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A VENDOR–BUYER INVENTORY MODEL WITH LOT-SIZE AND PRODUCTION RATE DEPENDENT LEAD TIME UNDER TIME VALUE OF MONEY

Sumon Sarkar*, Bibhas Chandra Giri and Ashis Kumar Sarkar

Abstract. The paper studies an integrated vendor–buyer model with shortages under stochastic lead time which is assumed to be variable but depends on the buyer's order size and the vendor's production rate. The replenishment lead time and the market demand uncertainty are assumed to be reduced by changing the regular production rate of the vendor at the risk of paying additional cost. Shortages are partially backlogged and the backlogging rate depends on the length of the buyer's replenishment lead time. The proposed model is formulated to obtain the net present value (NPV) of the expected total cost of the integrated system through optimization of (i) the buyer's order quantity, (2) the buyer's safety factor, and (3) the vendor's production rate. Theoretical results are derived to demonstrate the existence and uniqueness of the optimal solution. Through extensive numerical study, some valuable managerial insights are obtained.

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1. Introduction

Today's uncertain economy forces the supply chain managers to search for an alternative way to stay one step ahead from their competitors. It becomes very difficult for big retail companies to stand a chance without appropriate inventory control model. For many years, economic order quantity (EOQ) and reorder point have been used to make their decisions. An EOQ could help the company managers in order to take decision about the best optimal order quantity. On the other hand, the reorder point instructs when to place an order for particular products based on historical demand (Ben-Daya and Hariga [3], Ho and Hsiao [13], Tiwari *et al.* [49], Sarkar and Giri [43]). Additionally, the reorder point enables sufficient stock of products at hand *i.e.*, safety stock to fulfill the customer's demand while the next order arrives due to the lead time. Almost all integrated inventory models are developed based on the assumption that replenishment lead time is either zero or constant (Wee and Widyadana [51]; Das [7]) or a stochastic variable (Sajadieh and Jokar [42]; Zhou *et al.* [53]; Hossain *et al.* [14]) which is not subjected to control. According to Tersine [48], lead time involves order preparation time, order shipment/delivery time, set-up time, *etc.* Recognizing that manufacturing lead time is so much

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dependent on lot size, Kim and Benton [17] questioned on the assumption of fixed lead time, and established a relationship between lot size and lead time and showed that significant savings can be made by considering the interrelationships between lot size and safety stock decisions. Thereafter, Hariga [10], Hsiao [15], Rad *et al.* [41], and Khan *et al.* [16] used the concept of Kim and Benton [17] to develop their models.

Due to various reasons such as bad weather, labor strike, machine failure, unavailability of the raw materials, human errors, transcribing, etc., sometimes it becomes very difficult for the vendor to supply products to the buyer in the desired time. As a result, the buyer faces stock-out situation. The longer the length of lead time is, the larger the amount of shortages is, the smaller the proportion of customers can wait, and hence the smaller the backorder rate would be (see Ouyang and Chuang [34], Lee [21], Lee *et al.* [22]). In reality, for fashionable goods such as certain brand gum shoes, hi-fi equipment, cosmetics and clothes, some customers may wait up to a certain period for backorder and some may not wait at all. Replenishment lead time plays an important role when a supply chain system faces stock-out situation. A larger replenishment lead time increases the lost sale quantity, whereas a smaller lead time increases the backorder quantity. Therefore, consideration of lead time dependent backlogging rate is quite realistic. In several practical situations, however, lead time is controllable; it can be shortened by some additional cost. In doing so, one can reduce huge inventory asset in safety stock (held to fulfill the unanticipated demand during lead time), reduce opportunity loss due to stock out and increase customer satisfaction level.

Ordering and lead time decisions in a production–deliver supply chain under net present value (NPV) framework are important and challenging to company managers today because of changing inflation and time value of money. In this paper, we investigate a two-layer supply chain under NPV framework by contributing to the literature in several aspects. Firstly, it identifies the relationship between lead time, lot-size, and production rate by assuming replenishment lead time as a function of the buyer's order quantity and production rate of the vendor. Secondly, it describes how production rate regulation affects ordering and lead time decisions in a two-layer supply chain. Thirdly, it investigates how replenishment lead time affects the backorder quantity and possible way to increase the backorder rate during shortage. The rest of the paper is organized as follows: Section 2 reviews the relevant literature. Motivation of the research, notation, and assumptions are given in Section 3. In Section 4, the proposed model is formulated mathematically. Numerical example and sensitivity analysis are given in Section 5. Finally, the paper is concluded in Section 6.

2. Literature review

In reality, the backorder rate is treated as a variable in many situations and it is also assumed to depend on the replenishment lead time. Many researchers dealt inventory models with fixed backorder rate (Ouyang and Wu [33], Pan *et al.* [38], Tiwari *et al.* [49], Braglia *et al.* [5]). Ouyang and Chuang [7] were the first authors to consider shortage quantity dependent backorder rate with normally distributed lead time demand. However, when the lead time demand of the different customers are not identical, the model of Ouyang and Chuang [34] can not be used. Lee [21] modified Ouyang and Chuang's [34] model by considering the mixture of normal distribution. Lee *et al.* [22] developed a computational algorithm to solve an inventory model where backorder rate is dependent on the length of lead time through the amount of shortages. Ho and Hsiao [13] investigated a single-supplier single-retailer inventory model controlled by the reorder and shipping points with sharing information. They have shown that significant cost savings can be done if the supply chain members use both the reorder and shipping points to control the logistics and the inventory. Taleizadeh *et al.* [47] studied a joint single vendor–buyer supply chain problem with stochastic demand and fuzzy lead-time, which minimized the expected total cost by determining the reorder point and order quantity.

During shortage, motivating the customers to wait for backorder is a challenging task for the buyers. Discount policy on backordered items is a well known factor which can motivate the customers for backorder as well as increase the rate of backorder. In this direction, many researchers like Pan and Hsiao [37], Pan *et al.* [39], Lin [26], Sarkar *et al.* [44], and Sett *et al.* [45] considered various inventory models to incorporate the concept of backorder price discount. Kim *et al.* [18] considered an stochastic inventory model with backorder price discount and solved the model using distribution free approach. Kumar and Uthayakumar [20] developed a two-echelon supply chain model with imperfect production assuming lead time and price discount dependent backorder rate where buyer face stochastic lead time demand. Recently, Sarkar and Giri [43] formulated an integrated supply chain model with backorder price discount by assuming the replenishment lead time as a linear function of order size, setup time, and transportation time.

Since the backorder rate is much dependent on replenishment lead time, reduction of lead time becomes a vital issue to secure more backorder. Liao and Shyu [25] were the first researchers to study lead time reduction strategy in stochastic environment. They assumed that the lead time can be decomposed into several components having different crashing costs for reducing to a specified minimum duration. Thereafter Ouyang and Chang [35], Ouyang *et al.* [36], Li *et al.* [23] and Mandal and Giri [27] contributed significantly in the literature of controllable lead time. Glock [8] extended Hsiao's [15] model by assuming that lead time can be reduced by crashing the setup time and transportation time. Heydari [12] developed a supply chain model for a seller and a buyer by using lead time reduction strategy as a coordination mechanism. They showed that lead time reduction can motivate a buyer sufficiently to coordinate the supply chain. Mou *et al.* [32] wrote a note on Glock's [8] paper to introduce a more realistic lead time crashing cost. They proposed a modified integrated inventory model by considering setup and transportation time as independent decision variables and assuming that there are two different safety stocks. Hossain *et al.* [14] developed a vendor–buyer integrated inventory model with generalized lead time distribution and penalty cost for delivery lateness. They introduced a delivery tolerance factor to determine a tolerable latitude for delivery in terms of reorder time. Ponte *et al.* [40] investigated the effects of the mean and the variability of production and shipping lead times on multi-echelon supply chain models. They showed that lead time reduction strategy is not only profitable for the supply chain, but also increases the satisfactory level of the consumers.

NPV is usually used in business settlements to demonstrate the time value of money. The use of NPV is mainly relevant when interest rates are high. A survey by Klammer *et al.* [19] showed that NPV is the most frequently used method (80% for expansion/new operations decisions and 60% for replacement decisions) and has risen dramatically (tripling in 24 years) for capital budgeting and financial decision making. However, uses of NPV in production and inventory decisions are limited. Chao [6] investigated a single item EOQ model under NPV framework for the case of deterministic and stochastic demands. He was the first author to discuss NPV for stochastic demand. He showed that the optimal order quantity do not change with stochastic variability of the demand for the average cost model. However, for the case of high discount rate and the demand variability, it is necessary to use NPV. Moon and Yun [31] developed an finite planning horizon EOQ model in order to implement discount cash follow (DCF) approach to investigate the effects of time value of money on optimal order quantity. Bose *et al.* [4] and Hariga [9] developed two inventory models to examine the effects of inflation and time value of money in the presence of time varying demand and constant deterioration. Wee and Law [50] studied DCF approach to examine an EOQ model in the presence of price dependent demand and constant rate of deterioration. They presented a heuristic approach to find the near-optimal replenishment and pricing policy that maximize the total net present value profit. Yu *et al.* [52] considered a production inventory model to show the effect of inflation and time value of money with a random product life cycle. Mondal *et al.* [28] developed a production-repairing inventory model with fuzzy rough coefficients under inflation and time value of money. Hasan *et al.* [11] considered discounted cash flow (DCF) analysis of two greenhouse-type solar kilns (Oxford and Boral) for hardwood drying processes. Li *et al.* [24] analyzed pricing and lot-sizing policies for perishable items with discount cash flow on both sales revenue and all costs. The position of the paper with respect to the existing literature is shown in Table 1.

3. Motivation of the research, notation, and assumptions

Our study is motivated by the fact that many industries have devoted efforts to improve customer service and reduce order frequencies and costs with their business partners. In reality, it is very difficult to predict customers' demand in advance. Therefore, assumption of uncertain demand may be appropriate across all industries in this

TABLE 1. A comparison of the present model with some related works in the literature.

Author/(S)	Model type		Variable lead time	Lead time reduction	Variable production rate	Lead time dependent backlogging rate	Variable safety factor	NPV
	Integrated	Buyer's						
Ben-Daya and Hariga (2004)	–	$\surd$	$\surd$	–	–	$\times$	$\times$	$\times$
Chao (1992)	–	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$
Dey and Giri (2014)	$\surd$	–	$\surd$	$\surd$	$\times$	$\times$	$\surd$	$\times$
Glock (2012)	$\surd$	–	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
Hasan *et al.* (2016)	–	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$
Hsiao (2008)	–	$\surd$	$\surd$	–	–	$\times$	$\surd$	$\times$
Ho and Hsiao (2012)	–	$\surd$	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$
Khan *et al.* (2017)	$\surd$	–	$\surd$	$\times$	$\times$	$\times$	$\surd$	$\times$
Kim and Benton (1995)	–	$\surd$	$\surd$	$\times$	$\times$	$\times$	$\surd$	$\times$
Lee (2005)	$\surd$	–	$\surd$	$\surd$	$\times$	$\surd$	$\times$	$\times$
Lee *et al.* (2006)	$\surd$	–	$\surd$	$\surd$	$\times$	$\surd$	$\times$	$\times$
Li *et al.* (2017)	$\surd$	–	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$
Mandal and Giri (2015)	$\surd$	–	$\surd$	$\surd$	$\times$	$\times$	$\surd$	$\times$
Mondal *et al.* (2013)	–	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$
Moon and Cha (2005)	–	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
Moon and Yun (1993)	–	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$
Mou *et al.* (2017)	$\surd$	–	$\times$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
Ouyang and Chuang (2001)	$\surd$	–	$\surd$	$\surd$	$\times$	$\surd$	$\times$	$\times$
Red *et al.* (2014)	$\surd$	–	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$
Sarkar *et al.* (2015)	–	$\surd$	$\surd$	$\surd$	$\times$	$\times$	$\surd$	$\times$
Tiwari *et al.* (2017)	$\surd$	–	$\surd$	$\surd$	$\times$	$\times$	$\surd$	$\times$
Wee and Law (2001)	–	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$
This paper	$\surd$	–	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$

world. Additionally, lead time and safety stock play important roles when market demand is uncertain. Many industries are willing to invest additional capital in order to provide better services to their customers. In general, investment in lead time reduction might be a good idea through which one can generate high customer loyalty by quick service. One major implication of this policy is that this effort may result in higher degree of coordination between business parties. The aim of this paper is to develop a vendor–buyer integrated model in order to address the following issues:

(i) What should be the appropriate reorder point to place an order at the buyer's end?
(ii) In which condition the vendor should run the production system with maximum production rate and when with minimum production rate?
(iii) In which condition the buyer should invest money to reduce replenishment lead time?

We use the following notation to develop the proposed model.

Decision variables	Description
Q	Size of a shipment (units)
u	Safety factor
R	Production rate of the vendor (units)
Parameters	Description
D	Annual demand at the buyer (units/year)
R	Annual production rate of the vendor (units/year)
R_0	Regular production rate of the vendor (units/year)
$R_{\max}$	Maximum production rate of the vendor (units/year)
C_s	Vendor's setup cost per unit time ($/unit time)
C_o	Buyer's ordering cost per order ($/order)
H_b	Unit holding cost at the buyer ($/unit/year)

H_v	Unit holding cost at the vendor (\$/unit/year)
$l(Q,R)$	Buyer's replenishment lead time (time unit)
σ_l	Standard deviation of lead time demand (units)
r	Reorder point (units)
$\delta(l)$	Backorder rate during shortage period
b	Penalty cost at the buyer for unit short (\$/unit/year)
b_0	Buyer's marginal profit (\$/unit)
X	Lead time demand
j	Yearly interest rate (\$)
$E(X-r)^+$	Expected shortage quantity

We make the following assumptions to develop the model:

(i) This paper considers a supply chain consisting a single-vendor and a single-buyer to deal with a single type of item.
(ii) The buyer faces stochastic demand during replenishment lead time from his/her customers and the demand is normally distributed with a finite mean and standard deviation (Liao and Shyu [25]).
(iii) Following continuous review (Q,r) inventory policy, the buyer places an order of size Q whenever the inventory level falls to the reorder point and the vendor produces the items with a finite production rate $P(>D)$ in a single setup and transfers the entire quantity to the buyer over a single shipment.
(iv) The buyer's reorder point is defined as the sum of the expected demand during lead time and safety stock.
(v) Replenishment lead time between the buyer and the vendor is variable, which is directly proportional to the buyer's ordering size and inversely proportional to the vendor's production rate (Moon and Cha [30]). It is logical as larger order size will take longer production time than a smaller one.
(vi) Replenishment lead time is controllable which can be controlled by monitoring the vendor's production rate through some additional investment. The extra costs incurred by the vendor will be fully transferred to the buyer if shortened lead time is requested (Moon and Cha [30]).
(vii) Shortages are allowed in the buyer's inventory. The unsatisfied demand is backlogged, and the fraction of shortages backordered is (Abad [1,2])
$\delta(l)=\mathrm{e}^{-\alpha l}, 0\le\delta(l)\le 1$ with $\delta(0)=1$ where l is the lead time up to the next replenishment and α is a positive constant. Note that if $\delta(l)=1$(or 0) for all l, then shortages are completely backlogged (or lost).
(viii) Time value of money is considered.

4. Mathematical model

We suppose that the buyer follows (Q,r) inventory policy and places an order of size Q when the inventory level drops to the reorder point r. The vendor produces the entire order with a finite production rate $R(>D)$ in a single setup and transfers the entire quantity to the buyer over a single lot (see Fig. 1 for the inventory pattern). The lead time $l(Q,R)$ is directly proportional to the buyer's order quantity Q and inversely proportional to the vendor's production rate R, *i.e.*, $l(Q,R)=\frac{Q}{R}, 0<R_0\le R\le R_{\max}$, where R_0 is the vendor's regular production rate and $R_{\max}$ is the maximum production rate. During the replenishment lead time, the buyer may face stock-out situation because lead time and demand during lead time both are unpredictable. Consequently, it is necessary to calculate the safety stock level to prevent stock-outs. Therefore, the safety stock level is calculated by multiplying the safety stock risk factor (u) with the standard deviation (σ_l) and the square root of the lead time $(\sqrt{\frac{Q}{R}})$. Hence, the safety stock is $=u\sigma_l\sqrt{\frac{Q}{R}}$. Therefore, we have the reorder point (r) as the sum of the expected demand during lead time and safety stock, *i.e.*, $r=D\frac{Q}{R}+u\sigma_l\sqrt{\frac{Q}{R}}$. The buyer's expected shortage quantity at the end of the replenishment cycle is

$$B=E(X-r)^+=\int_r^\infty (x-r)\,\mathrm{d}f(x), \tag{4.1}$$

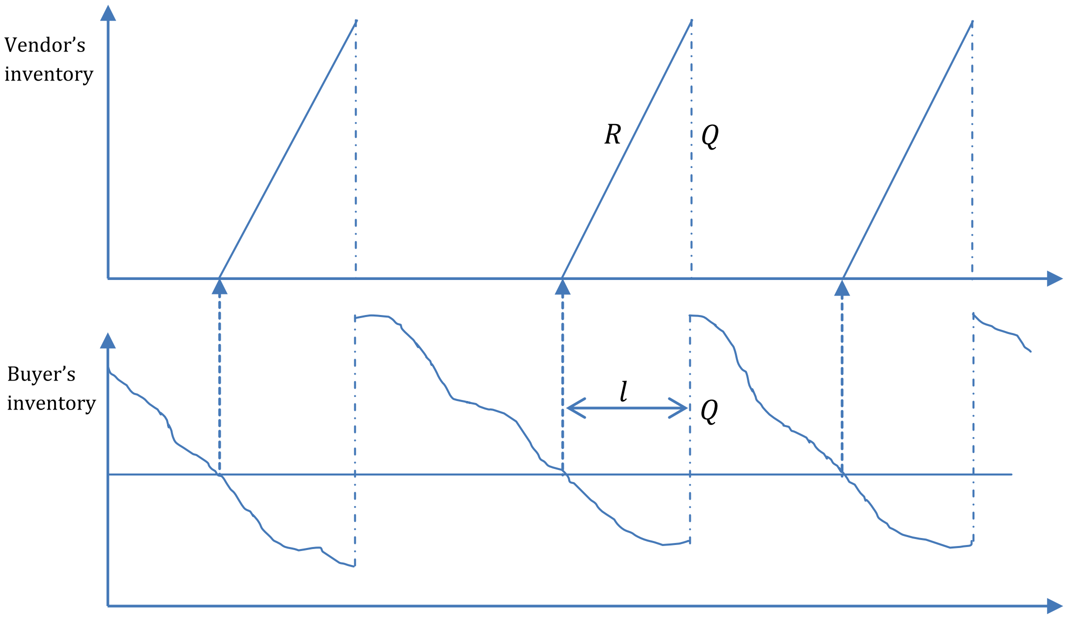

FIGURE 1. Inventory pattern for the vendor and the buyer.

where $f(x) = \frac{1}{\sigma_l\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma_l^2}}$ for mean μ and standard deviation σ_l.

The expected shortage quantity for a demand with mean $D\frac{Q}{R}$ and standard deviation $\sigma_l\sqrt{\frac{Q}{R}}$ during the lead time is given by

$$B = \int_r^\infty \frac{(x-r)}{\sqrt{2\pi}\sigma_l\sqrt{\frac{Q}{R}}}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-D\frac{Q}{R}}{\sigma_l\sqrt{\frac{Q}{R}}}\right)^2}\,\mathrm{d}x. \tag{4.2}$$

Assuming $z = \frac{x-D\frac{Q}{R}}{\sigma_l\sqrt{\frac{Q}{R}}}$ and $u = \frac{r-D\frac{Q}{R}}{\sigma_l\sqrt{\frac{Q}{R}}}$, (2) becomes

$$B = \sigma_l\sqrt{\frac{Q}{R}}\int_{z=u}^\infty (z-u)f(z)\,\mathrm{d}z, \tag{4.3}$$

where $f(z)$ is the standard normal probability density function.

Assuming

$$G(u) = \int_{z=u}^\infty (z-u)f(z)\,\mathrm{d}z, \tag{4.4}$$

(3) becomes

$$B = \sigma_l\sqrt{\frac{Q}{R}}G(u). \tag{4.5}$$

We assume that the backorder rate δ is a function of lead time $(l(Q,R))$ *i.e.*,

$$\delta(l) = \mathrm{e}^{-\alpha l(Q,R)} = \mathrm{e}^{-\frac{\alpha Q}{R}}. \tag{4.6}$$

Therefore, the expected backorder quantity is

$$\delta(l)E(X-r)^{+} = \mathrm{e}^{-\frac{\alpha Q}{R}}\sqrt{\frac{Q}{R}}G(u)\sigma_l \tag{4.7}$$

and hence the expected loss in sales per replenishment cycle is

$$\{1-\delta(l)\}E(X-r)^{+} = \left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)\sigma_l\sqrt{\frac{Q}{R}}G(u). \tag{4.8}$$

When the replenishment lead time is too long, *i.e.*, $l(Q,R)\to\infty$ then total backordered quantity is unsold whereas all backordered quantities are sold when $\alpha\to 0$, *i.e.*, the mean time of patience to wait $\left(\frac{1}{\alpha}\right)$ tends to infinity.

Further, at the beginning of each replenishment cycle, the retailer's expected net inventory is the safety stock $u\sigma_l\sqrt{\frac{Q}{R}}$ plus the previous replenishment cycle's lost sales $\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)\sigma_l\sqrt{\frac{Q}{R}}G(u)$, and the expected net inventory level immediately after a replenishment is $Q+u\sigma_l\sqrt{\frac{Q}{R}}+\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)\sigma_l\sqrt{\frac{Q}{R}}G(u)$.
Therefore, the expected average inventory over a replenishment cycle is

$$Q+u\sigma_l\sqrt{\frac{Q}{R}}+\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)\sigma_l\sqrt{\frac{Q}{R}}G(u)-Dt \text{ for } t\in\left[0,\frac{Q}{D}\right]\cdot \tag{4.9}$$

Therefore, the expected inventory holding cost for the buyer under time value of money is

$$\begin{aligned} I_c &= \int_{t=0}^{Q/D} H_b\left[Q+u\sigma_l\sqrt{\frac{Q}{R}}+\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)\sigma_l\sqrt{\frac{Q}{R}}G(u)-Dt\right]\mathrm{e}^{-jt}\,\mathrm{d}t \\ &= \frac{H_b}{j}\left[\left\{Q+u\sigma_l\sqrt{\frac{Q}{R}}+\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)\sigma_l\sqrt{\frac{Q}{R}}G(u)\right\}\left(1-\mathrm{e}^{-\frac{Qj}{D}}\right)\right. \\ &\quad \left. +Q\,\mathrm{e}^{-\frac{Qj}{D}}+\left(\mathrm{e}^{-\frac{Qj}{D}}-1\right)\frac{D}{j}\right]. \end{aligned} \tag{4.10}$$

The backorder cost is

$$S_c = \left[b+\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)b_0\right]\sigma_l\sqrt{\frac{Q}{R}}G(u) \tag{4.11}$$

The buyer's expected total cost is

$$\begin{aligned} \mathrm{ETC}_b &= \text{ordering cost } + \text{ holding cost } + \text{ backorder cost} \\ &= C_o+\frac{H_b}{j}\left[\left\{Q+u\sigma_l\sqrt{\frac{Q}{R}}+\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)\sigma_l\sqrt{\frac{Q}{R}}G(u)\right\}\left(1-\mathrm{e}^{-\frac{Qj}{D}}\right)\right. \\ &\quad \left.+Q\mathrm{e}^{-\frac{Qj}{D}}+\left(\mathrm{e}^{-\frac{Qj}{D}}-1\right)\frac{D}{j}\right]+\left[b+\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)b_0\right]\sigma_l\sqrt{\frac{Q}{R}}G(u). \end{aligned} \tag{4.12}$$

The vendor's expected total cost is

$$\mathrm{ETC}_v = \text{setup cost } + \text{ holding cost} \tag{4.13}$$

where vendor's holding cost is $H_v\frac{QD}{2R}\int_{t=0}^{Q/D}\mathrm{e}^{-jt}\,\mathrm{d}t$

$$= \frac{H_v}{j}\left(1-\mathrm{e}^{-\frac{Qj}{D}}\right)\frac{QD}{2R}\cdot \tag{4.14}$$

Therefore, the vendor's expected total cost is

$$\mathrm{ETC}_v = C_s + \frac{H_v}{j}\left(1 - \mathrm{e}^{-\frac{Qj}{D}}\right)\frac{QD}{2R}\cdot \tag{4.15}$$

We now use discounted cash flow approach (Moon and Yun [29]). There are cash outflows for the ordering cost, lead time crashing cost, and stockout cost at the beginning of each cycle. Therefore, the expected total relevant cost of the supply chain is

$$\begin{aligned}\mathrm{ETC} = C_o &+ \frac{H_b}{j}\left[\left\{Q + u\sigma_l\sqrt{\frac{Q}{R}} + \left(1 - \mathrm{e}^{-\frac{\alpha Q}{R}}\right)\sigma_l\sqrt{\frac{Q}{R}}G(u)\right\}\left(1 - \mathrm{e}^{-\frac{Qj}{D}}\right)\right.\\ &\left. + Q\mathrm{e}^{-\frac{Qj}{D}} + \left(\mathrm{e}^{-\frac{Qj}{D}} - 1\right)\frac{D}{j}\right] + \left[b + \left(1 - \mathrm{e}^{-\frac{\alpha Q}{R}}\right)b_0\right]\sigma_l\sqrt{\frac{Q}{R}}G(u) + C_s\\ &+ \frac{H_v}{j}\left(1 - \mathrm{e}^{-\frac{Qj}{D}}\right)\frac{QD}{2R}\cdot\end{aligned} \tag{4.16}$$

Our goal is to reduce the replenishment lead time by increasing the production rate of the vendor. If the buyer requests the vendor to increase the production rate, the buyer will be asked by the vendor for added cost in order to achieve this. In this case, the extra cost that is induced by the difference between the desired production rate and the regular production rate is given by (see Moon and Cha [30])

$$(R - R_0)l(Q, R)S = \left(1 - \frac{R_0}{R}\right)QS. \tag{4.17}$$

Hence, incorporating productivity improvement cost, (4.16) becomes

$$\begin{aligned}\mathrm{ETC} = C_o &+ \frac{H_b}{j}\left[\left\{Q + u\sigma_l\sqrt{\frac{Q}{R}} + \left(1 - \mathrm{e}^{-\frac{\alpha Q}{R}}\right)\sigma_l\sqrt{\frac{Q}{R}}G(u)\right\}\left(1 - \mathrm{e}^{-\frac{Qj}{D}}\right)\right.\\ &\left. + Q\mathrm{e}^{-\frac{Qj}{D}} + \left(\mathrm{e}^{-\frac{Qj}{D}} - 1\right)\frac{D}{j}\right] + \left[b + \left(1 - \mathrm{e}^{-\frac{\alpha Q}{R}}\right)b_0\right]\sigma_l\sqrt{\frac{Q}{R}}G(u) + C_s\\ &+ \frac{H_v}{j}\left(1 - \mathrm{e}^{-\frac{Qj}{D}}\right)\frac{QD}{2R} + \left(1 - \frac{R_0}{R}\right)QS.\end{aligned} \tag{4.18}$$

Then the net present value of expected total cost of the supply chain is (Silver *et al.* [46]),

$$\begin{aligned}&\mathrm{PVETC}(Q, u, R)\\ &= \frac{1}{1 - \mathrm{e}^{-\frac{Qj}{D}}}\left[\overbrace{C_o}^{\text{buyer's ordering cost}} + \overbrace{C_s}^{\text{vendor's setup cost}} + \overbrace{\frac{H_v}{j}(1 - \mathrm{e}^{-\frac{Qj}{D}})\frac{QD}{2R}}^{\text{vendors holding cost}}\right.\\ &+ \overbrace{\frac{H_b}{j}\left\{\left(Q + u\sigma_l\sqrt{\frac{Q}{R}} + \left(1 - \mathrm{e}^{-\frac{\alpha Q}{R}}\right)\sigma_l\sqrt{\frac{Q}{R}}G(u)\right)(1 - \mathrm{e}^{-\frac{Qj}{D}}) + Q\mathrm{e}^{-\frac{Qj}{D}} + (\mathrm{e}^{-\frac{Qj}{D}} - 1)\frac{D}{j}\right\}}^{\text{buyers holding cost}}\\ &\left. + \overbrace{\left[b + \left(1 - \mathrm{e}^{-\frac{\alpha Q}{R}}\right)b_0\right]\sigma_l\sqrt{\frac{Q}{R}}G(u)}^{\text{shortages cost}} + \overbrace{\left(1 - \frac{R_0}{R}\right)QS}^{\text{productivity improvement cost}}\right].\end{aligned} \tag{4.19}$$

4.1. Solution methodology

As the cost function given by (4.19) is highly nonlinear, it is not possible to prove the convexity of the cost function with respect to all the decision variables jointly. Therefore, an iterative algorithm is developed to find the optimal solution of the developed model. Some convexity and concavity properties with respect to the control parameters are also derived. This shows that the solution obtained from the algorithm is a global minimum.

Now, for a fixed value of R, the first order partial derivatives of PVETC in (4.19) with respect to u and Q yield

$$\frac{\partial PVETC}{\partial u} = \frac{H_b}{j}\sigma_l\sqrt{\frac{Q}{R}} - \sigma_l\sqrt{\frac{Q}{R}}\left[\frac{H_b}{j}\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right) + \frac{b+\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)b_0}{1-\mathrm{e}^{-\frac{Qj}{D}}}\right][1-F(u)], \tag{4.20}$$

and

$$\begin{aligned}\frac{\partial \mathrm{PVETC}}{\partial Q} = {}& \frac{(1-\mathrm{e}^{-\frac{Qj}{D}} - \frac{Qj}{D}\mathrm{e}^{-\frac{Qj}{D}})\left[\frac{H_b}{j} + S\left(1-\frac{R_0}{R}\right)\right]}{\left(1-\mathrm{e}^{-\frac{Qj}{D}}\right)^2} - \frac{je^{-\frac{Qj}{D}}\left(\frac{C_o+C_s}{D}\right)}{\left(1-\mathrm{e}^{-\frac{Qj}{D}}\right)^2} + \frac{u\sigma_l\sqrt{\frac{Q}{R}}H_b}{2Qj}\\ & + \frac{\sqrt{\frac{Q}{R}}\left(1-\mathrm{e}^{-\frac{Qj}{D}} - 2\frac{Qj}{D}\mathrm{e}^{-\frac{Qj}{D}}\right)\sigma_l G(u)\left(b+\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)b_0\right)}{2Q\left(1-\mathrm{e}^{-\frac{Qj}{D}}\right)^2} + \frac{DH_v}{2Rj}\\ & + \left[\frac{1-\mathrm{e}^{-\frac{\alpha Q}{R}}}{2R\sqrt{\frac{Q}{R}}} + \frac{\alpha\mathrm{e}^{-\frac{\alpha Q}{R}}\sqrt{\frac{Q}{R}}}{R}\right]\frac{H_b}{j}\sigma_l G(u) + \frac{b_0\alpha\mathrm{e}^{-\frac{\alpha Q}{R}}}{R(1-\mathrm{e}^{-\frac{Qj}{D}})}\sigma_l\sqrt{\frac{Q}{R}}G(u).\end{aligned} \tag{4.21}$$

Now, for fixed R and Q, PVETC is convex in u, since

$$\frac{\partial^2\mathrm{PVETC}}{\partial u^2} = \sigma_l\sqrt{\frac{Q}{R}}\left[\frac{H_b}{j}\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right) + \frac{b+\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)b_0}{1-\mathrm{e}^{-\frac{Qj}{D}}}\right]f(u) > 0\cdot \tag{4.22}$$

for all $u > 0$. Solving $\frac{\partial\mathrm{PVETC}}{\partial u} = 0$ for u, we get the optimal safety factor for a given lot-size (Q) and production rate (R) as

$$u = \overline{F}^{-1}\left(\frac{H_b\left(1-\mathrm{e}^{-\frac{Qj}{D}}\right)}{\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)\left[H_b\left(1-\mathrm{e}^{-\frac{Qj}{D}}\right)+jb_0\right]+jb}\right). \tag{4.23}$$

Next, differentiating (4.21) with respect to Q we get

$$\begin{aligned}&\frac{\partial^2\mathrm{PVETC}}{\partial Q^2}\\ &= \left(\frac{H_b}{D} + \frac{Sj}{D}\left(1-\frac{R_0}{R}\right)\right)f_1 + \frac{j^2(C_0+C_s)}{D^2}f_2 + \frac{2\sigma_l\overline{\pi}j\sqrt{\frac{Q}{R}}G(u)}{DQf_3^3}f_4 + \frac{j^2\mathrm{e}^{-\frac{Qj}{D}}\sigma_l\sqrt{\frac{Q}{R}}G(u)\overline{\pi}}{D^2f_3^2}\\ &\quad + \frac{\alpha\sigma\mathrm{e}^{-\frac{\alpha Q}{R}}G(u)\sqrt{\frac{Q}{R}}}{RQ}\left(\frac{H_b(R-Q\alpha)}{Rj} + \frac{b_0f_5}{f_3^2}\right) + \frac{\sigma_l\sqrt{\frac{Q}{R}}\frac{Qj}{D}^{\frac{Qj}{D}}\overline{\pi}G(u)}{Q^2f_3^2} - \frac{\sigma_l\sqrt{\frac{Q}{R}}\overline{\pi}G(u)}{4f_3Q^2}\\ &\quad - \frac{\mathrm{e}^{-\frac{\alpha Q}{R}}\mathrm{e}^{-\frac{Qj}{D}}(Df_4\alpha+Rj)\sigma_l\sqrt{\frac{Q}{R}}G(u)b_0\alpha}{R^2Df_3^2} - \frac{\sigma_l\sqrt{\frac{Q}{R}}}{4Q^2}\left(u+(1-\mathrm{e}^{-Q\alpha/R})G(u)\right)\frac{H_b}{j}\end{aligned} \tag{4.24}$$

where $\overline{\pi} = b + \left(1 - \mathrm{e}^{-\frac{\alpha Q}{R}}\right) b_0 > 0$, $f_1 = \mathrm{e}^{-\frac{Qj}{D}} \frac{(2+\frac{Qj}{D})\mathrm{e}^{-\frac{Qj}{D}} - 2 + \frac{Qj}{D}}{(1-\mathrm{e}^{-\frac{Qj}{D}})^3} > 0$, $f_2 = \frac{\mathrm{e}^{-\frac{Qj}{D}}(1+\mathrm{e}^{-\frac{Qj}{D}})}{(1-\mathrm{e}^{-\frac{Qj}{D}})^3} > 0$, $f_3 = 1 - \mathrm{e}^{-\frac{Qj}{D}} > 0$, $f_4 = 1 - \mathrm{e}^{\frac{Qj}{D}} + \frac{Qj}{D}\mathrm{e}^{\frac{Qj}{D}} > 0$, and $f_5 = 1 - \mathrm{e}^{-\frac{Qj}{D}} - \frac{Qj}{D}\mathrm{e}^{-\frac{Qj}{D}} > 0$.
From (25) it is difficult to check the sign of $\frac{\partial^2 \mathrm{PVETC}}{\partial Q^2}$ analytically. We will check the sign of the second order derivative in the numerical section later.
Further, for given Q and u, PVETC can not be shown to be convex in R. We develop the following lemma to obtain the optimal value of production rate R.

Lemma 4.1. *For fixed values of Q and u, the cost function PVETC is decreasing or increasing or concave function in R when $R_0 \leq R \leq R_{\max}$. Therefore, the optimal value of R that minimizes PVETC is either R_0 or $R_{\max}$.*

Proof. The first order partial derivative of PVETC with respect to R is

$$\begin{aligned}
\frac{\partial \mathrm{PVETC}}{\partial R} &= \frac{1}{1-\mathrm{e}^{-\frac{Qj}{D}}}\Bigg[-\frac{Q\sigma_l G(u)}{2R^2}\left\{b_0\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)+b_1\right\}\left(\frac{Q}{R}\right)^{-1/2} - \frac{b_0\alpha\sigma_l Q\mathrm{e}^{-\frac{\alpha Q}{R}}\sqrt{\frac{Q}{R}}G(u)}{R^2} \\
&\quad - \frac{H_b}{j}(1-\mathrm{e}^{-\frac{Qj}{D}})\left\{\frac{uQ\sigma_l}{2R^2\sqrt{\frac{Q}{R}}} + \frac{\alpha\sigma_l Q\mathrm{e}^{-\frac{\alpha Q}{R}}\sqrt{\frac{Q}{R}}G(u)}{R^2} + \frac{Q\sigma_l\left(1-\mathrm{e}^{-\frac{\alpha Q}{R}}\right)G(u)}{2R^2\sqrt{\frac{Q}{R}}}\right\} \\
&\quad - \frac{DQH_v(1-\mathrm{e}^{-\frac{Qj}{D}})}{2R^2 j} + \frac{QSR_0}{R^2}\Bigg] \\
&= \frac{1}{2R^2}\left(a_1 - a_2\sqrt{R} + a3\frac{(R-2\alpha Q)\mathrm{e}^{-\frac{\alpha Q}{R}}}{\sqrt{R}}\right), \qquad (4.25)
\end{aligned}$$

where $a_1(Q) = \frac{2QSR_0}{1-\mathrm{e}^{-\frac{Qj}{D}}} - \frac{DQH_v}{j}$, $a_2(Q) = \sqrt{Q}\left(\frac{\sigma_l(b_1+b_0)G(u)}{1-\mathrm{e}^{-\frac{Qj}{D}}} + \frac{\sigma_l H_b}{j}[u+G(u)]\right)$,
$a_3(Q) = \sqrt{Q}\left(\frac{\sigma_l b_0 G(u)}{1-\mathrm{e}^{-\frac{Qj}{D}}} + \frac{\sigma_l H_b}{j}G(u)\right)$. □

Case (i) $a_1 - a_2\sqrt{R_0} + a3\frac{(R_0-2\alpha Q)\mathrm{e}^{-\frac{\alpha Q}{R_0}}}{\sqrt{R_0}} < 0$.
In this case, PVETC is a strictly decreasing function of R. Therefore, the minimum total cost will occur at the maximum point $R_{\max}$. Hence $R_{\max}$ is the optimal solution minimizing PVETC.

Case (ii) $a_1 - a_2\sqrt{R_{\max}} + a3\frac{(R_{\max}-2\alpha Q)\mathrm{e}^{-\frac{\alpha Q}{R_{\max}}}}{\sqrt{R_{\max}}} > 0$.
In this case, PVETC is a strictly increasing function of R. Therefore, the minimum cost will occur at the minimum point R_0. Hence R_0 is the optimal solution minimizing PVETC.

Case (iii) $a_1 - a_2\sqrt{R_0} + a3\frac{(R_0-2\alpha Q)\mathrm{e}^{-\frac{\alpha Q}{R_0}}}{\sqrt{R_0}} > 0$ and $a_1 - a_2\sqrt{R_{\max}} + a3\frac{(R_{\max}-2\alpha Q)\mathrm{e}^{-\frac{\alpha Q}{R_{\max}}}}{\sqrt{R_{\max}}} < 0$.
In this case, PVETC is a concave function of R for all $R \in [R_0, R_{\max}]$. Therefore, the optimal production rate that minimizes PVETC for fixed Q and u can be selected as either R_0 or $R_{\max}$ by comparing PVETC(Q, u, R_0) and PVETC$(Q, u, R_{\max})$.

5. Numerical experiments

In this section, we provide four numerical examples using different data sets to investigate how the optimal decision variables change with the model-parameters.

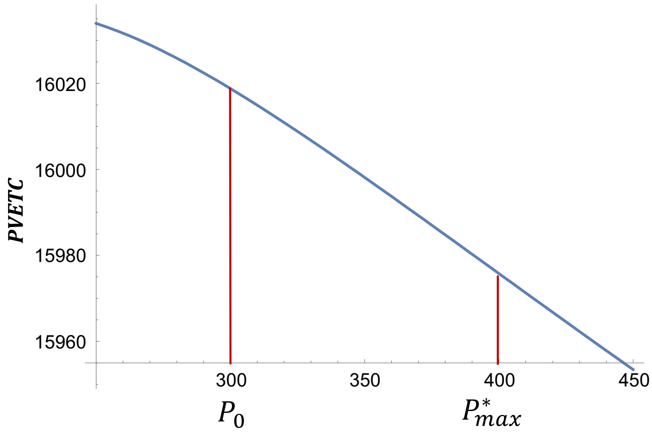

FIGURE 2. Graph of PJETC for Example 5.1 (optimal solution is $R_{\max}$).

TABLE 2. Numerical results of Example 5.1.

$R = R_0(= 300)$							$R = R_{\max}(= 400)$							LT reduction(%)
Q	u	r	SS	l	$\delta(l)$	TC	Q	u	r	SS	l	$\delta(l)$	TC	
183	1.85	144	22	0.6097	0.5956	15 700	**190**	**1.80**	**114**	**19**	**0.4758**	**0.6673**	**15 648**	21.96

Example 5.1. $D = 200$ units/year, $R_0 = 300$ units/year, $R_{\max} = 400$ units/year $C_o = \$300/$ order, $C_s = \$500/$ order, $H_b = \$6$/unit/unit time, $H_v = \$4$/unit/unit time, $\sigma_l = 15$ units, $S = \$1.5$, $b_0 = \$150$, $b = \$100$, $j = \$0.12$, $\alpha = 0.85$.

Using the solution algorithm, the optimal results are found for the case when the lead time demand follows normal distribution. Here we have $a_1 - a_2\sqrt{R_0} + a3\frac{(R_0-2\alpha Q)\mathrm{e}^{-\frac{\alpha Q}{R_0}}}{\sqrt{R_0}} = -70\,127 < 0$. Therefore, following Lemma 4.1(i), we can say that PVETC is strictly decreasing function of R and the minimum cost will occur when the production rate is maximum *i.e.*, $R = R_{\max}$ (see Fig. 2). The detailed results are given in Table 2. From Table 2, we have the optimal results as follows: production rate $R(= R_{\max}) = 400$, order quantity $Q = 190$ units, safety factor $u = 1.8045$, reorder point $r = 114$ units, safety stock SS $= 19$ units, lead time $l = 0.4758$ time units, backorder rate $\delta(l) = 0.6673$ and the net present value of the expected total cost PVETC $= \$15\,648$. It is observed that reductions in lead time by 21.96% and NPV of the expected total cost by \$52 are possible by running the production system at it's maximum level.

Example 5.2. $D = 150$ units/year, $R_0 = 300$ units/year, $R_{\max} = 400$ units/year $C_o = \$60/$ order, $C_s = \$300/$ order, $H_b = \$12$/unit/unit time, $H_v = \$9$/unit/unit time, $\sigma_l = 5$ units, $S = \$1.5$, $b_0 = \$100$, $b = \$20$, $j = \$0.11$, $\alpha = 0.85$.

Here we have $a_1 - a_2\sqrt{R_{\max}} + a3\frac{(R_{\max}-2\alpha Q)\mathrm{e}^{-\frac{\alpha Q}{R_{\max}}}}{\sqrt{R_{\max}}} = 171\,234 > 0$. Therefore, from Lemma 4.1(ii), we can say that PVETC is a strictly increasing function of R and the minimum cost will occur when the production rate is minimum *i.e.*, $R = R_{\min}$ (see Fig. 3). The detailed results are given in Table 3. From Table 3, we have the optimal results as follows: production rate $R(= R_{\min}) = 300$, order quantity $Q = 79$ units, safety factor $u = 1.04$, reorder point $r = 42$ units, safety stock SS $= 3$ units, lead time $l = 0.2645$ time units, backorder rate $\delta(l) = 0.7986$ and PVETC $= \$12\,799$.

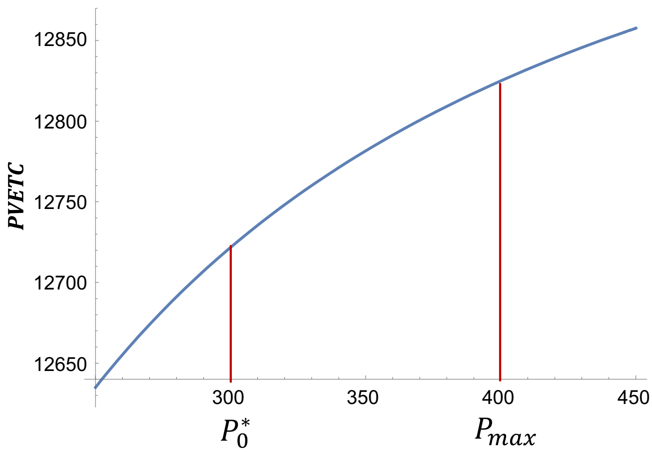

FIGURE 3. Graph of PJETC for Example 5.2 (optimal solution is R_0).

TABLE 3. Numerical results of Example 5.2.

$R = R_0(= 300)$							$R = R_{\max}(= 400)$							LT reduction(%)
Q	u	r	SS	l	$\delta(l)$	TC	Q	u	r	SS	l	$\delta(l)$	TC	
79	**1.04**	**42**	**3**	**0.2645**	**0.7986**	**12 799**	82	0.94	33	2	0.2056	0.8396	12 837	–

TABLE 4. Numerical results of Example 5.3.

$R = R_0(= 300)$							$R = R_{\max}(= 400)$							LT reduction(%)
Q	u	r	SS	l	$\delta(l)$	TC	Q	u	r	SS	l	$\delta(l)$	TC	
97	**1.92**	**82**	**49**	**0.3246**	**0.7589**	**7741**	100	1.89	68	43	0.2509	0.8079	7753	22.70

Example 5.3. $D = 100$ units/year, $R_0 = 300$ units/year, $R_{\max} = 400$ units/year $C_o = \$100/$ order, $C_s = \$200/$ order, $H_b = \$4$/unit/unit time, $H_v = \$1$/unit/unit time, $\sigma_l = 45$ units, $S = \$1.5$, $b_0 = \$150$, $b = \$100$, $j = \$0.1$, $\alpha = 0.85$.

Here we have $a_1 - a_2\sqrt{R_0} + a3\frac{(R_0-2\alpha Q)\mathrm{e}^{-\frac{\alpha Q}{R_0}}}{\sqrt{R_0}} = 82\,034 > 0$ and $a_1 - a_2\sqrt{R_{\max}} + a3\frac{(R_{\max}-2\alpha Q)\mathrm{e}^{-\frac{\alpha Q}{R_{\max}}}}{\sqrt{R_{\max}}} = -31\,848 < 0$. Therefore, from Lemma 4.1(iii), we can say that PVETC is a concave function of R and the minimum present value of joint expected total cost will occur either at the minimum production rate or maximum production rate. Now, we have $\text{PVETC}|_{R=R_0} = 7741 < \text{PVETC}|_{R=R_{\max}} = 7753$. Therefore, the minimum present value of joint expected total cost will occur when the production rate is minimum (see Fig. 4). The detailed results are given in Table 4. From Table 4, we have the optimal results as follows: production rate $R(= R_{\min}) = 300$, order quantity $Q = 97$ units, safety factor $u = 1.92$, reorder point $r = 82$ units, safety stock SS = 49 units, lead time $l = 0.3246$ time units, backorder rate $\delta(l) = 0.7589$, and PVETC = \$7741.

Example 5.4. $D = 180$ units/year, $R_0 = 300$ units/year, $R_{\max} = 400$ units/year $C_o = \$150/$ order, $C_s = \$250/$ order, $H_b = \$4$/unit/unit time, $H_v = \$1$/unit/unit time, $\sigma_l = 62$ units, $S = \$1.5$, $b_0 = \$200$, $b = \$100$, $j = \$0.1$, $\alpha = 0.85$.

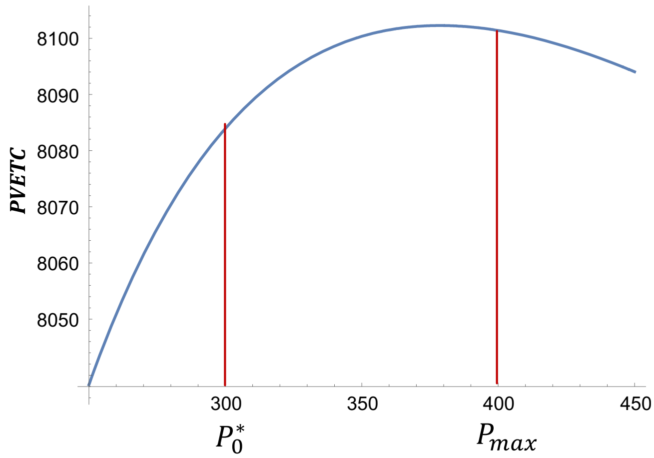

FIGURE 4. Graph of PJETC for Example 5.3 (optimal solution is R_0).

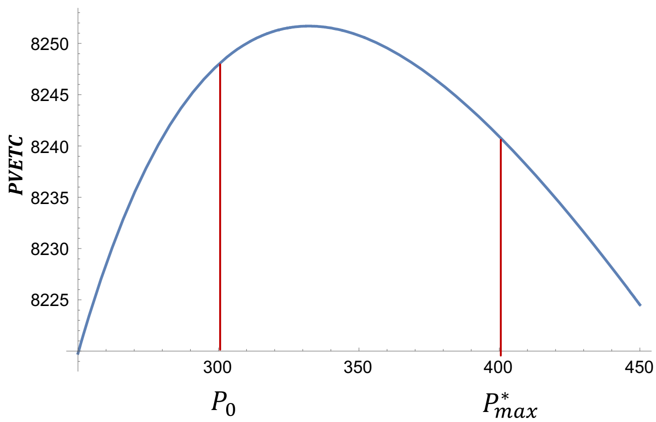

FIGURE 5. Graph of PJETC for Example 5.4 (optimal solution is $R_{\max}$).

Here we have $a_1 - a_2\sqrt{R_0} + a3\frac{(R_0-2\alpha Q)\mathrm{e}^{-\frac{\alpha Q}{R_0}}}{\sqrt{R_0}} = 48\,642 > 0$ and $a_1 - a_2\sqrt{R_{\max}} + a3\frac{(R_{\max}-2\alpha Q)\mathrm{e}^{-\frac{\alpha Q}{R_{\max}}}}{\sqrt{R_{\max}}} = -169\,350 < 0$. Therefore, from Lemma 4.1(iii), we can say that PVETC is a concave function of R and the minimum cost will occur either at the minimum production rate or maximum production rate. Now, we have $\text{PVETC}|_{R=R_0} = 12768 > \text{PVETC}|_{R=R_{\max}} = 12\,745$. Therefore, the minimum present value of joint expected total cost will occur when the production rate is maximum (see Fig. 5). The detailed results are given in Table 5. From Table 5, we find the optimal results as follows: production rate $R(= R_{\max}) = 400$, order quantity $Q = 148$ units, safety factor $u = 2.04$, reorder point $r = 144$ units, safety stock SS $= 77$ units, lead time $l = 0.3701$ time units, backorder rate $\delta(l) = 0.7301$ and PVETC $= \$12\,745$.

In Figure 6, the convexity of PVETC with respect to Q is shown.

5.1. Sensitivity analysis

To obtain insights of the behavior of the model, a brief sensitivity analysis is conducted in this section by varying several model-parameters. The sensitivity analysis is performed based on the parameter-values of

TABLE 5. Numerical results of Example 5.4.

$R = R_0 (= 300)$							$R = R_{max} (= 400)$							LT reduction(%)
Q	u	r	SS	l	$\delta(l)$	TC	Q	u	r	SS	l	$\delta(l)$	TC	
142	2.09	175	89	0.4744	0.6682	12 768	**148**	**2.04**	**144**	**77**	**0.3701**	**0.7301**	**12 745**	21.98

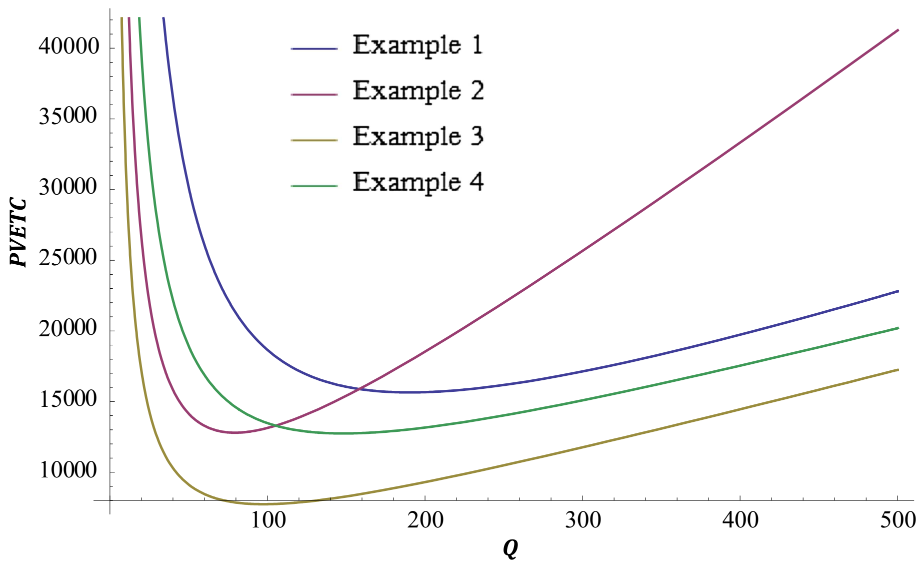

FIGURE 6. Graphical representation of JETC with respect to Q for Examples 5.1–5.4.

Example 5.3. The optimal decisions that minimize the net present value of the expected total cost of the supply chain are found for three different scenarios – high demand deviation ($\sigma_l = 50$) scenario, medium demand deviation ($\sigma_l = 30$) scenario, and low demand deviation ($\sigma_l = 10$) scenario. For these scenarios, we study the cost minimizing decision variables and the expected total cost for a varying productivity improvement cost (S). An increasing value of demand deviation increases the chance of stock-out probability. Therefore, the present value of the expected total cost increases for all three scenarios. In Figure 7, it is observed that the order quantity is insensitive to productivity improvement cost for low demand deviation. This is due to the fact that for low demand deviation, there is no need to invest in order to improve the productivity of the system (see Fig. 8). However, for high and medium demand deviations, order quantities are sensitive to productivity improvement cost and a step-wise decrease in order quantity can be seen in Figure 7. Therefore, for the case of medium and high demand deviations, it is beneficial to decrease the replenishment lead time by ordering less quantity.

The effect of variation in productivity improvement cost (S) on the production rate (R) is illustrated in Figure 8. It is observed that for low demand deviation, the production rate is kept constant at it's minimum level, *i.e.*, $R = 300$, which means that there is no need to increase the production rate for low demand deviation as in this case the chance of stock-out is less. However, for the the case of medium and high demand deviations, the production rate is kept at it's maximum level *i.e.*, $R = 400$ until S reaches 1 and 1.5 and R is decreased afterwards as well. This result is practical because, as the demand deviation increases, the stock-out probability increases which pushes the supply chain manager to run the production system with maximum production rate to reduce the replenishment lead time and prevent inventory stock-out.

Figures 9 and 10 exhibit the effects of productivity improvement cost (S) on safety stock level and replenishment lead time. From Figure 9, we see that, for the case of high demand uncertainty, the safety stock level should

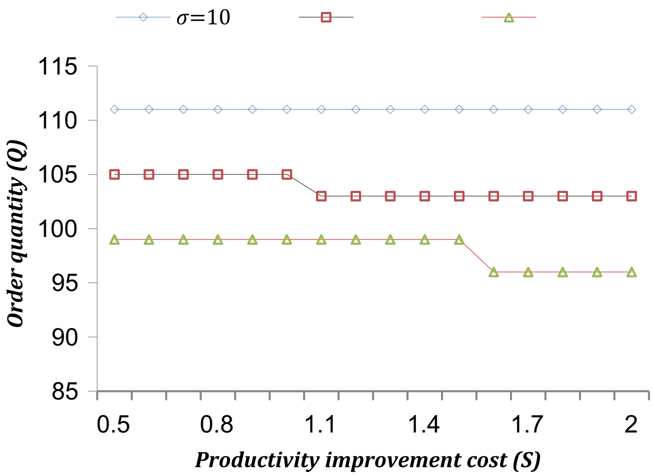

FIGURE 7. S *vs.* Q for different value of σ.

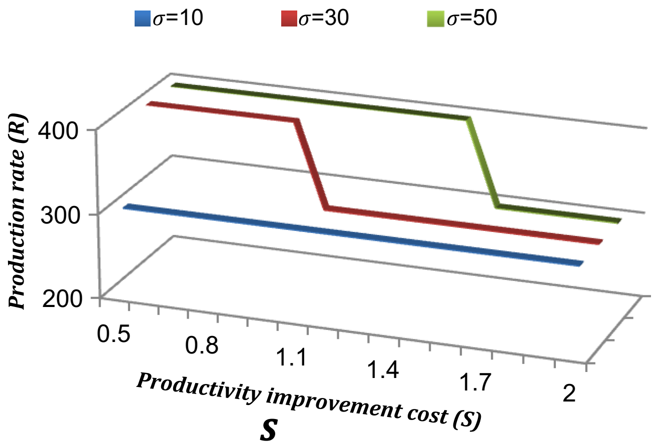

FIGURE 8. S *vs.* R for different value of σ

be higher than medium and low demand uncertainties in order to defend the system from the risk of stock-out probability. Figure 10 illustrates that replenishment lead time for low demand uncertainty is high and constant compared to medium and high demand uncertainties, respectively. This behavior is due to the fact that, for low demand uncertainty, the order quantity is high (see Fig. 7) and production rate is minimum (see Fig. 8) which leads to higher replenishment lead time. On the other hand, for the case of medium and high demand uncertainties, the replenishment lead time increases after a certain time due to minimum production rate.

Figure 11 presents the effect of coefficient of backorder rate (α) ranging from 0 to 15 on the expected total cost and backorder rate. Figure 11 shows that the coefficient of backorder rate is proportional to the total supply chain cost and inversely proportional to the backorder rate. A higher value of α increases the mean waiting time, thereby it decreases the backorder rate and hence it increases the total supply chain cost.

Figure 12 illustrates the effects of buyer's holding cost on optimal decisions. Figure 12 depicts that, with higher holding cost, both the order quantity and safety stock level decrease. The buyer's replenishment lead time decreases due to decrease in order quantity. It is observed that the buyer's holding cost plays an important

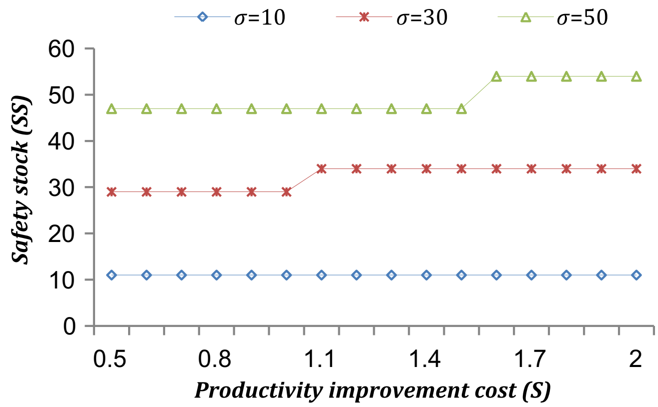

FIGURE 9. S *vs.* SS for different value of σ.

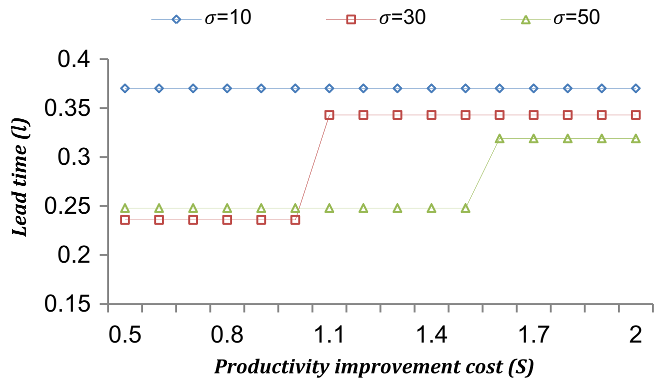

FIGURE 10. S *vs.* l for different value of σ.

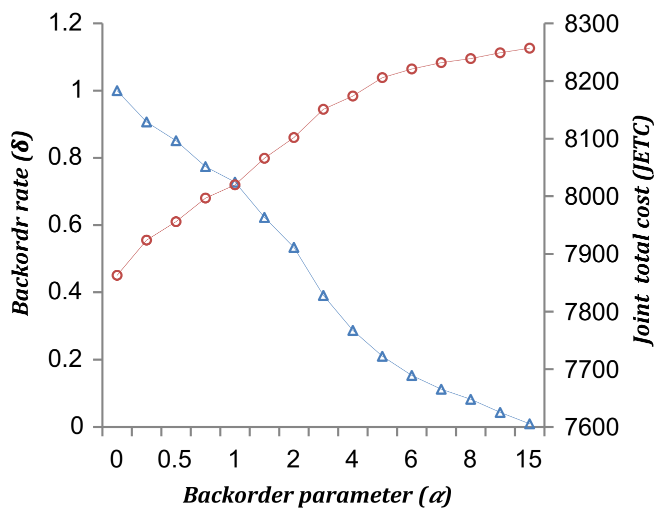

FIGURE 11. α *vs.* δ and $\boldsymbol{JETC}$.

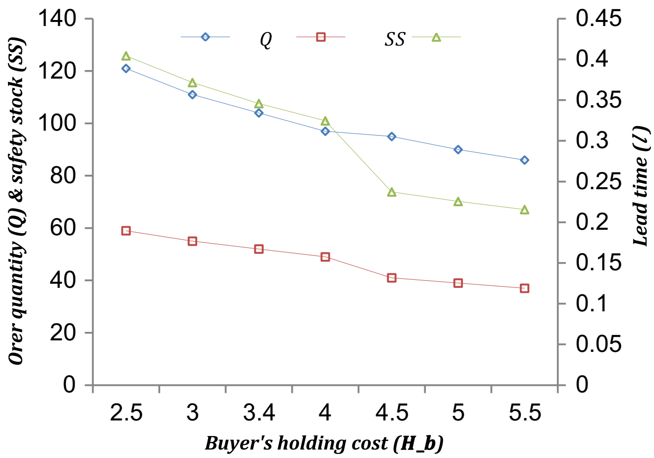

FIGURE 12. $\boldsymbol{H\_b}$ *vs.* $\boldsymbol{Q}$, $\boldsymbol{SS}$, and $\boldsymbol{l}$.

role in deciding the optimal safety stock level. The more the safety stock is, the higher is the holding cost, and hence higher is the supply chain cost.

5.2. Managerial insights

I. In real life situations, it is quite natural that the demand is random. When the demand deviation is high, the chance of inventory stock-out increases. To overcome this situation, it is advisable to the supply chain managers to store more safety stocks. Moreover, when the demand deviation is high, it is always preferable to cut down the replenishment lead time through increasing the production rate.

II. Determining reorder point is an important decision in continuous review inventory system. A very early order placement can increase the inventory holding cost whereas a very late order placement can put the system in stock-out situation. Supply chain managers can decide the optimum reorder point following the strategies of the proposed model.

III. From the computational results, it is observed that if the holding cost starts to increase, the total cost of the supply chain shoots rapidly. So, the supply chain managers must monitor the safety stock and the order quantity for this condition. When the buyer's holding cost is high, it is preferable to store less safety stock.

IV. It is profitable to run the production system with low production rate during low demand deviation. However, for the case of medium and high demand deviations, it is suggested to run the production system with high production rate.

6. Conclusions

In reality, the market demand is highly dependent on the delivery lead time; a little change in lead time affects extremely on the market demand. So, deciding the optimal delivery lead time plays a vital role in optimizing the total cost of a supply chain. In this paper, we have developed a two-echelon supply chain model where the buyer faces stochastic lead time demand from the customers and the lead time is assumed to be a function of order quantity and production rate. The vendor has the option that (s)he can produce the order quantity through maximum or minimum production rate. The backlogiing rate at the buyer is a function of replenishment lead time. Therefore, if the buyer wants to increase the backorder rate, (s)he can reduce the replenishment lead time by some additional investment. From the numerical study, we have found that lead time demand deviation has impressive effect on selecting the optimal production rate. The results of sensitivity analysis reveal that the waiting time of the customer decreases the backorder rate and increases the supply chain cost. Additionally, it is seen that, for comparatively high additional cost, it is not profitable to increase the production rate. Providing price discount on backordered items would be an interesting point to extend the proposed model (Sarkar and Giri [43], Kim *et al.* [18]). Another possible extension of our study could consider multiple buyers purchasing a single (multiple) item(s) from a supplier (Kumar and Uthayakumar [20], Mandal *et al.* [27]).

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