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THE DISASTERS QUEUE WITH WORKING BREAKDOWNS AND IMPATIENT CUSTOMERS *

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Abstract. We consider the $M/M/1$ queue with disasters and impatient customers. Disasters only occur when the main server being busy, it not only removes out all present customers from the system, but also breaks the main server down. When the main server is down, it is sent for repair. The substitute server serves the customers at a slow rate(working breakdown service) until the main server is repaired. The customers become impatient due to the working breakdown. The system size distribution is derived. We also obtain the mean queue length of the model and mean sojourn time of a tagged customer. Finally, some performance measures and numerical examples are presented.

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1. Introduction

Since the concept of negative customers introduced by Gelenbe [5], queue with negative customers (also called G-queues) have attracted considerable interests due to some practical applications, such as computer, communication networks and manufacturing systems. When a negative customer arrives at the system, it immediately removes the positive customer in service if present. About detailed studies on negative customers, readers may referred to Artalejo [2], Artalejo and Economou [3], Li and Zhao [12], Wang *et al.* [17], and reference therein. If a negative customer removes all the positive customers in the queue at once, then it is called a disaster. The queue with disasters are characterized by the phenomenon in which the occurrence of disasters not only destroy all unfinished jobs but also break down the processor. Towsley and Tripathi [16] first studied the $M/M/1$ queue with disasters in order to describe the behavior of distributed database systems with site failure. Then $M/M/1$ queue with disasters was extended to the $M/G/1$ queue by Jain and Sigman [6] and to the $GI/M/1$ queue by Yang and Chae [18]. Atencia and Moreno [4] studied the queue with negative customers and disasters. Lee *et al.* [11] discussed the discrete-time queues with disasters and general repair times. Recently, Jiang *et al.* [7] discussed $M/G/1$ queue in multi-phase random environment with disasters.

In all the models considered so far of queue systems with server breakdown, the underlying assumption has been that a server breakdown disrupts the service completely in the system. Such a system with repair has been

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studied as a reliability model by many authors. However, in practical situations, the breakdown of a server may not stop the service of a customer completely. For example, the presence of virus in the system may slow down the performance of the computers system. Another example is provided by the machine replace problem. When the machine (main server) break down, it is immediately replaced by another machine (substitute server). The substitute machine works at a slower rate until the main machine is repaired. This is the concept of working breakdowns first introduced by Kalidass and Kasturi [8]. Kim and Lee [9] discussed the $M/G/1$ queueing system with disasters and working breakdown services, and obtained the system size distribution and the sojourn time distribution.

Queue models with customers' impatience have drawn significant attention in the past, where the source of impatience was too long wait already experienced in the queue. Altman and Yechiali [1] studied the queue where customers become impatient due to the absence of server, more precisely, due to the server vacation. Yechiali [19] considered the $M/M/1$ queue with disasters where customers are impatient since no server is available. Sudhesh [15] discussed the transient probability of $M/M/1$ queue with disasters and customer impatience successively.

Recently, Perel and Yechiali [13] studied the queue where the customer's impatience is due to a slow service. They have studied M/M/c queues ($c = 1, 1 < c < \infty$, and $c = \infty$) in a 2-phase (fast and slow) Markovian random environment, with impatient customers. Yue *et al.* [20] have analyzed customer's impatience in working vacation queue where customers impatient is due to working vacation. Selvaraju and Goswami [14] discussed an $M/M/1$ impatient customer queue with single and multiple working vacations. Further, Laxmi and Jyothsna [10] studied the queue with impatient customers which incorporated the features of customers balking and Bernoulli schedule vacation interruption.

In this paper, we study the $M/M/1$ queue with disasters and impatient customers, where the customer's impatient is due to working breakdowns.

The paper organizes as follows. In the next section, some basic assumption and the model are described. In Section 3, we analyze the steady-state distribution of the model. Some performance measures and the mean sojourn time of a tagged customer are obtained in Sections 4 and 5, respectively. In Section 6, we give some numerical results. The conclusion and some suggestions for future research are given in Section 7.

2. Description of the model

In this paper, we consider a queue system with the following features. Customers arrive at a single server system, according to Poisson process at a rate λ. The customers are served according to FCFS discipline and the service times, denoted by S_1, rendered by the main server are assumed to be exponentially distributed with parameter μ_1. Disasters only occur when the main server being busy, they not only remove out all present customers from the system, but also break the main server down. The inter-arrival times between the successive disasters, denoted by D, are exponentially distributed with parameter δ. As soon as the main server down, the server is sent for repair immediately. The repair times have an exponential distribution with a parameter γ. The server is as good as new after repair.

The substitute server renders service to customers while the main server is repaired. The service rendered by the substitute server is defined as the working breakdown service. The working breakdown service times are also assumed to be exponentially distributed with a parameter $\mu_0(\leq \mu_1)$. During a repaired period, the stream of new customers arrival continuously. If the main server returns from its repair and finds there are customers in the system, the substitute server stops service and the main server restarts and operates at its service rate μ_1. Meanwhile, if there are no customers in the system at the end of the repair, the main server returns to system, stays idle, and waits for arriving customers.

The customers are assumed to be impatient during the period of working breakdown. That is, whenever a customer arrives to system and realizes that the system is in working breakdown, each customer activates an "impatient timer" T independently, which is exponential with rate η. If the substitute server is available during working breakdown before the time T expires, the customer is served with rate μ_0. If the working breakdown finishes (i.e. the main server returns from his repair) before the time T expires, the main server restarts and the

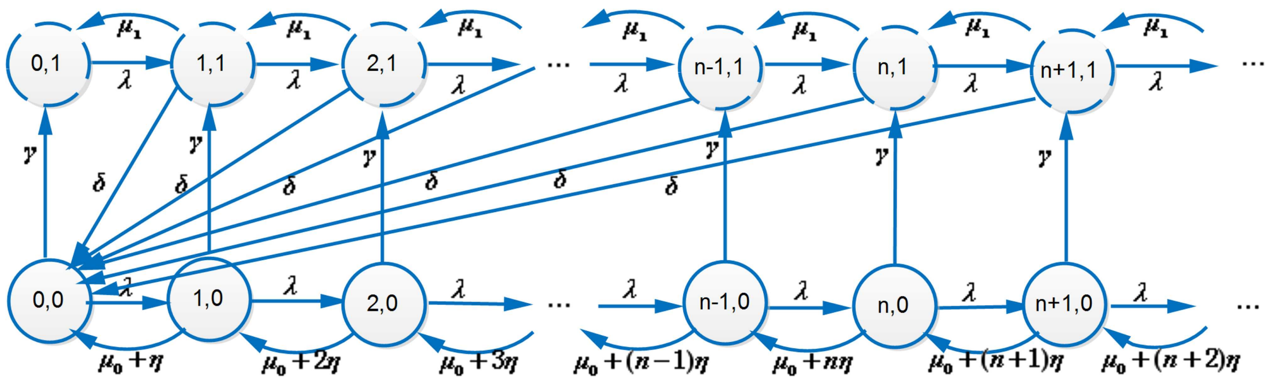

FIGURE 1. State transition rate diagram.

customer is served with rate μ_1. However, if T expires while the main server is still under repair, the customer abandons the system and never returns.

3. Analysis of the model

In this section, we will carry out the analysis of $M/M/1$ queue with disasters and inpatient customers.

3.1. Balance equations

Let $N(t)$ be the number of customers in the system and $J(t)$ be the state of the server with

$$J(t) = \begin{cases} 0, & \text{the main server is under repair at } t, \\ 1, & \text{the main server is available at } t. \end{cases}$$

Then, the two-dimensional continuous-time discrete-state process $\{(N(t), J(t)), t \geq 0\}$ becomes a Markov chain with state space $E = \{(i,j), i \geq 0, j = 0, 1\}$. Figure 1 provides with the state transition rate diagram of the system.

Let $p_{i,j} = \lim_{t\to\infty} P(N(t) = i, J(t) = j)$ denote the stationary probabilities of Markov process $\{(N(t), J(t)), t \geq 0\}$. Then, the balance equations are

$$(\lambda + \gamma)p_{0,0} = (\mu_0 + \eta)p_{1,0} + \delta \sum_{n=1}^{\infty} p_{n,1}, \tag{1}$$

$$(\lambda + \gamma + \mu_0 + n\eta)p_{n,0} = [\mu_0 + (n+1)\eta]p_{n+1,0} + \lambda p_{n-1,0}, \quad n \geq 1, \tag{2}$$

$$\lambda p_{0,1} = \mu_1 p_{1,1} + \gamma p_{0,0}, \tag{3}$$

$$(\lambda + \delta + \mu_1)p_{n,1} = \mu_1 p_{n+1,1} + \lambda p_{n-1,1} + \gamma p_{n,0}, \quad n \geq 1. \tag{4}$$

Define the Probability Generating Functions

$$P_0(z) = \sum_{n=0}^{\infty} z^n p_{n,0}, \qquad P_1(z) = \sum_{n=1}^{\infty} z^n p_{n,1},$$

with $P_0(1) + p_{0,1} + P_1(1) = 1$ and $P_0'(z) = \sum_{n=1}^{\infty} n z^{n-1} p_{n,0}$.

Multiplying equations (1), and (2) by $1, z^n$, respectively, summing over n and rearranging the terms, we get

$$\eta z(1-z)P_0^{'}(z) - [\lambda z(1-z) - \mu_0(1-z) + \gamma z]P_0(z) = \mu_0(1-z)p_{0,0} - \delta z P_1(1). \tag{5}$$

If $z \neq 0$ and $z \neq 1$, (5) can be written as

$$P_0^{'}(z) = \frac{[\lambda z(1-z) - \mu_0(1-z) + \gamma z]P_0(z) + \mu_0(1-z)p_{0,0} - \delta z P_1(1)}{\eta z(1-z)}. \tag{6}$$

In a similar manner we obtain from (3)–(4)

$$[\lambda z(1-z) - \mu_1(1-z) + \delta z]P_1(z) = -\lambda z(1-z)p_{0,1} + \gamma z P_0(z). \tag{7}$$

We get

$$P_1(z) = \frac{-\lambda z(1-z)p_{0,1} + \gamma z P_0(z)}{\lambda z(1-z) - \mu_1(1-z) + \delta z}. \tag{8}$$

Setting $z = 1$ in (8), we obtain

$$\delta P_1(1) = \gamma P_0(1). \tag{9}$$

Remark 3.1. If $\eta = 0$, the current model reduce to $M/M/1$ queue with disasters and working breakdowns, which is a special case of Kim and Lee [9].

3.2. Solution of the differential equation

Multiplying both sides of (6) by $e^{\int[\frac{-\lambda}{\eta} + \frac{\mu_0}{\eta z} - \frac{\gamma}{\eta(1-z)}]dz} = Ce^{-\frac{\lambda}{\eta}z}(1-z)^{\frac{\gamma}{\eta}}z^{\frac{\mu_0}{\eta}}$, where C is a constant, we give the following

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[e^{-\frac{\lambda}{\eta}z}(1-z)^{\frac{\gamma}{\eta}}z^{\frac{\mu_0}{\eta}}P_0(z)\right] = \frac{\mu_0(1-z)p_{0,0} - \delta z P_1(1)}{\eta z(1-z)}e^{-\frac{\lambda}{\eta}z}(1-z)^{\frac{\gamma}{\eta}}z^{\frac{\mu_0}{\eta}}. \tag{10}$$

Integrating both sides of (10) from 0 to z and rearranging terms gives

$$P_0(z) = (1-z)^{-\frac{\gamma}{\eta}}z^{-\frac{\mu_0}{\eta}}\int_0^z \frac{\mu_0(1-t)p_{0,0} - \delta t P_1(1)}{\eta t(1-t)}e^{\frac{\lambda}{\eta}(z-t)}(1-t)^{\frac{\gamma}{\eta}}t^{\frac{\mu_0}{\eta}}\,\mathrm{d}t. \tag{11}$$

Set

$$U = \int_0^1 e^{\frac{\lambda}{\eta}(1-t)}(1-t)^{\frac{\gamma}{\eta}}t^{\frac{\mu_0}{\eta}-1}\mathrm{d}t,$$
$$V = \int_0^1 e^{\frac{\lambda}{\eta}(1-t)}(1-t)^{\frac{\gamma}{\eta}-1}t^{\frac{\mu_0}{\eta}}\,\mathrm{d}t.$$

Taking limit as $z \to 1$ in (11), we get

$$P_0(1) = \left[\frac{\mu_0}{\eta}p_{0,0}\,U - \frac{\delta}{\eta}P_1(1)V\right]\lim_{z\to1}(1-z)^{-\frac{\gamma}{\eta}}.$$

Since $P_0(1)=\sum_{n=0}^{\infty} p_{n,0}<1$ and $\lim_{z\to 1}(1-z)^{-\frac{\gamma}{\eta}}=\infty$, we must have that

$$\frac{\mu_0}{\eta}p_{0,0}U-\frac{\delta}{\eta}P_1(1)V=0. \tag{12}$$

From (12), we get

$$P_1(1)=\frac{\mu_0 U}{\delta V}p_{0,0}. \tag{13}$$

Substituting (13) into (11) we get

$$P_0(z)=\frac{\mu_0}{\eta}p_{0,0}(1-z)^{-\frac{\gamma}{\eta}}z^{-\frac{\mu_0}{\eta}}\int_0^z \frac{V-(U+V)t}{V}e^{\frac{\lambda}{\eta}(z-t)}(1-t)^{\frac{\gamma}{\eta}-1}t^{\frac{\mu_0}{\eta}-1}\mathrm{d}t. \tag{14}$$

Consider the denominator of $P_1(z)$, we define $\beta(z)=\lambda z(1-z)-\mu_1(1-z)+\delta z$. Since

$$\beta(0)=-\mu_1<0,\quad \beta(1)=\delta>0,\quad \beta(+\infty)<0.$$

The two roots of $\beta(z)=0$ lie in $(0,1)$ and $(1,+\infty)$. Hence $\beta(z)=0$ has only one root between 0 and 1. Let the root be z_1. Since $P_1(z)\geq 0$ for $0\leq z\leq 1$, the numerator of $P_1(z)$ must vanish at $z=z_1$. From (8) we have

$$p_{0,1}=\frac{\gamma P_0(z_1)}{\lambda(1-z_1)}. \tag{15}$$

From the normalizing condition $P_0(1)+p_{0,1}+P_1(1)=1$, we have

$$p_{0,0}=\frac{\lambda\eta\gamma\delta V}{\mu_0\big[\lambda\eta(\delta+\gamma)U+\delta\gamma^2 V k(z_1)\big]}, \tag{16}$$

where

$$k(z_1)=(1-z_1)^{-\frac{\gamma}{\eta}-1}z_1^{-\frac{\mu_0}{\eta}}\int_0^{z_1}\left(1-\frac{U+V}{V}t\right)e^{\frac{\lambda}{\eta}(z_1-t)}(1-t)^{\frac{\gamma}{\eta}-1}t^{\frac{\mu_0}{\eta}-1}\mathrm{d}t.$$

The probabilities $p_{n,0}(n\geq 1)$ and $p_{n,1}(n\geq 1)$ can be evaluated in terms of $p_{0,0}$.

$$p_{n+1,0}=\frac{\lambda+\gamma+\mu_0+n\eta}{\mu_0+(n+1)\eta}p_{n,0}-\frac{\lambda}{\mu_0+(n+1)\eta}p_{n-1,0}, n\geq 2,$$
$$p_{1,0}=\frac{(\lambda+\gamma)V-\mu_0 U}{(\mu_0+\eta)V}p_{0,0}.$$

where $p_{0,0}$ is given in (16).

$$p_{n+1,1}=\frac{\lambda+\delta+\mu_1}{\mu_1}p_{n,1}-\frac{\lambda}{\mu_1}p_{n-1,0}-\frac{\gamma}{\mu_1}p_{n,0}, n\geq 1,$$
$$p_{1,1}=\frac{\gamma\mu_0 k(z_1)-\gamma\mu_1\eta}{\mu_1\eta}p_{0,0},$$

and $p_{0,1}$ is given in (15).

Remark 3.2. In our study, the inequality $\delta>0$ is necessary and sufficient condition for the system to be stable. This result can be seen in Kim and Lee [9]. In fact, all customers in the system are flushed out wherever the disaster arriving, which means that the number of customers at arbitrary epochs does not go to infty.

4. Performance measures

In this section, we give expressions for some useful performance measures of the system. The probability that the server is under repair and available, denote by P_0, P_1, respectively, is given by

$$P_0 = P_0(1) = \frac{\mu_0 U}{\gamma V} p_{0,0},$$
$$P_1 = p_{0,1} + P_1(1) = 1 - \frac{\mu_0 U}{\gamma V} p_{0,0}.$$

Let $E(L_0)$, $E(L_1)$ denote the average number of customers in the system when the main server is under repair and available, respectively. Setting $z \to 1$ and using L'Hospital rule in the right of equation (6), the expression for $E(L_0)$ is obtained as

$$E(L_0) = P_0^{'}(1) = \frac{(\lambda - \mu_0)U + \gamma V}{\gamma(\gamma + \eta)V} \mu_0 p_{0,0}.$$

Differentiation of z in both sides of equation (7) and taking $z = 1$ yield $E(L_1)$ as

$$\begin{aligned} E(L_1) &= \frac{\lambda - \mu_1}{\delta} P_1(1) + \frac{\gamma}{\delta} P_0^{'}(1) + \frac{\lambda}{\delta} p_{0,1} \\ &= \frac{\lambda - \mu_1}{\delta^2} \frac{U}{V} \mu_0 p_{0,0} + \frac{\gamma k(z_1)}{\delta \eta} \mu_0 p_{0,0} + \frac{\gamma}{\delta} E(L_0). \end{aligned}$$

Therefore, the average numbers of customer in the system, denoted by $E(L)$, is give by

$$E(L) = E(L_0) + E(L_1) = \left(\frac{\lambda - \mu_1}{\delta^2} \frac{U}{V} + \frac{\gamma k(z_1)}{\delta \eta} + \frac{\delta + \gamma}{\delta} \frac{(\lambda - \mu_0)U + \gamma V}{\gamma(\gamma + \eta)V} \right) \mu_0 p_{0,0}.$$

The average reneging rate is

$$R.R. = \sum_{n=1}^{\infty} n \eta p_{n,0} = \eta E(L_0).$$

5. Sojourn time

In this section, we turn our attention to the sojourn time W of a test customer (TC), which is defined as the overall time since the arrival till the departure from the system, either after completion of service, the occurrence of a disaster or as the result of an abandonment.

Let $W_{n,j}$ denote the conditional sojourn time of the TC, given that upon arrival he observes the state $(n, j), n \geq 0, j = 0, 1$. Let D_1 be the times between the TC arrival and the next disaster arrival. Clearly, it has the same distribution with D. So the conditional mean sojourn time of the TC arriving at the system in state $(0, 1)$ is follow as

$$E(W_{0,1}) = E(\min\{S_1, D_1\}) = \frac{1}{\mu_1 + \delta}. \tag{17}$$

Assume that the TC arrives at the system in state $(n, 1)$. Let X be the time of the unfinished work immediately after the arrival epoch of the TC. Then X is equal to the remaining normal service time plus the sum of n normal service times. Note that the remaining normal service time is stochastically equal to a new normal service due to memoryless property. So, we have

$$E(W_{n,1}) = E(\min\{X, D_1\}) = \frac{(\mu_1 + \delta)^{n+1} - \mu_1^{n+1}}{\delta(\mu_1 + \delta)^{n+1}}. \tag{18}$$

Assume that the TC arrives at the system in state $(n,0)$. By conditioning on the next future event, we find that

$$\begin{aligned}E(W_{n,0}) &= \frac{1}{\lambda+\mu_0+\gamma+(n+1)\eta} + \frac{\lambda}{\lambda+\mu_0+\gamma+(n+1)\eta}E(W_{n,0})\\ &+ \frac{\mu_0}{\lambda+\mu_0+\gamma+(n+1)\eta}E(W_{n-1,0}) + \frac{\gamma}{\lambda+\mu_0+\gamma+(n+1)\eta}E(W_{n,1})\\ &+ \frac{(n+1)\eta}{\lambda+\mu_0+\gamma+(n+1)\eta}\big(\frac{1}{n}\times 0 + \frac{n}{n+1}E(W_{n-1,0})\big),\end{aligned}$$

This expression can be further rewritten as

$$E(W_{n,0}) = \frac{1+(\mu_0+n\eta)E(W_{n-1,0})+\gamma E(W_{n,1})}{\mu_0+\gamma+(n+1)\eta}. \tag{19}$$

We also have

$$E(W_{0,0}) = \frac{1}{\lambda+\gamma+\eta} + \frac{\lambda}{\lambda+\gamma+\eta}E(W_{0,0}) + \frac{\gamma}{\lambda+\gamma+\eta}E(W_{0,1}),$$

implying that

$$E(W_{0,0}) = \frac{\mu_1+\delta+\gamma}{(\mu_1+\delta)(\gamma+\eta)}. \tag{20}$$

Iterating (19) and using (18) and (20), we obtain, for $n \geq 1$

$$\begin{aligned}E(W_{n,0}) &= I_{n+1} + \sum_{k=2}^{n}\prod_{i=k}^{n}\frac{\mu_0+i\eta}{\mu_0+\gamma+(i+1)\eta}I_k + \prod_{i=1}^{n}\frac{\mu_0+i\eta}{\mu_0+\gamma+(i+1)\eta}E(W_{0,0})\\ &+ \frac{1}{\mu_0+\gamma+(n+1)\eta}\left(1+\sum_{k=2}^{n}\prod_{i=k}^{n}\frac{\mu_0+i\eta}{\mu_0+\gamma+i\eta}\right),\end{aligned}$$

where

$$I_k = \frac{\gamma\left[(\delta+\ \mu_1)^k - \mu_1^k\right]}{\delta(\delta+\mu_1)^{k+1}[\mu_0+\gamma+(k+1)\eta]},$$

and with the conventions $\sum_{k=2}^{n} a_k = 0$ for $n=1$.

Finally, we get the mean sojourn time of the TC as

$$E(W) = \sum_{n=0}^{\infty}p_{n,0}E(W_{n,0}) + \sum_{n=0}^{\infty}p_{n,1}E(W_{n,1}).$$

However, more important measure of system performance is S_{served}, defining the total sojourn time of a customer who completes his service. Let $S_{n,j}$ denote the conditional sojourn time of a TC who does not leave system before completing his service, given that the state upon arrival is (n,j). Then, for $j=1$ and $n=0$,

$$E(S_{0,1}) = P(S_1 < D_1)E(S_1|S_1 < D_1) = \frac{\mu_1}{(\mu_1+\delta)^2}.$$

and for $n \geq 1$,

$$E(S_{n,1}) = \frac{\mu_1}{\mu_1+\delta}\left(\frac{1}{\mu_1+\delta} + E(S_{n-1,1})\right). \tag{21}$$

Iterating (21) we get

$$E(S_{n,1}) = \frac{\mu_1\left[(\mu_1+\delta)^{n+1} - \mu_1^{n+1}\right]}{\delta(\mu_1+\delta)^{n+2}}. \tag{22}$$

We now turn to calculate $E(S_{0,n})$ for $n = 0, 1, 2, \ldots$

$$E(S_{0,0}) = \frac{\mu_0}{(\mu_0+\gamma+\eta)^2} + \frac{\gamma}{\mu_0+\gamma+\eta}\left(\frac{1}{\mu_0+\gamma+\eta} + E(S_{0,1})\right). \tag{23}$$

For $n \geq 1$,

$$\begin{aligned} E(S_{n,0}) = {} & \frac{\mu_0}{\mu_0+\gamma+(n+1)\eta}\left(\frac{1}{\mu_0+\gamma+(n+1)\eta} + E(S_{n-1,0})\right) \\ & + \frac{\gamma}{\mu_0+\gamma+(n+1)\eta}\left(\frac{1}{\mu_0+\gamma+(n+1)\eta} + E(S_{n,1})\right) \\ & + \frac{(n+1)\eta}{\mu_0+\gamma+(n+1)\eta}\frac{n}{n+1}\left(\frac{1}{\mu_0+\gamma+(n+1)\eta} + E(S_{n-1,0})\right). \end{aligned} \tag{24}$$

Iterating (24) and using (22) yields

$$E(S_{n,0}) = \alpha_n + \sum_{k=2}^{n}\alpha_{k-1}\prod_{i=k}^{n}\frac{\mu_0+i\eta}{\mu_0+\gamma+(i+1)\eta} + \prod_{i=1}^{n}\frac{\mu_0+i\eta}{\mu_0+\gamma+(i+1)\eta}E(S_{0,0}),$$

where

$$\alpha_k = \frac{\mu_0+k\eta}{(\mu_0+\gamma+(k+1)\eta)^2} + \mu_1 I_k.$$

Finally, the expected sojourn time of the TC that is served may be calculated using the expression

$$E(W) = \sum_{n=0}^{\infty} p_{n,0}E(S_{n,0}) + \sum_{n=0}^{\infty} p_{n,1}E(S_{n,1}).$$

6. Sensitivity analysis

In this section, we present the effect of the model parameters on the system performance measures through some numerical results.

In the first two figures, we have examined the effect of λ on the probabilities $p_{0,0}, p_{0,1}$ for various values of μ_0. From the Figure 2, it is seen that, the three curves displayed in Figure 2(a) up to some special points, $p_{0,0}$ increases. After these points, the value of the $p_{0,0}$ gradually decreases. This because as λ increases, the number of the customers n increases, so does the instantaneous reneging rate $n\eta$ during the working breakdown period. The transition rate during the working breakdown period equal the service rate μ_0 plus the instantaneous reneging rate. If the transition rate is smaller than λ, $p_{0,0}$ increases with the increase of λ, otherwise, it decreases with the increase of λ. It is also seen that for a fixed λ, $p_{0,0}$ is increases with μ_0. This is to expected, since μ_0 is the service rate in the working breakdown period and $p_{0,0}$ is the probability of server is idle in working breakdown period. Figure 2(b) presents that the probability of the server being idle decreases with increase values of the arrival rate λ. For the same values of λ, $p_{0,0}$ is increases with μ_0. However, the working breakdown service rate μ_0 has the little effect on $p_{0,1}$.

From Figure 3, we can observe that the mean numbers of customers in the system $E(L)$ increases monotonously with the increasing of arrive rate λ and decreases with the increasing of μ_0, respectively.

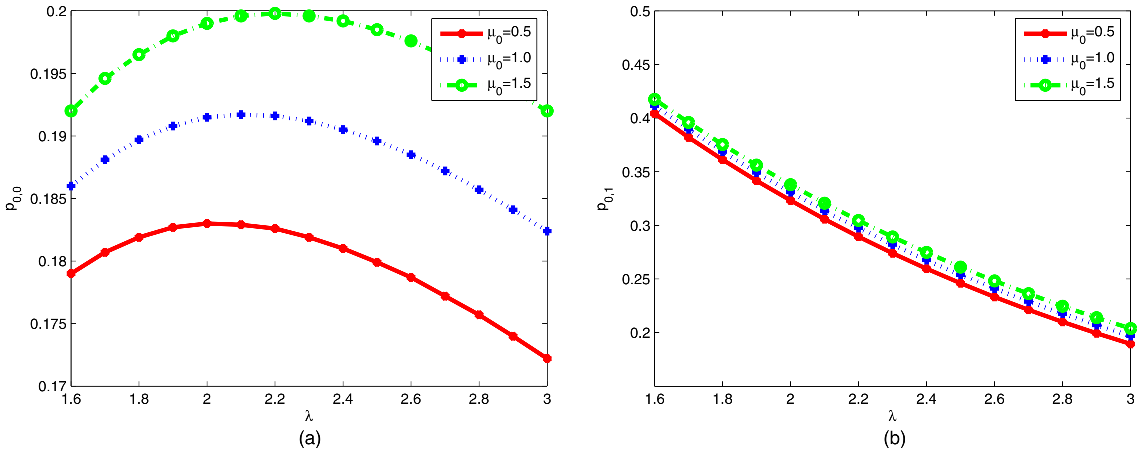

FIGURE 2. λ *vs.* $p_{0,0}, p_{0,1}$ with $\mu_1 = 2, \delta = 1.5, \gamma = 1.6, \eta = 1$. (a) the effect λ on $p_{0,0}$, (b) the effect λ on $p_{0,1}$.

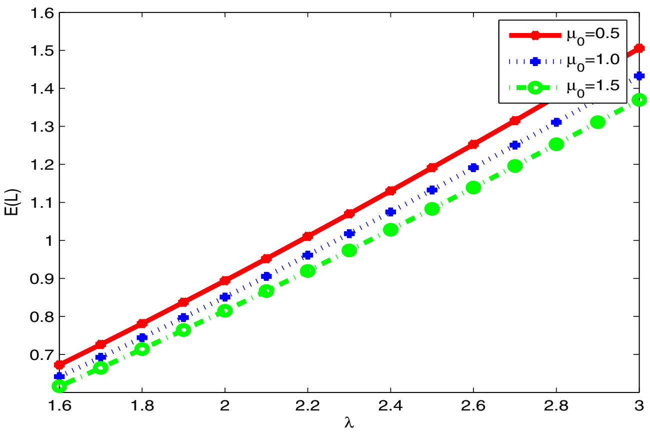

FIGURE 3. λ *vs.* $E(L)$ with $\mu_1 = 2, \delta = 1.5, \gamma = 1.6, \eta = 1$.

Figure 4(a) examines the variations in the mean numbers of customers $E(L)$ for the increasing values of γ. It is seen that $E(L)$ decreases with increasing values of the γ for $\mu_1 = 2.5$ and $\mu_1 = 3$ respectively. But $E(L)$ increases slowly with the increase of γ for $\mu_1 = 2$. And $E(L)$ increases more faster for $\mu_1 = 1.8$ than it does for $\mu_1 = 2$ when the the repair rate γ increases. From Figure 4(b), we can observe that the ratio of μ_1 and μ_0 is less than a special point, $E(L)$ is increase when γ increase, otherwise it decrease when the repair rate γ increase. Clearly, the increasing rate of $E(L)$ is depend on the ratio of μ_1 and μ_0.

Figure 5 present the effect of γ on the probabilities $p_{0,0}, p_{0,1}$ for various values of μ_1. From Figure 5(a) we can observe that as increase of γ, $p_{0,0}$ and $p_{0,1}$ decreases and increases, respectively. This is intuitive because the lager repair rate γ leads to a smaller number of customers who are provided with the working breakdown service at a rate that is lower than that of the normal service. For a fixed γ, $p_{0,0}, p_{0,1}$ decreases and increases with the increase of μ_1, respectively.

Figure 6(a) reveals that $p_{0,0}$ increases with an increase of δ. This indicates that the larger value of δ causes more customers to be forced leaving the system by a disaster, thus increasing the probability of server is idle

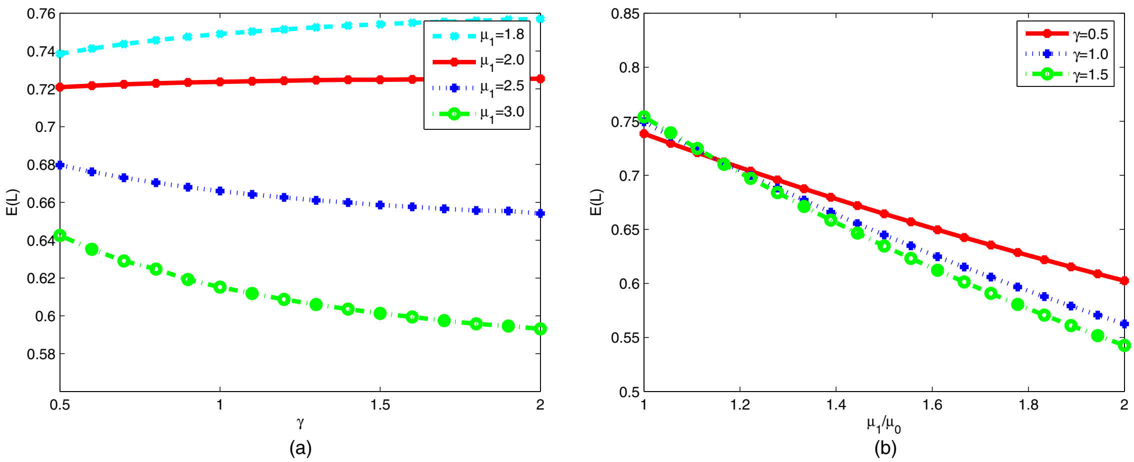

FIGURE 4. $\gamma, \mu_1/\mu_0$ *vs.* $E(L)$ with $\lambda = 2, \mu_0 = 1.8, \delta = 1.6, \eta = 1.6$. (a) the effect γ on $E(L)$, (b) the effect μ_1/mu_0 on $E(L)$.

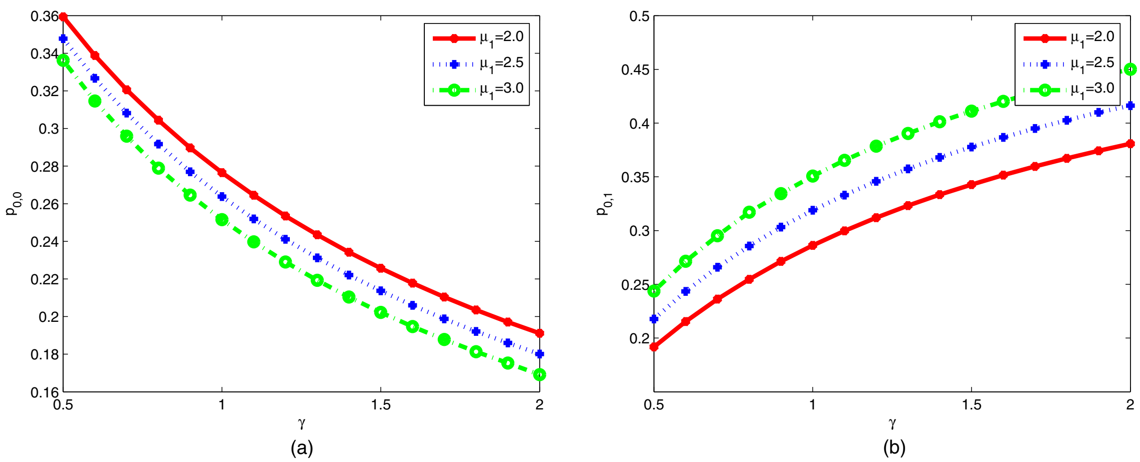

FIGURE 5. γ *vs.* $p_{0,0}, p_{0,1}$ with $\lambda = 2, \mu_0 = 1.8, \delta = 1.6, \eta = 1.6$. (a) the effect γ on $p_{0,0}$, (b) the effect γ on $p_{0,1}$.

during the working breakdown period. The effect of δ on $E(L)$ is presented in Figure 6(b) For any η, the averages system length $E(L)$ decreases with the increase of δ. For fixed δ, $E(L)$ decreases with the increase of η and it is highest for $\eta = 0$ implying that model without impatient customers yield longer system lengths when compared to models with impatient customers ($\eta > 0$).

7. CONCLUSION

In this paper, we have carried out an analysis $M/M/1$ queue with working breakdowns and impatient customers. We have obtained the queue length distribution. Various performance measures such as the probability of server state, the average queue length, the sojourn time in the queue are also carried out. For future, we can extend this paper to complex models such as queues with batch arrivals or $M/G/1$ case with generally distributed impatience times.

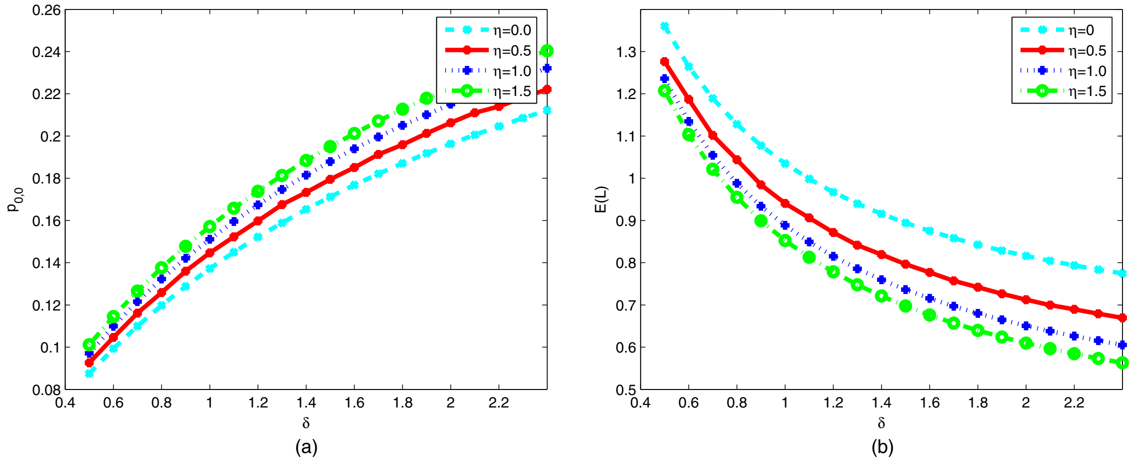

FIGURE 6. δ *vs.* $p_{0,0}, E(L)$ with $\lambda = 2, \mu_1 = 2.5, \mu_0 = 1.5, \gamma = 1.6$. (a) the effect δ on $p_{0,0}$, (b) the effect δ on $E(L)$.

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