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NEW SECOND-ORDER RADIAL EPIDERIVATIVES AND APPLICATIONS TO OPTIMALITY CONDITIONS*

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Abstract. In this paper, we introduce the second-order weakly composed radial epiderivative of set-valued maps, discuss its relationship to the second-order weakly composed contingent epiderivative, and obtain some of its properties. Then we establish the necessary optimality conditions and sufficient optimality conditions of Benson proper efficient solutions of constrained set-valued optimization problems by means of the second-order epiderivative. Some of our results improve and imply the corresponding ones in recent literature.

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1. Introduction

Optimality conditions are an important topic in optimization theory. To study this topic, the concept of derivative plays the most fundamental role in formulating optimality conditions for set-valued optimization problems. In the last few decades, many kinds of (higher-order) derivatives or epiderivatives for set-valued maps have been proposed and used for the formulation of optimality conditions in set-valued optimization problems, see e.g. [1–5, 7, 8, 10, 11, 14, 17, 20, 23, 24, 26–28] and references therein.

In recent years, second-order optimality conditions have been studied for set-valued optimization problems since this is very helpful for recognizing optimal solutions as well as for designing numerical algorithms for computing them. By second-order tangent (or contingent) set, Jiménez and Novo [16] derived the necessary conditions of efficient solutions for vector optimization problems. Since the epigraph of a set-valued mapping has better properties than the graph of a set-valued mapping in general, it is advantageous to employ the epiderivatives in set-valued optimization, Jahn *et al.* [15] introduced the second-order contingent epiderivatives for set-valued maps and applied these concepts to establish second-order optimality conditions for set-valued optimization problems. It is worth noting that the second-order contingent set is only a closed set and may not be a cone in general cases, and the second-order contingent set may not be a convex set even though the set

Keywords. Set-valued optimization problems, second-order weakly composed radial epiderivatives, Benson proper efficient solutions, second-order optimality conditions.

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is a convex set. Thus, comparing with the (generalized) contingent epiderivative introduced by Chen and Jahn [8], the (generalized) second-order contingent epiderivative does not have the strictly positive homogeneity and subadditivity. To solve this problem, Li *et al.* [18] introduced the concept of generalized second-order composed contingent epiderivatives for set-valued mappings and obtained a unified second-order sufficient and necessary optimality condition of weak minimizers for set-valued optimization problems. Wang *et al.* [27] introduced the concept of second-order weak composed contingent epiderivative for set-valued mappings and established the second-order sufficient optimality conditions and the necessary optimality conditions of weak minimizers for set-valued optimization problems by virtue of the second-order epiderivative. Peng and Xu [22] introduced a new second-order tangent epiderivative for a set-valued mapping and obtained second-order Fritz John and Kuhn-Tucker necessary optimality conditions and a second-order Kuhn-Tucker sufficient optimality condition for a weak minimizer of set-valued optimization problems. It is worth noting that the second-order sufficient optimality conditions are obtained under the assumption conditions of cone-convex set-valued mappings in [15, 18, 22, 27]. All these (higher-order) generalized contingent (tangent) epiderivatives capture local information of concerning maps (using $h_n \to 0^+$ in their formulations). To reflect global information, Flores-Bazán introduced the radial epiderivative in [12]. Thus the second-order optimality conditions for set-valued optimization problems still need to be addressed.

Our aim is to weaken the hypotheses of some results of [18,24,27]. Motivated by the above views and the work reported in [2, 4, 12, 18, 20, 27], we first introduce the second-order weakly composed radial epiderivative of set-valued mappings and discuss some of its properties. Then, by virtue of the second-order weakly composed radial epiderivative, we investigate second-order necessary optimality conditions and sufficient optimality conditions to set-valued optimization problems. Some results obtained improve and imply the corresponding ones in recent literature.

The organization of this paper is as follows. In Section 2, we recall some basic concepts and introduce a kind of set-valued optimization problem model. In Section 3, we introduce the second-order weakly composed radial epiderivative for set-valued mappings and discuss some of its properties. In Section 4, we establish the necessary optimality conditions and sufficient optimality conditions of Benson proper efficient solutions for constrained set-valued optimization problems.

2. Preliminaries and notations

Throughout this paper, let X, Y and Z be three real normed spaces, $C \subset Y$ be a closed convex pointed cone with a nonempty interior and $D \subset Z$ be a closed convex pointed cone. Let Y and Z be partially ordered by C and D, respectively. 0_X and 0_Y denote the origins of X and Y, respectively. Y^* denotes the topological dual space of Y. Let $\mathrm{cl}C$ and C^+ denote the closure and the topological dual cone of C, respectively. Let M be a nonempty subset of Y. The cone hull of M is defined by

$$\mathrm{cone}M = \{ty : y \in M, t \geq 0\}.$$

The quasi-interior C^{+i} of C^+ is defined by $C^{+i} = \{f \in Y^* | f(c) > 0, \forall c \in C \setminus \{0_Y\}\}$. A base of a cone C is a convex subset B of C which satisfies $0_Y \notin \mathrm{cl}B$ and $C = \mathrm{cone}B$. Suppose that C is a closed convex pointed cone in Y. Then $C^{+i} \neq \emptyset$ if and only if C has a base.

Let E be a nonempty subset of X and let $F : E \to 2^Y$ be a set-valued mapping. The domain, graph, epigraph of F are defined by, respectively,

$$\mathrm{dom}(F) := \{x \in E | F(x) \neq \emptyset\}, \quad \mathrm{graph}(F) := \{(x, y) \in X \times Y | x \in E, y \in F(x)\},$$

$$\mathrm{epi}(F) := \{(x, y) \in X \times Y | x \in E, y \in F(x) + C\}.$$

Let $y_0 \in Y$. Denote

$$F(E) = \bigcup_{x \in E} F(x), \quad \text{and } (F - y_0)(x) = F(x) - \{y_0\}.$$

Definition 2.1. (See [13, 21]) Let $C \subset Y$ be a convex cone and $F : E \to 2^Y$ be a set-valued mapping. F is said to be C-convex (C-function) on E if for every $x, y \in E$ and every $\lambda \in (0, 1)$,

$$\lambda F(x) + (1 - \lambda)F(y) \subset F(\lambda x + (1 - \lambda)y) + C.$$

Clearly, F is C-convex (C-function) on a convex set E if and only if $\text{epi}(F)$ is a convex subset in $X \times Y$.

Definition 2.2. (See [21]) Let M be a nonempty subset of Y. A point $y \in M$ is said to be a minimal point (resp. a weakly minimal point) of M if

$$M \bigcap (\{y\} - C) = \{y\} (\text{resp.} M \bigcap (\{y\} - \text{int}C) = \emptyset).$$

As usual, the set of all minimal points (resp. weakly minimal points) of M is denoted by $\text{Min}_C M$ (resp. $\text{WMin}_C M$). Specially, we set $\text{Min}_C M = \emptyset$ (resp. $\text{WMin}_C M = \emptyset$) if $M = \emptyset$.

Definition 2.3. (See [6]) Let M be a nonempty subset in Y. A point $y \in M$ is said to be a Benson proper efficient point of M if $\text{clcone}(M + C - \{y\}) \bigcap (-C) = \{0_Y\}$. The set of all Benson proper efficient points of M is denoted by $\text{PrMin}_C M$.

Definition 2.4. (See [28]) Let $E \subset Y$ be a nonempty subset, $F : E \to 2^Y$ be a set-valued mapping, $x_0 \in E$ and $y_0 \in F(x_0)$. F is called generalized C-convex at (x_0, y_0) on E if $\text{cone}(\text{epi}(F) - \{(x_0, y_0)\})$ is convex.

Remark 2.1. Obviously, if F is C-convex on a convex set E, then F is generalized C-convex at $(x_0, y_0) \in \text{graph}(F)$ on E. But the converse may not hold, the following example shows the case.

Example 2.1. Let $X = \mathbb{R}, Y = \mathbb{R}, C = \mathbb{R}_+ = \{y : y \geq 0\}, E = \{0, 1\}$. $F : E \to 2^Y$ is defined by

$$F(x) = \{y : y \geq x^2\}, \forall x \in E.$$

Take $(x_0, y_0) = (0, 0) \in \text{graph}(F)$. Then F is generalized C-convex at (x_0, y_0) on E, but F isn't C-convex on E since E is not a convex set.

Now, we recall notions of the contingent cone and the radial cone.

Definition 2.5. (see [4]) Let K be a nonempty subset of X and $x_0 \in \text{cl}K$. The contingent cone $T(K, x_0)$ to K at x_0 is defined by

$$T(K, x_0) := \limsup_{h \to 0^+} \frac{K - x}{h} \cdot$$

Definition 2.6. (see [12, 26]) Let $K \subset X$ be a nonempty subset of X and $x_0 \in \text{cl}K$. The closed radial cone $R(K, x_0)$ to K at x_0 is the set of all $v \in X$ for which there exist a sequence $\{h_n\}$ of positive real numbers and a sequence $\{x_n\}$ in K such that

$$\lim_{n \to \infty} h_n(x_n - x_0) = v.$$

Note that the closed radial cone $R(K, x_0)$ to K at x_0 can be defined as follows.

Definition 2.7. (see [20]) Let $K \subset X$ be a nonempty subset of X and $x_0 \in \text{cl}K$. The closed radial cone $R(K, x_0)$ to K at x_0 is the set of all $v \in X$ for which there exist a sequence $\{\lambda_n\}$ of positive real numbers and a sequence $\{z_n\}$ in X with $\lim_{n \to \infty} z_n = v$ such that $x_0 + \lambda_n z_n \in K$, for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the natural number set.

By Definitions 2.5 and 2.6, the following two results hold.

Theorem 2.1. *Let $K \subseteq X$ and $x \in \text{cl}K$. Then $R(K, x)$ is a closed cone and $R(K, x) = \text{clcone}(K - \{x\})$.*

From [16, Prop. 2.3] and [17, Prop. 3.1], we can obtain the following result.

Theorem 2.2. *If K is a convex subset of X and $x \in K$, then*

$$T(K,x) = R(K,x) = \text{clcone}(K - \{x\}).$$

Definition 2.8. (see [8]) Let X be a real linear space, and let Y be a real linear space and partially ordered by a convex cone $C \subset Y$ with apex at the origin. A set-valued map $H : X \to 2^Y$ is said to be

(i) strictly positive homogeneous if

$$H(\alpha x) = \alpha H(x), \forall \alpha > 0, \forall x \in X.$$

(ii) subadditive if

$$H(x_1) + H(x_2) \subseteq H(x_1 + x_2) + C.$$

Iff the property (i) holds with $\alpha \geq 0$ and (ii) holds, H is said to be sublinear.

Definition 2.9. (See [13, 21]) Let Y be a topological linear space and be partially ordered by a convex cone $C \subset Y$ with apex at the origin.

(i) A sequence $\{y_n\} \subseteq Y$ is said to be C-decreasing iff $\forall i, j \in N, i \leq j$ implies $y_j \leq_C y_i$.
(ii) A subset $D \subset Y$ is said to be C-lower bounded iff there exists a $y \in Y$ such that $D \subset \{y\} + C$.
(iii) The cone C is called Daniell iff every C-decreasing and C-lower bounded sequence in Y converges to its infimum.
(iv) The weak domination property is said to be held for a subset M of Y iff $M \subset \text{ WMin}_C M + \text{int}C \bigcup \{0_Y\}$.

In this paper, we consider the following constrained set-valued optimization problem:

$$\text{(SP)} \quad \begin{cases} \text{Min}_C \ F(x) \\ \text{s.t.} \quad G(x) \cap (-D) \neq \emptyset, x \in E, \end{cases}$$

where $E \subseteq X$, $F : E \to 2^Y$ and $G : E \to 2^Z$ are set-valued maps. The set $U := \{x \in E | G(x) \cap (-D) \neq \emptyset\}$ is a feasible set of (SP). A pair $(x_0, y_0) \in \text{graph}(F)$ is said to be a Benson proper efficient solution of (SP) iff $x_0 \in U$ and $y_0 \in \text{PrMin}_C F(U)$.

3. New second-order radial epiderivatives

In this section, we propose a new notion of second-order epiderivatives for set-valued mappings, and then investigate some properties of the derivative.

Definition 3.1. Let $(x_0, y_0) \in \text{graph}(F)$, $(u, v) \in X \times Y$. The second-order weakly composed radial epiderivative $D''_{wcr}F(x_0, y_0, u, v)$ of F at (x_0, y_0) in the direction (u, v) is the set-valued map from X to Y defined by

$$D''_{wcr}F(x_0, y_0, u, v)(x) = \text{ WMin}_C\{y \in Y | (x, y) \in R(R(\text{epi}(F), (x_0, y_0)), (u, v))\}.$$

To compare our epiderivative with other second-order epiderivatives, we recall some notions in [18, 27].

Definition 3.2. (see [27]) Let $(x_0, y_0) \in \text{graph}(F)$ and $(u, v) \in X \times Y$. The second-order weakly composed contingent epiderivative $D''_w F(x_0, y_0, u, v)$ of F at (x_0, y_0) in the direction (u, v) is a set-valued map from X to Y defined by

$$D''_w F(x_0, y_0, u, v)(x) = \text{ WMin}_C\{y \in Y | (x, y) \in T(T(\text{epi}(F), (x_0, y_0)), (u, v))\}.$$

Definition 3.3. (see [18]) Let $(x_0, y_0) \in$ graph(F) and $(u, v) \in X \times Y$. The generalized second-order composed contingent epiderivative $D''_g F(x_0, y_0, u, v)$ of F at (x_0, y_0) in the direction (u, v) is a set-valued map from X to Y defined by

$$D''_g F(x_0, y_0, u, v)(x) = \text{Min}_C\{y \in Y | (x, y) \in T(T(\text{epi}(F), (x_0, y_0)), (u, v))\}.$$

By Definitions 3.1, 3.2 and 3.3 and Theorem 2.1, we can obtain the following result.

Theorem 3.1. *Let E be a nonempty subset of X, $F : E \to 2^Y$ be a set-valued map, $(x_0, y_0) \in$ graph(F), and $(u, v) \in R(\text{epi}(F), (x_0, y_0))$. Suppose that the set*

$$P(x) := \{y \in Y | (x, y) \in R(R(\text{epi}(F), (x_0, y_0)), (u, v))\},$$

fulfills the weak domination property for all $x \in M := \text{dom}(P)$. Then

$$D''_w F(x_0, y_0, u, v)(x) \subseteq D''_{wcr} F(x_0, y_0, u, v)(x) + C, \forall x \in M.$$

Now we establish an existence theorem of $D''_{wcr} F(x_0, y_0, u, v)$.

Theorem 3.2. *Let C be a closed convex pointed cone and let C be Daniell and $(u, v) \in \text{cl}R(\text{epi}(F), (x_0, y_0))$. If the set*

$$Q(x) := \{y \in Y | (x, y) \in R(R(\text{epi}(F), (x_0, y_0)), (u, v))\},$$

is C-lower bounded for every $x \in \text{dom}(Q)$, then $D''_{wcr} F(x_0, y_0, u, v)(x)$ exists for all $x \in \text{dom}(Q)$.

Proof. From Theorem 2.1, $R(R(\text{epi}(F), (x_0, y_0)), (u, v))$ is a closed cone. Since $Q(x)$ is C-lower bounded for every $x \in \text{dom}(Q)$, from the existence theorem of minimal points (see [18]), $\text{Min}_C Q(x)$ is nonempty. Since $\text{Min}_C Q(x) \subset \text{WMin}_C Q(x)$, for every $x \in \text{dom}(Q)$, $D''_{wcr} F(x_0, y_0, u, v)(x) = \text{WMin}_C Q(x)$ is nonempty, and the proof is complete. □

Now we establish a few important properties of the second-order weakly composed radial epiderivatives.

Theorem 3.3. *Let E be a nonempty subset of X, $(x_0, y_0) \in$ graph(F), and $(u, v) \in \text{cl}R(\text{epi}(F), (x_0, y_0))$. If the set $Q(x) := \{y \in Y : (x - x_0 - u, y) \in R(R(\text{epi}(F), (x_0, y_0)), (u, v))\}$ fulfills the weak domination property for all $x \in E$, then*

$$(F(x) - \{y_0\} - \{v\}) \bigcap (Y \setminus (D''_{wcr} F(x_0, y_0, u, v)(x - x_0 - u) + C)) = \emptyset, \forall x \in E.$$

Proof. By virtue of Theorem 2.1, we have

$$R(R(\text{epi}(F), (x_0, y_0)), (u, v)) = \text{clcone}(\text{cone}(\text{epi}(F) - \{(x_0, y_0)\}) - \{(u, v)\}). \tag{3.1}$$

Since for every $x \in E$ and $y \in F(x)$, it is obvious that

$$(x - x_0 - u, y - y_0 - v) \in \{x\} \times (F(x) + C) - \{(x_0, y_0)\} - \{(u, v)\}.$$

Then it follows from (3.1) that

$$(x - x_0 - u, y - y_0 - v) \in R(R(\text{epi}(F), (x_0, y_0)), (u, v)).$$

Therefore, from Definition 3.1 and the weak domination property of $Q(x)$, we get

$$y - y_0 - v \in D''_{wcr} F(x_0, y_0, u, v)(x - x_0 - u) + C.$$

Therefore

$$(F(x) - \{y_0\} - \{v\}) \bigcap (Y \setminus (D''_{wcr} F(x_0, y_0, u, v)(x - x_0 - u) + C)) = \emptyset, \forall x \in E,$$

and this completes the proof. □

Remark 3.1. It is under the assumption of cone-convexity that authors established ([27], Prop. 5), however, Theorem 3.3 is obtained without the assumption of cone-convexity. Thus, combining with Theorem 2.2, we know that Theorem 3.3 improves and implies ([27], Prop. 5).

Now we provide an example to show Theorem 3.3 and Remark 3.1.

Example 3.1. Let $X = \mathbb{R}, Y = \mathbb{R}^2, E = \mathbb{R}_+, C = \mathbb{R}^2_+$, $F(x) = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 \geq x^{\frac{4}{5}}, y_2 \geq 0\}, \forall x \in E$. Let $(x_0, y_0) = (0, (0, 0))$, and $(u, v) = (1, (0, 0))$. Then

$$R(\mathrm{epi}(F), (x_0, y_0)) = \mathbb{R}_+ \times C$$

and

$$R(R(\mathrm{epi}(F), (x_0, y_0)), (u, v)) = \mathbb{R} \times C.$$

Therefore,

$$D''_{wcr}F(x_0, y_0, u, v)(x) = \{(y_1, y_2) \in C | y_1 y_2 = 0\}, \forall x \in \mathbb{R},$$

and then, for any $x \in \mathbb{R}$, we have

$$(F(x) - \{y_0\} - \{v\}) \bigcap (Y \setminus (D''_{wcr}F(x_0, y_0, u, v)(x - x_0 - u) + C)) = \emptyset.$$

So Theorem 3.3 holds here.

Since F is not cone-convex, ([27], Prop. 5) does not hold here.

By Theorem 3.3, we have the following corollary.

Corollary 3.1. *Let $E \subset X$ and $(x_0, y_0) \in \mathrm{graph}(F)$, $(u, v) \in \mathrm{cl}R(\mathrm{epi}(F), (x_0, y_0))$ with $v \in C$. If the set $Q(x) := \{y \in Y : (x - x_0, y) \in R(R(\mathrm{epi}(F), (x_0, y_0)), (u, v))\}$ fulfills the weak domination property for all $x \in E$, then for all $x \in E$,*

$$(F(x) - \{y_0\}) \bigcap (Y \setminus (D''_{wcr}F(x_0, y_0, u, v)(x - x_0 - u) + C)) = \emptyset.$$

Theorem 3.4. *Let $(x_0, y_0) \in \mathrm{graph}(F), (u, v) \in \mathrm{cl}R(\mathrm{epi}(F), (x_0, y_0))$. Suppose that the second-order weakly composed radial epiderivative $D''_{wcr}F(x_0, y_0, u, v)(x) \neq \emptyset$ for every $x \in X$. Then*

(i) $D''_{wcr}F(x_0, y_0, u, v)$ is strictly positive homogeneous.

Moreover, if F is generalized C-convex at (x_0, y_0) on E, and $Q(x) := \{y \in Y : (x, y) \in R(R(\mathrm{epi}(F), (x_0, y_0)), (u, v))\}$ satisfies the weak domination property for each $x \in X$, then

(ii) $D''_{wcr}F(x_0, y_0, u, v)$ is subadditive.

Proof. (i) Take $\lambda > 0$ and $x \in X$. Let $y \in D''_{wcr}F(x_0, y_0, u, v)(x) = \mathrm{WMin}_C\{y \in Y | (x, y) \in R(R(\mathrm{epi}(F), (x_0, y_0)), (u, v))\}$. Then

$$(x, y) \in R(R(\mathrm{epi}(F), (x_0, y_0)), (u, v)).$$

It follows from Theorem 2.1 that

$$(\lambda x, \lambda y) \in R(R(\mathrm{epi}(F), (x_0, y_0)), (u, v)).$$

We shall prove that

$$\lambda y \in D''_{wcr}F(x_0, y_0, u, v)(\lambda x). \tag{3.2}$$

To prove the result (3.2) by contradiction, suppose that there exists some $\bar{y} \in Y$ such that $(\lambda x, \bar{y}) \in R(R(\text{epi}(F), (x_0, y_0)), (u, v))$ and $\bar{y} - \lambda y \in -\text{int}C$. Since $\text{int}C \bigcup \{0_Y\}$ is a cone, it follows from Theorem 2.1 that

$$(x, \frac{1}{\lambda}\bar{y}) \in R(R(\text{epi}(F), (x_0, y_0)), (u, v)) \text{ and } \frac{1}{\lambda}\bar{y} - y \in -\text{int}C,$$

which contradicts that $y \in D''_{wcr}F(x_0, y_0, u, v)(x)$. Hence

$$\lambda D''_{wcr}F(x_0, y_0, u, v)(x) \subseteq D''_{wcr}F(x_0, y_0, u, v)(\lambda x). \tag{3.3}$$

The inclusion relationship of

$$D''_{wcr}F(x_0, y_0, u, v)(\alpha x) \subseteq \alpha D''_{wcr}F(x_0, y_0, u, v)(x),$$

is established in a similar fashion to (3.3). Thus $D''_{wcr}F(x_0, y_0, u, v)$ is strictly positive homogeneous.

(ii) Let $x_1, x_2 \in X$, $y_1 \in D''_{wcr}F(x_0, y_0, u, v)(x_1)$, $y_2 \in D''_{wcr}F(x_0, y_0, u, v)(x_2)$. Then we have

$$(x_1, y_1) \in R(R(\text{epi}(F), (x_0, y_0)), (u, v)) \quad \text{and} \quad (x_2, y_2) \in R(R(\text{epi}(F), (x_0, y_0)), (u, v)).$$

Since F is generalized C-convex at (x_0, y_0) on E, $\text{cone}(\text{epi}F - \{(x_0, y_0)\})$ is convex. Then, from Theorem 2.2, $R(R(\text{epi}(F), (x_0, y_0)), (u, v))$ is a closed convex cone. Thus we have

$$(x_1 + x_2, y_1 + y_2) \in R(R(\text{epi}(F), (x_0, y_0)), (u, v)),$$

which implies that

$$y_1 + y_2 \in Q(x_1 + x_2).$$

So, from the weak domination property of $Q(x)$ and the definition of $D''_{wcr}F(x_0, y_0, u, v)$, we have

$$y_1 + y_2 \in D''_{wcr}F(x_0, y_0, u, v)(x_1 + x_2) + C.$$

Hence

$$D''_{wcr}F(x_0, y_0, u, v)(x_1) + D''_{wcr}F(x_0, y_0, u, v)(x_2) \subset D''_{wcr}F(x_0, y_0, u, v)(x_1 + x_2) + C.$$

This completes the proof. □

Remark 3.2. In Theorem 3.4, we obtain the subadditivity of the second-order weakly composed radial epiderivative under the assumption of generalized cone-convexity. However, it is under the assumption of cone-convexity that authors established the subadditivity of generalized second-order composed contingent epiderivative in Theorem 3.2 of [18]. Since the generalized cone-convex set-valued mapping may not be cone-convex, and the cone-convex set-valued mapping must be generalized cone-convex, Theorem 3.4 improves ([18], Thm. 3.2).

By Theorem 3.4, we can get the following result.

Theorem 3.5. *Let $(x_0, y_0) \in \text{graph}(F), (u, v) \in \text{cl}R(\text{epi}(F), (x_0, y_0))$. Suppose that F is generalized C-convex at (x_0, y_0) on E, and $W(x) = \{y \in Y : (x, y) \in R(R(\text{epi}(F), (x_0, y_0)), (u, v))\}$ fulfills the weakly domination property for each $x \in \text{dom}(W)$. Then $D''_{wcr}F(x_0, y_0, u, v)(\Omega) + C$ is a convex cone, where $\Omega = \text{dom}(D''_{wcr} F(x_0, y_0, u, v))$.*

4. Optimality conditions of problem (SP)

In this section, we investigate the necessary optimality conditions and sufficient optimality conditions of Benson proper efficient solutions for (SP) by means of the second-order weakly composed radial epiderivatives.

According to [9, Thm. 4.1], we can obtain the following lemma.

Lemma 4.1. *Let $x_0 \in U$, $y_0 \in F(x_0)$. If there exists $\phi \in C^{+i}$ such that*

$$\phi(y) \geq \phi(y_0), \forall y \in F(U),$$

then (x_0, y_0) is a Benson proper efficient solution of (SP).

Theorem 4.1. *Let $x_0 \in U$, $y_0 \in F(x_0)$, $z_0 \in G(x_0)\bigcap(-D)$ and $(u, v, w) \in R(\mathrm{epi}(F, G), (x_0, y_0, z_0))$ with $(v, w) \in (-C) \times (-D)$. Suppose that (x_0, y_0) is a Benson proper efficient element of (SP). Then*

$$D''_{wcr}(F, G)(x_0, y_0, z_0, u, v, w)(x) \subset ((Y \times Z) \setminus (-((C \setminus \{0_Y\}) \times \mathrm{int}D))), \tag{4.1}$$

for all $x \in \triangle := \mathrm{dom}[D''_{wcr}(F, G)(x_0, y_0, z_0, u, v, w)]$.

Proof. Since (x_0, y_0) is a Benson proper efficient element of (SP),

$$\mathrm{clcone}(F(U) + C - \{y_0\}) \bigcap -C = \{0_Y\}. \tag{4.2}$$

To prove the result (4.1) by contradiction, suppose that there exist some $\bar{x} \in \triangle$ and

$$(\bar{y}, \bar{z}) \in D''_{wcr}(F, G)(x_0, y_0, z_0, u, v, w)(\bar{x}), \tag{4.3}$$

such that

$$\bar{y} \in -C \setminus \{0_Y\}, \tag{4.4}$$

and

$$\bar{z} \in -intD. \tag{4.5}$$

According to (4.3), there exist sequences $\{\lambda_n\}$ with $\lambda_n > 0$ and $\{(x_n, y_n, z_n)\}$ with

$$(x_n, y_n, z_n) \in R(\mathrm{epi}(F, G), (x_0, y_0, z_0)), \tag{4.6}$$

such that

$$\lambda_n((x_n, y_n, z_n) - (u, v, w)) \to (\bar{x}, \bar{y}, \bar{z}), \text{ as } n \to \infty. \tag{4.7}$$

It follows from (4.6) that, for any n, there exist sequences $\{t_{n_k}\}$ with $t_{n_k} > 0$ and $\{(x_{n_k}, y_{n_k}, z_{n_k})\}$ with $(x_{n_k}, y_{n_k}, z_{n_k}) \in \mathrm{epi}(F, G)$ such that

$$t_{n_k}((x_{n_k}, y_{n_k}, z_{n_k}) - (x_0, y_0, z_0)) \to (x_n, y_n, z_n), \text{ as } k \to \infty. \tag{4.8}$$

Then $\lambda_n\{t_{n_k}((x_{n_k}, y_{n_k}, z_{n_k}) - (x_0, y_0, z_0)) - (u, v, w)\}$

$$\to \lambda_n[(x_n, y_n, z_n) - (u, v, w)], \text{ as } k \to \infty.$$

Combining this with (4.7), we have

$$\lambda_n(t_{n_k}(y_{n_k} - y_0) - v) \to \bar{y}, \text{ as } k \to \infty, n \to \infty. \tag{4.9}$$

It follows from $(x_{n_k}, y_{n_k}, z_{n_k}) \in \mathrm{epi}(F, G)$ and $(v, w) \in -(C \times D)$ that

$\lambda_n t_{n_k}[(y_{n_k}, z_{n_k}) - (y_0, z_0) - \frac{1}{t_{n_k}}(v, w)]$

$$\in \lambda_n t_{n_k}[(F,G)(x_{n_k}) + C \times D - \{(y_0,z_0)\} - \tfrac{1}{t_{n_k}}(v,w)]$$
$$\subset \lambda_n t_{n_k}[(F,G)(x_{n_k}) + C \times D - \{(y_0,z_0)\}].$$

Then

$$\lambda_n(t_{n_k}(y_{n_k} - y_0) - v) \in \lambda_n t_{n_k}[F(x_{n_k}) + C - \{y_0\}]. \tag{4.10}$$

From (4.5) and (4.7), there exists a sufficiently large natural number N such that

$$\lambda_n(z_n - w) \in -\mathrm{int}D, \forall n > N.$$

Since $\lambda_n > 0$, $w \in -D$ and $\mathrm{int}D \bigcup \{0_Z\}$ is a cone,

$$z_n \in -\mathrm{int}D, \forall n > N.$$

Then, from (4.8), for any n with $n > N$, there exists a sufficiently large natural number $K(n)$ such that $z_{n_k} \in -intD, \forall k > K(n)$. Thus

$$x_{n_k} \in U, \forall k > K(n), n > N. \tag{4.11}$$

By (4.9), (4.10) and (4.11), we get

$$\bar{y} \in \mathrm{clcone}(F(U) + C - \{y_0\}).$$

Then it follows from (4.4) that

$$\bar{y} \in \mathrm{clcone}(F(U) + C - \{y_0\}) \bigcap -(C \setminus \{0_Y\}),$$

which contradicts (4.2). Thus (4.1) holds and this completes the proof. □

Theorem 4.2. *Let $x_0 \in U$, $y_0 \in F(x_0)$, $z_0 \in G(x_0) \bigcap (-D)$ and $(u,v,w) \in \mathrm{cl}R(\mathrm{epi}(F,G), (x_0,y_0,z_0))$ with $(v,w) \in (-C) \times (-D)$. Suppose that the following conditions are satisfied:*

(i) (F,G) is generalized $C \times D$-convex at (x_0,y_0,z_0) on E;
(ii) the set $V(x) = \{(y,z) \in Y \times Z | (x,y,z) \in R(R(\mathrm{epi}(F,G),(x_0,y_0,z_0)),(u,v,w))\}$ fulfills the weak domination property for all $x \in \mathrm{dom}(V)$;
(iii) (x_0,y_0) is a Benson proper efficient solution of (SP);
(iv) C has a compact base.

Then there exist $\phi \in C^{+i}$ and $\psi \in D^*$ such that

$$\inf\{\phi(y) + \psi(z) | (y,z) \in \Theta\} = 0,$$

where $\Theta := D''_{wcr}(F,G)(x_0,y_0,z_0,u,v,w)(\Delta)$ and $\Delta := \mathrm{dom}[D''_{wcr}(F,G)(x_0,y_0,z_0,u, v,w)]$.

Proof. Define

$$M = D''_{wcr}(F,G)(x_0,y_0,z_0,u,v,w)(\Delta) + C \times D.$$

Then it follows from Theorem 3.5 and conditions (i) and (ii) that M is a convex set. It follows from Theorem 4.1 that

$$M \bigcap (-((C \setminus \{0_Y\}) \times intD)) = \emptyset.$$

Thus, it follows from a standard separation theorem of convex sets and the similar proof method of Theorem 1 from [24] that there exist $\phi \in C^{+i}$ and $\psi \in D^*$ such that

$$\inf\{\phi(y) + \psi(z) | (y,z) \in \Theta\} = 0.$$

The proof is complete. □

Remark 4.1. Since cone-convex set-valued mappings must be generalized cone-convex ones and the converse may not hold, Theorem 4.2 improves Theorem 1 of [24].

Now, we provide an example to illustrate Theorem 4.2 and Remark 4.1.

Example 4.1. Suppose that $X = Z = E = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $D = \mathbb{R}_+$. Consider problem (SP) with $F : E \to 2^Y$ be a set-valued map and $G : E \to 2^Z$ defined by, respectively,

$$F(x) = \{(y_1, y_2) \in \mathbb{R}^2 | y_2 \geq y_1^{\frac{3}{4}}, y_1 \geq 0\}, \forall x \in E,$$

and

$$G(x) = \{z \in \mathbb{R} | z \geq x^2\}, \forall x \in E.$$

Let $(x_0, y_0, z_0) = (0, 0, 0)$ and $(u, v, w) = (1, (0, 0), -1) \in X \times (-C) \times (-D)$. Naturally, (F, G) is generalized $C \times D$-convex at (x_0, y_0, z_0) on E, and (x_0, y_0) is a Benson proper efficient solution of (SP). By directly calculating, we get that $\triangle = \mathbb{R}$ and

$$D''_{wcr}(F, G)(x_0, y_0, z_0, u, v, w)(x) = \{((y_1, y_2), z) \in C \times Z | y_1 y_2 = 0, z \in \mathbb{R}\}, \forall x \in \mathbb{R}.$$

Set $\phi = (1, 1)$ and $\psi = 0$. Then, for any $x \in \triangle$ and $((y_1, y_2), z) \in \Theta$, we have

$$\phi(y) + \psi(z) = y_1 + y_2 \geq 0.$$

Since $((0, 0), z) \in \Theta$,

$$\phi((0, 0)) + \psi(z) = 0.$$

Therefore

$$inf\{\phi(y) + \psi(z) | (y, z) \in \Theta\} = 0,$$

and Theorem 4.2 holds here.

Since (F, G) is not $C \times D$-convex on E, Theorem of [24] is unusable here.

Theorem 4.3. *Let $x_0 \in U$, $y_0 \in F(x_0)$, $z_0 \in G(x_0) \bigcap (-D)$ and $(u, v, w) \in \mathrm{cl}R(\mathrm{epi}(F, G), (x_0, y_0, z_0))$ with $(u, v, w) \in \{0_X\} \times C \times D$. If the set $\{(y, z) \in Y \times Z : (x - x_0, y, z) \in R(R(\mathrm{epi}(F, G), (x_0, y_0, z_0)), (u, v, w))\}$ fulfills the weak domination property for all $x \in U$, and there exist $\phi \in C^{+i}$ and $\psi \in D^*$ such that*

$$\inf\{\phi(y) + \psi(z) | (y, z) \in \Theta\} = 0 \textit{ and } \psi(z_0) = 0, \tag{4.12}$$

where $\Theta := \bigcup_{x \in U} D''_{wcr}(F, G)(x_0, y_0, z_0, u, v, w)(x - x_0)$, then (x_0, y_0) is a Benson proper efficient solution of (SP).

Proof. By Corollary 3.1, it is obvious that

$$(y - y_0, z - z_0) \in D''_{wcr}(F, G)(x_0, y_0, z_0, u, v, w)(x - x_0) + C \times D,$$

for all $y \in F(x), z \in G(x), x \in U$. Then, it follows from (4.12) that

$$\phi(y - y_0) + \psi(z - z_0) \geq 0, \forall (y, z) \in (F, G)(U). \tag{4.13}$$

Since, for any $x \in U$, there exists a $\bar{z} \in G(x) \bigcap (-D)$ such that $\psi(\bar{z}) \leq 0$, it follows from $\psi(z_0) = 0$ and (4.13) that

$$\phi(y) \geq \phi(y_0), \forall y \in F(U).$$

Therefore, it follows from Lemma 4.1 that (x_0, y_0) is a Benson proper efficient solution of (SP) and the proof of the theorem is complete. □

Remark 4.2. Since Theorem 4.3 does not involve the assumption of convexity, it improves and generalizes ([24], Thm. 2).

5. Concluding remarks

In this paper, we propose a new concept of a second-order derivative for set-valued mappings, which is called the second-order weakly composed radial epiderivative, and then investigate some of its properties. Finally, by virtue of the epiderivative, we establish second-order sufficient and necessary optimality conditions of Benson proper efficient solutions for constrained set-valued optimization problems. Some obtained results improve and imply the corresponding results in literature.

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