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COMPARISONS OF CUSTOMER BEHAVIOR IN MARKOVIAN QUEUES WITH VACATION POLICIES AND GEOMETRIC ABANDONMENTS

Wei Sun, Shiyong Li* and Naishuo Tian

Abstract. This paper mainly studies customers' equilibrium balking behavior in Markovian queues with single vacation and geometric abandonments. Whenever the system becomes empty, the server begins a vacation. If it is still empty when the vacation ends, the server stays idle and waits for new arrivals. During a vacation, abandonment opportunities occur according to a Poisson process, and at an abandonment epoch, customers decide sequentially whether they renege and leave the system or not. We consider four information levels: the fully/almost observable cases and the almost/fully unobservable cases, and get the customers' equilibrium balking strategies, respectively. Then we also get their optimal balking strategies for the almost observable and the almost/fully unobservable cases, and make comparisons of customer strategies and social welfare for the almost observable and the almost/fully unobservable queues with single vacation and multiple vacations. Because of abandonment, we find that the customers' equilibrium threshold in a vacation may exceed the one in a busy period in the fully observable queues. However, it has little effect on their equilibrium threshold in the almost observable queues, although frequent abandonment opportunity arrival inhibits their optimal threshold. Interestingly, for the almost unobservable queues, customers who arrive in a busy period are not affected by reneging that happened in the previous vacation when they make decisions of joining or balking, whereas the social planner expects that the customers can take it into consideration for social optimization. In the fully unobservable queues, because of no information, possible reneging surely influences customers' equilibrium and optimal balking behavior. For the almost observable and the almost/fully unobservable queues, the optimal social welfare is greater in the queues with single vacation than that in the queues with multiple vacations.

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1. Introduction

Queueing games models with various vacation policies have been studied by many literatures. For the classical vacation policies, for instance, Burnetas and Economou [3] first presented several Markovian queues with setup times and analyzed customers' equilibrium balking strategies under four information levels. Economou *et al.* [6] further discussed the unobservable and partially observable queues with general service time and vacation

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time. Then Guo and Hassin [10, 11] studied fully observable and unobservable queues with homogeneous and heterogeneous customers under N-policy. Liu and Wang [15] considered an Markovian queue with Bernoulli vacation, and Wang *et al.* [22] considered a constant retrial queue under N-policy. For the working vacation policies, Sun and Li [19] and Zhang *et al.* [25] first studied customers' equilibrium or optimal balking strategies in queues with multiple working vacations. Later, Sun *et al.* [20] focused on their equilibrium balking behavior in queues with two-stage working vacations, and Wang *et al.* [21] considered the same issue in a discrete-time queue with single working vacation. Then Guha *et al.* [8] investigated customers' equilibrium balking threshold strategies in a renewal input batch arrival queue with multiple and single working vacation, and Li *et al.* [14] studied an $M/M/1$ queue with working vacations and vacation interruptions.

In view of the impatience of customers waiting in a vacation, some researchers were interested in studying a reneging issue in the vacation queues. Altman and Yechiali [2] first comprehensively studied the customers' independent abandonments in a vacation in some single-server and multi-server queues with single vacation and multiple vacations, respectively. They assumed that the customers activate a random timer independently once a vacation starts, and they abandon immediately when the timer expires. Subsequently, Yechiali [23] discussed an $M/M/c\,(c \geq 1)$ queue with catastrophes and independent abandonments during a repair time. Guha *et al.* [9] considered an almost observable $GI/M/c/N$ queue with customer reneging with or without multiple synchronous vacations under state-dependent balking, and they derived customer's equilibrium balking strategy for constant balking as a special case of state-dependent balking. More recently, Panda *et al.* [18] studied customers' balking behavior in Markovian queues with independent abandonments and variant of working vacations, where they assumed that the impatience is due to slow service rate and the server is allowed to take multiple adaptive working vacations. The readers can also refer to Laxmi and Jyothsna [13], Maragathasundari and Srinivasan [16], and Yue *et al.* [24] for the queues with independent abandonments. Besides the case of independent abandonments, later, Adan *et al.* [1], Economou and Kapodistria [7] and Kapodistria [12] studied some vacation queues where the customers perform synchronized abandonments. That is, the abandonment opportunities occur according to a certain point process (*i.e.*, Poisson Process) in a vacation and then all present customers decide simultaneously but independently whether they abandon the system or not.

Based on the cases of independent and synchronized abandonments, Dimou *et al.* [4, 5] complemented those studies above by considering the case of geometric abandonments. The difference is that the customers decide sequentially whether they will leave the system or not when the abandonment opportunities occur in a vacation. They derived some classical system parameters, such as the queue length distribution, the customers' expected sojourn time and the expected busy period. On the other hand, from the economic viewpoint, Panda *et al.* [17] first studied the customer equilibrium and optimal balking behavior in Markovian queues with multiple vacations and sequential abandonment. Following [17], this paper studies customers' balking behavior in queues with single vacation and geometric abandonments.

Different with [17], however, we distinguish a customer's reward if he reneges from which if he does not, and actually, it is reasonable that the reward completing service is much higher than reneging. For example, in a perishable inventory system, which was introduced and illustrated in reference [17], the products may be fresh vegetables or fruits, and the value of a perished product is much lower than a qualified and served product, whereas it does not mean that the perished items are surely worthless and they also possibly can be used in other fields for earning, such as raw materials for livestock feed or organic fertilizer. Therefore, we provide a range for the reward of a reneging customer which is non-negative but lower than that of a customer who finally completes his service.

In this paper, we assume the server begins a vacation whenever the system becomes empty. If the system is still empty when the vacation ends, the server stays idle. During a vacation, abandonment opportunities occur according to a Poisson process and customers decide sequentially whether they renege or not. We consider four information levels: the fully/almost observable cases and the almost/fully unobservable cases, respectively. We obtain customers' equilibrium balking strategy under each information level, and also get their optimal balking strategy for the almost observable case and the almost/fully unobservable cases. Then we make comparisons of their strategies and the social welfare for the almost observable and the almost/fully unobservable queues with single vacation and multiple vacations.

In the fully observable queues, because of reneging behavior, customers' equilibrium threshold in a vacation may exceed the one in a busy period, which is obviously not affected by reneging. In the almost observable queues, surprisingly, we numerically find that what happened in a vacation that is related to reneging (*e.g.*, the abandonment opportunity arrival process and the reneging probability) also have no effect on customers' equilibrium threshold. However, frequent vacation or frequent abandonment opportunity arrival inhibits their optimal threshold.

In the almost unobservable queues, like the fully observable case, customers who arrive in a busy period are not affected by the reneging behavior that happened in the previous vacation when they make decisions of joining or balking, while the social planner expects that the customers can take it into consideration for social optimization. In the fully unobservable queues, because of no information, the abandonment opportunity arrival and the reneging probability surely influences customers' equilibrium and optimal behavior. Moreover, for both the almost/fully unobservable cases, including the almost observable case, the optimal social welfare is greater in the queues with single vacation than that in the queues with multiple vacations.

The paper is organized as follows: In Section 2, we describe the models and give some notations. Sections 3 and 4 are devoted to the customers' equilibrium threshold strategy for the fully/almost observable cases, respectively, and their optimal threshold strategy for the almost observable case is also derived. Then in Sections 5 and 6, besides customer's equilibrium mixed strategies, we also get their optimal mixed strategies for the almost/fully unobservable cases. In Section 7, we compare customers' behavior and optimal social welfare in the queues with single vacation and multiple vacations. Finally, brief conclusions of the paper are given in Section 8.

2. Model descriptions

In this paper, we mainly analyze an $M/M/1$ queue with single vacation and sequential abandonments. Assume that the service demand is infinite, *i.e.*, the customers' potential arrival rate Λ is large enough, and the actual arrival rate is λ and the server's service rate is μ. Whenever the system becomes empty, the server begins an exponentially distributed vacation with rate γ. If it is still empty when the vacation ends, the server stays idle and waits for new arrivals. During a vacation, abandonment opportunities occur according to a Poisson process with rate ξ. At an abandonment epoch, customers decide sequentially whether they will renege and leave the system or continue to stay, and their decision order is consistent with their position in the queue, *i.e.*, the customer in front of the queue first makes his decision (early arrivals abandon first). Each one of them abandons the system with probability p or remains in the system with probability q, where $p+q=1$. The reneging process stops at the first customer who continues to stay in the system, or when all present customers abandon the system. Thus, obviously, at an abandonment epoch, the number of customers who leave the system follows a geometric distribution with parameter p.

Let $(L(t), I(t))$ represent the system state at time t, where $L(t)$ denotes the system occupancy and $I(t)$ denotes the server state at time t, and

$$I(t) = \begin{cases} 0, & \text{the server is in a vacation;} \\ 1, & \text{the server is idle or busy.} \end{cases}$$

According to the information that customers can acquire before joining, we consider four information precision levels in this paper. In the fully observable case, customers can observe both $L(t)$ and $I(t)$ at time t, whereas they only can observe $L(t)$ in the almost observable case. On the other hand, in the fully unobservable case, customers can observe neither $L(t)$ nor $I(t)$ at time t, whereas they only can observe $I(t)$ in the almost unobservable case.

For each customer who joins the system, he can receive a reward R after service completion while receive a reward r $(0 \leq r < R)$ if he reneges at an abandonment epoch in a vacation, and he has to bear a cost c for waiting a time unit. We adopt a linear cost function with respect to his expected sojourn time. Because each customer can make individual decision to maximize his expected residual utility before joining, the system under each information level can be modeled as a noncooperative symmetric game among customers. We assume that the arrival process, the abandonment opportunity process, the service and vacation times are mutually independent, and the service order discipline is first in first out.

3. Fully observable queues

In the fully observable queues, we can easily get the customers' equilibrium balking threshold in the busy state. Define the expected sojourn time of a marked customer who observes $(n,1)$ and joins as $E[W_1^{fo}(n)]$, his expected residual utility after service as $U_1^{fo}(n)$, and his equilibrium balking threshold as n_1^e. So obviously,

$$U_1^{fo}(n) = R - cE[W_1^{fo}(n)] = R - \frac{c(n+1)}{\mu}, \tag{3.1}$$

which yields

$$n_1^e = \left\lfloor \frac{R\mu}{c} \right\rfloor - 1. \tag{3.2}$$

Otherwise, if he observes $(n,0)$ and joins, there are two possible outcomes he may encounter: completing his service finally or reneging at an abandonment epoch in the vacation. If he completes his service, two kinds of situation may arise: either no abandonment opportunity occurs or no more than n customers abandon during a whole vacation; If he reneges at an abandonment epoch, there must more than n customers abandon in the vacation. Define the probability that the marked customer completes his service as p_1^{fo} and the probability that he reneges as p_0^{fo}. Summarily, we get

$$p_1^{fo} = \int_0^\infty \mathrm{e}^{-\xi t}\gamma \mathrm{e}^{-\gamma t}\mathrm{d}t + \int_0^\infty \left(1-\mathrm{e}^{-\xi t}\right)\sum_{k=0}^{n} p^k q\gamma \mathrm{e}^{-\gamma t}\mathrm{d}t = 1 - \frac{\xi p^{n+1}}{\gamma+\xi}, \tag{3.3}$$

$$p_0^{fo} = \int_0^\infty \left(1-\mathrm{e}^{-\xi t}\right) p^{n+1}\gamma \mathrm{e}^{-\gamma t}\mathrm{d}t = \frac{\xi p^{n+1}}{\gamma+\xi}\cdot \tag{3.4}$$

Define the Laplace-Stieltjes transform (LST) of the marked customer's sojourn time if he observes $(n,0)$ and joins as $W_0^{*fo}(s)$. If no abandonment opportunity occurs in a vacation, based on equations (3.3) and (3.4), his sojourn time equals to a whole vacation and $n+1$ customers' service time; If k $(0 \le k \le n)$ customers abandon in a vacation, his sojourn time equals to a whole vacation and $n+1-k$ customers' service time; If he reneges at an abandonment epoch, his sojourn time equals to the passed vacation time before reneging. So we have

$$\begin{aligned} W_0^{*fo}(s) &= \int_0^\infty \mathrm{e}^{-\xi t}\gamma \mathrm{e}^{-\gamma t}\mathrm{d}t \left(\frac{\gamma}{\gamma+s}\right)\left(\frac{\mu}{\mu+s}\right)^{n+1} + \int_0^\infty \left(1-\mathrm{e}^{-\xi t}\right)\sum_{k=n+1}^{\infty} p^k q\gamma \mathrm{e}^{-\gamma t}\mathrm{d}t \left(\frac{\gamma}{\gamma+s}\right) \\ &\quad + \int_0^\infty \left(1-\mathrm{e}^{-\xi t}\right)\sum_{k=0}^{n} p^k q\gamma \mathrm{e}^{-\gamma t}\mathrm{d}t \left(\frac{\gamma}{\gamma+s}\right)\left(\frac{\mu}{\mu+s}\right)^{n+1-k} \\ &= \frac{\gamma}{\gamma+s}\left(\frac{\gamma}{\gamma+\xi}\left(\frac{\mu}{\mu+s}\right)^{n+1} + \frac{\xi q}{\gamma+\xi}\left(\frac{\mu}{\mu+s}\right)^{n+1}\frac{\mu^{n+1}-(p(\mu+s))^{n+1}}{\mu^{n+1}-\mu^n p(\mu+s)} + \frac{\xi p^{n+1}}{\gamma+\xi}\right)\cdot \end{aligned} \tag{3.5}$$

Then his expected sojourn time, denoted by $E[W_0^{fo}(n)]$, is

$$E[W_0^{fo}(n)] = -W_0^{'*fo}(0) = \frac{n+1}{\mu} + \frac{1}{\gamma} - \frac{\xi p(1-p^{n+1})}{\mu q(\gamma+\xi)}, \tag{3.6}$$

and his expected residual utility, denoted by $U_0^{fo}(n)$, is

$$U_0^{fo}(n) = p_1^{fo}R + p_0^{fo}r - cE[W_0^{fo}(n)] = \left(1-\frac{\xi p^{n+1}}{\gamma+\xi}\right)R + \frac{\xi r p^{n+1}}{\gamma+\xi} - c\left(\frac{n+1}{\mu} + \frac{1}{\gamma} - \frac{\xi p(1-p^{n+1})}{\mu q(\gamma+\xi)}\right)\cdot \tag{3.7}$$

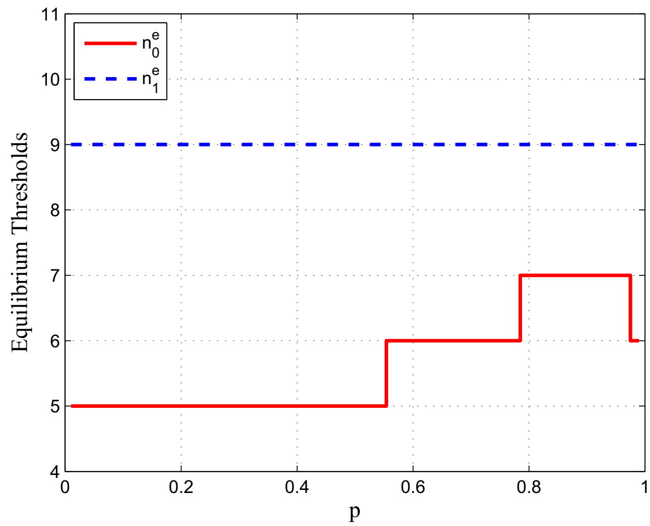

FIGURE 1. Equilibrium thresholds with respect to p when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$.

Based on equations (3.1) and (3.7), we can get the condition for the marked customer joining an empty system in the fully observable case is

$$R - \frac{c}{\mu} > \begin{cases} 0, & \text{if he observes } (0,1) \\ \dfrac{\xi p}{\gamma + \xi}(R - r) + c\left(\dfrac{1}{\gamma} - \dfrac{\xi p}{\mu(\gamma + \xi)}\right), & \text{if he observes } (0,0) \end{cases} \tag{3.8}$$

which must be satisfied. Otherwise, no customer will join an empty system. Define the customer's equilibrium balking threshold in the vacation state as n_0^e. Solving $U_0^{fo}(n) = 0$, we can get n_0^e. Because we only get a unique symmetric equilibrium result, customers avoid the crowd (ATC) in the fully observable case. As for the relationship of n_0^e and n_1^e, it does not necessarily hold that $n_0^e < n_1^e$ because of reneging, and it depends on the values of γ, ξ and p.

Figures 1 and 2 show that $n_0^e < n_1^e$, whereas it does not always hold in Figures 3 and 4. n_0^e shows bigger fluctuation when $p > 0.5$, and it may decrease when p is close to 1 because, after all, $0 \le r < R$. Similar to p, n_0^e also possibly decreases and $n_0^e = n_1^e$ when γ is large enough, *i.e.*, it nearly has no difference to customers that whether the server takes a vacation or not when the vacation time is too short. Moreover, Figure 4 shows that customers more prefer to join the queue although there is a fall in the number of customers present in the system with ξ. One reason is that the reward after reneging can be positive, and the other is customers' sojourn time in the queue can be expected to be shortened because of increasing abandonment rate ξ. So reneging is not intuitively a bad choice for customers, and the shorter queue length will encourage subsequent customers joining.

4. Almost observable queues

For the almost observable queues, we first try to derive the stationary queue length distribution for studying customer equilibrium behavior. Denote the customers' equilibrium balking threshold as n^e, and we get the transition probability equations in equilibrium based on the transition rate diagram depicted in Figure 5:

$$(\lambda + \gamma + \xi)\pi_{0,0} = \mu\pi_{1,1} + \xi \sum_{j=0}^{n^e+1} p^j \pi_{j,0}; \tag{4.1}$$

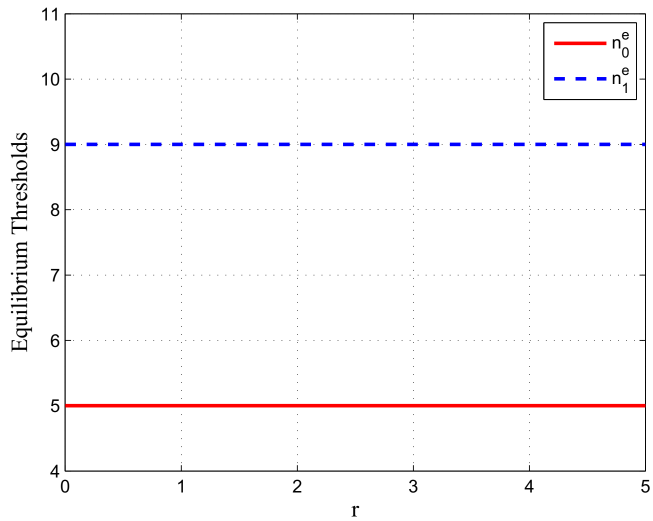

FIGURE 2. Equilibrium thresholds with respect to r when $R = 5$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$, $p = 0.1$.

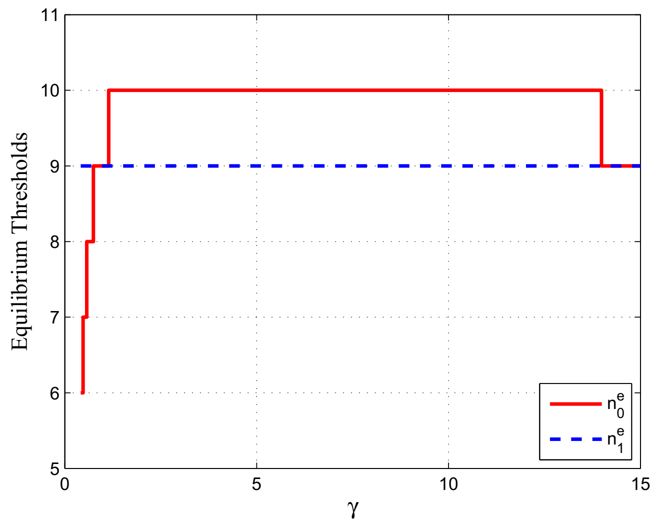

FIGURE 3. Equilibrium thresholds with respect to γ when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\xi = 8$, $p = 0.8$.

$$(\lambda+\gamma+\xi)\pi_{n,0} = \lambda\pi_{n-1,0} + \xi\sum_{j=n}^{n^e+1} p^{j-n} q\pi_{j,0}, \quad 1 \le n \le n^e; \tag{4.2}$$

$$(\gamma+\xi p)\pi_{n^e+1,0} = \lambda\pi_{n^e,0}; \tag{4.3}$$

$$\lambda\pi_{0,1} = \gamma\pi_{0,0}; \tag{4.4}$$

$$(\lambda+\mu)\pi_{n,1} = \lambda\pi_{n-1,1} + \mu\pi_{n+1,1} + \gamma\pi_{n,0}, \quad 1 \le n \le n^e; \tag{4.5}$$

$$\mu\pi_{n^e+1,1} = \lambda\pi_{n^e,1} + \gamma\pi_{n^e+1,0}. \tag{4.6}$$

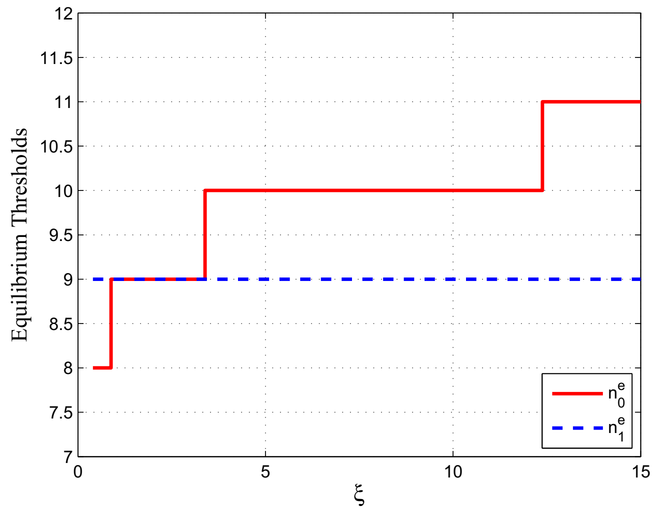

FIGURE 4. Equilibrium thresholds with respect to ξ when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 3$, $p = 0.8$.

Based on (4.3) and the recursion equation (4.2), we can successively get the probabilities $\{\pi_{n,0}, 0 \le n \le n^e\}$ in reverse order. That is,

$$\begin{cases} \pi_{n^e,0} = \dfrac{\gamma + \xi p}{\lambda} \pi_{n^e+1,0}, \\ \pi_{n-1,0} = \dfrac{\lambda + \gamma + \xi}{\lambda} \pi_{n,0} - \dfrac{\xi q}{\lambda} \displaystyle\sum_{j=n}^{n^e+1} p^{j-n} \pi_{j,0} \quad (n = n^e, n^e - 1, \ldots, 1). \end{cases} \tag{4.7}$$

From (4.1) and (4.4), we can get

$$\begin{cases} \pi_{0,1} = \dfrac{\gamma}{\lambda} \pi_{0,0}, \\ \pi_{1,1} = \dfrac{\lambda + \gamma + \xi}{\mu} \pi_{0,0} - \dfrac{\xi}{\mu} \displaystyle\sum_{j=0}^{n^e+1} p^j \pi_{j,0}. \end{cases} \tag{4.8}$$

Finally, based on (4.5), (4.6) and $\pi_{1,1}$, $\pi_{0,1}$, we can also get the probabilities $\{\pi_{n,1}, 2 \le n \le n^e + 1\}$ in normal order. That is,

$$\begin{cases} \pi_{n,1} = \left(\dfrac{\lambda + \gamma + \xi}{\mu} \pi_{0,0} - \dfrac{\xi}{\mu} \displaystyle\sum_{j=0}^{n^e+1} p^j \pi_{j,0} \right) + \displaystyle\sum_{k=1}^{n-1} \left(\left(\dfrac{\lambda + \gamma}{\mu} - \dfrac{\gamma}{\lambda} \right) \pi_{0,0} - \dfrac{\xi}{\mu} \displaystyle\sum_{j=0}^{n^e+1} p^j \pi_{j,0} \right) \rho^k \\ \qquad - \gamma \displaystyle\sum_{j=0}^{n-2} \sum_{k=0}^{j} \rho^k \pi_{j+1-k,0}, \quad (n = 2, 3, \ldots, n^e) \\ \pi_{n^e+1,1} = \rho \pi_{n^e,1} + \dfrac{\gamma}{\mu} \pi_{n^e+1,0}. \end{cases} \tag{4.9}$$

It is obvious that all the probabilities derived above are related to $\pi_{n^e+1,0}$. For $\pi_{n^e+1,0}$, it can be solved by the normalization equation:

$$\sum_{n=0}^{n^e+1} (\pi_{n,0} + \pi_{n,1}) = 1. \tag{4.10}$$

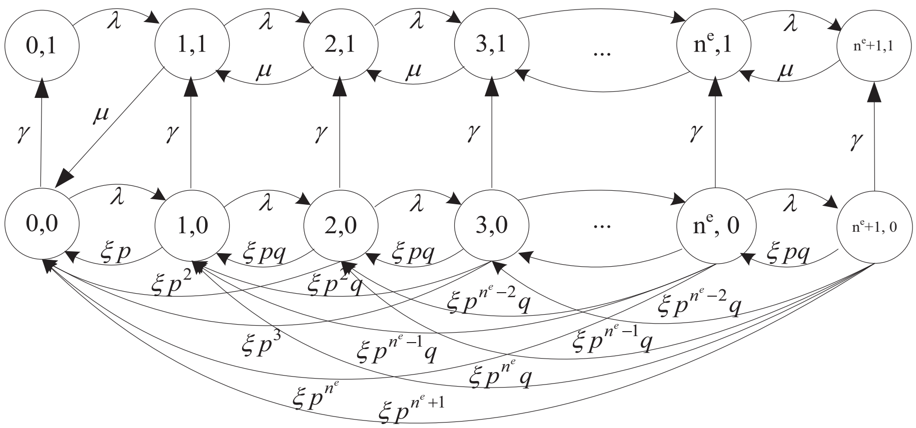

FIGURE 5. Transition rate diagram for the almost observable queues in equilibrium.

Given that all other customers follow the equilibrium threshold strategy n^e, we define the expected residual utility of a marked customer after service or after reneging who observes n customers and joins as $U_{n^e}^{ao}(n)$. So based on (3.1) and (3.7), we get

$$\begin{aligned} U_{n^e}^{ao}(n) &= \frac{\pi_{n,1}}{\pi_{n,1}+\pi_{n,0}}U_1^{fo}(n) + \frac{\pi_{n,0}}{\pi_{n,1}+\pi_{n,0}}U_0^{fo}(n) \\ &= \frac{\pi_{n,1}}{\pi_{n,1}+\pi_{n,0}}\left(R - \frac{c(n+1)}{\mu}\right) + \frac{\pi_{n,0}}{\pi_{n,1}+\pi_{n,0}}\left(\left(1 - \frac{\xi p^{n+1}}{\gamma+\xi}\right)R + \frac{\xi r p^{n+1}}{\gamma+\xi}\right. \\ &\quad \left. -c\left(\frac{n+1}{\mu} + \frac{1}{\gamma} - \frac{\xi p(1-p^{n+1})}{\mu q(\gamma+\xi)}\right)\right). \end{aligned} \tag{4.11}$$

Obviously, $U_{n^e}^{ao}(n)$ decreases with n because a customer joining only has negative effect on customers arriving later. Moreover, in order to make the analysis meaningful, we assure that the condition for joining an empty system in the almost observable case is satisfied, that is,

$$U_{n^e}^{ao}(0) = R - \frac{c}{\mu} - \frac{\lambda}{\gamma+\lambda}\left(\frac{\xi p}{\gamma+\xi}\left(R - r - \frac{c}{\mu}\right) + \frac{c}{\gamma}\right) > 0. \tag{4.12}$$

Otherwise, no customer will join an empty system.

Similar to the analysis in the almost observable case in Reference [17], we define the sequences $\{U_n^{ao}(n), n \geq 0\}$ and $\{U_{n-1}^{ao}(n), n \geq 1\}$. Obviously, $\lim_{n\to\infty} U_n^{ao}(n) = \lim_{n\to\infty} U_{n-1}^{ao}(n) = -\infty$. So there exists a finite non-negative integer n_U such that $U_0^{ao}(0), U_1^{ao}(1), U_2^{ao}(2), \ldots, U_{n_U}^{ao}(n_U) > 0$ and $U_{n_U+1}^{ao}(n_U+1) \leq 0$. Moreover, we have $U_{n-1}^{ao}(n) < U_n^{ao}(n)$ for $n \geq 1$ so that $U_{n_U}^{ao}(n_U+1) < U_{n_U+1}^{ao}(n_U+1) \leq 0$. Then for the sequence $\{U_{n-1}^{ao}(n),\ n \geq 1\}$, similarly, there also exists a finite non-negative integer $n_L(\leq n_U)$ such that $U_{n_U}^{ao}(n_U+1), U_{n_U-1}^{ao}(n_U), U_{n_U-2}^{ao}(n_U-1), \ldots, U_{n_L}^{ao}(n_L+1) \leq 0$ and $U_{n_L-1}^{ao}(n_L) > 0$.

If the marked customer decides joining when he finds $n(\leq n^e)$ customers, his expected residual utility is equal to $U_{n^e}^{ao}(n) \geq U_{n^e}^{ao}(n^e) > 0$ so that joining is beneficial for him. If he decides joining when he finds n^e+1 customers, his expected residual utility is equal to $U_{n^e}^{ao}(n^e+1) \leq 0$ so that balking is more beneficial for him. Therefore, $n^e \in \{n_L, n_L+1, \ldots, n_U\}$ which means the equilibrium threshold is generally not unique. Because we can get multiple symmetric equilibria, customers follow the crowd (FTC) in the almost observable case.

Based on (3.1) and (3.7), we get the expected social welfare per time unit, denoted by $\mathrm{SW}_s^{ao}(n)$, as

$$\mathrm{SW}_s^{ao}(n) = \lambda\left(\sum_{k=0}^{n}\left(\pi_{k,0}U_0^{fo}(k) + \pi_{k,1}U_1^{fo}(k)\right)\right). \tag{4.13}$$

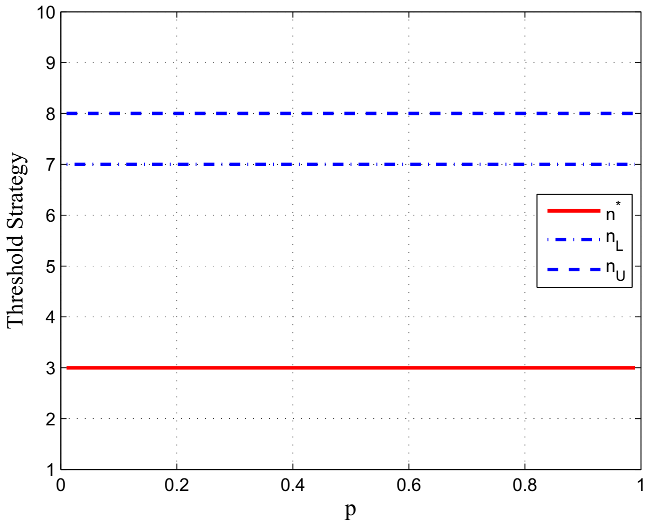

FIGURE 6. Equilibrium and optimal thresholds with respect to p when $R = 5$, $r = 2$, $c = 1$, $\lambda = 3$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$.

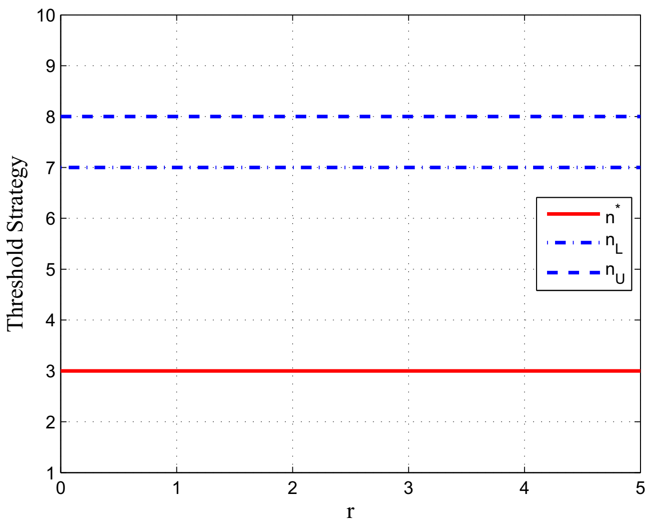

FIGURE 7. Equilibrium and optimal thresholds with respect to r when $R = 5$, $c = 1$, $\lambda = 3$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$, $p = 0.1$.

Solving the non-negative integer optimization problem $\mathbf{max}\ \mathrm{SW}_s^{ao}(n)$, we can obtain the customers' socially optimal threshold strategy n^*, which is substituted into (4.13) for the optimal social welfare SW_s^{*ao}.

Figures 6–9 show that the equilibrium balking thresholds are invariable with the parameters such as p, r, γ or ξ, and $n^* < n_L < n_U < n_1^e$ compared with Figures 1–4. So interestingly, it indicates that all the parameters related to the customer reneging behavior that may occur in the following vacation has little effect on customer balking threshold strategy decided before joining in the almost observable queues. That is, customers are only concerned about the queue length but nearly do not take the possible state of the server to consideration before

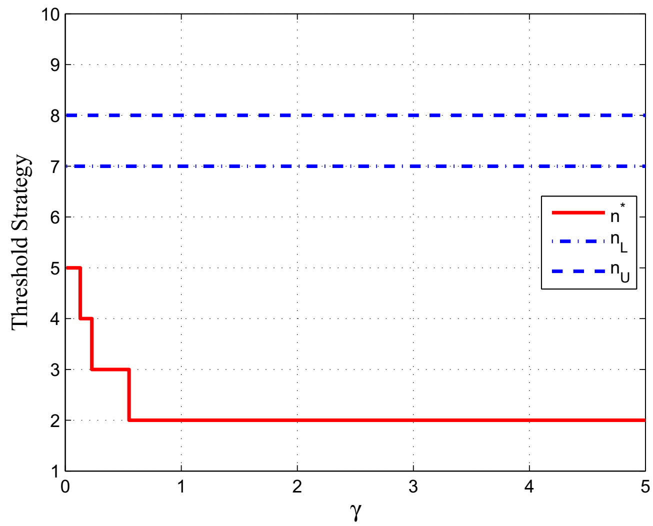

Figure 8. Equilibrium and optimal thresholds with respect to γ when $R = 5$, $r = 2$, $c = 1$, $\lambda = 3$, $\mu = 2$, $\xi = 4$, $p = 0.1$.

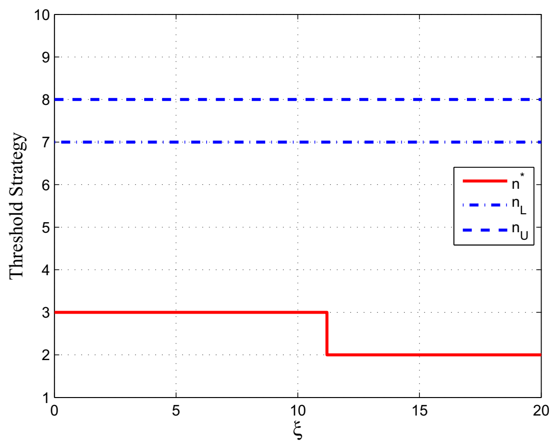

Figure 9. Equilibrium and optimal thresholds with respect to ξ when $R = 5$, $r = 2$, $c = 1$, $\lambda = 3$, $\mu = 2$, $\gamma = 0.5$, $p = 0.1$.

making decision. The possible reason may be that customers are more likely to believe that the server is busy in case they can not observe the server's state before joining (generally the probability that the server is busy is much larger), which dilutes the effect of reneging on customers' decision. Moreover, Figures 8 and 9 show that n^* decreases with γ or ξ, *i.e.*, the social planner expects that frequent vacation or frequent occurrence of abandonment opportunity can inhibit customers from joining.

5. Almost unobservable queues

Similar to the almost observable queues, for the almost unobservable queues, we still first derive the stationary queue length distribution for studying customer behavior. Because customers can observe the server's state at

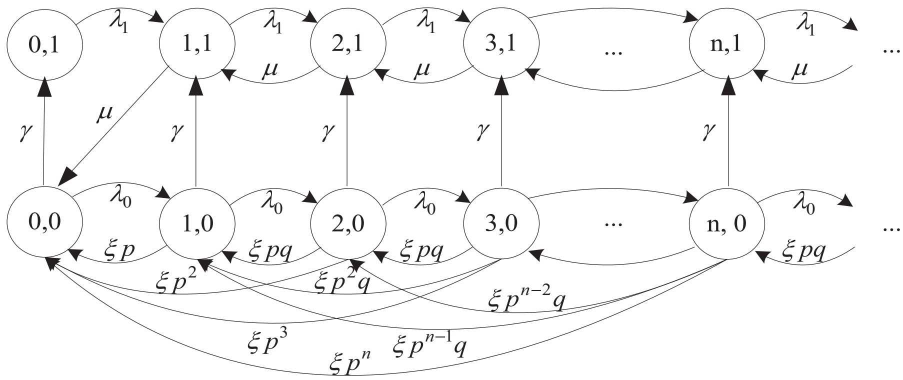

FIGURE 10. Transition rate diagram for the almost unobservable queues.

their arrival time, they have different joining rates under the two states, denoted by λ_0 and λ_1, respectively. Obviously, the process $\{L(t), I(t)\}$ is a quasi-birth-and-death process with the state space $\Omega = \{(k,j) : k \geq 0, j = 0, 1\}$. Let (L, I) be the limit of the process $\{L(t), I(t)\}$. Denote the stationary queue length distribution as

$$\pi_{k,j} = P\{L = k, I = j\} = \lim_{t\to\infty} P\{L(t) = k, I(t) = j\}, \quad (k,j) \in \Omega.$$

So we get the transition probability equations based on the transition rate diagram depicted in Figure 10:

$$(\lambda_0 + \gamma + \xi)\pi_{0,0} = \mu\pi_{1,1} + \xi \sum_{j=0}^{\infty} p^j \pi_{j,0}; \tag{5.1}$$

$$(\lambda_0 + \gamma + \xi)\pi_{n,0} = \lambda_0\pi_{n-1,0} + \xi \sum_{j=n}^{\infty} p^{j-n} q\pi_{j,0}, \quad n \geq 1; \tag{5.2}$$

$$\lambda_1\pi_{0,1} = \gamma\pi_{0,0}; \tag{5.3}$$

$$(\lambda_1 + \mu)\pi_{n,1} = \lambda_1\pi_{n-1,1} + \mu\pi_{n+1,1} + \gamma\pi_{n,0}, \quad n \geq 1. \tag{5.4}$$

Theorem 5.1. *For an almost unobservable queue with single vacation and geometric abandonments, if* $\rho = \lambda_1/\mu < 1$*, the stationary queue length distribution is*

$$\begin{cases} \pi_{n,0} = \pi_{0,0} x_1^n, \quad n \geq 0; \\ \pi_{n,1} = \left(\left(\dfrac{\gamma}{\lambda_1} - \dfrac{\gamma x_1}{(1-x_1)(\mu x_1 - \lambda_1)} \right) \rho^n + \dfrac{\gamma x_1}{(1-x_1)(\mu x_1 - \lambda_1)} x_1^n \right) \pi_{0,0}, \quad n \geq 0, \end{cases} \tag{5.5}$$

where

$$x_1 = \frac{(\lambda_0(1+p) + \gamma + \xi p) - \sqrt{(\lambda_0(1+p) + \gamma + \xi p)^2 - 4p\lambda_0(\lambda_0 + \gamma + \xi)}}{2p(\lambda_0 + \gamma + \xi)}, \tag{5.6}$$

and

$$\pi_{0,0} = \left(\frac{1}{1-x_1} + \left(\frac{\gamma}{\lambda_1} - \frac{\gamma x_1}{(1-x_1)(\mu x_1 - \lambda_1)} \right) \frac{1}{1-\rho} + \frac{\gamma x_1}{(1-x_1)^2(\mu x_1 - \lambda_1)} \right)^{-1} \tag{5.7}$$

Proof. First, we can rewrite (5.2) as follows:

$$\xi q \sum_{j=n+1}^{\infty} p^{j-n} \pi_{j,0} - (\lambda_0 + \gamma + \xi p)\pi_{n,0} + \lambda_0\pi_{n-1,0} = 0, \quad n \geq 1. \tag{5.8}$$

Obviously, its characteristic equation is a homogeneous linear difference equation with constant coefficients:

$$\xi q \sum_{j=n+1}^{\infty} p^{j-n} x^{j-n+1} - (\lambda_0 + \gamma + \xi p)x + \lambda_0 = 0. \tag{5.9}$$

Simplifying (5.9), we obtain a second-order characteristic equation:

$$p(\xi + \lambda_0 + \gamma)x^2 - (\lambda_0(1+p) + \xi p + \gamma)x + \lambda_0 = 0. \tag{5.10}$$

Solving (5.10), we get the two roots as:

$$x_{1,2} = \frac{(\lambda_0(1+p) + \gamma + \xi p) \mp \sqrt{(\lambda_0(1+p) + \gamma + \xi p)^2 - 4p\lambda_0(\lambda_0 + \gamma + \xi)}}{2p(\lambda_0 + \gamma + \xi)}, \tag{5.11}$$

where $0 \le x_1 < 1$ and $x_2 > 1$. The general solution of (5.2) is

$$\pi_{n,0} = A_1 x_1^n + B_1 x_2^n, \quad n \ge 0,$$

where A_1 and B_1 can be determined. Because $x_2 > 1$, it must hold that $B_1 = 0$. So $A_1 = \pi_{0,0}$, and

$$\pi_{n,0} = \pi_{0,0} x_1^n, \quad n \ge 0. \tag{5.12}$$

Similarly, the characteristic equation of (5.4) is a nonhomogeneous linear difference equation with constant coefficients:

$$\mu x^2 - (\lambda_1 + \mu)x + \lambda_1 = -\gamma \pi_{n,0} = -\gamma x_1^n \pi_{0,0}. \tag{5.13}$$

Therefore, the general solution of (5.4) is $\pi_{n,1} = \pi_{n,1}^{\text{hom}} + \pi_{n,1}^{\text{spec}}$ $(n \ge 0)$, where $\pi_{n,1}^{\text{hom}} = A_2 + B_2\rho^n$ and $\pi_{n,1}^{\text{spec}}$ is a specific solution of (5.4). Because the nonhomogeneous part of (5.13) is geometric with parameter x_1, we consider a specific solution $\pi_{n,1}^{\text{spec}} = Cx_1^n$. Substituting it into (5.4), we get

$$C = \frac{\gamma x_1}{(1-x_1)(\mu x_1 - \lambda_1)} \pi_{0,0}. \tag{5.14}$$

Based on (5.1) and (5.3), we get $A_2 = 0$ and

$$B_2 = \left(\frac{\gamma}{\lambda_1} - \frac{\gamma x_1}{(1-x_1)(\mu x_1 - \lambda_1)} \right) \pi_{0,0}. \tag{5.15}$$

So we get

$$\pi_{n,1} = B_2 \rho^n + C x_1^n, \quad n \ge 0, \tag{5.16}$$

where B_2 and C are given by (5.14) and (5.15), and $\pi_{0,0}$ can be solved by the normalization equation:

$$\sum_{n=0}^{\infty} (\pi_{n,0} + \pi_{n,1}) = 1, \tag{5.17}$$

which is given in (5.7). □

Now define the partial probability generating functions of the stationary queue length distribution in the vacation state and in the busy state as $\Pi_0(z)$ and $\Pi_1(z)$ $(|z| \le 1)$, respectively. Based on the result given in Theorem 5.1, we get

$$\begin{aligned} \Pi_0(z) &= \sum_{n=0}^{\infty} \pi_{n,0} z^n = \pi_{0,0} \sum_{n=0}^{\infty} (x_1 z)^n \\ &= \frac{1}{1-x_1 z} \left(\frac{1}{1-x_1} + \left(\frac{\gamma}{\lambda_1} - \frac{\gamma x_1}{(1-x_1)(\mu x_1 - \lambda_1)} \right) \frac{1}{1-\rho} + \frac{\gamma x_1}{(1-x_1)^2(\mu x_1 - \lambda_1)} \right)^{-1}, \end{aligned} \tag{5.18}$$

$$\begin{aligned}\Pi_1(z) = \sum_{n=0}^{\infty}\pi_{n,1}z^n &= \pi_{0,0}\sum_{n=0}^{\infty}\left(\left(\frac{\gamma}{\lambda_1}-\frac{\gamma x_1}{(1-x_1)(\mu x_1-\lambda_1)}\right)(\rho z)^n+\frac{\gamma x_1}{(1-x_1)(\mu x_1-\lambda_1)}(x_1 z)^n\right)\\ &= \left(\left(\frac{\gamma}{\lambda_1}-\frac{\gamma x_1}{(1-x_1)(\mu x_1-\lambda_1)}\right)\frac{1}{1-\rho z}+\frac{\gamma x_1}{(1-x_1)(\mu x_1-\lambda_1)}\frac{1}{1-x_1 z}\right)\\ &\quad\times\left(\frac{1}{1-x_1}+\left(\frac{\gamma}{\lambda_1}-\frac{\gamma x_1}{(1-x_1)(\mu x_1-\lambda_1)}\right)\frac{1}{1-\rho}+\frac{\gamma x_1}{(1-x_1)^2(\mu x_1-\lambda_1)}\right)^{-1}.\end{aligned} \tag{5.19}$$

Next we begin to study the customers' balking strategy. If a marked customer joins the system during a busy period, his conditional expected sojourn time, denoted by $E[W_1^{au}(\lambda_0,\lambda_1)]$, is

$$\begin{aligned}E[W_1^{au}(\lambda_0,\lambda_1)] &= \sum_{n=0}^{\infty}\pi_{n,1}\left(\frac{n+1}{\mu}\right)=\frac{1}{\mu}\left(\Pi_1'(1)+\Pi_1(1)\right)\\ &= \frac{1}{\mu}\left(\left(\frac{\gamma}{\lambda_1}-\frac{\gamma x_1}{(1-x_1)(\mu x_1-\lambda_1)}\right)\frac{\rho}{(1-\rho)^2}+\frac{\gamma x_1}{(1-x_1)(\mu x_1-\lambda_1)}\frac{x_1}{(1-x_1)^2}\right)\\ &\quad\times\left(\frac{1}{1-x_1}+\left(\frac{\gamma}{\lambda_1}-\frac{\gamma x_1}{(1-x_1)(\mu x_1-\lambda_1)}\right)\frac{1}{1-\rho}+\frac{\gamma x_1}{(1-x_1)^2(\mu x_1-\lambda_1)}\right)^{-1}.\end{aligned} \tag{5.20}$$

On the other hand, if the marked customer joins the system during a vacation, similar to the fully observable case, there are also two possible outcomes he may encounter: completing his service finally or reneging at an abandonment epoch in the vacation. Define the LST of the customer's sojourn time as $W_0^{*au}(s)$, and we have

$$\begin{aligned}W_0^{*au}(s) &= \sum_{n=0}^{\infty}\pi_{n,0}\left(\int_0^{\infty}\left(1-\mathrm{e}^{-\xi t}\right)\sum_{k=n+1}^{\infty}p^k q\gamma\mathrm{e}^{-\gamma t}\mathrm{d}t\left(\frac{\gamma}{\gamma+s}\right)+\int_0^{\infty}\mathrm{e}^{-\xi t}\gamma\mathrm{e}^{-\gamma t}\mathrm{d}t\left(\frac{\gamma}{\gamma+s}\right)\left(\frac{\mu}{\mu+s}\right)^{n+1}\right.\\ &\quad\left.+\int_0^{\infty}\left(1-\mathrm{e}^{-\xi t}\right)\sum_{k=0}^{n}p^k q\gamma\mathrm{e}^{-\gamma t}\mathrm{d}t\left(\frac{\gamma}{\gamma+s}\right)\left(\frac{\mu}{\mu+s}\right)^{n+1-k}\right)\\ &= \frac{\gamma}{(\gamma+s)(\gamma+\xi)}\left(\frac{\gamma\mu}{\mu+s}\Pi_0\left(\frac{\mu}{\mu+s}\right)+\frac{\xi\mu q}{\mu q-sp}\left(\frac{\mu}{\mu+s}\Pi_0\left(\frac{\mu}{\mu+s}\right)-p\Pi_0(p)\right)+\xi p\Pi_0(p)\right).\end{aligned} \tag{5.21}$$

So his conditional expected sojourn time, denoted by $E[W_0^{au}(\lambda_0)]$, is

$$E[W_0^{au}(\lambda_0)] = \frac{-W_0^{'*au}(0)}{\Pi_0(1)} = \frac{1}{\gamma}+\frac{1}{\mu(1-x_1)}-\frac{p\xi}{\mu q(\gamma+\xi)}\left(1-\frac{p(1-x_1)}{1-px_1}\right). \tag{5.22}$$

Define the probability that the marked customer completes his service finally as p_{01}^{au} and the probability that he reneges at an abandonment epoch in the vacation as p_{00}^{au}. So we have

$$p_{01}^{au} = \sum_{n=0}^{\infty}\frac{\pi_{n,0}}{\Pi_0(1)}\left(\int_0^{\infty}\mathrm{e}^{-\xi t}\gamma\mathrm{e}^{-\gamma t}\mathrm{d}t+\int_0^{\infty}\left(1-\mathrm{e}^{-\xi t}\right)\sum_{k=0}^{n}p^k q\gamma\mathrm{e}^{-\gamma t}\mathrm{d}t\right) = 1-\frac{p\xi}{\gamma+\xi}\frac{1-x_1}{1-px_1}, \tag{5.23}$$

$$p_{00}^{au} = \sum_{n=0}^{\infty}\frac{\pi_{n,0}}{\Pi_0(1)}\left(\int_0^{\infty}\left(1-\mathrm{e}^{-\xi t}\right)\sum_{k=n+1}^{\infty}p^k q\gamma\mathrm{e}^{-\gamma t}\mathrm{d}t\right) = \frac{p\xi}{\gamma+\xi}\frac{1-x_1}{1-px_1}. \tag{5.24}$$

Obviously, $p_{01}^{au}=1-p_{00}^{au}$. The expected residual utility of the marked customer who joins in the vacation state, denoted by $U_0^{au}(\lambda_0)$, is

$$U_0^{au}(\lambda_0) = p_{01}^{au}R+p_{00}^{au}r-cE[W_0^{au}(\lambda_0)], \tag{5.25}$$

and the expected residual utility of the customer who joins in the busy (or idle) state, denoted by $U_1^{au}(\lambda_0,\lambda_1)$, is

$$U_1^{au}(\lambda_0,\lambda_1) = R-cE[W_1^{au}(\lambda_0,\lambda_1)]. \tag{5.26}$$

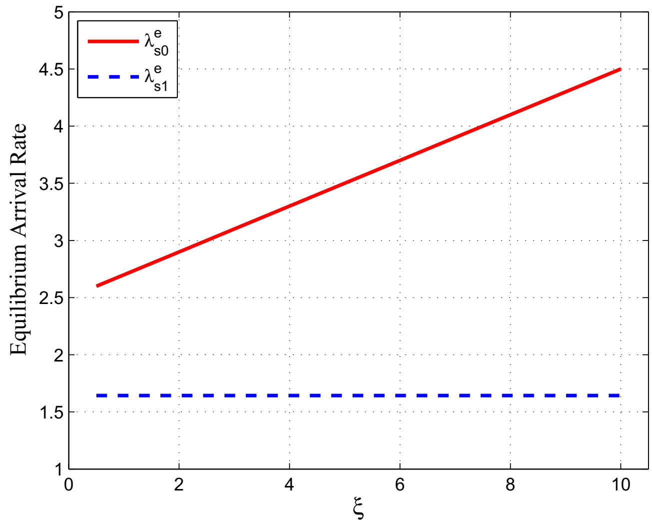

FIGURE 11. Equilibrium arrival rates with respect to ξ when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $p = 0.2$.

Define the positive equilibrium arrival rate of customers in the busy (or idle) state and in the vacation state as λ_{s1}^e and λ_{s0}^e[2], respectively. Solving $U_0^{au}(\lambda_0) = 0$ and $U_1^{au}(\lambda_0, \lambda_1) = 0$, we can get λ_{s1}^e and λ_{s0}^e, and the expected social welfare per time unit, denoted by $\mathrm{SW}_s^{au}(\lambda_0, \lambda_1)$, is

$$\mathrm{SW}_s^{au}(\lambda_0, \lambda_1) = \lambda_0 \Pi_0(1) \left(p_{01}^{au} R + p_{00}^{au} r - cE[W_0^{au}(\lambda_0)]\right) + \lambda_1 \Pi_1(1) \left(R - cE[W_1^{au}(\lambda_0, \lambda_1)]\right). \quad (5.27)$$

Solving the optimization problem **max** $\mathrm{SW}_s^{au}(\lambda_0, \lambda_1)$ subject to $\lambda_1 < \mu$, we can obtain the optimal arrival rates λ_{s0}^* and λ_{s1}^*, which is substituted into (5.27) for the optimal social welfare SW_s^{*au}.

Figures 11 and 12 show that λ_{s0}^e increases linearly with ξ, whereas it increases convexly with p. Comparing Figure 12 with Figure 1, we find that λ_{s0}^e does not decrease under the same conditions when p is close to 1. This indicates that the information precision indeed affects customers' equilibrium strategies. As for λ_{s1}^e, we numerically observe that it is not influenced by ξ or p ($\lambda_{s1}^e \equiv 1.643$ given that λ_{s0}^e is first solved)[3]. That is, in an almost unobservable queue, all the things happened in a vacation are not taken into consideration by the customers who arrive in the next busy period when they make decisions of joining or balking, although they may affect the queue length at the beginning of the busy period. Moreover, $\lambda_{s0}^e > \lambda_{s1}^e$, which is generally impossible in case reneging is prohibited.

6. Fully unobservable queues

In the fully unobservable queues, customers can not observe the server's state at their arrival time, so we can get the stationary queue length distribution based on Theorem 5.1 by replacing λ_0 and λ_1 with λ, and the transition rate diagram is depicted in Figure 13.

Similar to the fully observable and almost unobservable cases, if a marked customer joins, there are also two outcomes he may encounter: completing service or reneging. Define the probability that he completes service as

[2]The subscript "s" of λ_{s1}^e and λ_{s0}^e is to differentiate the notations of the customers' equilibrium (or optimal) arrival rates in queues with single vacation and in queues with multiple vacations.

[3]Denote the customers' equilibrium arrival rates in queues with multiple vacations as λ_{m1}^e and λ_{m0}^e. We have verified that under the same conditions with Figures 11 and 12, $\lambda_{m0}^e = \lambda_{s0}^e$ and $\lambda_{m1}^e \equiv 1.634 < \lambda_{s1}^e \equiv 1.643$ given that λ_{m0}^e is first solved.

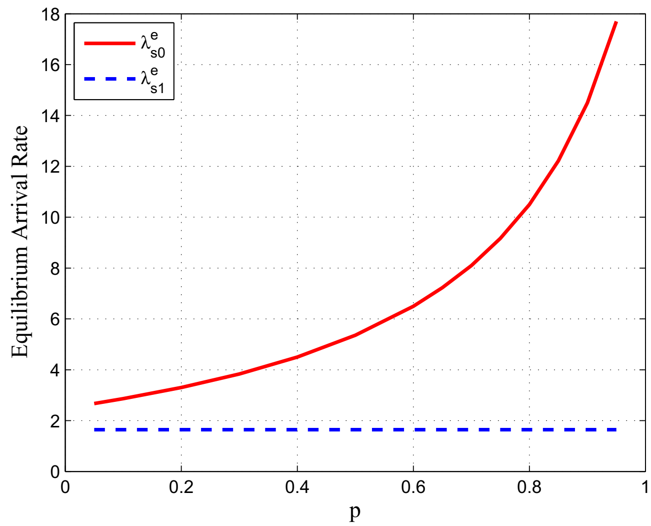

FIGURE 12. Equilibrium arrival rates with respect to p when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$.

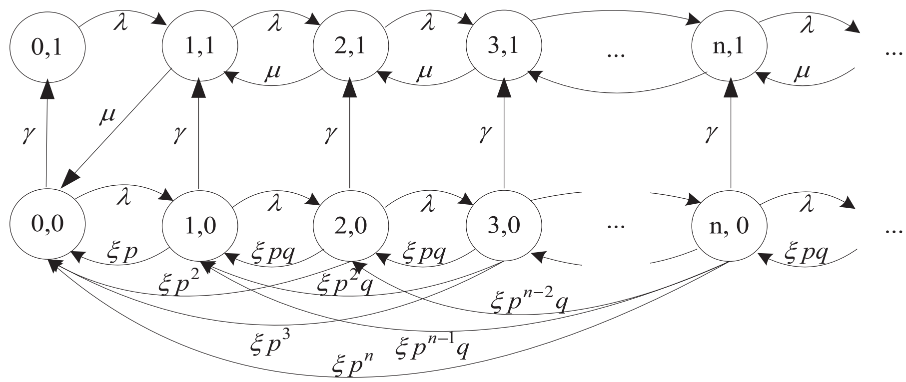

FIGURE 13. Transition rate diagram for the fully unobservable queues.

p_1^{fu} and the probability that he reneges as p_0^{fu}. Based on (3.3) and (3.4),

$$\begin{aligned} p_1^{fu} &= \sum_{n=0}^{\infty} \pi_{n,1} + \sum_{n=0}^{\infty} \pi_{n,0} p_1^{fo} \\ &= 1 - \frac{p\xi}{\gamma+\xi} \frac{1}{1-px} \left(\frac{1}{1-x} + \left(\frac{\gamma}{\lambda} - \frac{\gamma x}{(1-x)(\mu x - \lambda)} \right) \frac{1}{1-\rho} + \frac{\gamma x}{(1-x)^2(\mu x - \lambda)} \right)^{-1}, \end{aligned} \tag{6.1}$$

$$\begin{aligned} p_0^{fu} &= \sum_{n=0}^{\infty} \pi_{n,0} p_0^{fo} \\ &= \frac{p\xi}{\gamma+\xi} \frac{1}{1-px} \left(\frac{1}{1-x} + \left(\frac{\gamma}{\lambda} - \frac{\gamma x}{(1-x)(\mu x - \lambda)} \right) \frac{1}{1-\rho} + \frac{\gamma x}{(1-x)^2(\mu x - \lambda)} \right)^{-1}, \end{aligned} \tag{6.2}$$

where $\rho = \lambda/\mu$ and

$$x = \frac{(\lambda(1+p)+\gamma+\xi p) - \sqrt{(\lambda(1+p)+\gamma+\xi p)^2 - 4p\lambda(\lambda+\gamma+\xi)}}{2p(\lambda+\gamma+\xi)}. \tag{6.3}$$

Obviously, $p_1^{fu} = 1 - p_0^{fu}$. Define the LST of the customer's sojourn time as $W^{*fu}(s)$, and

$$\begin{aligned}
W^{*fu}(s) &= \sum_{n=0}^{\infty} \pi_{n,1} \left(\frac{\mu}{\mu+s}\right)^{n+1} + \sum_{n=0}^{\infty} \pi_{n,0} \left(\int_0^{\infty} (1-\mathrm{e}^{-\xi t}) \sum_{k=0}^{n} p^k q \gamma \mathrm{e}^{-\gamma t} \mathrm{d}t \left(\frac{\gamma}{\gamma+s}\right) \left(\frac{\mu}{\mu+s}\right)^{n+1-k} \right. \\
&\quad \left. + \int_0^{\infty} \mathrm{e}^{-\xi t} \gamma \mathrm{e}^{-\gamma t} \mathrm{d}t \left(\frac{\gamma}{\gamma+s}\right) \left(\frac{\mu}{\mu+s}\right)^{n+1} + \int_0^{\infty} (1-\mathrm{e}^{-\xi t}) \sum_{k=n+1}^{\infty} p^k q \gamma \mathrm{e}^{-\gamma t} \mathrm{d}t \left(\frac{\gamma}{\gamma+s}\right) \right) \\
&= \left(\frac{\mu}{\mu+s} \left(\left(\frac{\gamma}{\lambda} - \frac{\gamma x}{(1-x)(\mu x - \lambda)} \right) \frac{\mu+s}{\mu+s-\lambda} + \frac{\gamma x}{(1-x)(\mu x-\lambda)} \frac{\mu+s}{\mu(1-x)+s} \right) \right. \\
&\quad \left. + \frac{\gamma}{(\gamma+s)(\gamma+\xi)} \left(\frac{\gamma\mu}{\mu+s} \frac{\mu+s}{\mu(1-x)+s} + \frac{\xi\mu q}{\mu q - sp} \left(\frac{\mu}{\mu+s} \frac{\mu+s}{\mu(1-x)+s} - \frac{p}{1-px} \right) + \frac{\xi p}{1-px} \right) \right) \\
&\quad \times \left(\frac{1}{1-x} + \left(\frac{\gamma}{\lambda} - \frac{\gamma x}{(1-x)(\mu x-\lambda)} \right) \frac{1}{1-\rho} + \frac{\gamma x}{(1-x)^2(\mu x - \lambda)} \right)^{-1}.
\end{aligned} \tag{6.4}$$

So his expected sojourn time, denoted by $E[W^{fu}(\lambda)]$, is

$$\begin{aligned}
E[W^{fu}(\lambda)] = -W'^{*fu}(0) &= \left(\frac{1}{\mu} \left(\left(\frac{\gamma}{\lambda} - \frac{\gamma x}{(1-x)(\mu x-\lambda)} \right) \frac{\rho}{(1-\rho)^2} + \frac{\gamma x}{(1-x)(\mu x - \lambda)} \frac{x}{(1-x)^2} \right) \right. \\
&\quad \left. + \left(\frac{1}{\gamma(1-x)} + \frac{1}{\mu(1-x)^2} - \frac{p\xi}{\mu q(\gamma+\xi)} \left(\frac{1}{1-x} - \frac{p}{1-px} \right) \right) \right) \\
&\quad \times \left(\frac{1}{1-x} + \left(\frac{\gamma}{\lambda} - \frac{\gamma x}{(1-x)(\mu x-\lambda)} \right) \frac{1}{1-\rho} + \frac{\gamma x}{(1-x)^2(\mu x-\lambda)} \right)^{-1},
\end{aligned} \tag{6.5}$$

and his expected residual utility, denoted by $U^{fu}(\lambda)$, is

$$U^{fu}(\lambda) = p_1^{fu} R + p_0^{fu} r - cE[W^{fu}(\lambda)]. \tag{6.6}$$

Define the positive equilibrium arrival rate of customers as λ_s^e. Solving $U^{fu}(\lambda) = 0$, we can get λ_s^e. As for ATC or FTC behavior of customers in the fully unobservable case, including the almost unobservable case, we numerically find a unique symmetric mixed equilibrium, so we conjecture that customers adopt ATC.

Then the expected social welfare per time unit, denoted by $\mathrm{SW}_s^{fu}(\lambda)$, is

$$\mathrm{SW}_s^{fu}(\lambda) = \lambda \left(p_1^{fu} R + p_0^{fu} r - cE[W^{fu}(\lambda)] \right). \tag{6.7}$$

Solving the optimization problem **max** $\mathrm{SW}_s^{fu}(\lambda)$ subject to $\lambda < \mu$, we can get the optimal arrival rate λ_s^*, which is substituted into (6.7) for the optimal social welfare SW_s^{*fu}.

7. Numerical comparisons

In this section, for the almost/fully unobservable queues, we compare customers' behavior and optimal social welfare between the queues with single vacation and the queues with multiple vacations[4]. Moreover, for the almost observable queues, we also compare optimal social welfare under the two vacation policies, and discuss the price of anarchy.

[4]For the fully observable case, the customers' equilibrium behavior have no difference under the two types of vacation policies; For the almost observable case, the difference is also can be neglected.

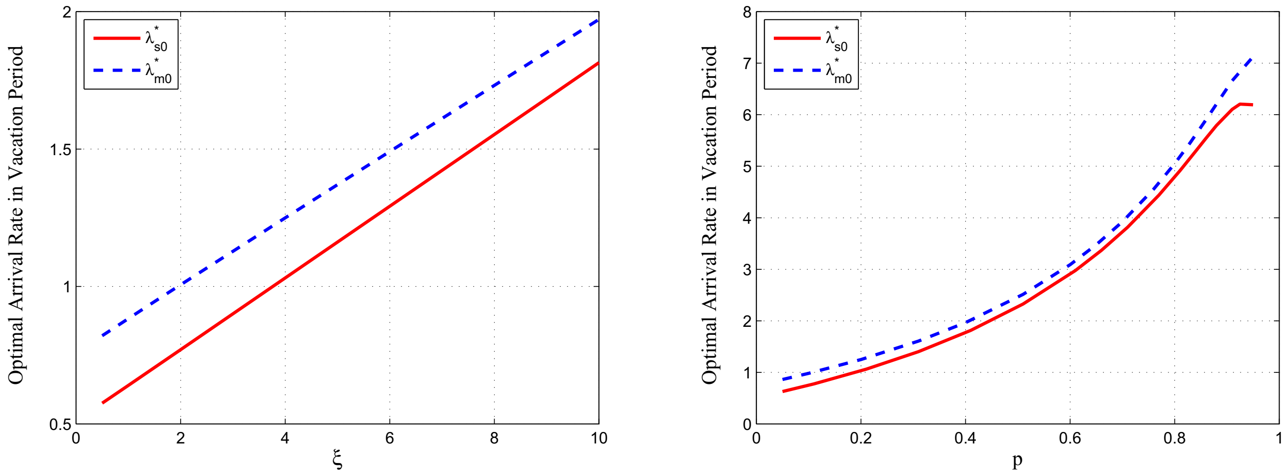

FIGURE 14. Comparison of λ_{s0}^* and λ_{m0}^*: (1) with respect to ξ when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $p = 0.2$; (2) with respect to p when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$.

Similar to the definitions in the queues with single vacation, we define the customers' optimal arrival rates in the vacation state and in the busy state as λ_{m0}^* and λ_{m1}^* (λ_{m0}^e and λ_{m1}^e have been defined in footnote 2), respectively, and the optimal social welfare as SW_m^{*au} in the almost unobservable queues with multiple vacations. Define the customers' equilibrium and optimal arrival rates as λ_m^e and λ_m^*, respectively, and the optimal social welfare as SW_m^{*fu} in the fully unobservable queues with multiple vacations. Then define the optimal social welfare as SW_m^{*ao} in the almost observable queues with multiple vacations[5].

Figure 14 shows that λ_{s0}^* and λ_{m0}^* all increase with ξ or p. Comparing it with Figures 11 and 12, we find that the optimal arrival rates have the similar increasing trend with the equilibrium arrival rates, whereas λ_{s0}^* and λ_{m0}^* are no longer equal to each other but $\lambda_{s0}^* < \lambda_{m0}^*$. It indicates that higher value of ξ or p could have positive effect on the social welfare, *i.e.*, the social planner and customers could have the same preference (λ_{s0}^* has lower increasing rate than λ_{s0}^e). This is resulted by the assumption $0 \le r < R$. That is, although reneging customers receive lower reward, high probability of reneging cuts down customers' expected sojourn time meanwhile. On the other hand, Figure 15 shows that both λ_{s1}^* and λ_{m1}^* decrease with ξ or p, except that p is large, whereas both λ_{s1}^e and λ_{m1}^e are invariable with ξ or p shown in Figures 11 and 12. This indicates that different with the customers' equilibrium behavior, for social optimization, the social planner expects customers arriving in the busy state to consider their reneging behavior happened in the previous vacation before joining. Obviously, it holds that $\lambda_{si}^* < \lambda_{si}^e$ and $\lambda_{mi}^* < \lambda_{mi}^e$ ($i = 0, 1$).

For customers' behavior in the fully unobservable queues, Figures 16 and 17 show that $\lambda_m^e < \lambda_s^e$ while $\lambda_m^* > \lambda_s^*$. Hence, customers and the social planner have disagreement on the preference of vacation policies. And it also holds that $\lambda_s^* < \lambda_s^e$ and $\lambda_m^* < \lambda_m^e$. Comparing customer equilibrium behavior between the almost unobservable queues and the fully unobservable queues, we find that $\lambda_{s1}^e < \lambda_s^e < \lambda_{s0}^e$ and $\lambda_{m1}^e < \lambda_m^e < \lambda_{m0}^e$. For the almost unobservable queues, the social planner expects that the customers prefer the multiple vacation policy to the single vacation policy, which coincides with his expectation in the fully unobservable queues, in the vacation state while the opposite in the busy state.

Next, we compare the optimal social welfare for the almost observable and almost/fully unobservable queues. Figures 18–20 show that the optimal social welfare per time unit in the queues with multiple vacations is always lower than that in the queues with single vacation, especially in the fully unobservable queues. So it is best for the social planner to persuade (or command) the server to select single vacation policy before taking

[5] The analysis process on the queues with multiple vacations are omitted.

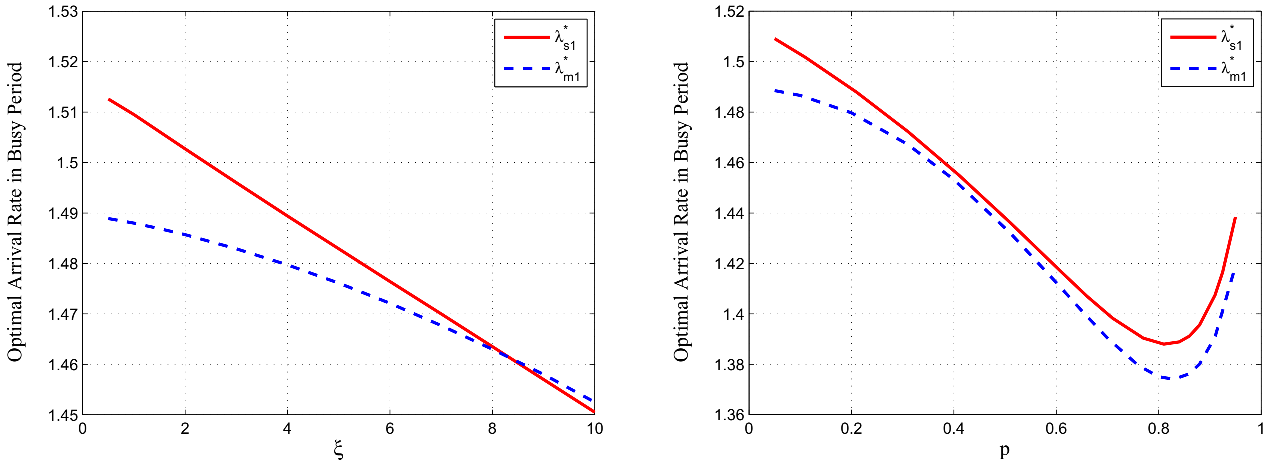

FIGURE 15. Comparison of λ_{s1}^* and λ_{m1}^*: (1) with respect to ξ when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $p = 0.2$; (2) with respect to p when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$.

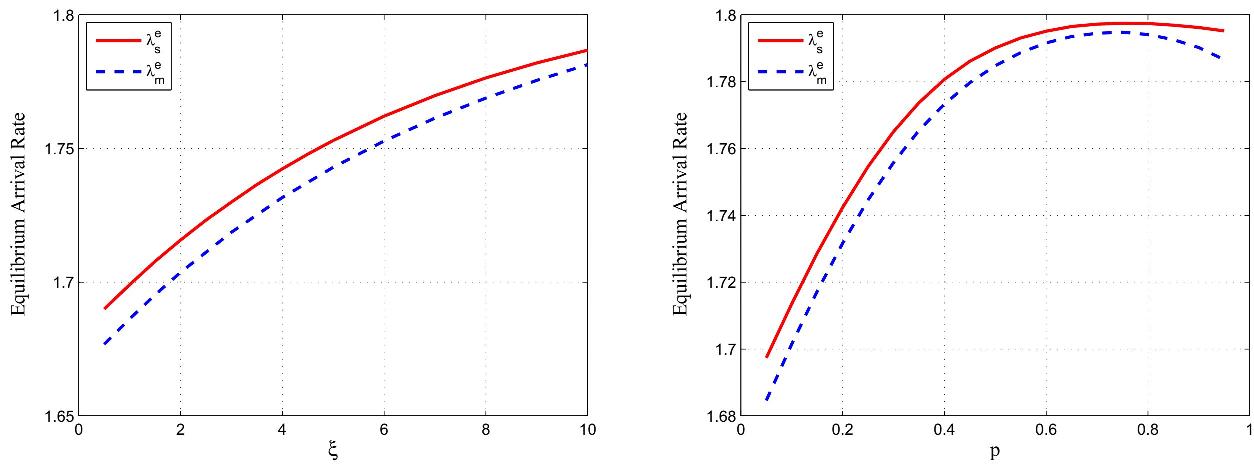

FIGURE 16. Comparison of λ_s^e and λ_m^e: (1) with respect to ξ when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $p = 0.2$; (2) with respect to p when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$.

measures to reduce customers' equilibrium arrival rate(s). It is obvious that $\mathrm{SW}_s^{*fu} < \mathrm{SW}_s^{*au} < \mathrm{SW}_s^{*ao}$ and $\mathrm{SW}_m^{*fu} < \mathrm{SW}_m^{*au} < \mathrm{SW}_m^{*ao}$, which is determined by the extent of the information asymmetry.

Finally, we discuss the price of anarchy in the almost observable case, which is herein defined as the ratio between the optimal social welfare and the worst residual utilities in equilibrium per time unit. We observe from Figure 21 that it monotonically increases with ξ whereas first increases then decreases with p. This indicates that generally customers' frequent reneging behavior indeed rapidly degrades the system efficiency, except that the optimal social welfare itself decreases with high reneging probability.

8. Conclusions

Under four levels of system information precision, this paper studied customers' equilibrium balking behavior in Markovian queues with single vacation and geometric abandonments. For the fully observable queues, we

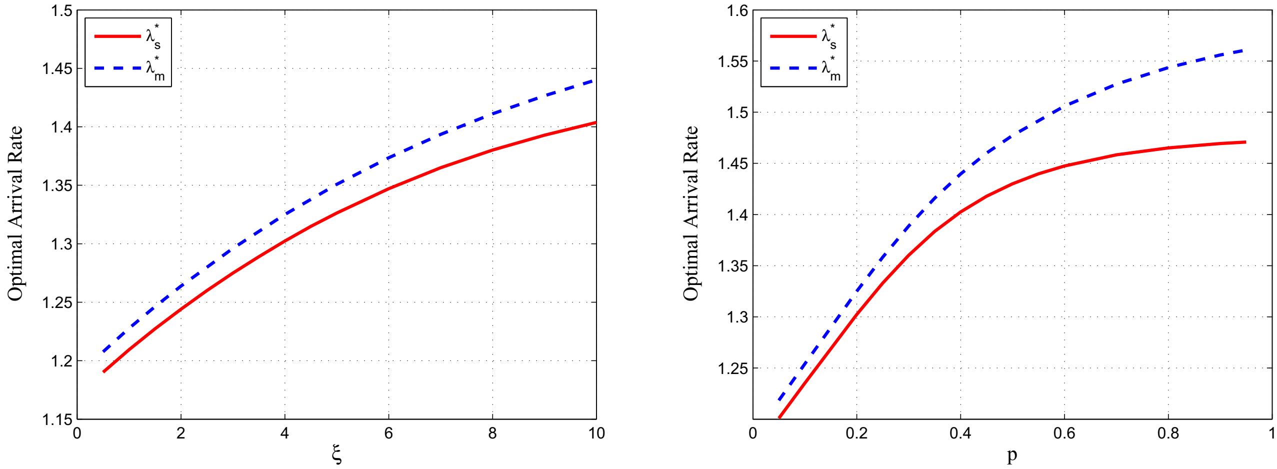

FIGURE 17. Comparison of λ_s^* and λ_m^*: (1) with respect to ξ when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $p = 0.2$; (2) with respect to p when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$.

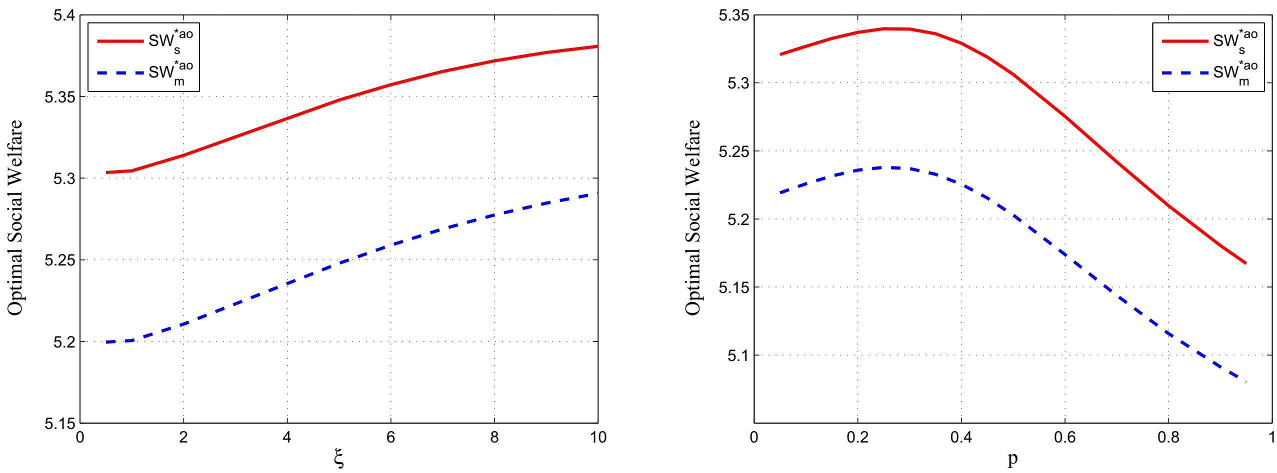

FIGURE 18. Comparison of SW_s^{*ao} and SW_m^{*ao}: (1) with respect to ξ when $R = 5$, $r = 2$, $c = 1$, $\lambda = 3$, $\mu = 2$, $\gamma = 0.5$, $p = 0.2$; (2) with respect to p when $R = 5$, $r = 2$, $c = 1$, $\lambda = 3$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$.

found that customers' equilibrium threshold in the vacation state possibly exceeds the one in the busy state because of customers' reneging, while it has little effect on their equilibrium threshold strategy in the almost observable queues. Moreover, generally, the customers' frequent reneging behavior raises the price of anarchy in the almost observable case. On the other hand, for the almost unobservable queues, customers' equilibrium arrival rate in the vacation state also can exceed the one in the busy state, and customers who arrive in a busy period are not influenced by the reneging behavior (such as the abandonment opportunity arrival process and the reneging probability) that happened in the previous vacation time when they select joining or balking in equilibrium, whereas the social planner expects that the customers can take it into consideration for social optimization. Furthermore, for the almost observable and the almost/fully unobservable queues, we found that the single vacation policy is more beneficial to maximizing social welfare than the multiple vacation policy.

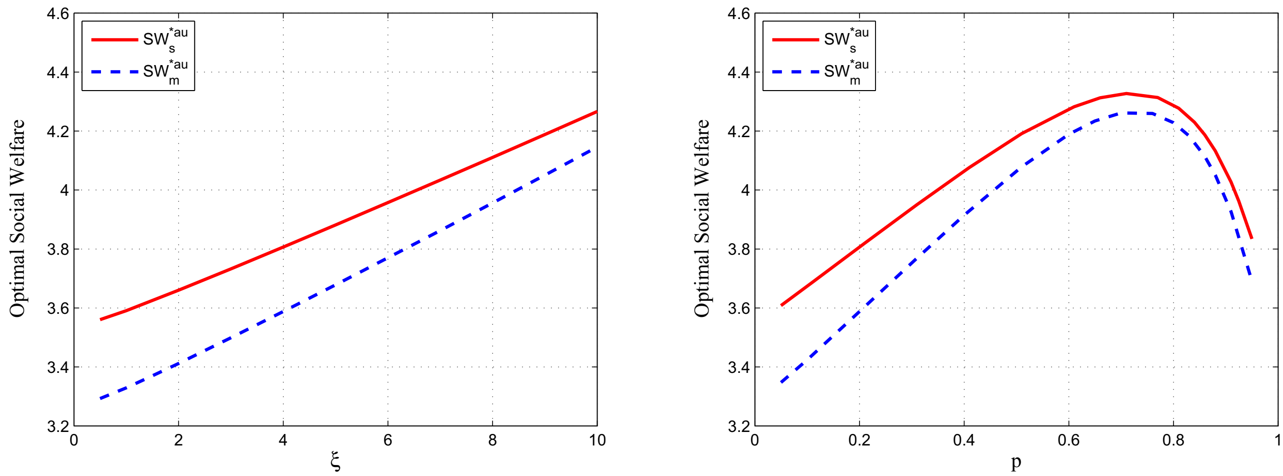

FIGURE 19. Comparison of SW_s^{*au} and SW_m^{*au}: (1) with respect to ξ when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $p = 0.2$; (2) with respect to p when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$.

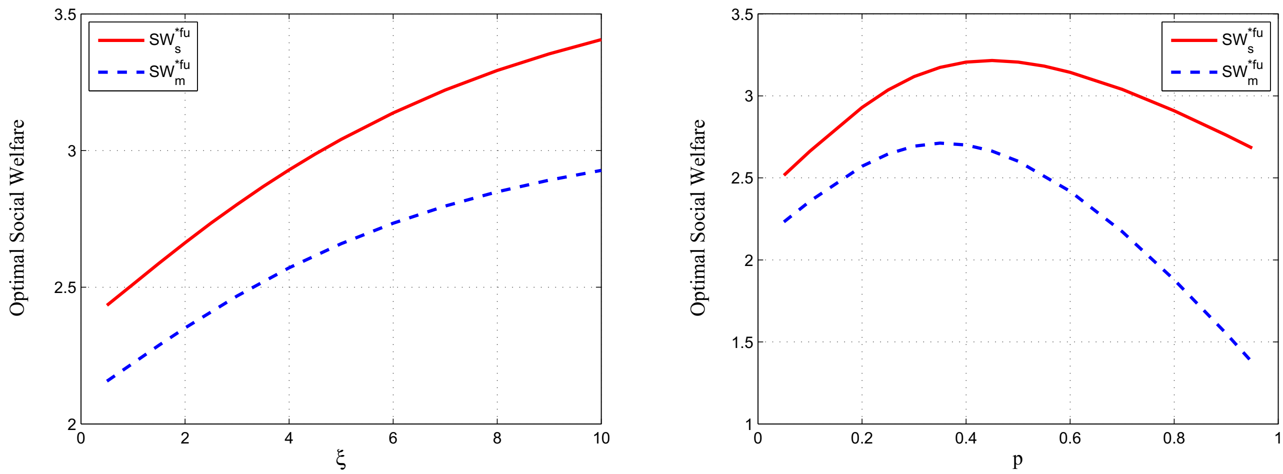

FIGURE 20. Comparison of SW_s^{*fu} and SW_m^{*fu}: (1) with respect to ξ when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $p = 0.2$; (2) with respect to p when $R = 5$, $r = 2$, $c = 1$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$.

Summarily, we list out the main contributions and the differences of this paper with those of other literatures below:

- From the economic viewpoint, we discussed customers' equilibrium and optimal balking behavior based on most literatures on reneging studying from the classical viewpoint. Moreover, we comprehensively considered four information levels and compared customers' equilibrium and optimal behavior under different vacation policies.
- We distinguished a reneging customer's reward from which of a customer completing service, and gave a range for their reward representing their remaining value. This assumption involves many types of practical queueing systems in which reneging is not necessarily unrewarded.

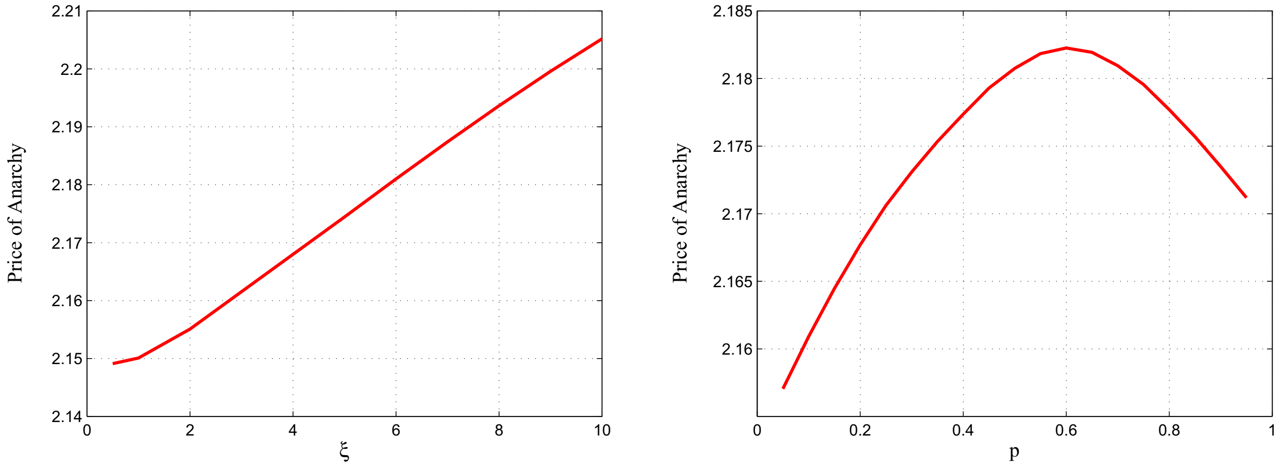

FIGURE 21. Price of anarchy: (1) with respect to ξ when $R = 5$, $r = 2$, $c = 1$, $\lambda = 3$, $\mu = 2$, $\gamma = 0.5$, $p = 0.2$; (2) with respect to p when $R = 5$, $r = 2$, $c = 1$, $\lambda = 3$, $\mu = 2$, $\gamma = 0.5$, $\xi = 4$.

– We found that reneging has obvious effect on customer' equilibrium behavior if they have no information. However, it has no effect on their equilibrium behavior in busy period as long as they can observe the server state. Furthermore, it also has no effect on their equilibrium behavior although they can access the queue length information but not the server state.

In this paper, we assumed that the customers who arrive in a vacation time make their decisions of whether reneging or not in normal order (early arrivals abandon first) at an abandonment epoch, and the number of customers who leave the system follows a geometric distribution (sequential abandonments). On the other hand, if customers make their decisions in reverse order (late arrivals abandon first) or in random order at an abandonment epoch, or if the number of leaving customers follows a binomial distribution (synchronized abandonments), whether these factors can distinctly influence the customers' balking strategies? So this type of topics would become future work.

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