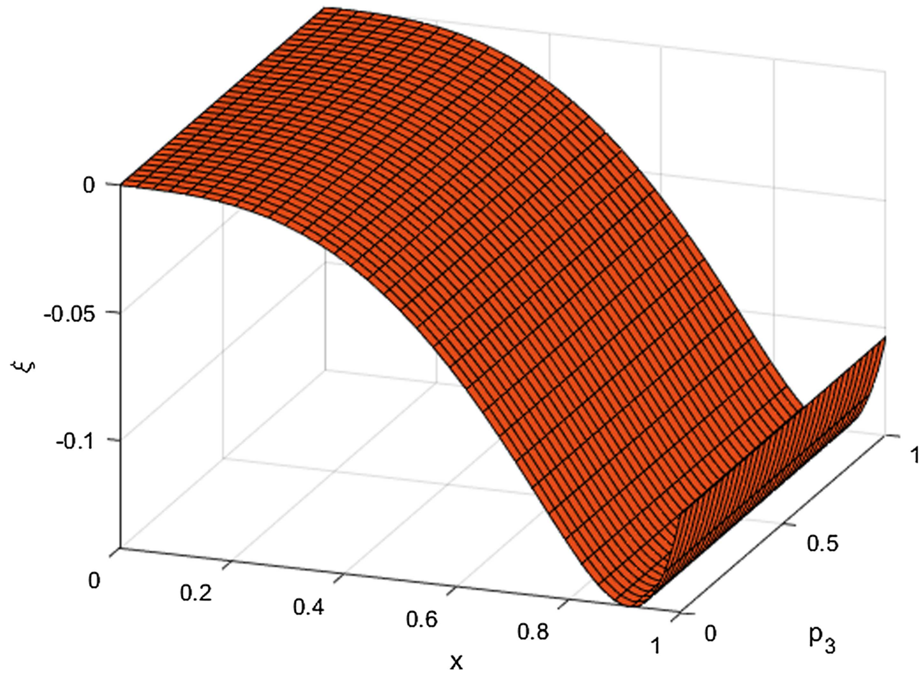

FIGURE 3. $\xi = arc(\tan x) + \frac{1}{9}x^{14} - x, \forall x \in [0,1]$

or

$$\delta = \text{arc (tanx)} - \text{arc (tanu)} - \left(-\frac{1}{9}x^{14} + x - \frac{1}{12}x^7u^3 - \frac{1}{4}x^2u^2 + u \right)\frac{1}{1+u^2},$$

$$\delta = \text{arc (tanx)} + \frac{1}{9}x^{14} - x \quad \text{at } u = 0,$$

$$\delta = \frac{\pi}{4} + \frac{1}{9} - 1 < 0 \quad \text{at } x = 1.$$

Therefore, f_3 is not η-invex at $u = 0$. Hence, $f = (f_1,\ f_2,\ f_3,\ f_4)$ is not η-invex at $u = 0$.
We now give an example of G_f-pseudobonvex with respect to η, which is neither η-pseudobonvex nor η-pseudoinvex.

Example 2.2. Let $f : [-1, 1] \to R^2$ be defined as

$$f(x) = \{f_1(x), f_2(x)\},$$

where $f_1(x) = (e^x - e^{-x}),\ f_2(x) = x^3$ and $G_f = \{G_{f_1}, G_{f_2}\} : R \to R^2$ be defined as:

$$G_{f_1}(t) = t^4,\ G_{f_2}(t) = t^2 + 2.$$

Let $\eta : [-1, 1] \times [-1, 1] \to R$ be given as:

$$\eta(x, u) = x^2 + u^2.$$

For showing that f is G_f-pseudobonvex at $u = 0$ with respect to η, for this we have to claim that, for $i = 1, 2$

$$\zeta_i = \eta^T(x,u)\left[G'_{f_i}(f_i(u))\nabla_x f_i(u) + \left\{G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\right\} p_i\right]$$
$$\geq 0 \Rightarrow G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) + \frac{1}{2}p_i^T\left[G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\right] p_i \geq 0,$$

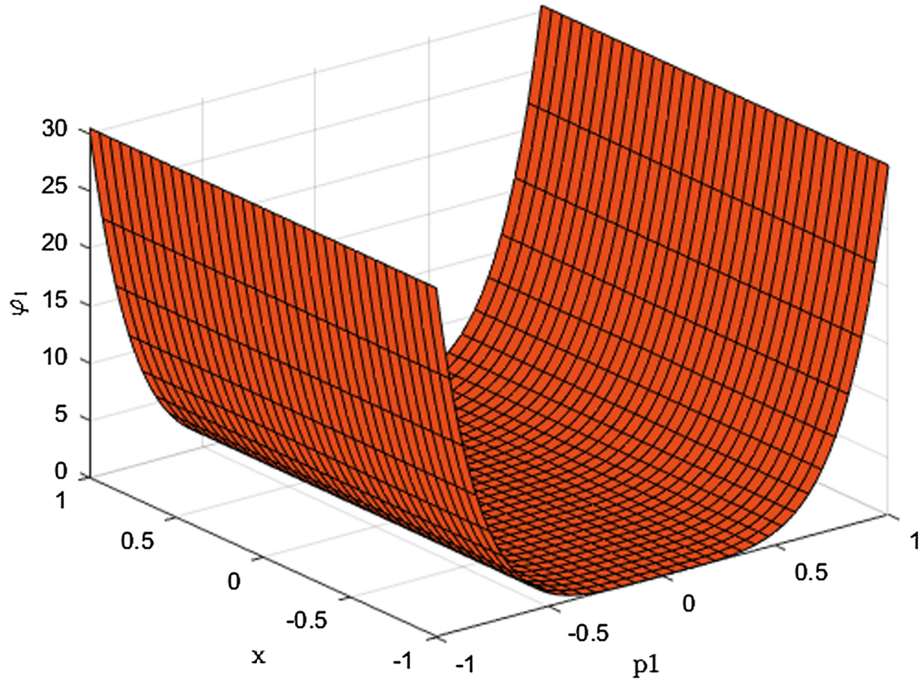

FIGURE 4. $\varphi_1 = (e^x - e^{-x})^4, \forall x \in [-1, 1]$

or $\zeta_i = \phi_i \geq 0 \Rightarrow \varphi_i \geq 0$, for $i = 1, 2$,
where, for $i = 1, 2$,

$$\phi_i = \eta^T(x,u)\left[G'_{f_i}(f_i(u))\nabla_x f_i(u) + \left\{G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\right\} p_i\right]$$

and

$$\varphi_i = G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) + \frac{1}{2}p_i^T\left[G''_{f_i}(f_i(u))\nabla_x f_i(u)(\nabla_x f_i(u))^T + G'_{f_i}(f_i(u))\nabla_{xx} f_i(u)\right] p_i.$$

Now $\phi_1 = \eta^T(x,u)\left[G'_{f_1}(f_1(u))\nabla_x f_1(u) + \{G''_{f_1}(f_1(u))\nabla_x f_1(u)(\nabla_x f_1(u))^T + G'_{f_1}(f_1(u))\nabla_{xx} f_1(u)\}p_1\right]$,

$$\phi_1 = (x^2+u^2)(4(e^u - e^{-u})^3 \times (e^u + e^{-u}) + \{12(e^u - e^{-u})^2 \times (e^u + e^{-u})^2 + 4(e^u - e^{-u})^4\}p_1).$$

At the point $u = 0$, we have

$$\phi_1 \geq 0,\ \forall\ x \in [-1, 1],\ \forall\ p_1.$$

Also,

$$\varphi_1 = G_{f_1}(f_1(x)) - G_{f_1}(f_1(u)) + \frac{1}{2}p_1^T\left[G''_{f_1}(f_1(u))\nabla_x f_1(u)(\nabla_x f_1(u))^T + G'_{f_1}(f_1(u))\nabla_{xx} f_1(u)\right] p_1$$

$$\varphi_1 = (e^x - e^{-x})^4 - (e^u - e^{-u})^4 + \frac{p_1^2}{2}\left\{12(e^u - e^{-u})^2 \times (e^u + e^{-u})^2 + 4(e^u - e^{-u})^4\right\}$$

which at $u = 0$, we get

$\varphi_1 = (e^x - e^{-x})^4$

$\varphi_1 \geq 0,\ \forall\ x \in [-1, 1],\ \forall\ p_1$ (see from the Fig. 4).

$\phi_2 = \eta^T(x,u)\left[G'_{f_2}(f_2(u))\nabla_x f_2(u) + \left\{G''_{f_2}(f_2(u))\nabla_x f_2(u)(\nabla_x f_2(u))^T + G'_{f_2}(f_2(u))\nabla_{xx} f_2(u)\right\} p_2\right]$

$\phi_2 = (x^2 + u^2)(6u^5 + 30u^4 p_2)$.

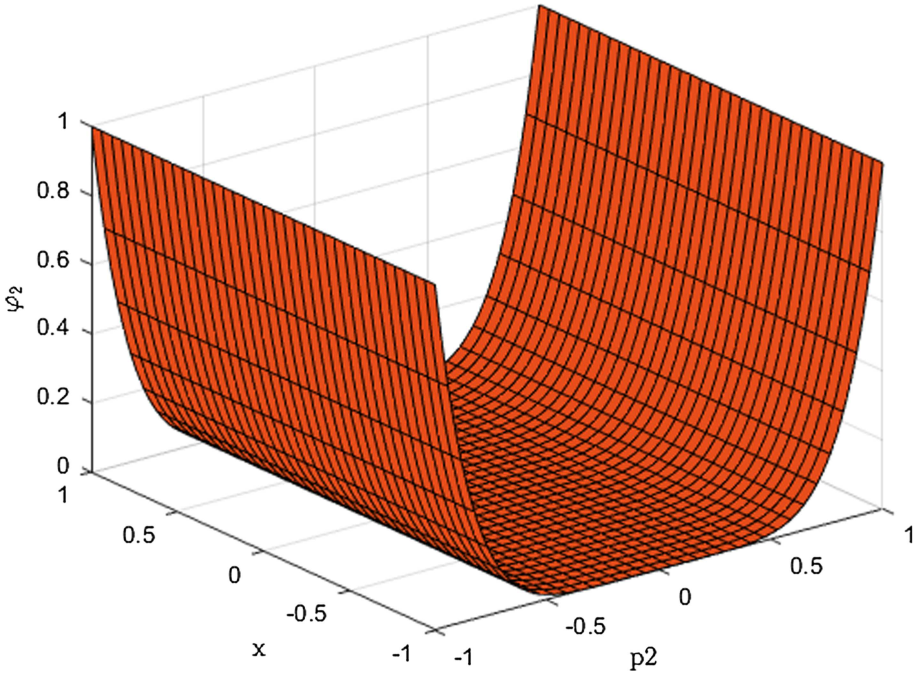

FIGURE 5. $\varphi_2 = x^6, \forall x \in [-1,1]$

At the point $u=0$, we get

$$\phi_2 \geq 0,\ \forall\ x \in [-1,1],\ \forall\ p_2.$$

Also,

$$\varphi_2 = G_{f_2}(f_2(x)) - G_{f_2}(f_2(u)) + \frac{1}{2}p_2^T\left[G''_{f_2}(f_2(u))\nabla_x f_2(u)(\nabla_x f_2(u))^T + G'_{f_2}(f_2(u))\nabla_{xx} f_2(u)\right]p_2$$
$$\varphi_2 = x^6 - u^6 + 15p_2^2u^4.$$

From the Figure 5, which at the point $u=0$, we obtain

$$\varphi_2 \geq 0,\ \forall\ x \in [-1,1],\ \forall\ p_2.$$

Clearly, $\zeta_i = \phi_i \geq 0 \Rightarrow \varphi_i \geq 0$, for $i=1,2$.
Hence, from the above expressions $\phi_1,\ \phi_2,\ \varphi_1$ and φ_2 indicate that f is G_f-pseudobonvex at $u=0$ with respect to η.

Next,

$$\phi_3 = \eta^T(x,u)[\nabla_x f_2(u) + \nabla_{xx} f_2(u)p_2]$$
$$\phi_3 = (x^2+u^2)[3u^2 + 6up_2].$$

At the point $u=0$, we have

$$\phi_3 \geq 0,\ \forall\ x \in [-1,1],\ \forall\ p_2 \in R.$$

Also,

$$\varphi_3 = f_1(x) - f_2(u) + \frac{1}{2}p_2^2\nabla_{xx} f_2(u)$$
$$\varphi_3 = x^3 - u^3 + 3p_2^2u.$$

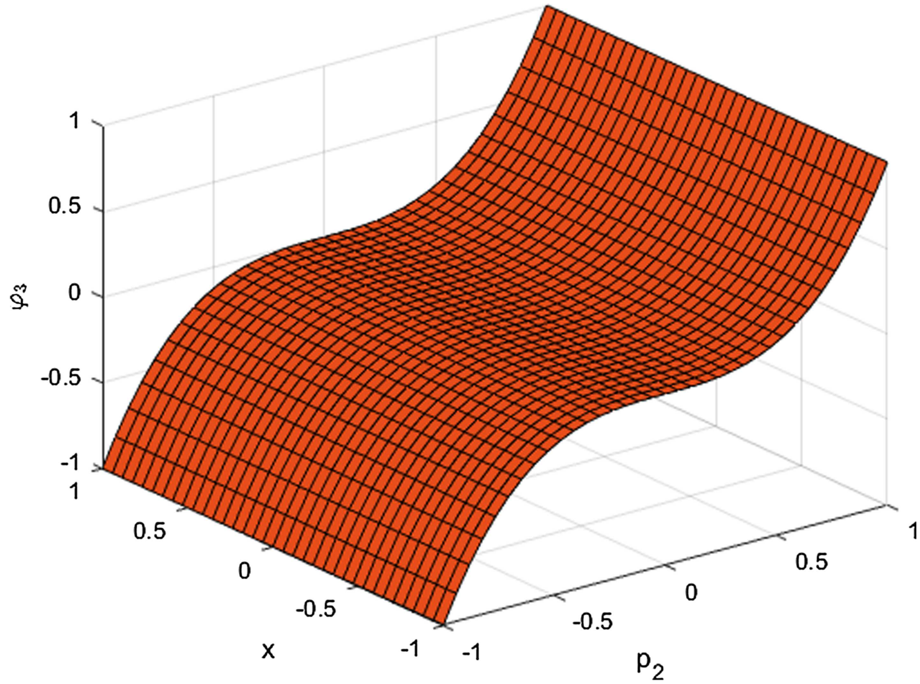

FIGURE 6. $\varphi_3 = x^3, \forall x \in [-1, 1]$

At the point $u = 0$, we obtain

$$\varphi_3 \ngeq 0, \ \forall \ x \in [-1, 1], \ \forall \ p_2 \ \ \text{(from Fig. 6)}.$$

Hence, f_2 is not η-pseudobonvex at $u = 0 \in [-1, 1]$. Hence, $f = (f_1, \ f_2)$ is not η-pseudobonvex at $u = 0 \in [-1, 1]$.

Finally,

$$\phi_4 = \eta^T(x, u)\nabla_x f_2(u)$$
$$\phi_4 = 3(x^2 + u^2)u^2.$$

At the point $u = 0$, we have

$$\phi_4 \geq 0, \ \forall \ x \in [-1, 1], \ \forall \ p_2.$$

Also,

$$\varphi_4 = f_2(x) - f_2(u)$$
$$\varphi_3 = x^3 - u^3.$$

At the point $u = 0$, we obtain

$$\varphi_4 \ngeq 0, \ \forall \ x \in [-1, 1].$$

Hence, f_2 is not η-pseudoinvex at $u = 0 \in [-1, 1]$. Hence, $f = (f_1, \ f_2)$ is not η-pseudoinvex at $u = 0 \in [-1, 1]$.

Definition 2.7. Let C be a compact convex set in R^n. The support function of C is defined by

$$s\,(x|C) = \max\{x^T y : y \in C\}.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For any convex set $S \subset R^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n : y^T(z - x) \leqq 0 \text{ for all } z \in S\}.$$

It is readily verified that for a compact convex set S, y is in $N_S(x)$ if and only if

$$s(y|S) = x^T y.$$

Suppose that $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$.

3. N-G-Mond-Weir type symmetric dual program

Suneja *et al.* [19] formulated a pair of symmetric dual multiobjective programs over arbitrary cones in which the objective function is optimized with respect to an arbitrary closed convex cone by assuming the function involved to be cone convex. Mishra and Lai [17] extended the results of Khurana [14] to second-order and proved duality theorems under cone-second-order pseudoinvexity assumptions. Recently, Dubey *et al.* [6, 8]-extended the concept of G_f-bonvexity and proved the duality theorems under generalized assumptions.
In this section, we formulate a nondifferentiable multiobjective Mond- Weir type primal-dual model over arbitrary cones:

(MPP) Minimize $U(x,y,p) = \Big(U_1(x,y,p_1), U_2(x,y,p_2), U_3(x,y,p_3), \ldots, U_k(x,y,p_k)\Big)^T$

Subject to

$$-\sum_{i=1}^{k} \lambda_i \big[G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y) - r_i + \big\{G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y) \times (\nabla_y f_i(x,y))^T + G'_{f_i}(f_i(x,y))\nabla_{yy} f_i(x,y)\big\} p_i\big] \in C_2^*, \tag{3.1}$$

$$y^T\Bigg(\sum_{i=1}^{k} \lambda_i \big[[G'_{f_i}(f_i(x,y))\nabla_y f_i(x,y) - r_i + \big\{G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y) \times (\nabla_y f_i(x,y))^T + G'_{f_i}(f_i(x,y))\nabla_{yy} f_i(x,y)\big\} p_i\big]\Bigg) \geq 0, \tag{3.2}$$

$$\lambda_i > 0,\ r_i \in F_i,\ x \in C_1,\ i = 1,2,\ldots,k. \tag{3.3}$$

(MDP) Maximize $V(u,v,q) = \Big(V_1(u,v,q_1), V_2(u,v,q_2), V_3(u,v,q_3), \ldots, V_k(u,v,q_k)\Big)^T$

Subject to

$$\sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + t_i + \{G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v) \times (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v)\}q_i] \in C_1^*, \tag{3.4}$$

$$u^T\Bigg(\sum_{i=1}^{k} \lambda_i [G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + t_i + \{G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v)$$

$$\times (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v)\}q_i]\Big) \leq 0, \tag{3.5}$$

$$\lambda_i > 0,\ t_i \in B_i,\ v \in C_2,\ i = 1, 2, \ldots, k, \tag{3.6}$$

where, for $i = 1, 2, \ldots, k$,

$$U_i(x,y,r,p_i) = G_{f_i}(f_i(x,y)) + S(x|B_i) - y^T r_i - \frac{1}{2} p_i^T [G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y) \\ \times (\nabla_y f_i(x,y))^T + G'_{f_i}(f_i(x,y))\nabla_{yy} f_i(x,y)]p_i$$

and

$$V_i(x,y,t,q_i) = G_{f_i}(f_i(u,v)) - S(v|F_i) + u^T t_i - \frac{1}{2} q_i^T [G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v) \\ \times (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v)]q_i,$$

where $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$, C_1 and C_2 are arbitrary cones in R^n and R^m, respectively such that $C_1 \times C_2 \subseteq S_1 \times S_2$, $f_i : S_1 \times S_2 \to R$ is differentiable function,$G_{f_i} : I_{f_i} \to R$ is differentiable strictly increasing function on its domain, C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively.
Next, we prove weak, strong and converse duality theorems for (MFP) and (MFD), respectively. Let $r = (r_1, r_2, \ldots, r_k)$ and $t = (t_1, t_2, \ldots, t_k)$.

Theorem 3.1. *(Weak duality theorem). Let (x, y, r, λ, p) and (u, v, t, λ, q) be feasible solution of (MPP) and (MDP), respectively. Let*

(i) *$f(., v)$ be G_f-bonvex and $(.)^T t$ be invex at u with respect to η_1,*
(ii) *$f(x, .)$ be G_{f_i}-boncave and $(.)^T r$ be invex at y with respect to η_2,*
(iii) *$\eta_1(x, u) + u \in C_1$ and $\eta_2(v, y) + y \in C_2$.*

Then, the following cannot hold:

$$U_i(x,y,r,p) \leq V_i(u,v,t,q),\ \textit{for all } i = 1, 2, \ldots, k \tag{3.7}$$

and

$$U_r(x,y,r,p) < V_r(u,v,t,q),\ \textit{for some } r = 1, 2, \ldots, k. \tag{3.8}$$

Proof. Suppose that (3.7) and (3.8) hold, then using $\lambda > 0$, we have

$$\sum_{i=1}^{k} \lambda_i \Big[G_{f_i}(f_i(x,y)) - r_i - \frac{1}{2} p_i^T [G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y)(\nabla_y f_i(x,y))^T + G'_{f_i}(f_i(x,y)) \\ \times \nabla_{yy} f_i(x,y)]p_i - \Big(G_{f_i}(f_i(u,v)) + t_i - \frac{1}{2} q_i^T\ [G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v) \\ \times (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v)]\ q_i \Big) \Big] < 0. \tag{3.9}$$

For the dual constraint (3.4) and hypothesis (iii), we get

$$(\eta_1(x,u) + u)^T \sum_{i=1}^{k} \big[G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + t_i + (G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v) \\ \times (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v))q_i \big] \geq 0.$$

Using the constraint (3.5) in the above inequality, we have

$$\eta_1^T(x,u)\sum_{i=1}^k \lambda_i\left[G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)+t_i+(G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v)\right.$$
$$\left.\times(\nabla_x f_i(u,v))^T+G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v))q_i\right]\geq 0.$$

From hypothesis (i), yield $G_{f_i}(f_i(x,v)) - G_{f_i}(f_i(u,v)) \geq \eta_1^T(x,u)[G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v) + \{G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v)(\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v)\}p_i] - \frac{1}{2}p_i^T[G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v)$

$$(\nabla_x f_i(u,v))^T+G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v)]p_i,\ \text{for all } i=1,2,\ldots,k,$$

and

$$x^Tt_i-u^Tt_i\geq \eta_1^T(x,u)t_i,\ \text{for all } i=1,2,\ldots,k.$$

Further, it follows from $\lambda>0$ that

$$\sum_{i=1}^k\lambda_i\left[G_{f_i}(f_i(x,v))+x^Tt_i-G_{f_i}(f_i(u,v))-u^Tt_i\right]$$
$$\times\eta_1^T(x,u)\sum_{i=1}^k\lambda_i\Big[[G'_{f_i}(f_i(u,v))\nabla_x f_i(u,v)+t_i+\{G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v)$$
$$\times(\nabla_x f_i(u,v))^T+G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v)\}p_i]-\frac{1}{2}p_i^T[G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v)$$
$$\times(\nabla_x f_i(u,v))^T+G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v)]p_i\Big].$$

Using (3.5) and hypothesis (iii) in above inequality becomes

$$\sum_{i=1}^k\lambda_i\Big[G_{f_i}(f_i(x,v))+x^Tt_i-G_{f_i}(f_i(u,v))-u^Tt_i+\frac{1}{2}p_i^T\left[G''_{f_i}(f_i(u,v))\right.$$
$$\left.\times\nabla_x f_i(u,v)(\nabla_x f_i(u,v))^T+G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v)\right]p_i\Big]\geq 0. \tag{3.10}$$

Similarly, by hypotheses (ii)-(iii), the constraints (3.1)–(3.2), we obtain

$$\sum_{i=1}^k\lambda_i\Big[-G_{f_i}(f_i(x,v))+v^Tr_i+G_{f_i}(f_i(x,y))-y^Tr_i-\frac{1}{2}p_i^T\left\{G''_{f_i}(f_i(x,y))\right.$$
$$\left.\times\nabla_y f_i(x,y)(\nabla_y f_i(x,y))^T+G'_{f_i}(f_i(x,y))\nabla_{yy} f_i(x,y)\right\}p_i\Big]\geq 0. \tag{3.11}$$

Adding (3.10) and (3.11), we get

$$\sum_{i=1}^k\lambda_i\Big[G_{f_i}(f_i(x,y))+x^Tt_i-y^Tr_i-\frac{1}{2}p_i^T\left[G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y)(\nabla_y f_i(x,y))^T\right.$$
$$\left.+G'_{f_i}(f_i(x,y))\nabla_{yy} f_i(x,y)\right]p_i\Big]\geq\sum_{i=1}^k\lambda_i\Big[G_{f_i}(f_i(u,v))-v^Tr_i+u^Tt_i$$

$$- \frac{1}{2} q_i^T \left[G''_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v)) \nabla_{xx} f_i(u,v) \right] q_i \Big] \cdot \tag{3.12}$$

Finally, $x^T t_i \leq s(x|B_i)$ and $v^T r_i \leq s(v|F_i),\ 1 \leq i \leq n$, we have

$$\sum_{i=1}^{k} \lambda_i \Big[G_{f_i}(f_i(x,y)) + s(x|B_i) - y^T r_i - \frac{1}{2} p_i^T \left[G''_{f_i}(f_i(x,y)) \nabla_y f_i(x,y) (\nabla_y f_i(x,y))^T \right.$$
$$\left. + G'_{f_i}(f_i(x,y)) \nabla_{yy} f_i(x,y) \right] p_i \Big] \geq \sum_{i=1}^{k} \lambda_i \Big[G_{f_i}(f_i(u,v)) - s(v|F_i) + u^T t_i$$
$$- \frac{1}{2} q_i^T \left[G''_{f_i}(f_i(u,v)) \nabla_x f_i(u,v) (\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v)) \nabla_{xx} f_i(u,v) \right] q_i \Big], \tag{3.13}$$

which contradicts the inequality (3.9). Thus, we get desired conclusion. □

Example 3.1. Let $n = m = 1,\ k = 2$ and $S_1 = S_2 = R$. The functions $f_1,\ f_2 : S_1 \times S_2 \to R$ be defined as:

$$f_1(x,y) = x^5,\ \ f_2(x,y) = -y^6.$$

Suppose $G_{f_1}(t) = t^4,\ G_{f_2}(t) = t^3$ and $B_1 = B_2 = F_1 = F_2 = \{0\}$. Further, let $\eta_1, \eta_2 : S_1 \times S_2 \to R$ be defined as:

$$\eta_1(x,u) = x^2 u^2 + u,\ \ \eta_2(v,y) = v^2 + y^2 + 1.$$

Assuming that $C_1 = C_2 = R_+$, then $C_1^* = C_2^* = R_+$. Obviously, $C_1 \times C_2 \subseteq S_1 \times S_2$. Substituting these expressions in the problems (GMP) and (GMD), we obtain

(ENGMP) Minimize $\chi(x,y,\lambda,p) = \left\{ x^{20},\ -y^{18} + 153 p_2^2 y^{16} \right\}$
subject to
$\lambda_2 y^{16}(y - 51 p_2) \geqq 0,$
$\lambda_2 y^{17}(y - 51 p_2) \leqq 0,$
$\lambda_1,\ \lambda_2 > 0,\ x \geq 0.$

(EGMD) Maximize $\psi(u,v,\lambda,q) = \left(u^{20} - 190 q_1^2 u^{18},\ -v^{18} \right)$
Subject to
$\lambda_1 u^{18}(u + 19 q_1) \geqq 0,$
$\lambda_1 u^{19}(u + 19 q_1) \leqq 0,$
$\lambda_1,\ \lambda_2 > 0,\ v \geq 0.$

First, we will claim that the functions defined above satisfy the hypotheses of the Theorem 3.1.
(i) $f_1(.,v)$ is G_{f_1}-bonvex at $u = 0$ with respect to η_1 since

$$G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(u,v)) - \eta_1^T(x,u) \left[G'_{f_1}(f_1(u,v)) \nabla_x f_1(u,v) + \left\{ G''_{f_1}(f_1(u,v)) \right. \right.$$
$$\left. \left. \times\ \nabla_x f_1(u,v) (\nabla_x f_2(u,v))^T + G'_{f_1}(f_1(u,v)) \nabla_{xx} f_1(u,v) \right\} q_1 \right] + \frac{1}{2} q_1^T \left[G''_{f_1}(f_1(u,v)) \right.$$
$$\left. \times\ \nabla_x f_1(u,v) (\nabla_x f_1(u,v))^T + G'_{f_1}(f_1(u,v)) \nabla_{xx} f_1(u,v) \right] q_1$$
$$= x^{20} - u^{20} - (u^2 x^2 + u^2) \left[20 u^{19} + 380 u^{18} q_1 \right] + 190 q_1^2 u^{18}$$

$$= x^{20} \text{ at } u = 0$$
$$\geqq 0 \ \forall \ q_1.$$

Obviously, $(.)^T w_1 = 0$ is invex at $u = 0$ with respect to η_1.
Next, we prove that $f_2(., v)$ is G_{f_2}-bonvex at $u = 0$ with respect to η_1.

$$\begin{aligned}
&G_{f_2}(f_2(x,v)) - G_{f_2}(f_2(u,v)) - \eta_1^T(x,u)\left[G'_{f_2}(f_2(u,v))\nabla_x f_2(u,v) + \left\{G''_{f_2}(f_2(u,v))\right.\right.\\
&\quad \times\ \nabla_x f_2(u,v)(\nabla_x f_2(u,v))^T + G'_{f_2}(f_2(u,v))\nabla_{xx} f_2(u,v)\left.\right\} q_2\left.\right] + \frac{1}{2} q_2^T\left[G''_{f_2}(f_2(u,v))\right.\\
&\quad \times \nabla_x f_2(u,v)(\nabla_x f_2(u,v))^T + G'_{f_2}(f_2(u,v))\nabla_{xx} f_2(u,v)\left.\right] q_2\\
&= \ - v^{18} + v^{18} - (u^2x^2 + u^2) \times 0\\
&= 0 \text{ at } u = 0 \text{ and } \forall \ q_2.
\end{aligned}$$

Clearly, $(.)^T w_2 = 0$ is invex u with respect to η_1, Hence, hypothesis (i) of the theorem holds.
(ii) $f_1(x, .)$ is G_{f_1}-boncave at $y = 0$ with respect to η_2 since

$$\begin{aligned}
&G_{f_1}(f_1(x,v)) - G_{f_1}(f_1(x,y)) - \eta_2^T(v,y)\left[G'_{f_1}(f_1(x,y))\nabla_y f_1(x,y) + \left\{G''_{f_1}(f_1(x,y))\right.\right.\\
&\quad \times\ \nabla_y f_1(x,y)(\nabla_y f_1(x,y))^T + G'_{f_1}(f_1(x,y))\nabla_{yy} f_1(x,y)\left.\right\} p_1\left.\right] + \frac{1}{2} p_1^T\left[G''_{f_1}(f_1(x,y))\right.\\
&\quad \times \nabla_y f_1(x,y)(\nabla_y f_1(x,y))^T + G'_{f_1}(f_1(x,y))\nabla_{yy} f_1(x,y)\left.\right] p_1\\
&= x^{20} - u^{20} - (v^2 + y^2 + 1) \times 0\\
&= 0 \text{ at } y = 0 \text{ and } \forall \ p_1.
\end{aligned}$$

Now, we show that $f_2(x, .)$ is G_{f_2}-boncave at $y = 0$ with respect to η_2. The expression

$$\begin{aligned}
&G_{f_2}(f_2(x,v)) - G_{f_2}(f_2(x,y)) - \eta_2^T(v,y)\left[G'_{f_2}(f_2(x,y))\nabla_y f_2(x,y) + \left\{G''_{f_2}(f_2(x,y))\right.\right.\\
&\quad \times\ \nabla_y f_2(x,y)(\nabla_y f_2(x,y))^T + G'_{f_2}(f_2(x,y))\nabla_{yy} f_2(x,y)\left.\right\} p_2\left.\right] + \frac{1}{2} p_2^T\left[G''_{f_2}(f_2(x,y))\right.\\
&\quad \times \nabla_y f_2(x,y)(\nabla_y f_2(x,y))^T + G'_{f_2}(f_2(x,y))\nabla_{yy} f_2(x,y)\left.\right] p_2\\
&= \ - v^{18} + y^{18} - (v^2 + y^2 + 1)[-18y^{17} - 306y^{16}p_2] - 153y^{16}p_2^2\\
&= \ - v^{18} \text{ at } y = 0\\
&\leqq 0 \ \forall \ p_2.
\end{aligned}$$

(iii) Since $X \subseteq R_+$ therefore $\eta_1(x,u) + u \geqq 0$ and $\eta_2(v,y) + y \geqq 0$.
Hence, all the hypotheses of Theorem 3.1 are satisfied.

Validation: The point $(x = 2, y = 0, \lambda_1 = \frac{1}{2}, \lambda_2 = 1, p_1 = 1, p_2 = 1)$ is a feasible solution of primal problem and $(u = 0, v = 1, \lambda_1 = \frac{1}{2}, \lambda_2 = 1, q_1 = 3, q_2 = 2)$ is a feasible solution of dual problem. To validate our result, it's sufficient to prove that the inequality (3.9) does not hold. Now, the expression

$$\begin{aligned}
&\sum_{i=1}^{2} \lambda_i\big[G_{f_i}(f_i(x,y)) - \frac{1}{2} p_i^T\left[G''_{f_i}(f_i(x,y))\nabla_y f_i(x,y)(\nabla_y f_i(x,y))^T + G'_{f_i}(f_i(x,y))\nabla_{yy} f_i(x,y)\right] p_i\\
&\qquad - \left(G_{f_i}(f_i(u,v)) - \frac{1}{2} q_i^T\left[G''_{f_i}(f_i(u,v))\nabla_x f_i(u,v)(\nabla_x f_i(u,v))^T + G'_{f_i}(f_i(u,v))\nabla_{xx} f_i(u,v)\right] q_i\right)\big]\\
&= \lambda_1\left[x^{20} - u^{20} - 190q_1^2u^{18}\right] + \lambda_2\left[-y^{18} + 108p_2^2y^{16} + v^{18}\right]\\
&= 2^{20} + 1 \not< 0 \text{ (at the above mentioned feasible points).}
\end{aligned}$$

Hence, the Theorem 3.1 has been verified.

Theorem 3.2. *(Weak duality theorem). Let (x, y, λ, r, p) and (u, v, λ, t, q) be feasible solution of (MPP) and (MDP), respectively. Let*

(i) *$f(., v)$ be G_f-pseudobonvex and $(.)^T t$ be pseudoinvex at u with respect to η_1,*
(ii) *$f(x, .)$ be G_{f_i}-pseudoboncave and $(.)^T r$ be pseudoinvex at y with respect to η_2,*
(iii) *$\eta_1(x, u) + u \in C_1$ and $\eta_2(v, y) + y \in C_2$.*

Then, the following cannot hold:

$$U_i(x, y, r, p) \leq V_i(u, v, t, q), \text{ for all } i = 1, 2, \ldots, k$$

and

$$U_r(x, y, r, p) < V_r(u, v, t, q), \text{ for some } r = 1, 2, \ldots, k.$$

Proof. The proof follows on the lines of Theorem 3.1. □

Theorem 3.3. *(Strong duality theorem). Let $(\bar{x}, \bar{y}, \bar{r}, \bar{\lambda}, \bar{p})$ be an efficient solution of (MPP); fix $\lambda = \bar{\lambda}$ in (MDP) and suppose that*

(i) *either $\left\{G''_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})(\nabla_y f_i(\bar{x}, \bar{y}))^T + G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_{yy} f_i(\bar{x}, \bar{y})\right\}$ is positive definite, for all $i = 1, 2, ..., k$ and $\sum_{i=1}^{k} \bar{\lambda}_i \bar{p}_i^T \left[G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y}) - \bar{r}_i\right] \geq 0$ or $\left\{G''_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})(\nabla_y f_i(\bar{x}, \bar{y}))^T + G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_{yy} f_i(\bar{x}, \bar{y})\right\}$ is negative definite, for all $i = 1, 2, ..., k$ and $\sum_{i=1}^{k} \bar{\lambda}_i \bar{p}_i^T \left[G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y}) - \bar{r}_i\right] \leq 0$,*
(ii) *$\left\{G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y}) - \bar{r}_i + \left[G''_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})(\nabla_y f_i(\bar{x}, \bar{y}))^T + G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_{yy} f_i(\bar{x}, \bar{y})\right] \bar{p}_i\right\}_{i=1}^{k}$ is linearly independent.*

Then, there exists $\bar{t}_i \in B_i, i = 1, 2, \ldots, k$ such that $(\bar{x}, \bar{y}, \bar{t}, \bar{\lambda}, \bar{q}_1 = \bar{q}_2 = \ldots = \bar{q}_k = 0)$ is feasible solution of (MDP) and the objective values of (MPP) and (MDP) are identical. Moreover, if the hypotheses of Theorem 3.1 or 3.2 are satisfied for all feasible solutions of (MPP) and (MDP), then $(\bar{x}, \bar{y}, \bar{t}, \bar{\lambda}, \bar{q}_1 = \bar{q}_2 = \ldots = \bar{q}_k = 0)$ is an efficient solution of (MDP).

Proof. Since $(\bar{x}, \bar{y}, \bar{r}, \bar{\lambda}, \bar{p})$ is an efficient solution of (MPP), by the Fritz John necessary conditions [20], then there exist $\alpha \in R_+, \beta \in R^m_+,\ \gamma \in R_+,\ \delta \in R^k_+$ and $\bar{t}_i \in R^n,\ i = 1, 2, \ldots, k$ such that

$$\begin{aligned}(x - \bar{x})^T \sum_{i=1}^{k} \alpha_i \Bigg[& G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x f_i(\bar{x}, \bar{y}) + \bar{t}_i - \frac{1}{2}\bar{p}_i^T \nabla_x \left[\{G''_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})\right. \\ & \times\ \left.(\nabla_y f_i(\bar{x}, \bar{y}))^T + G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_{yy} f_i(\bar{x}, \bar{y})\}\right] \bar{p}_i\Bigg] + (\beta - \gamma\bar{y})^T \sum_{i=1}^{k} \bar{\lambda}_i \Bigg[G''_{f_i}(f_i(\bar{x}, \bar{y})) \\ & \times \nabla_x f_i(\bar{x}, \bar{y})\nabla_y f_i(\bar{x}, \bar{y}) + G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_{xy} f_i(\bar{x}, \bar{y}) + \nabla_x \left[\{G''_{f_i}(f_i(\bar{x}, \bar{y}))\right. \\ & \left.\times \nabla_y f_i(\bar{x}, \bar{y})(\nabla_y f_i(\bar{x}, \bar{y}))^T + G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_{yy} f_i(\bar{x}, \bar{y})\} \bar{p}_i\right]\Bigg] \geq 0,\ \forall x \in C_1,\end{aligned} \tag{3.14}$$

$$\sum_{i=1}^{k} \alpha_i \Bigg[G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y}) - \bar{r}_i - \frac{1}{2}\bar{p}_i^T \nabla_y \left[\{G''_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y})(\nabla_y f_i(\bar{x}, \bar{y}))^T\right.$$

$$
\begin{aligned}
&+ G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\}\big]\,\bar{p}_i\Big] + \sum_{i=1}^{k}\bar{\lambda}_i\Big[G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T \\
&+ G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y}) + \nabla_y\left[\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T\right. \\
&\left. + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\}\,\bar{p}_i\right]\Big](\beta - \gamma\bar{y}) - \gamma\sum_{i=1}^{k}\bar{\lambda}_i\big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) \\
&+ \{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\}\,\bar{p}_i\big] = 0, \qquad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
(\beta - \gamma\bar{y})^T\big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{r}_i + \{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T \\
+ G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\}\,\bar{p}_i\big] - \delta_i = 0, \quad i = 1,2,\ldots,k, \qquad (3.16)
\end{aligned}
$$

$$
\begin{aligned}
\big[(\beta - \gamma\bar{y})\bar{\lambda}_i - \alpha_i\bar{p}_i\big]^T\left[G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T\right. \\
\left. + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\right] = 0, \quad i = 1,2,\ldots,k, \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
\beta^T\sum_{i=1}^{k}\bar{\lambda}_i\left[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{r}_i + \{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})\right. \\
\left. \times\ (\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\}\,\bar{p}_i\right] = 0, \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
\gamma\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i\left[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) - \bar{r}_i + \{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})\right. \\
\left. \times\ (\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\}\,\bar{p}_i\right] = 0, \qquad (3.19)
\end{aligned}
$$

$$\delta^T\bar{\lambda} = 0, \qquad (3.20)$$

$$\bar{t}_i \in B_i,\ \bar{r}_i \in F_i,\ \bar{x}^T\bar{t}_i = s(\bar{x}/\bar{B}_i),\ i = 1,2,\ldots,k, \qquad (3.21)$$

$$\alpha_i\bar{y} + \bar{\lambda}_i(\beta - \gamma\bar{y})\bar{\lambda}_i \in N_{F_{(\bar{r}_i)}},\ i = 1,2,\ldots,k, \qquad (3.22)$$

$$(\alpha,\beta,\gamma,\delta) \neq (0,0,0,0), \quad (\alpha,\beta,\gamma,\delta) \geq (0,0,0,0). \qquad (3.23)$$

As $\bar{\lambda} > 0$, then from (3.20), that $\delta = 0$. Therefore from (3.16), we obtain

$$
\begin{aligned}
(\beta - \gamma\bar{y})^T\big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) + \bar{r}_i + \{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T \\
+ G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy}f_i(\bar{x},\bar{y})\}\,\bar{p}_i\big] = 0,\ i = 1,2,\ldots,k. \qquad (3.24)
\end{aligned}
$$

By hypothesis (i), it follows from (3.17) that

$$(\beta - \gamma\bar{y})\bar{\lambda}_i = \alpha_i\bar{p}_i,\ \forall\ i = 1,2,\ldots,k. \qquad (3.25)$$

Now, we claim that $\alpha_i \neq 0,\ \forall\ i = 1,2,\ldots,k$. Indeed, if for some $k_0,\ \alpha_{k_0} = 0$, then it follows from $\bar{\lambda}_{k_0} > 0$ and (3.25) that

$$\beta = \gamma\bar{y}. \qquad (3.26)$$

From (3.15), we get

$$\sum_{i=1}^{k}(\alpha_i-\gamma\bar{\lambda}_i)\big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})-\bar{r}_i\big]+\sum_{i=1}^{k}\bar{\lambda}_i\Big[\big\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})\big\}\,\bar{p}_i\Big](\beta-\gamma\bar{y}-\gamma\bar{p}_i)+\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y\Big[\big\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y}) \times(\nabla_y f_i(\bar{x},\bar{y}))^T+G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})\big\}\,\bar{p}_i\Big]\Big[(\beta-\gamma\bar{y})\bar{\lambda}_i-\frac{1}{2}\alpha_i\bar{p}_i\Big]=0.$$

Using (3.25), it follows that

$$\sum_{i=1}^{k}(\alpha_i-\gamma\bar{\lambda}_i)\big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})-\bar{r}_i+\big\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})\big\}\,\bar{p}_i\big]+\frac{1}{2}\sum_{i=1}^{k}\bar{\lambda}_i\nabla_y\Big[\big\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})\big\}\,\bar{p}_i\Big](\beta-\gamma\bar{y})=0.$$

The above equation together with (3.26) yields

$$\sum_{i=1}^{k}(\alpha_i-\gamma\bar{\lambda}_i)\big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})-\bar{r}_i+\big\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})\big\}\,\bar{p}_i\big]=0,$$

which from hypothesis (ii) yields

$$\alpha_i=\gamma\bar{\lambda}_i,\ \forall\ i=1,2,\ldots,k. \tag{3.27}$$

From $\bar{\lambda}_i>0,\ i=1,2,\ldots,k$ and $\alpha_{k_0}=0$, for some k_0, it follows that $\gamma=0$. Now from (3.26), (3.27) and $\gamma=0$, we have $\beta=0,\ \alpha_i=0,\ i=1,2,\ldots,k$, which contradicts (3.23). Therefore, $\alpha_i,\ i\in K$. Pre-multiplying by $\bar{\lambda}_i$ in (3.24), using (3.25) and noting $\alpha_i>0,\ i=1,2,\ldots,k$, we get

$$\bar{p}_i^T\big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})-\bar{r}_i+\big\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})\big\}\,\bar{p}_i\big]=0.$$

that is

$$\sum_{i=1}^{k}\bar{p}_i^T\big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})-\bar{r}_i+\big\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})\big\}\,\bar{p}_i\big]=0. \tag{3.28}$$

We now claim that $\bar{p}_i=0,\ \forall\ i=1,2,\ldots,k$. From hypotheses (i), we have

$$\sum_{i=1}^{k}\bar{p}_i^T\big[G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})-\bar{r}_i+\big\{G''_{f_i}(f_i(\bar{x},\bar{y}))\nabla_y f_i(\bar{x},\bar{y})(\nabla_y f_i(\bar{x},\bar{y}))^T + G'_{f_i}(f_i(\bar{x},\bar{y}))\nabla_{yy} f_i(\bar{x},\bar{y})\big\}\,\bar{p}_i\big]\neq 0, \tag{3.29}$$

which contradicts (3.28). Hence, $\bar{p}_i = 0,\ \forall\ i = 1, 2, \ldots, k$. It follows that from $\bar{\lambda}_i > 0,\ \bar{p}_i = 0$, for $i = 1, 2, \ldots, k$ and (3.25) that

$$\beta = \gamma\bar{y}. \tag{3.30}$$

Using (3.30) and $\bar{p}_i = 0,\ i = 1, 2, \ldots, k$ in (3.15), it follows that

$$\sum_{i=1}^{k}(\alpha_i - \gamma\bar{\lambda}_i)\big[G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_y f_i(\bar{x}, \bar{y}) - \bar{r}_i\big] = 0.$$

By condition (iii), we get

$$\alpha_i = \gamma\bar{\lambda}_i,\ i = 1, 2, \ldots, k. \tag{3.31}$$

Therefore, $\gamma > 0$.
Using (3.30), (3.31) and $\gamma > 0$ in (3.14), we have

$$(x - \bar{x})^T\left[\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x f_i(\bar{x}, \bar{y})) + \bar{t}_i\right] \geq 0,\ \forall x \in C_1. \tag{3.32}$$

Let $x \in C_1$. Then $x + \bar{x} \in C_1$ as C_1 is a closed convex cone. On substituting $x + \bar{x}$ in place of x in (3.32), we get

$$x^T\left[\sum_{i=1}^{k}\bar{\lambda}_i(G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x)f_i(\bar{x}, \bar{y}) + \bar{t}_i\right] \geq 0.$$

Hence,

$$\sum_{i=1}^{k}\bar{\lambda}_i\left(G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x)f_i(\bar{x}, \bar{y}) + \bar{t}_i\right) \in C_1^*. \tag{3.33}$$

Also, by letting $x = 0$ and $x = 2\bar{x}$ simultaneously in (3.32), we have

$$\bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i[(G'_{f_i}(f_i(\bar{x}, \bar{y}))\nabla_x)f_i(\bar{x}, \bar{y}) + \bar{t}_i] = 0. \tag{3.34}$$

Since $\beta = \gamma\bar{y}$ and $\gamma > 0$, we get

$$\bar{y} = \frac{\beta}{\gamma} \in C_2. \tag{3.35}$$

Next, $\alpha > 0$, by (3.22) and the fact that $\beta = \gamma\bar{y}$, we get $\bar{y} \in N_{F_{(\bar{r}_i)}},\ i = 1, 2, 3, \ldots, k$. This gives

$$\bar{y}^T\bar{r}_i = s(\bar{y}/F_i),\ i = 1, 2, 3, \ldots, k. \tag{3.36}$$

Hence, we find that $(\bar{x}, \bar{y}, \bar{t}, \bar{\lambda}, \bar{q}_1 = \bar{q}_2 =, \ldots, = \bar{q}_k = 0)$ satisfies (3.4) and (3.5) which is feasible solution for (MDP). Using, (3.30), (3.36), $\bar{p}_i = 0,\ i = 1, 2, \ldots, n$ and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{t}, \bar{q}_1 = \bar{q}_2 =, \ldots, = \bar{q}_k = 0)$, we get

$$U(\bar{x}, \bar{y}, \bar{r}, \bar{p}) = V(\bar{x}, \bar{y}, \bar{t}, \bar{q}). \tag{3.37}$$

Hence, we get the desired result. □

Theorem 3.4. *(Converse duality). Let $(\bar{u}, \bar{v}, \bar{t}, \bar{\lambda}, \bar{q})$ be an efficient solution of (MDP); fix $\lambda = \bar{\lambda}$ in (MPP) and* *suppose that*

(i) *either* $\left\{G''_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v})(\nabla_x f_i(\bar{u},\bar{v}))^T + G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_{xx} f_i(\bar{u},\bar{v})\right\}$ *is positive definite, for all* $i = 1,2,...,k$ *and* $\sum_{i=1}^{k}\bar{\lambda}_i\bar{q}_i^T\left[G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v}) + \bar{t}_i\right] \geq 0$ *or* $\left\{G''_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v})(\nabla_x f_i(\bar{u},\bar{v}))^T + G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_{xx} f_i(\bar{u},\bar{v})\right\}$ *is negative definite, for all* $i = 1,2,...,k$ *and* $\sum_{i=1}^{k}\bar{\lambda}_i\bar{q}_i^T\left[G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v}) + \bar{t}_i\right] \leq 0$,

(ii) *the set* $\left\{G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v}) + [G''_{f_i}(f_i(\bar{u},\bar{v}))\nabla_x f_i(\bar{u},\bar{v})(\nabla_x f_i(\bar{u},\bar{v}))^T + G'_{f_i}(f_i(\bar{u},\bar{v}))\nabla_{xx} f_i(\bar{u},\bar{v})]\bar{q}_i\right\}_{i=1}^{k}$ *is linearly independent.*

Then, $(\bar{u},\bar{v},\bar{\lambda},\bar{r},\bar{p}_1 = \bar{p}_2 =,\ldots,= \bar{p}_k = 0)$ is feasible solution for (MPP) and the objective values of (MDP) and (MPP) are equal. Moreover, Weak duality Theorem 3.1 or 3.2 hold, then $(\bar{u},\bar{v},\bar{r},\bar{\lambda},\bar{p}_1 = \bar{p}_2 =,\ldots,= \bar{p}_k = 0)$ is an efficient solution of (MPP).

Proof. It follows on the lines of Theorem 3.3. □

4. Concluding remarks

In this paper, we have formulate a second-order symmetric nondifferentiable Mond-Weir type dual for a nonlinear multiobjective optimization problem. Number of duality relations are further established under G_f-bonvexity/G_f-pseudobonvexity assumptions on the function f. We have discussed various numerical examples to show the existence of G_f-bonvex/G_f-pseudobonvex functions. Also, we have justified weak duality theorem by a suitable example. The question arises as to whether the duality results developed in this paper hold for Mond-Weir type higher-order nondifferentiable multiobjective optimization problems. This may be the future direction for the researchers working in this area.

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