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VARIANT IMPATIENT BEHAVIOR OF A MARKOVIAN QUEUE WITH BALKING RESERVED IDLE TIME AND WORKING VACATION

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Abstract. The customers' impatience and its effect plays a major role in the economy of a country. It directly affects the sales of products and profit of a trading company. So, it is very important to study various impatient behaviors of customers and to analyze different strategies to hold such impatient customers. This situation is modeled mathematically in this research work along with working vacation and reserved idle time of server, balking and re-service of customers. This paper studies the transient analysis of an $M/M/1$ queueing model with variant impatient behavior, balking, re-service, reserved idle time and working vacation. Whenever the system becomes empty, the server resumes working vacation. When he is coming back from the working vacation and finding the empty system, he stays idle for a fixed time period known as reserved idle time and waits for an arrival. If an arrival occurs before the completion of reserved idle time, the server starts a busy period. Otherwise, he resumes another working vacation after the completion of reserved idle time. During working vacation, the arriving customers may either join or balk the queue. The customers waiting in the queue for service, during working vacation period, become impatient. But, the customer who is receiving the service in the slow service rate, does not become impatient. After each service, the customer may demand for immediate re-service. The transient system size probabilities for the proposed model are derived using generating function and continued fraction. The time-dependent mean and variance of system size are also obtained. Finally, numerical illustrations are provided to visualize the impact of various system parameters.

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1. Introduction

Usually trading centers concentrate to increase their sales as well as profit. Now a days, many trading centers face a unique problem of handling impatient customers. Because, due to the long waiting time in the queue, many customers leave the queue before their turn come for service. It is seen as the most important scenario among the business sectors. In this research work, the analysis of one such situation is carried out.

In recent years, queueing models with working vacation (WV) are mainly focused by the researchers due to their importance and practical applications. In the past few decades, many researchers studied such a queueing models as they have plenty of applications in many fields such as service systems, manufacturing

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industry, communication systems, etc. Whenever the server becomes free after finishing the service of all the customers, he resumes WV. During the WV period, the server serves the arriving customers with a slow service rate than the regular service rate [20,21,23,24].

If the server is in WV period, the arriving customers become impatient due to the long waiting time and may decide to either join the queue or balk the queue [7,8,11,12]. During the WV period, the arriving customers after joining the queue, may also decide to activate an impatient timer, independent of each other. This is called the impatient behavior of customers [1,2,13,24,25]. If there is no customer in the system at the completion time of vacation, the server stays idle for a reserved idle time period and waits for an arriving customers to provide service. If a customer arrives within the reserved idle time period, the server starts a busy period. Otherwise, he resumes another WV after the completion of reserved idle time period. The idea of reserved idle time was initiated by Zhang and Wang [26].

The working vacation concept was introduced by Servi and Finn [19] in which they have explained the notion of WV with numerous examples. The steady state and time-dependent system distributions were derived by Sudhesh and Raj [20] for a single server Markovian queueing model. Selvaraju and Goswami [18] obtained the distributions of size of the system for the $M/M/1$ queueing model with WV under stationary case. They considered the situation that the customer in service does not become impatient whereas the customers in the queue only become impatient during WV. Sudhesh and Azhagappan [21] investigated an $M/M/1$ queue with vacation of server, impatient customers and a waiting server. They obtained the transient solution, mean and variance for that model.

Sudhesh *et al.* [24] analyzed the $M/M/1$ model in which the server commences WV (single and multiple) whenever the server becomes free after completing the service of all the customers in the queue and the customers possess impatient behavior during the WV period. They have derived the system size distributions under transient case. Sudhesh and Azhagappan [22] studied the $M/M/1$ queueing model with working vacation and variant impatient behavior of customers. They also derived the time-dependent system distributions, expectation and variance of system size. Sudhesh and Azhagappan [23] considered an $M/M/\infty$ queueing model with server's additional task and impatient behavior of customers. They derived the system size probabilities for the transient case using continued fraction approach. Azhagappan [3] investigated an $M/M/1$ queue with a waiting server, working vacation and variant impatient behavior of customers. He computed the transient solution for that model, time-dependent mean and variance od size of the system.

Altman and Yechiali [1] studied the impatient behavior of various Markovian and non-Markovian queueing models with multiple and single vacation policies. They derived the system size distributions under steady state. Perel and Yechiali [13] considered a single server Markovian queueing model wherein the server gives two different modes of service such as slow and fast. During the slow service mode, the arriving customers become impatient due to the long waiting time in the queue and decide to activate an independent impatience timer individually. Ammar [2] presented an $M/M/1$ queuing model with multiple vacation and impatient customers. He obtained the time-dependent system distributions, mean and variance of the size of the system. Yu and Liu [25] investigated a Markovian multi server queueing model with impatient behavior of customers. They computed the probability generating functions for different values of the number of server "c".

Zhang and Wang [26] studied a non-Markovian single server retrial queueing model with reserved idle time of server and setup time. Haight [6] introduced the notion of balking behavior of customers in queueing models. Kumar *et al.* [10] considered a single server queueing model and obtained the system probabilities for that queueing model for the transient case. Jain [7] studied a batch arrival queueing model with unreliable server and balking. She derived the probability generating function of the queue size distributions at an arbitrary epoch. Laxmi and Jyothsna [11] analyzed the impatient behavior of customers such as balking and reneging during the working vacation duration of servers in a multi server queueing model. Dshalalow and Merie [4] carried out the fluctuation analysis of queues which were operated under several operational modes and priority customers.

Ganguly *et al.* [5] analyzed the influence of controllable lead time, premium price, and unequal shipments under environmental effects in a supply chain management. Kim and Sarkar [8] considered a complex multi-stage imperfect manufacturing process to clean the production system. They developed an improved algorithm and some theorems to proved to determine the global optimal solution. Kim and Sarkar [9] investigated a supply chain model with stochastic lead time, trade-credit financing and transportation discounts. Malik and Sarkar [12] studied an optimizing a multi-product continuous-review inventory model with uncertain demand, quality improvement, setup cost reduction and variation control in lead time. Sarkar and Majumder [14] studied about the integrated vendor-buyer supply chain. They focused on the reduction of total system cost by reducing the setup cost of the vendor. Sarkar *et al.* [15] analyzed an inventory model in which the customers were offered credit period to buy more products and by this way the suppliers can sell more products by increasing the profit. Sarkar *et al.* [16] studied a quality improvement and backorder price discount under controllable lead time in an inventory model. Sarkar *et al.* [17] investigated a two-echelon supply chain model with manufacturing quality improvement and setup cost reduction.

In this research work, the transient system size probabilities of an $M/M/1$ queueing model with balking, variant impatient behavior, reserved idle time and working vacation are derived using the method of continued fraction and generating function. The system measures like average and variance of number present in the system are obtained. The effects of various parameters are visualized numerically. This research work is a deviated version of the work carried out by Azhagappan [3] and a new version of Sudhesh and Azhagappan [21].

The rest of the sections are organized as follows. In Section 2, problem definition, notations and assumptions to derive the transient system distributions for the $M/M/1$ queueing model with variant impatient behavior, balking, re-service, reserved idle time and working vacation are presented. In Section 3, the governing equations are presented and transient solutions are derived for the proposed model with a special case. In Section 4, the time-dependent mean and variance of system size are obtained. In Section 5, numerical illustrations are presented. In Section 6, the conclusion of the work and its future directions are given.

2. Problem definition, notations and assumptions

This section presents the problem definition, notations and assumption for the model under consideration for the better understanding to the readers.

2.1. Problem definition

The aim of this analysis is to derive the time-dependent system size probabilities of an $M/M/1$ queueing model with variant impatient customers, balking, re-service, working vacations and reserved idle time using the method of generating function and continued fraction. The time-dependent mean and variance of number of customers present in the system are also obtained. Figures 1–4 depict the variations of transient probabilities, mean and variance against time graphically.

2.2. Notations

The following are the notations used in this research work.

λ_0	Arrival rate during working vacations
λ_1	Arrival rate during busy period
μ_0	Service rate during working vacations
μ_1	Service rate during busy period
γ	Working vacation rate
σ	Probability of customers joining the queue
$(1-\sigma)$	Probability of customer's balking the queue
η	Reserved idle time of server
ξ	Customers' impatience rate
α	Probability of customer's leaving the system after the completion of service without demanding re-service
$(1-\alpha)$	Probability of customer's re-service
$Y(t)$	Number of customers in the system at time t
$G(t)$	State of the server at time t

$\{Y(t), G(t), t \geq 0\}$	Two dimensional Markov chain
$\pi_{j,0}(t)$	Probability of "j" customers in the system and the server is in working vacation state
$\pi_{j,1}(t)$	Probability of "j" customers in the system and the server is in busy state
$E(Y(t))$	Expected system size at time t
$E(Y^2(t))$	Second order moment of the system size at time t
$V(Y(t))$	Variance of system size at time t
$G(z,t)$	Probability generating function of $\pi_{j,1}(t)$

2.3. Assumptions

The assumptions to derive the transient system size probabilities for the model under consideration are presented in this section.

- Arrival of customers to join the queue follows a Poisson process with rate λ_0 (λ_1) when the server is in working vacation (busy).
- Customers receive their service in an exponentially distributed manner with rate μ_1.
- Whenever the system becomes empty, the server resumes working vacation which is exponentially distributed with rate γ.
- When he is returning from the working vacation and finding the system empty, the server stays idle for a fixed time period (reserved idle time) which follows an exponential distribution with rate η and waits for an arrival. If an arrival occurs before the completion of reserved idle time, he starts the busy period. Otherwise, he resumes another working vacation after the completion of reserved idle time.
- During WV, the arrivals are served with a lower service rate μ_0 which is exponentially distributed.
- During WV, the customers may either join the queue with probability σ or balk with probability $1-\sigma$.
- After the service, customers may get immediate re-service with probability $1-\alpha$ or leave the system with probability α
- During WV, a customer who is getting the service does not become impatient. But the only arrivals who find the server busy during WV, become impatient due to the long waiting time for the service. The customers who are waiting in the queue during WV, activate a timer which is independent and exponentially distributed with parameter ξ. If the service is not started before the timer expires, he abandons the system and never returns.
- Assume that inter-arrival times, service times during WV, service times during non-WV period, reserved idle time and WV times are all independent.
- The service discipline is first-come first-served (FCFS).
- $G(t) = 0$, if the server is on WV at time t
- $G(t) = 1$, if the server is busy at time t
- $\{Y(t), G(t), t \geq 0\}$ is a Markov chain with the state space $S = \{j, r : j = 0, 1, 2, \ldots ; r = \ 0 \text{ or } 1\}$.
- $\pi_{j,0}(t) = P\{Y(t) = j, G(t) = 0\}, \ j = 0, 1, 2, \ldots$
- $\pi_{j,1}(t) = P\{Y(t) = j, G(t) = 1\}, \ j = 0, 1, 2, \ldots$

3. Model description

We consider an $M/M/1$ queuing model with variant impatient behavior, balking, re-service, reserved idle time and working vacation. The governing equations for the proposed model and the derivation of time-dependent system distributions are provided in this section.

3.1. Governing equations

The probabilities $\pi_{n,1}(t)$ and $\pi_{n,0}(t)$, for $n \geq 0$ satisfy the following forward Kolmogorov differential difference equations:

$$\pi'_{0,1}(t) = -(\lambda_1 + \eta)\pi_{0,1}(t) + \gamma\pi_{0,0}(t), \tag{3.1}$$

$$\pi'_{j,1}(t) = -(\lambda_1 + \alpha\mu_1)\pi_{j,1}(t) + \lambda_1\pi_{j-1,1}(t) + \alpha\mu_1\pi_{j+1,1}(t) + \gamma\pi_{j,0}(t), \ j \geq 1, \tag{3.2}$$

$$\pi'_{0,0}(t) = -(\sigma\lambda_0+\gamma)\pi_{0,0}(t) + \mu_0\pi_{1,0}(t) + \eta\pi_{0,1}(t) + \alpha\mu_1\pi_{1,1}(t), \tag{3.3}$$

$$\pi'_{j,0}(t) = -(\sigma\lambda_0+\gamma+\mu_0+(j-1)\xi)\pi_{j,0}(t) + \sigma\lambda_0\pi_{j-1,0}(t) + (\mu_0+j\xi)\pi_{j+1,0}(t), \quad j\geq 1, \tag{3.4}$$

with $\pi_{0,0}(0)=1$, *i.e.*, there is no customer in the system at time $t=0$.

3.2. Transient probabilities

In this section, using the method of generating function and continued fraction, the time-dependent probabilities of the number of customers in the system are derived for the model under investigation.

The probabilities $\pi_{j,1}(t)$ are derived in terms of $\pi_{j,0}(t)$, for $j=1,2,3,\ldots$, in the following theorem.

Theorem 3.1. *The probabilities $\pi_{j,1}(t)$ are obtained, for $j=1,2,3,\ldots$, from (3.2) as*

$$\pi_{j,1}(t) = \gamma\int_0^t \sum_{m=1}^{\infty} \pi_{m,0}(\theta) w^{j-m}\left[I_{j-m}(u(t-\theta)) - I_{j+m}(u(t-\theta))\right] \mathrm{e}^{-(\lambda_1+\alpha\mu_1)(t-\theta)}\mathrm{d}\theta, \tag{3.5}$$

where $I_j(t)$ is the modified Bessel function of the first kind of order j, $u=2\sqrt{\lambda_1\alpha\mu_1}$ and $w=\sqrt{\frac{\lambda_1}{\alpha\mu_1}}$.

The proof of Theorem 3.1 is given in Appendix A. The following theorem expresses $\pi_{j,0}(t)$ in terms of $\pi_{0,0}(t)$, for $j=1,2,3,\ldots$ and gives $\pi_{0,0}(t)$ explicitly.

Theorem 3.2. *The probabilities $\pi_{j,0}(t)$, for $j=1,2,3,\ldots$ and $\pi_{0,1}(t)$ are obtained from (3.4) and (3.1) respectively as*

$$\pi_{j,0}(t) = \chi_j(t) * \pi_{0,0}(t), \tag{3.6}$$

$$\pi_{0,1}(t) = \gamma\mathrm{e}^{-(\lambda_1+\eta)t} * \pi_{0,0}(t), \tag{3.7}$$

where

$$\pi_{0,0}(t) = \sum_{k=0}^{\infty}\sum_{i+j+l=1} \frac{k!}{i!j!l!}{\mu_0}^i (\gamma\eta)^j \gamma^l \mathrm{e}^{-(\sigma\lambda_0+\gamma)t}\frac{t^{k-1}}{(k-1)!} * \mathrm{e}^{-(\lambda_1+\eta)t}\frac{t^{j-1}}{(j-1)!} \times (\chi_1(t))^{*i} * \left[\sum_{m=1}^{\infty}\chi_m(t) * \frac{\lambda_1}{h^{1+m}}\mathrm{e}^{-(\lambda_1+\alpha\mu_1)t}\left\{I_{m-1}(ut) - I_{m+1}(ut)\right\}\right]^{*l}, \tag{3.8}$$

$$\chi_j(t) = \left(\frac{\lambda_1}{\xi}\right)^j \sum_{n=0}^{\infty}(-\lambda_1)^n b_{j+n}(t) * \sum_{k=0}^{\infty}(\lambda_1)^k a_k(t), \tag{3.9}$$

$$b_k(t) = \frac{\xi^{-2k+1}}{k!}\prod_{i=0}^{k-1}(\mu_0+i\xi)\sum_{n=1}^{k}\frac{(-1)^{n-1}}{(k-n)!(n-i)!}\mathrm{e}^{-(\gamma+\mu_0+(n-1)\xi)t}, \ k=1,2,3,\ldots,$$

$$a_k(t) = \sum_{n=1}^{k}(-1)^{n-1}b_n(t) * a_{k-n}(t), \ k=2,3,4,\ldots; \ a_1(t)=b_1(t),$$

where "$*m$" denotes the m-fold convolution.

The proof of Theorem 3.2 is given in Appendix B.

3.3. Special case

When the reserved idle time is infinite, probability of balking the queue is one, probability of re-service is one and the customers' impatience rate is zero, that is, $\eta=\infty$, $\sigma=1, \alpha=1$ and $\xi=0$, then for the proposed model the probability of "j" customers in the system and the server is in working vacation state, $\pi_{j,0}(t)$, for $j\geq 1$, become

$$\pi_{j,0}(t) = \gamma w^{j-1}\mathrm{e}^{-(\lambda_1+\gamma+\mu_0)t}[I_{j-1}(ut) - I_{j+1}(ut)] * \pi_{0,0}(t),$$

which coincides with the equation (16) of Sudhesh and Raj [20].

4. Performance measures

The mean and variance of number of customers in the system at time t are derived in this section for the time-dependent case.

4.1. Mean number of customers in the system

Let $E[Y(t)]$ be the average number of customers in the system at time t.

$$E[Y(t)] = \sum_{j=1}^{\infty} j\left(\pi_{j,0}(t) + \pi_{j,1}(t)\right).$$

From (3.2) and (3.4), we obtain

$$E[Y(t)] = \sigma\lambda_0 \sum_{j=0}^{\infty} \int_0^t \pi_{j,0}(\theta)\mathrm{d}\theta + \lambda_1 \sum_{j=0}^{\infty} \int_0^t \pi_{j,1}(\theta)\mathrm{d}\theta - \mu_0 \sum_{j=1}^{\infty} \int_0^t \pi_{j,0}(\theta)\mathrm{d}\theta$$
$$- \alpha\mu_1 \sum_{j=1}^{\infty} \int_0^t \pi_{j,1}(\theta)\mathrm{d}\theta - \xi \sum_{j=1}^{\infty}(j-1) \int_0^t \pi_{j,0}(\theta)\mathrm{d}\theta. \tag{4.1}$$

4.2. Variance of the number of customers in the system

Let $V[Y(t)]$ be the variance of the number of customers in the system at time t.

$$V[Y(t)] = E[Y^2(t)] - (E[Y(t)])^2,$$

where
$E[Y^2(t)] = \sum_{j=1}^{\infty} j^2\left(\pi_{j,0}(t) + \pi_{j,1}(t)\right)$.
From (3.2) and (3.4), we obtain

$$E(Y^2(t)) = \sigma\lambda_0 \sum_{j=0}^{\infty} \int_0^t \pi_{j,0}(\theta)\mathrm{d}\theta + \lambda_1 \sum_{j=0}^{\infty} \int_0^t \pi_{j,1}(\theta)\mathrm{d}\theta + 2\lambda_1 \int_0^t E[Y(\theta)]\mathrm{d}\theta$$
$$- \mu_0 \sum_{j=1}^{\infty}(2j-1) \int_0^t \pi_{j,0}(\theta)\mathrm{d}\theta - \alpha\mu_1 \sum_{j=1}^{\infty}(2j-1) \int_0^t \pi_{j,1}(\theta)\mathrm{d}\theta$$
$$- \xi \sum_{j=1}^{\infty}(2j^2-3j+1) \int_0^t \pi_{j,0}(\theta)\mathrm{d}\theta, \tag{4.2}$$

where $\pi_{j,0}(t)$ and $\pi_{j,1}(t)$ are given by (3.6) and (3.5) respectively.

5. Numerical illustration

The model under consideration have many real world applications such as the scenario in customers service centers, super markets, manufacturing industries, etc. In this section, the variations of transient probabilities, average and variance against time are studied graphically. In Figure 1, the graphs corresponding to the transient probabilities of the system size during the vacation state of the server against time are plotted for $\lambda_0 = 1, \lambda_1 = 1.25, \mu_1 = 1.5, \mu_0 = 1.25, \xi = 0.8, \eta = 0.1, \gamma = 0.1, \sigma = 0.5, \alpha = 0.2$. The transient probability curves are plotted for the same set of parametric values in Figure 2 when the server is in busy state. From Figures 1 and 2, it is clear that the transient probability curves, except $\pi_{0,0}(t)$, increase initially and converge to their corresponding steady-state probability values with the increment of time t. Figures 3 and 4 present the variations of average and variance values against time for various values of the impatient rate. As the time increases, average and variance values increase. When the impatient rate of customers increases, the mean and variance number of customers in the system fall down which is clearly shown in Figures 3 and 4.

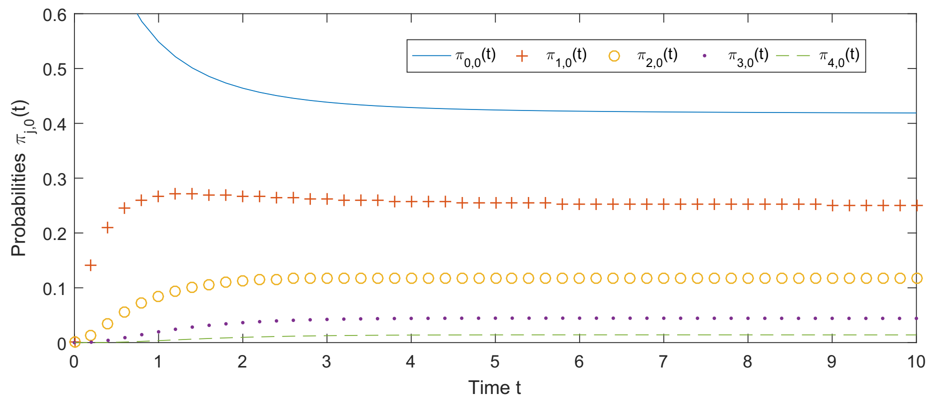

FIGURE 1. Transient probabilities for the working vacation state of the server.

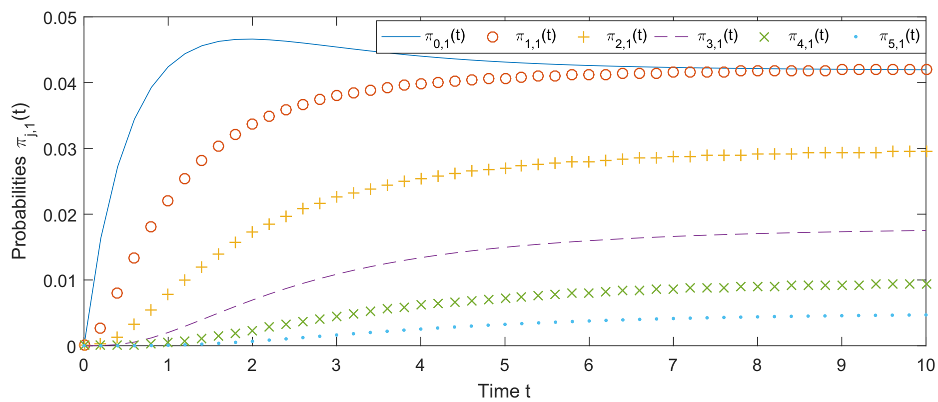

FIGURE 2. Time-dependent probabilities for the busy state of the server.

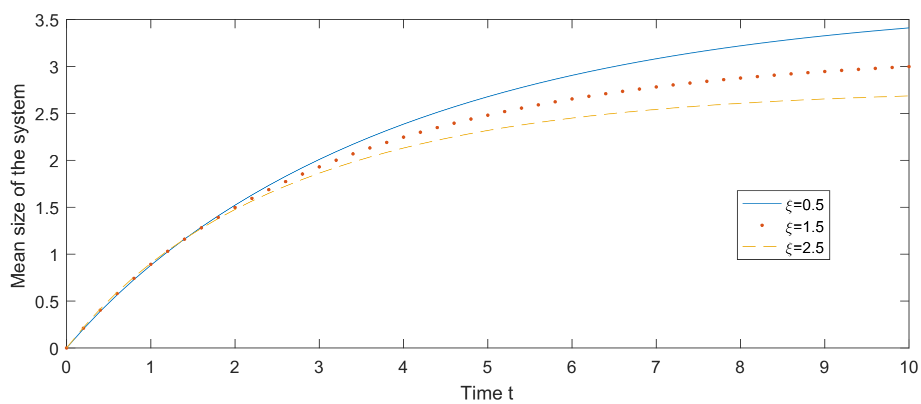

FIGURE 3. Expected system size against time for different values of ξ.

Table 1 presents the variations of mean and variance of system size against the time t for different values of the impatient rate ξ with $\lambda_0 = 1, \lambda_1 = 1.25, \mu_1 = 1.5, \mu_0 = 1.25, \eta = 0.1, \gamma = 0.1, \sigma = 0.5, \alpha = 0.2$. When comparing the values of mean and variance of system size for the similar set of ξ and t values to the corresponding set of values of mean and variance of system size of [21], it is clear that less number of customers leave the system with the increment of ξ in the proposed model. Table 2 provides the contributions of different authors.

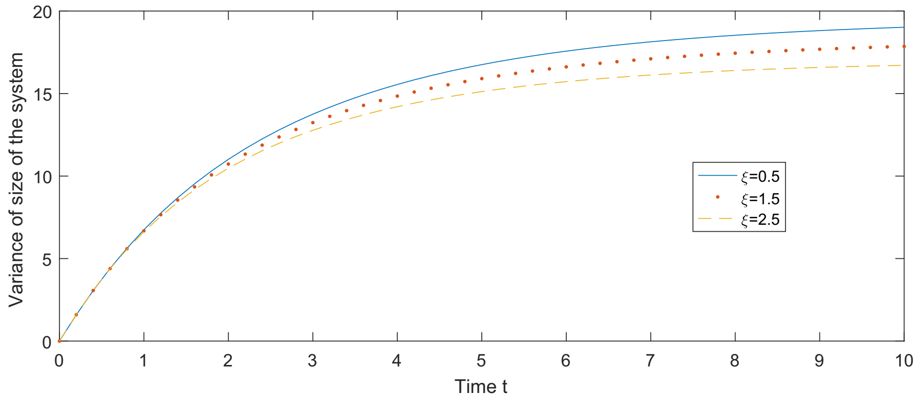

Figure 4. Variance of system size against time for various ξ varies.

Table 1. Variations of mean and variance against time with various values of impatient rate.

Impatient rate / time	$t = 5$		$t = 10$	
ξ	$E[Y(t)]$	$V[Y(t)]$	$E[Y(t)]$	$V[Y(t)]$
0.5	2.5124	16.9145	3.4851	19.2318
1.0	2.4019	16.0126	3.2141	18.8749
1.5	2.2481	15.5418	3.0154	18.2543
2.0	2.1982	14.4561	2.8569	17.5247
2.5	2.1025	13.8529	2.6952	16.9514

The *managerial insight* of the proposed model is as follows: Due to the slow service in the working vacation period, customers become impatient due the long waiting time. In order to reduce the customers' impatience, the management can increase the working vacation completion rate.

6. Conclusion and future work

In this analysis, we derived the time-dependent system distributions for an $M/M/1$ queuing model with variant impatient behavior, balking, re-service, reserved idle time and working vacation. The convergence of our results with the literature is shown as a special case. In addition, the time-dependent average and variance of number present in the system are obtained. Graphs are also drawn for the probability curves, mean as well as variance against the time in order to analyze their variations with respect to time. This model may be extended to a non-Markovian queueing model with general service time distribution or with general input arrival pattern.

Appendix A. Proof of Theorem 3.1

The probability generating function is defined as

$$\Phi(z,t) = \sum_{j=1}^{\infty} \pi_{j,1}(t) z^j.$$

From the equation (3.2) and doing some mathematical simplifications, we get

$$\frac{\partial}{\partial t}\Phi(z,t) = \left[-(\lambda_1 + \alpha\mu_1) + \lambda_1 z + \frac{\alpha\mu_1}{z}\right]\Phi(z,t) + \gamma \sum_{j=1}^{\infty} \pi_{j,0}(t) z^j - \alpha\mu_1 \pi_{1,1}(t). \tag{A.1}$$

TABLE 2. Contribution of different authors.

Author (year)	Number of servers	Solution type	Methodology*	Parameters used
Kumar *et al.* (1993)	1	Transient	GF	Balking
Altman and Yechiali (2006)	1	Stationary	GF	Vacations, impatient customers
Perel and Yechiali (2010)	1	Stationary	GF	Slow server, impatient customers
Sudhesh and Raj (2012)	1	Stationary, transient	GF, CF	working vacations
Selvaraju and Goswami (2013)	1	Stationary	GF	Working vacations, impatient customers
Ammar (2015)	1	Transient	GF, CF	Vacations, impatient customers
Sudhesh and Azhagappan (2016)	1	Transient	GF, CF	Vacations, impatient customers, waiting server
Sudhesh and Azhagappan (2017)	1	Transient	GF, CF	Working vacations, impatient customers, heterogeneous service
Sudhesh and Azhagappan (2018)	1	Transient	GF, CF	Working vacations, variant impatient customers
Azhagappan (2018)	1	Transient	GF, CF	Working vacations, variant impatient customers, waiting server
Sudhesh and Azhagappan (2019)	∞	Transient	GF, CF	Additional tasks, impatient customers
This model	1	Transient	GF, CF	Working vacations, variant impatient customers, reserved idle time, balking, re-service

Notes. * GF-Generating function, CF-Continued fraction.

Solving the equation (A.1), we get

$$\Phi(z,t) = \gamma \int_0^t \left[\sum_{m=1}^{\infty} \pi_{m,0}(\theta) z^m\right] \mathrm{e}^{-(\lambda_1+\alpha\mu_1)(t-\theta)} \mathrm{e}^{(\lambda_1 z + \frac{\alpha\mu_1}{z})(t-\theta)} \mathrm{d}\theta$$
$$- \alpha\mu_1 \int_0^t \pi_{1,1}(\theta) \mathrm{e}^{-(\lambda_1+\alpha\mu_1)(t-\theta)} \mathrm{e}^{(\lambda_1 z + \frac{\alpha\mu_1}{z})(t-\theta)} \mathrm{d}\theta. \tag{A.2}$$

Let us assume that

$$\mathrm{e}^{(\lambda_1 z + \frac{\alpha\mu_1}{z})t} = \sum_{j=-\infty}^{\infty} (wz)^j I_j(ut). \tag{A.3}$$

Using (A.3) in (A.2) and comparing the coefficients of z^j on both sides, for $j = 1, 2, 3, \ldots$, we get

$$\pi_{j,1}(t) = \gamma \int_0^t \sum_{m=1}^{\infty} \pi_{m,0}(\theta) w^{j-m} I_{j-m}(u(t-\theta)) \mathrm{e}^{-(\lambda_1+\alpha\mu_1)(t-\theta)} \mathrm{d}\theta$$
$$- \alpha\mu_1 \int_0^t \pi_{1,1}(\theta) w^j I_j(u(t-\theta)) \mathrm{e}^{-(\lambda_1+\alpha\mu_1)(t-\theta)} \mathrm{d}\theta. \tag{A.4}$$

The above equation also holds for negative values of "j". Then

$$0 = \gamma \int_0^t \sum_{m=1}^{\infty} \pi_{m,0}(\theta) w^{-j-m} I_{j+m}(u(t-\theta)) \mathrm{e}^{-(\lambda_1+\alpha\mu_1)(t-\theta)} \mathrm{d}\theta - \alpha\mu_1 \int_0^t \pi_{1,1}(\theta) w^{-j} I_j(u(t-\theta)) \mathrm{e}^{-(\lambda_1+\alpha\mu_1)(t-\theta)} \mathrm{d}\theta. \tag{A.5}$$

From the equations (A.4) and (A.5), we get (3.5), for $j = 1, 2, 3, \ldots$. □

Appendix B. Proof of Theorem 3.2

Laplace transform on (3.4) gives, for $j = 1, 2, 3, \ldots,$

$$\frac{\hat{\pi}_{j,0}(s)}{\hat{\pi}_{j-1,0}(s)} = \frac{\sigma\lambda_0}{(s + \sigma\lambda_0 + \gamma + \mu_0 + (j-1)\xi) - (\mu_0 + j\xi)\frac{\hat{\pi}_{j+1,0}(s)}{\hat{\pi}_{j,0}(s)}}. \tag{B.1}$$

Using the identity of confluent hypergeometric function given in the equation (2.3) of [24], the equation (B.1) can be written as

$$\begin{aligned} \hat{\pi}_{j,0}(s) &= \left(\frac{\sigma\lambda_0}{\xi}\right)^j \frac{1}{\prod_{i=0}^{j-1}\left(\frac{s+\gamma+\mu_0}{\xi} + i\right)} \frac{{}_1F_1\left(\frac{\mu_0}{\xi} + j; \frac{s+\gamma+\mu_0}{\xi} + j; -\frac{\sigma\lambda_0}{\xi}\right)}{{}_1F_1\left(\frac{\mu_0}{\xi}; \frac{s+\gamma+\mu_0}{\xi}; -\frac{\sigma\lambda_0}{\xi}\right)} \hat{\pi}_{0,0}(s), \\ &= \hat{\chi}_j(s)\hat{\pi}_{0,0}(s), \end{aligned} \tag{B.2}$$

where

$$\hat{\chi}_j(s) = \left(\frac{\sigma\lambda_0}{\xi}\right)^j \frac{1}{\prod_{i=0}^{j-1}\left(\frac{s+\gamma+\mu_0}{\xi} + i\right)} \frac{{}_1F_1\left(\frac{\mu_0}{\xi} + j; \frac{s+\gamma+\mu_0}{\xi} + j; -\frac{\sigma\lambda_0}{\xi}\right)}{{}_1F_1\left(\frac{\mu_0}{\xi}; \frac{s+\gamma+\mu_0}{\xi}; -\frac{\sigma\lambda_0}{\xi}\right)}. \tag{B.3}$$

On Laplace inversion of (B.2), we get (3.6). Laplace inversion of (B.3) yields (3.9). Taking Laplace transform of (3.1) and after some simplifications, we obtain

$$\hat{\pi}_{0,1}(s) = \frac{\gamma}{(s + \lambda_1 + \eta)} \hat{\pi}_{0,0}(s). \tag{B.4}$$

Laplace inversion of (B.4) yields (3.7). Taking Laplace transform on (3.3), we get

$$\hat{\pi}_{0,0}(s) = \frac{1}{(s + \sigma\lambda_0 + \gamma) - \mu_0\frac{\hat{\pi}_{1,0}(s)}{\hat{\pi}_{0,0}(s)} - \eta\frac{\hat{\pi}_{0,1}(s)}{\hat{\pi}_{0,0}(s)} - \alpha\mu_1\frac{\hat{\pi}_{1,1}(s)}{\hat{\pi}_{0,0}(s)}}. \tag{B.5}$$

Using (B.2) for $j = 1$, (3.5) and (B.4) in (B.5), after some mathematical manipulations, we obtain

$$\hat{\pi}_{0,0}(s) = \sum_{k=0}^{\infty} \sum_{i+j+l=1} \frac{k!}{i!j!l!} \frac{{\mu_0}^i (\gamma\eta)^j \gamma^l (\hat{\chi}_1(s))^i}{(s + \sigma\lambda_0 + \gamma)^k (s + \lambda_1 + \eta)^j} \left[\sum_{m=1}^{\infty} \hat{\chi}_m(s) d^m\right]^l. \tag{B.6}$$

where

$$d = \frac{c - \sqrt{c^2 - u^2}}{uw}, \quad c = s + \lambda_1 + \alpha\mu_1.$$

On Laplace inversion of (B.6), we have (3.8).

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