MULTI-ITEM OPTIMAL CONTROL PROBLEM WITH FUZZY COSTS AND CONSTRAINTS USING FUZZY VARIATIONAL PRINCIPLE

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Abstract. An imperfect multi-item production system is considered against time dependent demands for a finite time horizon. Here production is defective. Following [Khouja and Mehrez *J. Oper. Res. Soc.* **45** (1994) 1405–1417], unit production cost depends on production, raw-material and maintenance costs. Produced items have same fixed life-time. Warehouse capacity is limited and used as a constraint. Available space, production, stock and different costs are assumed as crisp or imprecise. With the above considerations, crisp and fuzzy constrained optimal control problems are formulated for the minimization of total cost consisting of raw-material, production and holding costs. These models are solved using conventional and fuzzy variational principles with equality constraint condition and no-stock as end conditions. For the first time, the inequality space constraint is converted into an equality constraint introducing a pseudo state variable following Bang Bang control. [Roul *et al.*, *J. Intell. Fuzzy Syst.* **32** (2017) 565–577], as stock is mainly controlled by production, for the control problems production is taken as the control variable and stock as state variable. The reduced optimal control problem is solved by generalised reduced gradient method using Lingo-11.0. The models are illustrated numerically. For the fuzzy model, optimum results are obtained as fuzzy numbers expressed by their membership functions. From fuzzy results, crisp results are derived using α-cuts.

1. Introduction

Here, one method is presented for the resolution of query regarding the imprecise nature of integrand for the optimal control work "Optimality of the multi-item cost functions under finite time horizon in crisp and fuzzy environments". As a consequence, Fuzzy Variational Principle (FVP; *cf.* Farhadinia [8]) is applied for the work. Generally, in variational problems, the task is to search a suitable curve or optimal path for which a given functional gets its optimum. The idea of fuzzy sets is widely spread to various optimization problems but solving

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procedure of fuzzy variational problems is seldom available in literatures. However, optimality conditions for fuzzy variational problems are reported by very few investigators. It looks that, it is a new idea to optimality conditions for both fuzzy constrained and unconstrained variational problems due to Buckley and Feuring [4].

In the conventional optimal control problems (*cf.* Benalia *et al.* [1], Benjafar *et al.* [2], Chahim *et al.* [5]), the input and output data or parameters are deterministic. In real-life problems, for a new parameter or variable, experts' opinions are sought and sometimes average value is taken as its value. Recently Nayeem [20], Jinquan *et al.* [14], Zhang *et al.* [28], Chai and Ngai [6], Huang *et al.* [11], and others modelled the fuzzy models with subjective imaginations of the decision maker as precisely as he could. The shelf products selection problems with fuzzy parameters have been developed using chance-constrained method. Recently Roul *et al.* [24] formulated an optimal control problem for a single item with fuzzy integrand. They assumed an imperfect production inventory models over a finite time horizon with dynamic demand and fuzzy parameters. The limitations if this investigation is that, being a single item problem, there was no resource constraint in crisp or fuzzy environment. Once the model is formulated for multi-items, there will be at least one resource constraint (crisp or fuzzy) connecting the items and it is of inequality form. Normally equality constraints are connected with the optimal control cost function using Lagrange multipliers and solved using the variational principle. If the constraint is of the inequality type, it is converted to a corresponding equality constraint introducing an additional state variable.

There are several dynamical systems such as defective production system, decaying environmental system *etc.* in which the objective function to be optimized is in the form of an integral with imprecise integrand and integration limits due to imprecise parameters, state and control variables. Objective is to minimize the said cost functional with respect to the variables so that value of the cost functional is maximum or minimum. This leads to fuzzy optimal control problem.

Let us consider an Economic Production Problem of a deteriorating item over a finite time period T. Here, stock depletes due to demand and deteriorated units and the corresponding inventory costs(they are holding, set-up, production cost, revenue or cost due to deterioration). The unit cost or revenue parameters involved in this system may not be deterministic rather are uncertain, may be imprecise (fuzzy or rough *etc.*). Objective of the study of this dynamical system is to find the optimum production rate so that total profit over the time period T is maximum or minimum, which is represented in the form of an finite integral with a constraint connecting the production rate, demand and deterioration. Here the integrand is a fuzzy expression with fuzzy parameters and variables. Thus the problem becomes a fuzzy optimal control problem. Like this model, there are several systems which can be expressed as integral functional with deterministic limits and fuzzy expressions for integrands. The system is solved using the Euler–Lagrange equations.

One of the weaknesses of some production-inventory models is the unrealistic assumption that all items produced are of good quality. But production of defective units is a natural phenomenon due to different difficulties faced in a long-run production process. Recently, Salameh and Jaber [25] formulated a model in a situation where all items are not perfect. In the present study, we have considered production inventory model with defective items. Defective items as a result of imperfect production process were initially considered by Porteus [22] and later by several researchers such as Maity and Maiti [17]; Panda *et al.* [21]; Buscher *et al.* [3]; Roul *et al.* [23]; Chen *et al.* [7] and others.

Among the above mentioned research papers, there are few investigations for dynamical inventory systems in uncertain environment. Nayeem [20] developed a purchasing inventory model with back-ordering, taking several inventory parameters as independent fuzzy variables. Author formulated the problem as a fuzzy expected value model using a linear combination of possibility and necessity measures. Jinquan *et al.* [14] proposed fractional-type robust control of a fuzzy dynamical system where control is deterministic and obtained a closed-form solution. Maity and Maiti [17] considered production-inventory problem for multi-items in the form of an optimal control problem with fuzzy resource constraints. They transformed fuzzy inequality constraints to deterministic ones using possibility and necessity measures. Here the integrand in the optimal control problem was deterministic. Roul *et al.* [24] formulated an optimal control problem with fuzzy integrand for a single item. The present investigation differs from the earlier one that the present model is formulated for multi-items with

fuzzy resource constraints. Here the integrand in the formulated optimal control problem is also fuzzy and fuzzy constraints are connected with the fuzzy integrand using Bang Bang control theory.

Thus the development of a dynamic imperfect production inventory control system is expressed as an optimal control problem with the fuzzy integrand and solved using variational principle under imprecise environment. The highlights of this paper are as follows.

- Formulation of a dynamic inventory control problem over a finite horizon in fuzzy environment and its solution using FVP is not available. The present investigation is an application of FVP.
- In numerical experiments, Mathematica-9.0 is taken as an another tool and used to compare the results.
- Very few works on FVP (*cf.* Roul *et al.* [24]) have been reported and it is for the first time where FVP is applied for multi-item imperfect production inventory model with the integrand of functional as fuzzy in the variational principle formulation.
- A pseudo state variable has been introduced to transform the inequality constraint on state variable into an equality constraint.
- Here Min cost, optimum productions and stocks are calculated for different values of α (α-cut), $\alpha = 0.25, 0.5, 0.75, 1.00$ and Min cost is presented by its appropriate membership function.

In multi-component systems, individual components must be assigned to the tasks that they are supposed to perform. In many applications, there are also limited resources available throughout the system. Keeping in mind that fact, Turner and Turner [26] and Huang *et al.* [10] developed a constraint-based approach to assign system components. Again the real-life resource constraints like available budget, available storage space, *etc.* are also imprecise, not deterministic. Suppose, a manufacturer has a godown of area $S\,ft^2$ initially. In the interest of business, this available space may be augmented, if required *i.e.*, the storage space may be a fuzzy variable.

Taking the above facts into account, in this paper, multi-item imperfect production inventory models are formulated over a finite time horizon with known dynamic demands in crisp and fuzzy environments subject to the space constraint. Here, imperfect unit production costs depend on unit production rate, directly proportional to the corresponding raw material cost and wear–tear costs. Unit production cost which is a part of the imperfect production is also a function of time and is taken as control variable. For this system, total production cost is considered as crisp and fuzzy. The general models are formulated as an optimal control problem over a finite time horizon with defective units for the cost minimisation. Then by the theory of FVP (*cf.* Farhadinia [8]), the imprecise integrand is converted into crisp one by using α measure techniques. The inequality space constraint on the state variable has been converted into an equality constraint introducing a pseudo state variable. Then, the equivalent crisp problem is solved by conventional Variational Principle for fixed-final time and free-final state system (*cf.* Naidu [19]) and Generalised Reduced Gradient (GRG) Technique (*cf.* Gabriel and Ragsdell [9]). Finally, the evaluated results have been compared by the introduction of a new tool Mathematica-9.0. Throughout the paper, all the parameters and variables are also executed by the tool Mathematica-9.0. But the results are almost same for two different tools. For these reasons, the behaviours of optimal production rates, stocks and demands with respect to α over the time are depicted for the single tool Lingo-11.0. We have shown the few results in Section 6 in the form of tabulation. The total Min Cost with respect to different values of α are graphically plotted. For $\alpha = 1.00$, the Min cost in imprecise environment is the same with the result in crisp environment.

2. Preliminaries: Basic concepts of Fuzzy Algebra and Calculus

2.1. Fuzzy algebra

Fuzzy number: A fuzzy number (*cf.* Zimmermann [29]), and its membership function is defined in R. Among the various shapes of fuzzy number, triangular fuzzy number is the most popular one.
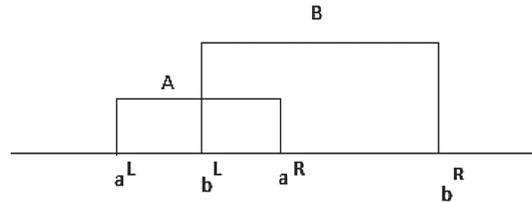
FIGURE 1. Order relation between fuzzy numbers.

Triangular fuzzy number (TFN): A TFN (*cf.* Yang *et al.* [27]) $\widetilde{A}$ is specified by the triplet (a_1, a_2, a_3) and is defined by its continuous membership function $\mu_{\widetilde{A}}(x) : X \to [0,1]$ as follows

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{if } a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & \text{if } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

α-cut of fuzzy number: A fuzzy number is represented with three points as follows: $\tilde{A} = (a_1, a_2, a_3)$. The α-cut of fuzzy number $\tilde{A}$ determines it crisp one. So $A_\alpha = [A_\alpha^l, A_\alpha^r] = [a_1 + (a_2 - a_1)\alpha, a_3 - (a_3 - a_2)\alpha]$ for $\alpha \epsilon [0,1]$.

Order relation between fuzzy numbers: Many researchers have searched the comparison and order relation of fuzzy numbers and intervals. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers and assume that $\tilde{A} \leq \tilde{B}$. Then the α-cut of that fuzzy numbers are $A(\alpha) = [A_\alpha^l, A_\alpha^r]$ and $B(\alpha) = [B_\alpha^l, B_\alpha^r]$. The order relation is depicted in Figure 1. Then order relation can be written as (*cf.* Okada and Gen [13], Ishibuchi and Tanaka [12])

$$A_\alpha^l \leq B_\alpha^l \quad \text{and} \quad A_\alpha^r \leq B_\alpha^r. \tag{2.1}$$

2.2. Fuzzy Calculus

Differentiability of a fuzzy function: (*cf.* Farhadinia [8]) Then the differentiable of fuzzy function $\tilde{f}(\tilde{x}, \alpha)$ with respect to $\tilde{x}$ is denoted by $\frac{d\tilde{f}(\tilde{x}, \alpha)}{d\tilde{x}} = [\frac{df^l(x^l, x^r, \alpha)}{dx^l}, \frac{df^r(x^l, x^r, \alpha)}{dx^r}]$ for $\tilde{x} \epsilon R; \alpha \epsilon [0,1]$

Integrability of a fuzzy function: (*cf.* Farhadinia [8]) The integral of fuzzy function $\tilde{f}(\tilde{x}, \alpha)$ with respect to $\tilde{x}$ is denoted by $\int \tilde{f}(\tilde{x}, \alpha) \, dx = [\int f^l(x^l, x^r, \alpha) \, dx,$
$\int f^r(x^l, x^r, \alpha) \, dx]$ for $\tilde{x} \epsilon R; \alpha \epsilon [0,1]$.

Fuzzy variational principle: Let $\tilde{x}^* = \tilde{x}^*(t)$ be an admissible fuzzy curve, *i.e.*, it is twice continuously differentiable fuzzy curve that joins the given endpoints. Then $\tilde{x}^*(t)$ gives a relative (local) minimum to the fuzzy functional $J(\tilde{x}, \dot{\tilde{x}}, t)$ in (FVP; *cf.* Farhadinia [8]). Here, FVP (Fuzzy Variational Principle) has been derived due to Roul *et al.* [24], for the functional $J(\tilde{x}, \dot{\tilde{x}}, t)$,

$$\frac{\partial}{\partial x^l} g^l \left(x^{*l}, x^{*r}, \dot{x}^{*l}, \dot{x}^{*r}, t, \alpha \right) - \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}^l} g^l \left(x^{*l}, x^{*r}, \dot{x}^{*l}, \dot{x}^{*r}, t, \alpha \right) \right) = 0; \tag{2.2}$$

$$\frac{\partial}{\partial x^r} g^l \left(x^{*l}, x^{*r}, \dot{x}^{*l}, \dot{x}^{*r}, t, \alpha \right) - \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}^r} g^l \left(x^{*l}, x^{*r}, \dot{x}^{*l}, \dot{x}^{*r}, t, \alpha \right) \right) = 0; \tag{2.3}$$

$$\frac{\partial}{\partial x^l} g^r \left(x^{*l}, x^{*r}, \dot{x}^{*l}, \dot{x}^{*r}, t, \alpha \right) - \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}^l} g^r \left(x^{*l}, x^{*r}, \dot{x}^{*l}, \dot{x}^{*r}, t, \alpha \right) \right) = 0; \tag{2.4}$$

$$\frac{\partial}{\partial x^r} g^r \left(x^{*l}, x^{*r}, \dot{x}^{*l}, \dot{x}^{*r}, t, \alpha \right) - \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}^r} g^r \left(x^{*l}, x^{*r}, \dot{x}^{*l}, \dot{x}^{*r}, t, \alpha \right) \right) = 0; \tag{2.5}$$

where α-cut of $J(\tilde{x}, \tilde{\dot{x}}, t)$ is $\left[g^l \left(x^l, x^r, \dot{x}^l, \dot{x}^r, t, \alpha \right), g^r \left(x^l, x^r, \dot{x}^l, \dot{x}^r, t, \alpha \right) \right]$ with $g^l \left(x^l, x^r, \dot{x}^l, \dot{x}^r, t, \alpha \right)$

stands for $g^l \left(x^l(t, \alpha), x^r(t, \alpha), \dot{x}^l(t, \alpha), \dot{x}^r(t, \alpha), t \right)$ and $g^r \left(x^l, x^r, \dot{x}^l, \dot{x}^r, t, \alpha \right)$ stands for

$g^r \left(x^l(t, \alpha), x^r(t, \alpha), \dot{x}^l(t, \alpha), \dot{x}^r(t, \alpha), t \right)$.

2.3. Transformation of inequality constraints to equality constraints

Lemma 2.1. *Here a lemma is introduced for transformation. If $f(x_i(t)) \leq A$ then the inequality constraints can be expressed as an equality constraints as*

$$Z^2 - \left(A - f(x_i(t)) \right) = 0, \tag{2.6}$$

where Z is real and a pseudo state variable.

Proof. If $A > f(x_i(t))$ then Z^2 is a negative quantity and z is not real which contradicts the proposition "Z is real". If $A \leq f(x_i(t))$ then Z^2 is zero of a positive quantity. Hence Z is real. □

2.4. Kuhn–Tucker's conditions

Here Kuhn–Tucker's necessary and sufficient conditions for optimality have been explained (Roul *et al.* [23]).

3. PROPOSED OPTIMAL CONTROL MODELS

For a multi-item imperfect production-inventory optimal control problem under finite time horizon, following assumptions and notations are used.

3.1. Assumptions

– It is a single period multi-item imperfect production inventory model with finite time horizon $t\epsilon[0, T]$;
– Defective rate is known; Shortages are not allowed;
– There is no repair or replacement of defective units over the whole time period;
– The development cost of the system increases the durability of the system;
– Unit production cost depends on produced quantity, raw material, wear–tear and development costs;
– Stock level and production rate is taken as deterministic as well as fuzzy.

3.2. Notations

$$(h_i^l(\alpha), h_i^r(\alpha)) = [h_{i1} + (h_{i2} - h_{i1})\alpha, h_{i3} - (h_{i3} - h_{i2})\alpha] \text{ for } \alpha\epsilon[0, 1].$$
$$(L_i^l(\alpha), L_i^r(\alpha)) = [L_{i1} + (L_{i2} - L_{i1})\alpha, L_{i3} - (L_{i3} - L_{i2})\alpha] \text{ for } \alpha\epsilon[0, 1].$$
$$(N_i^l(\alpha), N_i^r(\alpha)) = [N_{i1} + (N_{i2} - N_{i1})\alpha, N_{i3} - (N_{i3} - N_{i2})\alpha] \text{ for } \alpha\epsilon[0, 1].$$

$$\left(c_d(l)_i^l(\alpha), c_d(l)_i^r(\alpha) \right)$$
$$= [c_d(l)_{i1} + (c_d(l)_{i2} - c_d(l)_{i1})\alpha, c_d(l)_{i3} - (c_d(l)_{i3} - c_d(l)_{i2})\alpha] \text{ for } \alpha\epsilon[0, 1]$$
$$= [L_{i1} + (L_{i2} - L_{i1})\alpha + N_{i1} + (N_{i2} - N_{i1})\alpha, L_{i3} - (L_{i3} - L_{i2})\alpha + N_{i3} - (N_{i3} - N_{i2})\alpha]$$
$$= [L_{i1} + N_{i1} + (L_{i2} + N_{i2} - L_{i1} - N_{i1})\alpha, L_{i3} + N_{i3} + (L_{i2} + N_{i2} - L_{i3} - N_{i3}\alpha]$$

$T:$ total time for the system;

$A:$ space constraint of the system;

$u_i(t):$ production rate at time t which is taken as control variable;

$\tilde{u}_i(t):$ fuzzy production rate at time t

$u_i(t, \alpha):$ the α-cut of fuzzy number $\tilde{u}_i(t)$ *i.e.* $[u_i^l(t, \alpha), u_i^r(t, \alpha)]$ for $\alpha \epsilon [0, 1]$.

$x_i(t):$ stock level at time t which is a state variable;

$\tilde{x}_i(t):$ fuzzy stock level at time t

$x_i(t, \alpha):$ the α-cut of fuzzy number $\tilde{x}(t)$ *i.e.* $[x_i^l(t, \alpha), x_i^r(t, \alpha)]$ for $\alpha \epsilon [0, 1]$.

$h_i:$ holding cost per unit;

$\tilde{h}_i$ triangular fuzzy number.

$\tilde{h}_i$ $= (h_{i1}, h_{i2}, h_{i3})$

$h_i(\alpha):$ the α-cut of fuzzy number $\tilde{h}_i$.

$L_i:$ fixed cost like labour, energy, etc;

$\tilde{L}_i$ triangular fuzzy number.

$\tilde{L}_i$ $= (L_{i1}, L_{i2}, L_{i3})$.

$L_i(\alpha):$ the α-cut of fuzzy number $\tilde{L}_i$.

$N_i:$ cost of technology, design, complexity, resources, etc;

$\tilde{N}_i$ triangular fuzzy number.

$\tilde{N}_i$ $= (N_{i1}, N_{i2}, N_{i3})$.

$N_i(\alpha):$ the α-cut of fuzzy number $\tilde{N}_i$.

$c_{i0}:$ constant material cost;

$d_{i0}:$ constant demand at initial stage;

$c_d(l)_i$ development cost to improve the quality of the product;

$c_d(l)_i$ $= N_i + L_i$

$\tilde{c}_d(l)$ triangular fuzzy number.

$\tilde{c}_d(l)$ $= (c_d(l)_{i1}, c_d(l)_{i2}, c_d(l)_{i3})$

$c_d(l)_i(\alpha):$ the α-cut of fuzzy number $\tilde{c}_d(l)_i$

$\beta_{i0}:$ wear–tear cost for the system.

3.3. Model-1: Formulation of optimal control models for dynamic demand in crisp environment

In this model, differential equation for the stock level $x_i(t)$ regarding above system during a fixed time-horizon, T is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x_i(t)\right) = (1 - \delta_i)u_i(t) - d_i(t) \tag{3.1}$$

where $d_i(t) = d_{i0} - a_i \mathrm{e}^{-b_i t}$ with $(d_{i0} - a_i), a_i, b_i > 0$

The unit production cost has considered as a function of produced-quantity, raw material cost and wear–tear costs (*cf.* Khouja and Mehrez [15]). So the total production cost is

$$c_{ui}(t)u_i(t) = \left(c_{i0}u_i(t) + \beta_{i0}u_i^2(t) + c_d(l)_i\right). \tag{3.2}$$

The total holding cost over the finite time interval $[0, T]$ for the stock $x_i(t)$ is $\int_0^T \sum_{i=1}^n h_i x_i(t)\,\mathrm{d}t$. Let s_i be the area of space for per unit stock level $x_i(t)$ and A be the maximum allocation of space for warehouse. Then we have

$$\sum_{i=1}^n s_i x_i(t) \leq A. \tag{3.3}$$

Now, change the inequality constraint (3.3) into equality constraint introducing the variable z where $z^2(-\sum_{i=1}^{n} s_i x_i(t) + A) = 0$; where z being a pseudo real state variable and if inequality constraints (3.3) are satisfied then $z^2 \geq 0$. It can be written as

$$\int_0^T \sum_{i=1}^{n} (-s_i x_i(t) + A) \mathrm{d}t = \int_0^T z^2 \mathrm{d}t. \tag{3.4}$$

Thus the problem reduces to the minimization of the cost function J subject to the constraint satisfying the dynamic production–demand relation and allocation of space for warehouse

$$\text{Min} J = \int_0^T \sum_{i=1}^{n} \left(h_i\, x_i(t) + c_{ui}(t) u_i(t) - p_i \delta_i u_i(t) \right) \mathrm{d}t$$

$$\text{sub to } \frac{\mathrm{d}}{\mathrm{d}t}\left(x_i(t) \right) = (1 - \delta_i) u_i(t) - d_i(t) \tag{3.5}$$

$$\int_0^T \sum_{i=1}^{n} (-s_i x_i(t) + A) \mathrm{d}t = \int_0^T z^2 \mathrm{d}t \quad \text{where } u_i(t) \geq 0\,, x_i(t) \geq 0. \tag{3.6}$$

The above equations (3.5) and (3.6) are defined as an optimal control problem with control variable $u_i(t)$ and state variable $x_i(t)$. Then the expression (3.6) takes the form

$$J_{\min}\left(x(t), \dot{x}(t), t \right)$$

$$= \int_0^T \sum_{i=1}^{n} \left(h_i\, x_i(t) + c_d(l)_i + \left(\frac{c_{i0}}{(1 - \delta_i)} - p_i \delta_i \right)\left(\dot{x}_i(t) + d_i(t) \right) \right.$$

$$\left. + \frac{\beta_{i0}}{(1 - \delta_i)^2}\left(\dot{x}_i(t) + d_i(t) \right)^2 \right) \mathrm{d}t$$

$$\text{subject to } \int_0^T \sum_{i=1}^{n} (-s_i x_i(t) + A) \mathrm{d}t = \int_0^T z^2 \mathrm{d}t \quad \text{where } x_i(t) \geq 0 \tag{3.7}$$

Let us consider Lagrangian L as

$$L\left(x_i(t), \dot{x}_i(t), t, \lambda \right) = \int_0^T \left[\sum_0^{n} \left(h_i\, x_i(t) + \frac{c_{i0}}{(1 - \delta_i)} \dot{x}_i(t) + \frac{c_{i0}}{(1 - \delta_i)} d_i(t) + \frac{\beta_{i0}}{(1 - \delta_i)^2} \dot{x}_i{}^2(t) \right.\right.$$

$$\left. + \frac{\beta_{i0}}{(1 - \delta_i)^2} d_i^2(t) + \frac{2\beta_{i0}}{(1 - \delta_i)^2} \dot{x}_i(t) d_i(t) + c_d(l)_i \right)$$

$$\left. + \lambda\left(z^2 - A + \sum_{i=1}^{n} s_i x_i(t) \right) \right] \mathrm{d}t \quad \text{where } x_i(t) \geq 0, \tag{3.8}$$

where λ is an undetermined Lagrange multiplier. The above problem (3.8) is defined as an optimal control problem with state variables $x_i(t)$ only. Here, (3.8) contains $u_i(t)$ implicitly.

Using Euler–Lagrange's equation for Fixed-Final Time and Fixed State System (*i.e.*, Here final time T is specified and $x_i(t) \equiv x_i, \dot{x}_i(t) \equiv \dot{x}_i, x_i(0) = 0\ x_i(T) = 0$), we have

$$\frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i} \right) = 0 \tag{3.9}$$

$$\frac{\partial L}{\partial z_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}_i} \right) = 0 \tag{3.10}$$

$$\text{with Kuhn–Tucker's conditions} \quad \lambda \frac{\mathrm{d}L}{\mathrm{d}\lambda} = 0; \tag{3.11}$$

Using equation (3.8) in (3.9)–(3.11), we get

$$h_i + \lambda s_i - \frac{2\beta_{i0}}{(1-\delta_i)^2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}_i(t) - a_i \mathrm{e}^{-b_i t}\right) = 0 \tag{3.12}$$

$$2\lambda z = 0 \tag{3.13}$$

$$\text{with} \quad \lambda\left(z^2 + \sum_{i=1}^{n}(s_i x_i(t) - A)\right) = 0. \tag{3.14}$$

From (3.13), we have either $\lambda = 0$ or, $z = 0$. If $\lambda = 0$, then by Kuhn Tucker conditions and from (3.14), we have $\left(z^2 + \sum_{i=1}^{n}(s_i x_i(t) - A)\right) \leq 0$. which is impossible. Therefore $\lambda \neq 0$ and $z = 0$. Considering the positivity of λ. Then for $\lambda > 0$,

$$\sum_{i=1}^{n} s_i x_i(t) = A. \tag{3.15}$$

Considering the conditions $x_i(0) = 0$ and $x_i(T) = 0$ in (3.12), then the stock level becomes

$$x_i(t) = \frac{a_i}{b_i}(1 - \mathrm{e}^{-b_i t}) + \frac{a_i t}{b_i T}(\mathrm{e}^{-b_i T} - 1) + \frac{h_i t(1-\delta_i)^2 + \lambda s_i}{4\beta_{i0}}(t - T). \tag{3.16}$$

From (3.1), we have the production rate

$$u_i(t) = \frac{1}{1-\delta_i}\left(\frac{a_i}{b_i T}(\mathrm{e}^{-b_i T} - 1) + d_{i0} + \frac{h_i(1-\delta_i)^2 + \lambda s_i}{4\beta_{i0}}(2t - T)\right). \tag{3.17}$$

Then the cost function J takes the form $J_{\min}$

$$\begin{aligned}
&= \sum_{i=1}^{n}\int_0^T \left(h_i\, x_i(t) + c_{ui}(t)u_i(t) - p_i\delta_i u_i(t)\right)\mathrm{d}t \\
&= \sum_{i=1}^{n}\left[h_i\left(\frac{a_i}{b_i^2}(b_i T + \mathrm{e}^{-b_i T} - b_i) + TC_d(l)_i + \frac{a_i T}{2b_i}(\mathrm{e}^{-b_i T} - 1) - \frac{h_i(1-\delta_i)^2}{24\beta_{i0}}T^3\right)\right. \\
&\quad - \frac{\lambda s_i}{24\beta_{i0}}T^3 + \left(\frac{a_i}{b_i}(\mathrm{e}^{-b_i T} - 1) + d_{i0}T\right)\left(\frac{c_{i0}}{1-\delta_i} - p_i\delta_i\right) + \frac{\beta_{i0}}{(1-\delta_i)^2}\left(\frac{a_i^2}{Tb_i^2}(\mathrm{e}^{-b_i T}\right. \\
&\quad \left.\left. - 1)^2 + Td_{i0}^2 + \frac{(1-\delta_i)^2 h_i^2 + \lambda s_i}{12\beta_{i0}}T^3 + \frac{2a_i d_{i0}}{b_i}(\mathrm{e}^{-b_i T} - 1)\right)\right],
\end{aligned} \tag{3.18}$$

subject to the Space Constraint

$$A = \sum_{i=1}^{n}\left[s_i\left(\frac{a_i}{b_i}(1 - \mathrm{e}^{-b_i t}) + \frac{a_i t}{b_i T}(\mathrm{e}^{-b_i T} - 1) + \frac{h_i t(1-\delta_i)^2 + \lambda a_i}{4\beta_{i0}}(t - T)\right)\right]. \tag{3.19}$$

3.4. Model-2: Formulation of optimal control models for dynamic demand in fuzzy environment

In this model, stock variables $x_i(t)$ and control variables $u_i(t)$ are taken as fuzzy and putting them in (3.1), the differential equation for stock levels $\tilde{x}_i(t)$ regarding above system during a fixed time-horizon, T is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{x}_i(t)\right) = (1 - \delta_i)\tilde{u}_i(t) - d_i(t) \quad \text{with} \quad d_i(t) = d_{i0} - a_i \mathrm{e}^{-b_i t}. \tag{3.20}$$

The unit production cost has considered as a function of produced-quantity, raw material cost, wear–tear and development costs (*cf.* Khouja [15]). Here one parameters like $h_i(> 0)$ for holding cost per unit stock level and development cost $c_d(l)_i$ are considered as fuzzy. So the total production cost is

$$\tilde{c}_{ui}(t)u_i(t) = \left(c_{i0}u(t) + \tilde{c}_d(l)_i + \beta_{i0}\tilde{u}_i^2(t) \right), \tag{3.21}$$

and the total holding cost over the finite time interval $[0, T]$ for the stock $\tilde{x}_i(t)$ is $\int_0^T \sum_{i=1}^n \tilde{h}_i \tilde{x}_i(t)\, \mathrm{d}t$ Let s_i be the areas of space for per unit stock levels $\tilde{x}_i(t)$ and A be the maximum allocation of space for warehouse. Then we have

$$\sum_{i=1}^n s_i \tilde{x}_i(t) \le \tilde{A}. \tag{3.22}$$

Taking the α-cut of (3.22), we have

$$\sum_{i=1}^n [s_i x_i^l(t, \alpha), s_i x_i^r(t, \alpha)] \le [A^l(\alpha), A^r(\alpha)]. \tag{3.23}$$

Then (3.23) can be written as

$$\sum_{i=1}^n s_i x_i^l(t, \alpha) \le A^l(\alpha) \tag{3.24}$$

$$\sum_{i=1}^n s_i x_i^r(t, \alpha) \le A^r(\alpha). \tag{3.25}$$

Now, changing the inequality constraints (3.24) and (3.25) into equality constraints introducing the variables z_1 and z_2 where

$$z_1^2 - \left(-\sum_{i=1}^n s_i x_i^l(t, \alpha) + A^l(\alpha) \right) = 0; \tag{3.26}$$

$$z_2^2 - \left(-\sum_{i=1}^n s_i x_i^r(t, \alpha) + A^r(\alpha) \right) = 0; \tag{3.27}$$

where z_1 and z_2 being real variables and according lemma -4.2, (3.27) and (3.27) can be written as

$$\int_0^T \left(\sum_{i=1}^n -s_i x_i^l(t, \alpha) + A^l(\alpha) \right) \mathrm{d}t = \int_0^T z_1^2 \mathrm{d}t \tag{3.28}$$

$$\int_0^T \left(\sum_{i=1}^n -s_i x_i^r(t, \alpha) + A^r(\alpha) \right) \mathrm{d}t = \int_0^T z_2^2 \mathrm{d}t. \tag{3.29}$$

Thus the problem reduces to minimize the cost function $\tilde{J}$ subject to the constraint satisfying the dynamic production–demand relation and allocation of space for warehouse.

$$\text{Min } \tilde{J} = \int_0^T \sum_{i=1}^n \left(\tilde{h}_i \tilde{x}_i(t) + \tilde{c}_{ui}(t)\tilde{u}_i(t) - p_i \delta_i \tilde{u}_i(t) \right) \mathrm{d}t$$

$$\text{sub to } \frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{x}_i(t) \right) = (1 - \delta_i)\tilde{u}_i(t) - d_i(t) \tag{3.30}$$

$$\text{where } \tilde{u}_i(t) \;\; i.e. \; [u_i^l(t, \alpha), u_i^r(t, \alpha)] \text{ and } u_i^l(t, \alpha) \ge 0, u_i^r(t, \alpha) \ge 0;$$

$$\tilde{x}(t) \;\; i.e. \; [x_i^l(t, \alpha), x_i^r(t, \alpha)] \text{ and } x_i^l(t, \alpha) \ge 0, x_i^r(t, \alpha) \ge 0. \tag{3.31}$$

The above equations (3.30) and (3.31) are defined as an optimal control problem with control variables $\tilde{u}_i(t)$ and state variables $\tilde{x}_i(t)$. Then the expression (3.30) takes the form

$$\tilde{J}_{\min}\left(\tilde{x}_i(t), \dot{\tilde{x}}_i(t), t\right) = \int_0^T \sum_{i=1}^n \left[\tilde{h}_i\, \tilde{x}_i(t) + \left(\frac{c_{i0}}{(1-\delta_i)} - p_i\delta_i\right)\left(\dot{\tilde{x}}_i(t) + d_i(t)\right)\right.$$
$$\left. + \frac{\beta_{i0}}{(1-\delta_i)^2}\left(\dot{\tilde{x}}_i(t) + d_i(t)\right)^2 + \tilde{c}_d(l)_i\right]\mathrm{d}t. \tag{3.32}$$

The above integral can be taken as α-level set since it is Lebesgue Integrable. Now the α-level of $\tilde{J}$ is

$$\tilde{J}\left(\tilde{x}_i(t), \dot{\tilde{x}}_i(t), t\right)[\alpha]$$
$$= \left[J^l\left(x_i^l(t,\alpha), x_i^r(t,\alpha), \dot{x}_i^l(t,\alpha), \dot{x}_i^r(t,\alpha), t, \right), J^r\left(x_i^l(t,\alpha), x_i^r(t,\alpha), \dot{x}_i^l(t,\alpha), \dot{x}_i^r(t,\alpha), t\right)\right]$$
$$\text{taking as } \left[J^l\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \alpha\right), J^r\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \alpha\right)\right]. \tag{3.33}$$

Hence

$$J^l\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \alpha\right)$$
$$= \int_0^T \left[\sum_{i=1}^n \left(h_i^l(\alpha)\, x_i^l(t,\alpha) + \frac{c_{i0}}{1-\delta_i}\dot{x}_i^l(t,\alpha) + \frac{c_{i0}}{1-\delta_i}d_i(t) - p_i\delta_i d_i(t) - p_i\delta_i\dot{x}_i^r(t,\alpha)\right.\right.$$
$$\left.\left. + \frac{\beta_{i0}}{(1-\delta_i)^2}\dot{x}_i^{l2}(t,\alpha) + \frac{\beta_{i0}}{(1-\delta_i)^2}d_i^2(t) + \frac{2\beta_{i0}}{(1-\delta_i)^2}\dot{x}_i^l(t,\alpha)d_i(t) + c_d^l(l)_i(\alpha)\right)\right]\mathrm{d}t \tag{3.34}$$

$$J^r\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \alpha\right)$$
$$= \int_0^T \left[\sum_{i=1}^n \left(h_i^r(\alpha)\, x_i^r(t,\alpha) + \frac{c_{i0}}{1-\delta_i}\dot{x}_i^r(t,\alpha) + \frac{c_{i0}}{1-\delta_i}d_i(t) - p_i\delta_i d_i(t) - p_i\delta_i\dot{x}_i^l(t,\alpha)\right.\right.$$
$$\left.\left. + \frac{\beta_{i0}}{(1-\delta_i)^2}\dot{x}_i^{r2}(t,\alpha) + \frac{\beta_{i0}}{(1-\delta_i)^2}d_i^2(t) + \frac{2\beta_{i0}}{(1-\delta_i)^2}\dot{x}_i^r(t,\alpha)d_i(t) + c_d^r(l)_i(\alpha)\right)\right]\mathrm{d}t. \tag{3.35}$$

For left cut of α, the reduced optimal control problem is

$$J^l\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \alpha\right)$$
$$= \int_0^T \left[\sum_{i=1}^n \left(h_i^l(\alpha)\, x_i^l(t,\alpha) + \frac{c_{i0}}{1-\delta_i}\dot{x}_i^l(t,\alpha) + \frac{c_{i0}}{1-\delta_i}d_i(t) - p_i\delta_i d_i(t) - p_i\delta_i\dot{x}_i^r(t,\alpha)\right.\right.$$
$$\left.\left. + \frac{\beta_{i0}}{(1-\delta_i)^2}\dot{x}_i^{l2}(t,\alpha) + \frac{\beta_{i0}}{(1-\delta_i)^2}d_i^2(t) + \frac{2\beta_{i0}}{(1-\delta_i)^2}\dot{x}_i^l(t,\alpha)d_i(t) + c_d^l(l)_i(\alpha)\right)\right]\mathrm{d}t$$
$$\text{subject to } \int_0^T \left(\sum_{i=1}^n -s_i x_i^l(t,\alpha) + A^l(\alpha)\right)\mathrm{d}t = \int_0^T z_1^2\mathrm{d}t \tag{3.36}$$

Let us consider Lagrangian L as

$$L^l\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_1, \alpha\right) = \int_0^T \left[\sum_{i=1}^n \left(h_i^l(\alpha)\, x_i^l(t, \alpha) + \frac{c_{i0}}{1-\delta_i} \dot{x}_i^l(t, \alpha) + \frac{c_{i0}}{1-\delta_i} d_i(t) - p_i \delta_i d_i(t) \right. \right.$$
$$\left. - p_i \delta_i \dot{x}_i^r(t, \alpha) + \frac{\beta_{i0}}{(1-\delta_i)^2} \dot{x}_i^{l2}(t, \alpha) + \frac{\beta_{i0}}{(1-\delta_i)^2} d_i^2(t) + \frac{2\beta_{i0}}{(1-\delta_i)^2} \dot{x}_i^l(t, \alpha) d_i(t) \right)$$
$$\left. + c_d^l(l)_i(\alpha) + \lambda_1 \left(z_1^2 - \left(\sum_{i=1}^n -s_i x_i^l(t, \alpha) + A^l(\alpha) \right) \right) \right] dt \qquad (3.37)$$

where λ_1 is an undetermined Lagrange multiplier. Now, Fuzzy Euler–Lagrange's equations (2.2) and (2.3) have been considered as

$$\frac{\partial}{\partial x_i^l} L^l\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_1, \alpha\right) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i^l} L^l\left(x^l, x^r, \dot{x}^l, \dot{x}^r, t, \lambda_1, \alpha\right) \right) = 0; \qquad (3.38)$$

$$\frac{\partial}{\partial x_i^r} L^l\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_1, \alpha\right) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i^r} L^l\left(x^l, x^r, \dot{x}^l, \dot{x}^r, t, \lambda_1, \alpha\right) \right) = 0; \qquad (3.39)$$

$$\frac{\partial}{\partial z_1} L^l\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_1, \alpha\right) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{z}_1} L^l\left(x^l, x^r, \dot{x}^l, \dot{x}^r, t, \lambda_1, \alpha\right) \right) = 0; \qquad (3.40)$$

with Kuhn–Tucker's conditions $\quad \lambda_1 \frac{\partial}{\partial \lambda_1} L^l\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_1, \alpha\right) = 0. \qquad (3.41)$

Using (3.38) and (3.39); taking $B = \frac{2\beta_{i0}}{(1-\delta_i)^2}$, we have from (3.37)

$$h_i^l(\alpha) + \lambda_1 s_i - B \frac{d}{dt}\left(\dot{x}_i^l(t, \alpha) + d_i(t) \right) = 0 \qquad (3.42)$$

$$\text{and} \ \ 0 - \frac{d}{dt}\left(p_i \delta_i \right) = 0. \qquad (3.43)$$

Using (3.40) and (3.41) and from (3.37), we have

$$2\lambda_1 z_1 = 0 \qquad (3.44)$$

$$\text{with} \quad \lambda_1 \left(z_1^2 - \sum_{i=1}^n s_i x_i^l(t, \alpha) + A^l(\alpha) \right) = 0. \qquad (3.45)$$

From (3.45), we have either $\lambda_1 = 0$ or, $z_1 = 0$.

If $\lambda_1 = 0$, then by Kuhn Tucker conditions and from (3.45), we have $\left(z_1^2 + \sum_{i=1}^n (s_i x_i^l(t, \alpha) - A^l(\alpha)) \right) \leq 0$ which is impossible. Therefore $\lambda_1 \neq 0$ and $z_1 = 0$. Considering the positivity of λ_1. Then for $\lambda_1 > 0$,

$$\sum_{i=1}^n s_i x_i^l(t, \alpha) = A^l(\alpha). \qquad (3.46)$$

For right-cut of α, the reduced optimal control problem is

$$
J^r\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \alpha\right)
$$

$$
= \int_0^T \left[\sum_{i=1}^n \left(h_i^r(\alpha)\, x_i^r(t, \alpha) + \frac{c_{i0}}{1-\delta_i}\dot{x}_i^r(t,\alpha) + \frac{c_{i0}}{1-\delta_i}d_i(t) - p_i\delta_i d_i(t) \right.\right.
$$

$$
- p_i\delta_i\dot{x}_i^l(t,\alpha) + \frac{\beta_{i0}}{(1-\delta_i)^2}\dot{x}_i^{r2}(t,\alpha) + \frac{\beta_{i0}}{(1-\delta_i)^2}d_i^2(t)
$$

$$
+ \left.\left. \frac{2\beta_{i0}}{(1-\delta_i)^2}\dot{x}_i^r(t,\alpha)d_i(t) + c_d^r(l)_i(\alpha) \right) \right]\,\mathrm{d}t \tag{3.47}
$$

$$
\text{subject to } \int_0^T \left(\sum_{i=1}^n -s_i x_i^r(t,\alpha) + A^r(\alpha) \right)\,\mathrm{d}t = \int_0^T z_2^2\,\mathrm{d}t \tag{3.48}
$$

Let us consider Lagrangian L as

$$
L^r\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_2, \alpha\right) = \int_0^T \left[\sum_{i=1}^n \left(h_i^r(\alpha)\, x_i^r(t,\alpha) + \frac{c_{i0}}{1-\delta_i}\dot{x}_i^r(t,\alpha) + \frac{c_{i0}}{1-\delta_i}d_i(t) - p_i\delta_i d_i(t) \right.\right.
$$

$$
- p_i\delta_i\dot{x}_i^l(t,\alpha) + \frac{\beta_{i0}}{(1-\delta_i)^2}\dot{x}_i^{r2}(t,\alpha) + \frac{\beta_{i0}}{(1-\delta_i)^2}d_i^2(t) + c_d^r(l)_i(\alpha)
$$

$$
+ \left.\left. \frac{2\beta_{i0}}{(1-\delta_i)^2}\dot{x}_i^r(t,\alpha)d_i(t) \right) + \lambda_2\left(z_2^2 - \left(\sum_{i=1}^n -s_i x_i^r(t,\alpha) + A^r(\alpha) \right) \right) \right]\,\mathrm{d}t \tag{3.49}
$$

where λ_2 is an undetermined Lagrange multiplier. Similarly, (10) and (11) also have been considered as

$$
\frac{\partial}{\partial x_i^l}L^r\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_2, \alpha\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{x}_i^l}L^r\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_2, \alpha\right) \right) = 0; \tag{3.50}
$$

$$
\frac{\partial}{\partial x_i^r}L^r\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_2, \alpha\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{x}_i^r}L^r\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_2, \alpha\right) \right) = 0; \tag{3.51}
$$

$$
\frac{\partial}{\partial z_2}L^r\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_2, \alpha\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{z}_2}L^r\left(x^l, x^r, \dot{x}^l, \dot{x}^r, t, \lambda_2, \alpha\right) \right) = 0; \tag{3.52}
$$

$$
\text{with Kuhn–Tucker's conditions } \quad \lambda_2\,\frac{\partial}{\partial \lambda_2}L^r\left(x_i^l, x_i^r, \dot{x}_i^l, \dot{x}_i^r, t, \lambda_2, \alpha\right) = 0. \tag{3.53}
$$

Using (3.50) and (3.51); taking $B = \frac{2\beta_{i0}}{(1-\delta_i)^2}$, we have from (3.49)

$$
h_i^r(\alpha) + \lambda_2 s_i - B\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}_i^r(t,\alpha) + d_i(t) \right) = 0 \tag{3.54}
$$

$$
\text{and } \ 0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(p_i\delta_i \right) = 0 \tag{3.55}
$$

Using (3.52) and (3.53) in (3.49), we have

$$
2\lambda_2 z_2 = 0 \tag{3.56}
$$

$$
\text{with } \quad \lambda_2.\left(z_2^2 - \sum_{i=1}^n s_i x_i^r(t,\alpha) + A^r(\alpha) \right) = 0. \tag{3.57}
$$

From (3.56), we have either $\lambda_2 = 0$ or, $z_2 = 0$. If $\lambda_2 = 0$, then by Kuhn Tucker conditions, we have $\left(z_2^2 + \sum_{i=1}^n (s_i x_i^r(t, \alpha) - A^r(\alpha)\right) \le 0$. which is impossible. Therefore $\lambda_2 \ne 0$ and $z_2 = 0$. Considering the positivity of λ_2. Then for $\lambda_2 > 0$,

$$\sum_{i=1}^n s_i x_i^r(t, \alpha) = A^r(\alpha). \tag{3.58}$$

Here (3.42) and (3.54) are two governing equations. Using these with the conditions Fixed-Final Time and Fixed State System (*i.e.*, here final time T is specified and $x_i^l[0, \alpha] = 0, x_i^l[T, \alpha] = 0; x_i^r[0, \alpha] = 0; x_i^r[T, \alpha] = 0$), and substituting the value of B, we have

$$x_i^l(t, \alpha) = \frac{a_i}{b_i}(1 - e^{-b_i t}) + \frac{a_i t}{b_i T}(e^{-b_i T} - 1) + \frac{(1 - \delta_i)^2}{2\beta_{i0}} \frac{h_i^l(\alpha) + \lambda_1 s_i}{2} t(t - T) \tag{3.59}$$

$$x_i^r(t, \alpha) = \frac{a_i}{b_i}(1 - e^{-b_i t}) + \frac{a_i t}{b_i T}(e^{-b_i T} - 1) + \frac{(1 - \delta_i)^2}{2\beta_{i0}} \frac{h_i^r(\alpha) + \lambda_2 s_i}{2} t(t - T). \tag{3.60}$$

Using (3.59) in (3.20), we have the production rate

$$u_i^l(t, \alpha) = \frac{1}{1 - \delta_i} \left(\frac{a_i}{b_i T}(e^{-b_i T} - 1) + d_{i0} + \frac{(1 - \delta_i)^2}{2\beta_{i0}} \frac{h_i^l(\alpha) + \lambda_1 s_i}{2}(2t - T) \right) \tag{3.61}$$

and using (3.60) in (3.20), we have the production rate

$$u_i^r(t, \alpha) = \frac{1}{1 - \delta_i} \left(\frac{a_i}{b_i T}(e^{-b_i T} - 1) + d_{i0} + \frac{(1 - \delta_i)^2}{2\beta_{i0}} \frac{h_i^r(\alpha) + \lambda_2 s_i}{2}(2t - T) \right). \tag{3.62}$$

Let us consider the cost function $\tilde{J}$ as $\left[G^l(\alpha), G^r(\alpha) \right]$. Then the cost function J takes the form for left-cut of α as $G^l(\alpha)$

$$\begin{aligned}
&= \sum_{i=1}^n \int_0^T \left(h_i^l(\alpha) x_i^l(t, \alpha) + c_{ui}(t) u_i^l(t, \alpha) - p_i \delta_i u_i^r(t, \alpha) \right) dt \\
&= \sum_{i=1}^n \left[h_i^l(\alpha) \left(\frac{a_i}{b_i^2}(b_i T + e^{-b_i T} - b_i) + \frac{a_i T}{2b_i}(e^{-b_i T} - 1) - \frac{h_i^l(\alpha)(1 - \delta_i)^2 + \lambda_1 s_i}{24\beta_{i0} T^{-3}} \right) \right. \\
&\quad + T c_d^l(l)_i(\alpha) + \frac{\beta_{i0}}{(1 - \delta_i)^2} \left(\frac{a_i^2}{T b_i^2}(e^{-b_i T} - 1)^2 + T d_{i0}^2 + \frac{(1 - \delta_i)^2 h_i^{l2}(\alpha) + \lambda_1 s_i}{12\beta_{i0} T^{-3}} \right. \\
&\quad \left. + \frac{2 a_i d_{i0}}{b_i}(e^{-b_i T} - 1) \right) + \left(\frac{a_i}{b_i}(e^{-b_i T} - 1) + d_{i0} T \right) \left(\frac{c_{i0}}{1 - \delta_i} - p_i \delta_i \right) \right]. \tag{3.63}
\end{aligned}$$

Subject to the Space Constraint

$$A^l(\alpha) = \sum_{i=1}^n \left[s_i \left(\frac{a_i}{b_i}(1 - e^{-b_i t}) + \frac{a_i t}{b_i T}(e^{-b_i T} - 1) + \frac{h_i^l(\alpha) t (1 - \delta_i)^2 + \lambda_1 s_i}{4\beta_{i0}}(t - T) \right) \right]. \tag{3.64}$$

TABLE 1. Input data for the crisp environment.

i	a_i	b_i	δ_i	h_i	c_{i0}	β_{i0}	d_{i0}	p_i	T	s_i	L_i	N_i	$cd(l)_i$
1	105	0.7	0.01	0.5	0.7	0.5	211	10	12	2	14	16	30
2	100	0.8	0.02	0.7	0.6	0.4	214	12	12	3	13	15	28

Then the cost function J takes the form for right-cut of α as $G^r(\alpha)$

$$= \sum_{i=1}^{n} \int_0^T \left(h_i^r(\alpha)\, x_i^r(t,\alpha) + c_{ui}(t) u_i^r(t,\alpha) - p_i \delta_i u_i^l(t,\alpha) \right) \mathrm{d}t$$

$$= \sum_{i=1}^{n} \left[h_i^r(\alpha) \big(\frac{a_i}{b_i^2}(b_i T + \mathrm{e}^{-b_i T} - b_i) + \frac{a_i T}{2 b_i}(\mathrm{e}^{-b_i T} - 1) - \frac{h_i^l(\alpha)(1-\delta_i)^2 + \lambda_1 s_i}{24\beta_{i0} T^{-3}} \big) \right.$$

$$+ T C_d^l(l)_i(\alpha) + \frac{\beta_{i0}}{(1-\delta_i)^2} \big(\frac{a_i^2}{T b_i^2}(\mathrm{e}^{-b_i T} - 1)^2 + T d_{i0}^2 + \frac{(1-\delta_i)^2 h_i^{l2}(\alpha) + \lambda_1 s_i}{12\beta_{i0} T^{-3}}$$

$$\left. + \frac{2 a_i d_{i0}}{b_i}(\mathrm{e}^{-b_i T} - 1) \big) + \big(\frac{a_i}{b_i}(\mathrm{e}^{-b_i T} - 1) + d_{i0} T \big) \big(\frac{c_{i0}}{1-\delta_i} - p_i \delta_i \big) \right]. \tag{3.65}$$

Subject to the Space Constraint

$$A^r(\alpha) = \sum_{i=1}^{n} \left[s_i \left(\frac{a_i}{b_i}(1 - \mathrm{e}^{-b_i t}) + \frac{a_i t}{b_i T}(\mathrm{e}^{-b_i T} - 1) + \frac{h_i^r(\alpha) t (1-\delta_i)^2 + \lambda_2 s_i}{4\beta_{i0}}(t - T) \right) \right]. \tag{3.66}$$

In case of imprecise environment, the min cost lies in the following interval. So the required min cost is

$$\tilde{J}_{\min}\left(\tilde{x}(t), \tilde{\dot{x}}(t), t \right)[\alpha] = \left[G^l(\alpha), G^r(\alpha) \right]$$

$$\text{subject to } \left[A_1^l(\alpha), A_1^r(\alpha) \right] \leq A. \tag{3.67}$$

4. NUMERICAL EXPERIMENTS

4.1. Input data for crisp and fuzzy production models

To illustrate the models in the crisp (Model-1) and fuzzy environments (Model-2), the following input data are taken in Tables 1 and 2 respectively. Here maximum available space 500 square units have been taken. For fuzzy environment, taking the level of α is taken as 0.5. The unit of the quantities h_i, c_{i0}, β_{i0}, p_i, L_i, N_i, $cd(l)_i$ are taken as \$. For the simulation of the proposed model, Lingo-11.0 and Mathematica-9.0 have been used fot same inputs.

4.2. Output for the optimal control problems in crisp and fuzzy-environments

Using the above input data, the numerical values of optimal path of the system have been derived with the help of software LINGO-11.0 and optimal states of the unknowns $x_i(t)$, $u_i(t)$ have evaluated in crisp and fuzzy environments with respect to optimal $\lambda^* = 0.01 = \lambda_1^* = \lambda_2^*$.

For the defective units, we have $\delta_1 = 0.01$ for item-1 and $\delta_2 = 0.02$ for item-2. The required min costs for item-1 and-2 in **crisp environment** are \$243402.7 and \$260618.9 respectively; and for $i = 1, 2$ the values of production $u_i(t)$, demand $d_i(t)$ and stock $x_i(t)$ for the time period $[0, T]$ are given in Table 3. These are also graphically depicted in Figures 2–4.

TABLE 2. Input data for the fuzzy environment.

i	a_i	b_i	δ_i	s_i	c_{i0}	β_{i0}	d_{i0}	p_i	h_{i1}	h_{i2}	h_{i3}	$h_i^l(\alpha)$	$h_i^r(\alpha)$
1	105	0.7	0.01	3	0.7	0.5	211	10	0.4	0.5	0.6	0.45	0.55
2	100	0.8	0.02	2	0.6	0.4	214	12	0.5	0.7	0.8	0.6	0.75
i	L_{i1}	L_{i2}	L_{i3}	N_{i1}	N_{i2}	N_{i3}	$cd(l)_{i1}$	$cd(l)_{i2}$	$cd(l)_{i3}$	$L_i^l(\alpha)$	$L_i^r(\alpha)$	$N_i^l(\alpha)$	$N_i^r(\alpha)$
1	12	14	15	14	16	18	26	30	32	12	10	0.8	0.01
2	11	13	14	13	15	17	24	28	29	12	10	0.8	0.01

TABLE 3. Values of $u_i(t), d_i(t), x_i(t)$ for $i = 1, 2$ for the Model-1.

Time	$u_1(t)$	$u_2(t)$	$x_1(t)$	$x_2(t)$	$d_1(t)$	$d_2(t)$
$t = 0$	196.356	202.441	0	0	106	114
$t = 1$	197.881	203.324	60.237	53.699	158.858	169.067
$t = 2$	198.406	204.207	82.965	70.353	185.107	193.810
$t = 3$	198.932	205.928	87.322	70.831	198.142	204.928
$t = 4$	199.457	205.973	82.808	64.515	204.615	209.924
$t = 5$	199.983	206.855	74.146	55.618	207.829	212.168
$t = 6$	200.508	207.739	63.676	46.034	209.425	213.630
$t = 7$	201.033	208.623	52.564	36.614	210.218	213.177
$t = 8$	201.558	209.505	41.387	27.739	210.612	213.834
$t = 9$	202.083	210.388	32.432	19.582	210.807	213.925
$t = 10$	203.609	211.271	19.841	12.219	210.904	213.966
$t = 11$	203.134	212.154	09.685	5.687	210.952	213.984
$t = 12$	204.659	213.037	0	0	210.976	213.933

In **fuzzy environment**, for the item-1, the min cost is $J^l(\alpha = 0.5) = \$243340.7$ and $J^r(\alpha = 0.5) = \$243458.1$, *i.e.* $J = [\$243340.7, \$243458.1]$ and for the item-2, the min cost is $J^l(\alpha = 0.5) = \$260531.5$ and $J^r(\alpha = 0.5) = \$260656.7$ *i.e.* $J = [\$260531.5, \$260656.7]$ and the values of production $(u_1^l(t, \alpha), u_1^r(t, \alpha))$; stocks $(x_1^l(t, \alpha), x_1^r(t, \alpha))$; demand $d_1(t)$ and the values of production $(u_2^l(t, \alpha), u_2^r(t, \alpha))$; stocks $(x_2^l(t, \alpha), x_2^r(t, \alpha))$; demand $d_2(t)$ for the time period $[0, T]$ with α-level = 0.5 are given in Table 4.

For item-1, the Min cost for different level of α (α-left cut) are: $\mathbf{\$243309.9}_{(\alpha_{\text{left}}=0.25)}$
$< \mathbf{\$243340.7}_{(\alpha_{left}=0.5)} < \mathbf{\$243371.6}_{(\alpha_{\text{left}}=0.75)} < \mathbf{\$243402.7}_{(\alpha=1.00)}$
and same for the α (α-right cut)are: $\mathbf{\$243402.7}_{(\alpha=1.00)} < \mathbf{\$243430.2}_{(\alpha_{right}=0.75)} < \mathbf{\$243458.1}_{(\alpha_{right}=0.5)} <$
$\mathbf{\$243485.9}_{(\alpha_{right}=0.25)}$.

For item-2, the Min cost for different level of α (α-left cut) are: $\mathbf{\$260487.8}_{(\alpha_{left}=0.25)}$
$< \mathbf{\$260531.5}_{(\alpha_{left}=0.5)} < \mathbf{\$260575.2}_{(\alpha_{left}=0.75)} < \mathbf{\$260618.9}_{(\alpha=1.00)}$
and same for the α (α-right cut)are: $\mathbf{\$260618.9}_{(\alpha=1.00)} < \mathbf{\$260637.8}_{(\alpha_{right}=0.75)} < \mathbf{\$260656.7}_{(\alpha_{right}=0.5)}$
$< \mathbf{\$260675.6}_{(\alpha_{right}=0.25)}$.

4.2.1. *Outputs for the crisp environment*

Outputs for the crisp environment is given in Table 3 using the tool LINGO-11.0.

4.2.2. *Outputs for the fuzzy environment*

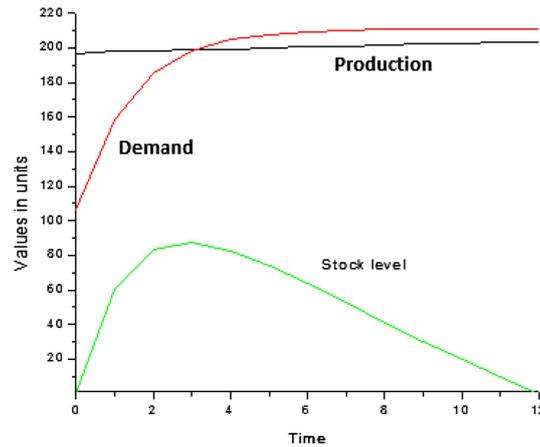Outputs for the fuzzy environment is given in Table 4 using the tool LINGO-11.0.

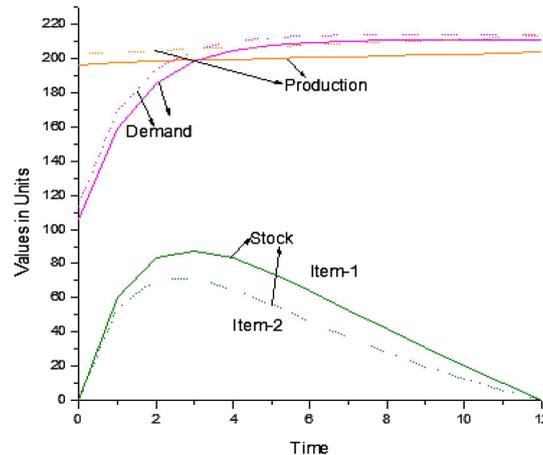FIGURE 2. Time *vs.* stock, production and demand in crisp environment for item-1.

FIGURE 3. Time *vs.* stock, production and demand in crisp environment for item-1 and 2; Dotted for item-2.

4.2.3. Crisp results vs. fuzzy results

As the formulation and solution methodologies for crisp and fuzzy models are different, results obtained for these models cannot compared in strict sense. However for the results of crisp model (*i.e.* fuzzy models at $\alpha = 1$), and those of fuzzy models are different for different α *i.e.*; for $\alpha = 0.25, 0.5, 0.75$. These results are as follows: It is noted that costs for item-1 and-2 in crisp environment are \$243402.7 and \$260618.9 respectively. In fuzzy environment, Min costs are:

For item-1, the Min cost for different level of α (α-left cut) are: $\mathbf{\$243309.9}_{(\alpha_{\text{left}}=0.25)}$ $<$ $\mathbf{\$243340.7}_{(\alpha_{left}=0.5)}$ $<$ $\mathbf{\$243371.6}_{(\alpha_{left}=0.75)}$ $<$ $\mathbf{\$243402.7}_{(\alpha=1.00)}$¡ $\mathbf{\$243402.7}_{(\alpha=1.00)}$ $<$ $\mathbf{\$243430.2}_{(\alpha_{right}=0.75)} < \mathbf{\$243458.1}_{(\alpha_{right}=0.5)} < \mathbf{\$243485.9}_{(\alpha_{right}=0.25)}$.

For item-2, the Min cost for different level of α (α-left cut) are: $\mathbf{\$260487.8}_{(\alpha_{\text{left}}=0.25)}$ $<$ $\mathbf{\$260531.5}_{(\alpha_{\text{left}}=0.5)}$ $<$ $\mathbf{\$260575.2}_{(\alpha_{\text{left}}=0.75)}$ $<$ $\mathbf{\$260618.9}_{(\alpha=1.00)}$¡ $\mathbf{\$260618.9}_{(\alpha=1.00)}$ $<$ $\mathbf{\$260637.8}_{(\alpha_{right}=0.75)} < \mathbf{\$260656.7}_{(\alpha_{\text{right}}=0.5)} < \mathbf{\$260675.6}_{(\alpha_{\text{right}}=0.25)}$.
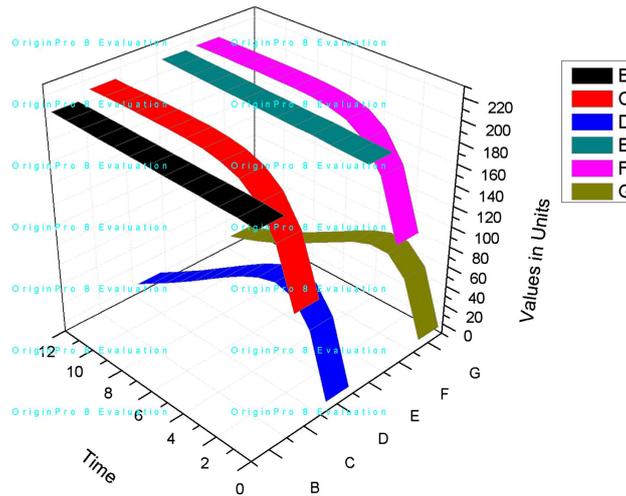
FIGURE 4. Time *vs.* stock (D,G), production (B,E) and demand (C,F) in crisp environment for item-1, 2 respectively; 3D approach to make prominent one.

TABLE 4. Values of $u_i^l(t, \alpha), x_i^l(t, \alpha), u_i^r(t, \alpha), x_i^r(t, \alpha), d_i(t)$ *at* $\alpha = 0.5$ for $i = 1, 2$ of the Model-2.

t	$u_1^l(t, \alpha)$	$u_2^l(t, \alpha)$	$x_1^l(t, \alpha)$	$x_2^l(t, \alpha)$	$u_1^r(t, \alpha)$	$u_2^r(t, \alpha)$	$x_1^r(t, \alpha)$	$x_2^r(t, \alpha)$	$d_1(t)$	$d_2(t)$
0	197.653	203.176	0	0	197.059	202.073	0	0	106	114
1	198.128	203.936	60.515	54.373	197.633	203.017	59.959	53.362	158.858	169.067
2	198.128	204.697	83.471	71.578	198.208	203.962	82.461	69.739	185.107	193.810
3	199.081	205.457	88.003	72.485	198.208	203.962	82.461	69.739	198.142	204.928
4	199.556	206.217	83.617	66.476	198.783	204.906	86.639	70.004	204.615	209.924
5	200.032	206.978	75.029	57.762	199.933	206.795	73.262	54.546	207.829	212.168
6	200.507	207.738	64.586	48.240	200.508	207.739	62.767	44.931	209.425	213.630
7	200.984	208.984	53.448	38.758	201.082	208.683	51.680	35.541	210.218	213.177
8	201.459	209.259	42.195	29.699	201.657	209.627	40.578	26.758	210.612	213.834
9	201.935	210.020	31.113	21.236	202.232	210.571	29.749	18.754	210.807	213.925
10	202.411	210.780	20.346	13.445	202.807	211.516	19.335	11.607	210.904	213.966
11	202.887	211.541	9.963	6.361	203.382	212.460	9.407	5.351	210.952	213.984
12	203.363	212.302	0	0	203.956	213.404	0	0	210.976	213.933

As a consequence, the nature of min cost at different level of α are pointed in Table 5 and also Figure 5. Here, for the item-1, the Min cost for the level of $\alpha = 1.00$ is equivalent to the min cost in crisp environment;

i.e. $243402.7_{(\textbf{fuzzyenvironment})}(\alpha = 1.00) = \$243402.7_{(\textbf{crispenvironment})}.$

Also, for the item-2, the Min cost for the level of $\alpha = 1.00$ is equivalent to the min cost in crisp environment;

i.e. $260618.9\$_{(\textbf{fuzzyenvironment})(\alpha=1.00)} = \$260618.9_{(\textbf{crispenvironment})}.$

4.3. Fuzzy representation of decision variables for item-1

Following Figure 5, the membership function (MF) of the total cost is defined by its continuous membership function $\mu_{\widetilde{cost}}(x)$ as follows

TABLE 5. Values of Min cost J in different values of α

i	1			2		
α − left	0.25	0.5	0.75	0.25	0.5	0.75
Min Cost in(\$)	243309.9	243340.7	243371.6	260487.8	260531.5	260575.2
α − right	0.75	0.5	0.25	0.75	0.5	0.25
Min Cost in (\$)	243430.2	243458.1	243485.9	260637.8	260656.7	260675.6

$$
\mu_{\widetilde{cost}}(x) = \begin{cases} \dfrac{x - J_1}{J_2 - J_1} & \text{if } J_1 \leq x \leq J_2 \\ \dfrac{J_3 - x}{J_3 - J_2} & \text{if } J_2 \leq x \leq J_3 \\ 0 & \text{otherwise} \end{cases}
$$

where the cost triplet $(J_1, J_2, J_3) = (\$243309.9, \$243402.7, \$243485.9)$

Similarly, the MFs of production at $t = 2$ and stock at $t = 2$ are respectively furnished as

$$
\mu_{\widetilde{prod}}(x) = \begin{cases} \dfrac{x - u_1}{u_2 - u_1} & \text{if } u_1 \leq x \leq u_2 \\ \dfrac{u_3 - x}{u_3 - u_2} & \text{if } u_2 \leq x \leq u_3 \\ 0 & \text{otherwise} \end{cases}
$$

where $(u_1, u_2, u_3) = (192.703, 198.001, 198.109)$

$$
\mu_{\widetilde{stock}}(x) = \begin{cases} \dfrac{x - x_1}{x_2 - x_1} & \text{if } x_1 \leq x \leq x_2 \\ \dfrac{x_3 - x}{x_3 - x_2} & \text{if } x_2 \leq x \leq x_3 \\ 0 & \text{otherwise} \end{cases}
$$

where $(x_1, x_2, x_3) = (81.955, 82.208, 83.723)$

5. Output for the optimal control problems in crisp and fuzzy-environments using Mathematica-9.0

It is noted that results of numerical experiment in Mathematica-9.0 are almost same as the results in Lingo-11.0. For this reason, some precise results are given in the present section and Tables 6 and 7.

Using the above input data, the numerical values of optimal path of the system is derived with the help of software Mathematica-9.0 and optimal states of the unknowns $x_i(t), u_i(t)$ are evaluated in crisp and fuzzy environments with respect to optimal $\lambda^* = 0.01$. For the defective units $\delta_1 = 0.01$ and $\delta_2 = 0.02$ the required min costs for item-1 and item-2 are \$243 403 and \$260 619 respectively.

For item-1, the Min cost for different levels of α are: $\$243\,309_{(\alpha_{\text{left}}=0.25)}$ < $\$243\,340_{(\alpha_{\text{left}}=0.5)}$ < $\$243\,371_{(\alpha_{\text{left}}=0.75)}$ < $\$243\,402_{(\alpha=1.00)}$ < $\$243\,430_{(\alpha_{\text{right}}=0.75)}$ < $\$243\,458_{(\alpha_{\text{right}}=0.5)}$ < $\$243\,485_{(\alpha_{\text{right}}=0.25)}$.

For item-2, the Min cost for different levels of α are: $\$260\,487_{(\alpha_{\text{left}}=0.25)}$ < $\$260\,531_{(\alpha_{\text{left}}=0.5)}$ < $\$260\,575_{(\alpha_{\text{left}}=0.75)}$ < $\$260\,618_{(\alpha=1.00)}$ < $\$260\,637_{(\alpha_{\text{right}}=0.75)}$ < $\$260\,656_{(\alpha_{\text{right}}=0.5)}$ < $\$260\,675_{(\alpha_{\text{right}}=0.25)}$.

For the item-1, the Min cost for the level of $\alpha = 1.00$ is equivalent to the min cost in crisp environment; *i.e.*

$\$243\,402_{(\textbf{fuzzyenvironment})(\alpha=1.00)} = \$243\,402_{(\textbf{crispenvironment})}$.

Similar results are for the item-2, *i.e.*

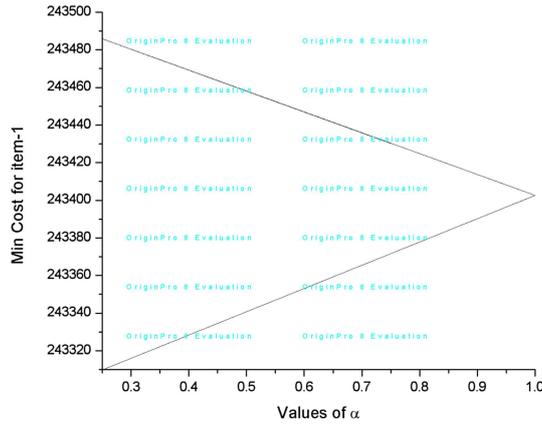$\$260\,618_{(\textbf{fuzzyenvironment})(\alpha=1.00)} = \$260\,618_{(\textbf{crispenvironment})}$.

FIGURE 5. Membership function of total min cost: Alpha (α) *vs.* minimum costs for fuzzy environment for item-1.

TABLE 6. Values of $u_i(t), d_i(t), x_i(t)$ for $i = 1, 2$ for the Model-1: (Outputs for the crisp environment).

Time	$u_1(t)$	$u_2(t)$	$x_1(t)$	$x_2(t)$	$d_1(t)$	$d_2(t)$
$t = 0$	196.3	202.4	0	0	106	114
$t = 1$	197.8	203.3	60	53	158	169
$t = 2$	198.4	204.2	82	70	185	193
$t = 3$	198.9	205.9	87	70	198	204
$t = 4$	199.4	205.9	82	64	204	209
$t = 5$	199.9	206.8	74	55	207	212
$t = 6$	200.5	207.7	63	46	209	213
$t = 7$	201.1	208.6	52	36	210	213.1
$t = 8$	201.5	209.5	41	27	210.6	213.3
$t = 9$	202.1	210.3	32	19	210.7	213.6
$t = 10$	203.6	211.2	19	12	210.8	213.8
$t = 11$	203.1	212.1	9	5	210.9	213.9
$t = 12$	204.6	213.1	0	0	211	214

5.1. Outputs for the crisp environment

Here outputs are given in Table 6 using the tool Mathematica-9.0.

5.2. Outputs for the fuzzy environment

Here outputs are given in Table 7 using the tool Mathematica-9.0.

6. Sensitivity analysis

For production system, a general belief is that cost of the produced item increases as defective units of the system increase. Consequently, Min cost of the production system increases. From the Table 8 and Figure 6, it is seen that min cost increases with defective rates. These are as per our expection. Here as defective units increases, cost increases not in linear form, it increases nonlinearly.

TABLE 7. Values of $u_i^l(t, \alpha), x_i^l(t, \alpha), u_i^r(t, \alpha), x_i^r(t, \alpha), d_i(t)$ at $\alpha = 0.5$ for $i = 1, 2$ of the Model-2: (Outputs for the fuzzy environment).

t	$u_1^l(t,\alpha)$	$u_2^l(t,\alpha)$	$x_1^l(t,\alpha)$	$x_2^l(t,\alpha)$	$u_1^r(t,\alpha)$	$u_2^r(t,\alpha)$	$x_1^r(t,\alpha)$	$x_2^r(t,\alpha)$	$d_1(t)$	$d_2(t)$
0	197.6	203.1	0	0	197.1	202.1	0	0	106.0	114.0
1	198.1	203.9	60	54	197.6	203.1	59	53	158.0	169.0
2	198.2	204.6	83	71	198.1	203.8	80	68	185.0	193.0
3	199.1	205.4	88	72	198.2	203.9	82	69	198.0	204.0
4	199.5	206.2	83	66	198.7	204.9	86	70	204.0	209.0
5	200.1	206.9	75	57	199.9	206.7	73	54	207.0	212.0
6	200.5	207.7	64	48	200.5	207.7	62	44	209.0	213.0
7	200.9	208.9	53	38	201.1	208.6	51	35	210.1	213.1
8	201.4	209.2	42	29	201.6	209.6	40	26	210.6	213.4
9	201.9	210.1	31	21	202.2	210.5	29	18	210.8	213.5
10	202.4	210.7	20	13	202.8	211.5	19	11	210.9	213.7
11	202.8	211.5	9	6	203.3	212.4	9	5	210.9	213.9
12	203.3	212.3	0	0	203.9	213.4	0	0	211.0	214.0

TABLE 8. Values of Min cost J in different values of δ_1 and δ_2.

	Item-1		Item-2
δ_1	Min Cost in \$	δ_2	Min Cost in \$
0.01	243402.6	0.02	260618.9
0.02	248130.1	0.03	265709.2
0.03	253011.9	0.04	270968.4
0.04	258054.4	0.05	276403.6
0.05	263264.2	0.06	282022.4
0.06	268648.7	0.07	287832.6
0.07	274215.2	0.08	293842.7
0.08	279971.9	0.09	300601.4
0.09	283011.9	0.10	306968.4

7. DISCUSSION

Here Lingo-11.0 and Mathematica-9.0 have been taken as tools for the simulation to validate the results. Most of the results by using the Lingo-11.0 are almost the same as with the results of Mathematica-9.0. The Min cost in imprecise environment $(\alpha = 1.00) = \$243402.7 = $ Min cost in crisp environment for item-1, which is established from Figure 4. This is a mathematically established result. In the Figures 2–4 for the crisp model, initially production is more than demand and hence stock is built up after satisfying the demand. As demand is here dynamic which increases with time, after some time, it surpasses the product and then excess demand is satisfied from the stock. Again, the deficit in demand occurs, so the stock starts to decrease at this point and gradually decreases up to the end of the cycle and finally becomes zero. Similarly, for the fuzzy model, it has been observed that initially stock is built up and after some time, it gradually decreases to zero following in the same process as in crisp model.

8. MANAGERIAL DECISION

For any real life problem, system parameters are very seldom deterministic rather, these are uncertain, may be imprecise. In these cases, manager takes the help of experts and get the mathematical expression of fuzzy
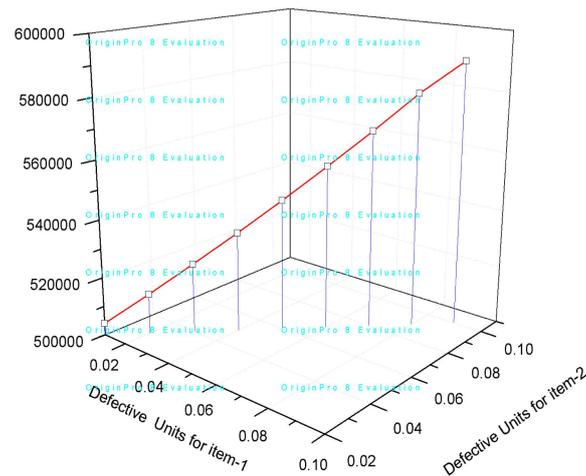
FIGURE 6. Defective units *vs.* Min cost (in $) in crisp environment for item-1.

parameters. Following the presented analysis, the management can evaluate the upper and lower values of the objective function i.e, total cost and the corresponding values of control and state variables. In reality, if management takes the decision against the fuzzy parameters within the fuzzy tolerances, the total cost or profit will lie within the evaluated upper and lower optimal values. Hence, it helps the management to have an idea about the return value. In the present investigation, in Section 4.2, fuzzy min cost for item-1 is [$243340.7, $243458.1] whereas that of crisp min cost for item-1 is $243402.7 which lies within the fuzzy min cost interval. This is true for item-2 also. Thus the present investigation is helpful for the management.

9. Conclusion

Here we have considered both crisp and fuzzy optimal control problems with Khouja and Merhez's [15] unit production cost. Here the crisp and fuzzy space constraints are expressed in the form of inequality equation. This investigation focuses on how to deal a real-life precise or imprecise inequality constraint for the decision making optimal control problems. For the first time, a crisp or fuzzy inequality constraint has been transformed to an equality constraint introducing an additional state variable following Bang Bang Control Theory. This type of inequality constraint occurs not only in the production inventory control problem, but also in several other areas such as in prey-predator bio-mathematical models, production with environmental pollution, uncertain switch systems *etc.* Hence the procedure of solving an optimal control problem by FVP with inequality source constraint will be helpful for the researchers in other fields. As the formulation and solution methods of the present problem are quite general, different new models can be formulated and solved with different forms of dynamic demand and other resources constraints such as budget constraint, etc.

References

[1] K. Benalia, C. David and B. Oukacha, An optimal time control problem for the one-dimensional, linear heat equation, in the presence of a scaling parameter. *RAIRO: OR* **51** (2017) 1289–1299.

[2] S. Benjaafar, J.P. Gayon and S. Tepex, Optimal control of a production-inventory system with customer impatience. *Oper. Res. Lett.* **38** (2010) 267–272.

[3] U. Buscher, S. Rudert and C. Schwarz, A Note on: An optimal batch size for an imperfect production system with quality assurance and rework. *Int. J. Prod. Res.* **47** (2009) 7063–7067.

[4] J.J. Buckley and T. Feuring, Introduction to fuzzy partial differential equations. *Fuzzy Set Syst.* **105** (1999) 241–248.

[5] M. Chahim, R.F. Hart and P.M. Kort, A tutorial on the deterministic impulse control maximum principle: Necessary and sufficient optimality conditions. *Eur. J. Oper. Res.* **219** (2012) 18–26.

[6] J. Chai and E.W.T. Ngai, Multi-perspective strategic supplier selection in uncertain environments. *Int. J. Prod. Econ.* **166** (2015) 215–225.

[7] C.K. Chen, C.C. Lo and T.C. Weng, Optimal production run length and warranty period for an imperfect production system under selling price dependent on war ranty period. *Eur. J. Oper. Res.* **259** (2017) 401–412.

[8] B. Farhadinia, Necessary optimality conditions for fuzzy variational problems. *Inf. Sci.* **181** (2011) 1348–1357.

[9] G.A. Gabriel and K.M. Ragsdell, The generalized reduced gradient method. *AMSE J. Eng. Ind.* **99** (1977) 384–400.

[10] Y.S. Huang, R.P. Wang and J.W. Ho, Determination of optimal lot size and production rate for multi-production channels with limited capacity. *Int. J. Syst. Sci.* **46** (2015) 1679–1691.

[11] Y.S. Huang, L.C. Liu and J.W. Ho, Decisions on new product development under uncertainties. *Int. J. Syst. Sci.* **46** (2015) 1010–1019.

[12] H. Ishibuchi and H. Tanaka, Multiobjective programming in optimization of the interval objective function. *Eur. J. Oper. Res.* **48** (1990) 219–225.

[13] S. Okada and M. Gen, Order relation between intervals and its application to shortest path problem. *Comput. Ind. Eng.* **25** (1993) 147–150.

[14] X. Jinqu, Y. Chen and H. Guo, Fractional robust control design for fuzzy dynamical systems: An optimal approach. *J. Intell. Fuzzy Syst.* **29** (2014) 553–569.

[15] M. Khouja and A. Mehrez, The economic production lot-size model with variable production rate and imperfect Quality. *J. Oper. Res. Soc.* **45** (1994) 1405–1417.

[16] K. Leng, X. Wang and Y. Wang, Research on inventory control policies for non stationary demand based on TOC. *Int. J. Comput. Intell. Syst.* **3** (2010) 114–128.

[17] K. Maity and M. Maiti, Possibility and Necessity Constraints and their Defuzzification – a multi-item production-inventory scenario via optimal control theory. *Eur. J. Oper. Res.* **177** (2007) 882–896.

[18] M.K. Mehlawat and P. Gupta, COTS products selection using fuzzy chance-constrained multi objective programming. *Appl. Intell.* **43** (2015) 732–751.

[19] D.S. Naidu, Optimal Control System. CRC Press, Pocatello, ID (2000).

[20] S.M.A. Nayeem, An expected value approach involving linear combination of possibility and necessity measures for a fuzzy EOQ model. *J. Intell. Fuzzy Syst.* **29** (2015) 761–767.

[21] D. Panda, S. Kar, K. Maity and M. Maiti, A single period inventory model with imperfect production and stochastic demand under chance and imprecise constraints. *Eur. J. Oper. Res.* **188** (2008) 121–139.

[22] E.L. Porteus, Optimal lot sizing, process quality improvement and setup cost reduction. *Oper. Res.* **34** (1986) 137–144.

[23] J.N. Roul, K. Maity, S. Kar and M. Maiti, Multi-item reliability dependent imperfect production inventory optimal control models with dynamic demand under uncertain resource constrain. *Int. J. Prod. Res.* **53** (2015) 4993–5016.

[24] J.N. Roul, K. Maity, S. Kar and M. Maiti, Optimal control problem for an imperfect production process using fuzzy variational principle. *J. Intell. Fuzzy Syst.* **32** (2017) 565–577.

[25] M.K. Salameh and M.Y. Jaber, Economic production quantity model for items with imperfect quality. *Int. J. Prod. Econ.* **64** (2000) 59–64.

[26] E.H. Turner and R.M. Turner, A constraint-based approach to assigning system components to tasks. *Appl. Intell.* **109** (1999) 155–172.

[27] T. Yang, J.P. Ignizio and H.J. Kim, Fuzzy programming with nonlinear membership functions: Piecewise linear approximation. *Fuzzy Sets Syst.* **41** (1991) 39–53

[28] X. Zhang, X. Deng, F.T. Chan and S. Mahadevan, A fuzzy extended analytic network process-based approach for global supplier selection. *Appl. Intell.* **43** (2015) 760–772.

[29] H.J. Zimmermann, Fuzzy Set Theory and its Applications. Kluwer-Nijhoff Publishing, Boston (1985).