RAIRO-Oper. Res. 54 (2020) 637–652
https://doi.org/10.1051/ro/2019019

RAIRO Operations Research
www.rairo-ro.org

AN EXACT MINIMAX PENALTY FUNCTION APPROACH TO SOLVE MULTITIME VARIATIONAL PROBLEMS

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Abstract. This paper aims to examine an appropriateness of the exact minimax penalty function method applied to solve the partial differential inequation (PDI) and partial differential equation (PDE) constrained multitime variational problems. The criteria for equivalence between the optimal solutions of a multitime variational problem with PDI and PDE constraints and its associated unconstrained penalized multitime variational problem is studied in this work. We also present some examples to validate the results derived in the paper.

Mathematics Subject Classification. 26B25, 65K10, 90C30.

Received July 3, 2018. Accepted February 2, 2019.

1. Introduction

Multitime variational problems find applications in various branches of mathematical, economical and engineering sciences, especially in mechanical engineering due to the fact that the curvilinear integrals objectives have a physical meaning of mechanical work. These integrals are also used in mathematical modeling of different processes arising in the areas of engines, tribology, robotics, etc. Furthermore, various real world and application oriented problems arising in diverse fields of science and engineering such as shape-optimization in fluid mechanics and medicine, material inversion in geophysics, structural optimization, optimal control of processes, data assimilation in regional weather prediction modeling, etc. require optimization problems with partial differential inequations and equations (PDI and PDE) as constraints. Therefore, PDI and PDE constrained multitime variational problems, which present significant reasoning and computational challenges, have been given considerable interest in the recent years.

Hanson [10], in 1964, observed that the scalar variational and control problems are continuous-time analogue of finite dimensional mathematical programming problems. Since then, the fields of mathematical programming and calculus of variations have merged together within optimization theory to some extent, hence enhancing the potential for continued research in both fields. Udrişte and Ţevy [20] have introduced an optimization problem of path independent curvilinear integrals with PDI and PDE constraints. Some other interesting results related to the variational and control problems with PDI and PDE constraints can refer to [16–19, 21, 22].

Keywords. Convexity, exact minimax penalty function method, multitime variational problem, PDI and PDE constraints.

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Pitea *et al.* [18] stated and proved the necessary optimality conditions for the problem of minimizing a vector of path independent curvilinear integral functionals with PDIs and PDEs as constraints. They also analyzed the necessary efficiency conditions for a class of multitime multiobjective fractional variational problems. In continuation to this work, Pitea and Postolache [17] studied the duals of Mond-Weir type and generalized Mond-Weir-Zalmai type for multitime multiobjective variational problems under the assumptions of generalized convexity. Recently, Pitea and Antczak [16] further continued the study of these kind of problems involving univex functionals to establish the sufficient optimality conditions for efficiency and proper efficiency and duality results in the sense of Mond-Weir and Wolfe.

However, PDI and PDE constrained optimization problems are generally infinite dimensional in nature, large and complex. In spite of the above stated studies, there is still a need of finding an effective way to solve these large and complex structured multitime variational problems. In the theory of global optimization, it is well known that the penalty function method proves to be an effective way to solve constrained optimization problem. In this method, the constrained optimization problem is transformed either into a single or a sequence of unconstrained optimization problems by adding a so-called penalty function to the objective function which reflects the measure of violation of constraints. An optimal solution of the unconstrained optimization problems coincides with an optimal solution of the original constrained optimization problem for a penalty parameter, which exceeds the threshold value.

Many authors (see *e.g.*, [1, 3–9, 11–14]) have studied the different types of penalty function approach to solve the various kinds of constrained optimization problems. Antczak [4] used the exact minimax penalty function method to solve a general nondifferentiable extremum problem with both inequality and equality constraints using convexity assumptions. Thereafter, Jayswal and Choudhury [12] applied the exact minimax penalty function and studied saddle point criteria for a class of nonsmooth convex vector optimization problems. Recently, Antczak [2] characterized the exactness property of the exact minimax penalty function and used it solve a nonconvex differentiable optimization problems involving both inequality and equality constraints.

Motivated from the ongoing research in this area, we test the appropriateness of the exact minimax penalty function method for a class of multitime variational problems with PDIs and PDEs constraints by using the concept of convexity for functionals. The paper is sectionally organized as follows: in Section 2, some preliminary definitions and the necessary optimality conditions for a PDI and PDE constrained multitime variational problem are presented, which will be used in the sequel of the paper. The equivalence between an optimal solutions of a PDI and PDE constrained multitime variational problem and its associated unconstrained penalized variational problem with the exact minimax penalty function are discussed in Section 3. Also, examples are provided to elucidate the established results. Finally, the paper is concluded in Section 4 along with some future perspectives.

2. Notations and preliminaries

Let $(T;h)$ and $(M;g)$ be the two Riemannian manifolds of dimensions p and n, respectively. The local coordinates of the Riemannian manifolds T and M are denoted as $t=(t^\alpha),\ \alpha=1,\dots,p$ and $x=(x^i),\ i=1,\dots,n$, respectively. Here, $t=(t^\alpha)$ is known as the multitime. Throughout this paper, we shall use the following convention for equalities and inequalities for any two vectors, $x=(x^i),\ y=(y^i),\ i=1\dots,n$:

(i) $x=y \Leftrightarrow x^i=y^i,\ \forall\ i=1,\dots,n$;
(ii) $x<y \Leftrightarrow x^i<y^i,\ \forall\ i=1,\dots,n$;
(iii) $x\leqq y \Leftrightarrow x^i\le y^i,\ \forall\ i=1,\dots,n$; (product order relation)
(iv) $x\le y \Leftrightarrow x\leqq y$ and $x\neq y$.

The hyperparallelepiped $\Omega_{t_0,t_1}\subset T$ with diagonal opposite points $t_0=(t_0^1,\dots,t_0^p)$ and $t_1=(t_1^1,\dots,t_1^p)$ can be written as being the interval $[t_0,t_1]$ using the product order relation. Let γ_{t_0,t_1} be a piecewise C^1-class curve joining the points t_0 and t_1. We denote $C^\infty(\Omega_{t_0,t_1},M)$ as the space of all functions $x:\Omega_{t_0,t_1}\mapsto M$ of C^∞-class with the norm

$$\|x\| = \|x\|_\infty + \sum_{\gamma=1}^{p} \|x_\gamma\|_\infty$$

where $x_\gamma(t) = \dfrac{\partial x(t)}{\partial t^\gamma}$, $\gamma = 1, \dots, p$, are the partial velocities. The first order jet bundle associated to T and M is denoted by $J^1(T, M)$, *i.e.* we consider $J^1(T, M) = \Omega_{t_0,t_1} \times \mathbb{R}^n \times \mathbb{R}^{np}$.

Let us consider a closed Lagrange 1-form density of C^∞-class,

$$f = (f_\alpha) : J^1(T, M) \mapsto \mathbb{R}^p,$$

which produces the following path-independent curvilinear functional, also known as "action":

$$F(x(\cdot)) = \int_{\gamma_{t_0,t_1}} f_\alpha(t, x(t), x_\gamma(t))\, \mathrm{d}t^\alpha.$$

The closedness conditions (complete integrability conditions) are given as $D_\alpha f_\beta = D_\beta f_\alpha$, α, $\beta = 1, \dots, p$, $\alpha \neq \beta$, where D_α is the total derivative.

The purpose of this work is to apply an exact minimax penalty function method to solve the following PDI and PDE constrained multitime variational problem:

$$\text{(MVP)} \qquad \min_{x(\cdot)} F(x(\cdot))$$

subject to

$$x(t_0) = x_0,\ x(t_1) = x_1$$

$$g(t, x(t), x_\gamma(t)) \leqq 0,\ t \in \Omega_{t_0,t_1}, \tag{2.1}$$

$$h(t, x(t), x_\gamma(t)) = 0,\ t \in \Omega_{t_0,t_1}, \tag{2.2}$$

where

$$g = (g_a^j) : J^1(T, M) \mapsto \mathbb{R}^{ms},\ a = 1, \dots, s,\ j = 1, \dots, m,\ m < n$$
$$\text{and } h = (h_a^l) : J^1(T, M) \mapsto \mathbb{R}^{ks},\ a = 1, \dots, s,\ l = 1, \dots, k,\ k < n$$

are the Lagrange matrix densities of C^∞-class which define the partial differential inequations (PDI) of evolution (2.1) and partial differential equations (PDE) of evolution (2.2), respectively.

The set of all feasible solutions to (MVP), denoted by $\mathcal{F}$, is given as

$$\mathcal{F}(\Omega_{t_0,t_1}) = \{x(t) \in C^\infty(\Omega_{t_0,t_1}, M) : x(t_0) = x_0,\ x(t_1) = x_1,\ g(t, x(t), x_\gamma(t)) \leqq 0,$$
$$h(t, x(t), x_\gamma(t)) = 0,\ t \in \Omega_{t_0,t_1}\}.$$

Motivated by the definition of convexity for variational problems used in Mond and Hanson [15], we present the following definition of convexity for a path-independent curvilinear functional which will be used to prove the main results of the paper. Let S be any nonempty subset of $C^\infty(\Omega_{t_0,t_1}, M)$ and $\bar{x}(\cdot) \in S$ be given.

Definition 2.1. A functional $F(x(\cdot))$ is said to be convex at $\bar{x}(\cdot)$ on S, if the following inequality

$$F(x(\cdot)) - F(\bar{x}(\cdot)) \geq \int_{\gamma_{t_0,t_1}} \left\{ \left\langle x(t) - \bar{x}(t), \frac{\partial f_\alpha}{\partial x}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle + \left\langle x_\gamma(t) - \bar{x}_\gamma(t), \frac{\partial f_\alpha}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle \right\} \mathrm{d}t^\alpha$$

holds for all $x(\cdot) \in S$.

Example 2.2. Let us consider the path independent curvilinear functional

$$F(x(\cdot)) = \int_{\gamma_{t_0,t_1}} f_\alpha(t, x(t), x_\gamma(t))\, \mathrm{d}t^\alpha,$$

where $f : J^1(T, M) \mapsto \mathbb{R}^2$ is defined as:

$$f_\alpha(t, x(t), x_\gamma(t)) = (x_1^2(t) + x_1(t) + x_2(t), x_2(t) + 1).$$

Here, $x_\gamma,\ \gamma = 1, 2$ are the partial velocities with respect to t^γ, respectively. Then, we can easily verify that $F(x(\cdot))$ is convex on any nonempty subset S of $C^\infty(\Omega_{t_0,t_1}, \mathbb{R}^2)$.

The necessary conditions for optimality of a feasible solution in a multitime variational problem are given as follows:

Theorem 2.3 ([18]). *Let $f = (f_\alpha)$ be a closed 1-form of C^∞-class. If $\bar{x}(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is a normal optimal solution to the multitime variational problem (*MVP*), then there exist Lagrange multiplier $\bar{\lambda} \in \mathbb{R}^+$ and smooth matrix functions $\bar{\mu}(t) = (\bar{\mu}_\alpha(t)) : \Omega_{t_0,t_1} \mapsto \mathbb{R}^{msp}$ and $\bar{\nu}(t) = (\bar{\nu}_\alpha(t)) : \Omega_{t_0,t_1} \mapsto \mathbb{R}^{ksp}$ such that*

$$\begin{aligned}
&\bar{\lambda}\frac{\partial f_\alpha}{\partial x}(t, \bar{x}(t), \bar{x}_\gamma(t)) + \left\langle \bar{\mu}_\alpha(t), \frac{\partial g}{\partial x}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle + \left\langle \bar{\nu}_\alpha(t), \frac{\partial h}{\partial x}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle \\
&\quad - D_\gamma \left(\bar{\lambda}\frac{\partial f_\alpha}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t)) + \left\langle \bar{\mu}_\alpha(t), \frac{\partial g}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle + \left\langle \bar{\nu}_\alpha(t), \frac{\partial h}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle \right) \\
&\quad = 0, \qquad t \in \Omega_{t_0,t_1},\ \alpha = 1, \ldots, p \text{ (Euler-Lagrange PDEs)},\ [\bar{\lambda} = 1], \qquad (2.3)
\end{aligned}$$

$$\langle \bar{\mu}_\alpha(t), g(t, \bar{x}(t), \bar{x}_\gamma(t)) \rangle = 0, \quad t \in \Omega_{t_0,t_1},\ \alpha = 1, \ldots, p, \tag{2.4}$$

$$\bar{\mu}_\alpha(t) \geqq 0, \quad t \in \Omega_{t_0,t_1},\ \alpha = 1, \ldots, p. \tag{2.5}$$

Remark 2.4. If $\bar{\lambda} \neq 0$, the optimal feasible solution $\bar{x}(\cdot)$ of the problem (MVP) is called normal. Further, without loss of generality, if $\bar{x}(\cdot)$ is an optimal normal solution of the problem (MVP), we can assume that $\bar{\lambda} = 1$.

3. Exact minimax penalty function method for multitime variational problem

It is well known that in the penalty function methods, a constrained optimization problem is transformed into unconstrained ones by adding the constraints to the objective function together with a penalty parameter such that it penalizes any violation of the constraints. In this section, we study the usefulness of the exact minimax penalty function method for solving a PDI and PDE constrained multitime variational problem.

Now, we construct a penalized unconstrained multitime variational problem (MVP$_\infty$(c)) involving the exact minimax penalty function, associated with the PDI and PDE constrained multitime variational problem (MVP) as follows:

$$(\mathrm{MVP}_\infty(\mathbf{c})) \qquad \min \quad F_\infty(x(\cdot), c(\cdot)) := \int_{\gamma_{t_0,t_1}} \Bigg(f_\alpha(t, x(t), x_\gamma(t)) + c(t) \max_{\substack{1\le a\le s\\ 1\le j\le m\\ 1\le l\le k}} \left\{ \left(g_a^j\right)^+(t, x(t), x_\gamma(t)), |h_a^l(t, x(t), x_\gamma(t))| \right\} e_\alpha \Bigg)\, \mathrm{d}t^\alpha,$$

where $e = (e_\alpha) := (1,\ldots,1) \in \mathbb{R}^p$ and $c(t) > 0$ is a penalty parameter. For a given constraint $g_a^j(t,x(t), x_\gamma(t)) \leqq 0$, $t \in \Omega_{t_0,t_1}$, the function $(g_a^j)^+(t,x(t),x_\gamma(t))$ is defined as follows:

$$(g_a^j)^+(t,x(t),x_\gamma(t)) := \begin{cases} 0, & \text{if } \; g_a^j(t,x(t),x_\gamma(t)) \leqq 0, \\ g_a^j(t,x(t),x_\gamma(t)), & \text{if } \; g_a^j(t,x(t),x_\gamma(t)) > 0. \end{cases} \tag{3.1}$$

We state the following lemma along the lines of Lemma 2.1 in Antczak [4] which will be helpful to prove the subsequent results.

Lemma 3.1. *Let $\phi_r(t,x(t),x_\gamma(t))$, $r=1,\ldots,q$ be real-valued functionals defined on $J^1(T,M)$. For each $x(\cdot) \in C^\infty(\Omega_{t_0,t_1},M)$, we have*

$$\max_{1\le r\le q} \phi_r(t,x(t),x_\gamma(t)) = \max_{\zeta\in Z} \sum_{r=1}^{q} \zeta_r \phi_r(t,x(t),x_\gamma(t)),$$

where $Z = \{\zeta = (\zeta_1,\ldots,\zeta_q) \in \mathbb{R}_+^q : \sum_{r=1}^{q} \zeta_r = 1\}$.

Firstly, we shall prove that any feasible solution to (MVP) satisfying the necessary optimality conditions is equivalent to a minimizer of the associated unconstrained penalized problem (MVP_∞(c)) when the penalty parameter exceeds a certain threshold value under the assumptions of convexity.

Theorem 3.2. *Let $\bar{x}(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ be a feasible solution in the* PDI *and* PDE *constrained multitime variational problem (*MVP*) at which the necessary optimality conditions (2.3)–(2.5) are satisfied with the Lagrange multipliers $\bar{\lambda} \in R^+$, $\bar{\mu}(\cdot)$ and $\bar{\nu}(\cdot)$. Assume that $F(x(\cdot))$, $\int_{\gamma_{t_0,t_1}} \langle \bar{\mu}_{\alpha j}(t), g^j(t,x(t),x_\gamma(t))\rangle \,\mathrm{d}t^\alpha$, $j=1,\ldots,m$ and $\int_{\gamma_{t_0,t_1}} \langle \bar{\nu}_{\alpha l}(t), h^l(t,x(t),x_\gamma(t))\rangle \,\mathrm{d}t^\alpha$, $l=1,\ldots,k$ are convex at $\bar{x}(\cdot)$ on $C^\infty(\Omega_{t_0,t_1},M)$. Furthermore, we assume that the penalty parameter $c(\cdot)$ is sufficiently large (it is sufficient to set $c(\cdot) \geq (m+k)s \max_{1\le\alpha\le p} \{\bar{\mu}^a_{\alpha j}(\cdot), |\bar{\nu}^a_{\alpha l}(\cdot)| : 1\le a\le s,\ 1\le j\le m,\ 1\le l\le k\}$), then $\bar{x}(\cdot)$ is a minimizer of the penalized unconstrained multitime variational problem (*MVP_∞*(c)) involving the exact minimax penalty function.*

Proof. Since, by hypotheses, $F(x(\cdot))$, $\int_{\gamma_{t_0,t_1}} \langle \bar{\mu}_{\alpha j}(t), g^j(t,x(t),x_\gamma(t))\rangle \,\mathrm{d}t^\alpha$, $j = 1\ldots,m$ and $\int_{\gamma_{t_0,t_1}} \langle \bar{\nu}_{\alpha l}(t), h^l(t,x(t),x_\gamma(t))\rangle \,\mathrm{d}t^\alpha$, $l=1,\ldots,k$ are convex at $\bar{x}(\cdot)$ on $C^\infty(\Omega_{t_0,t_1},M)$, we have for all $x \in C^\infty(\Omega_{t_0,t_1},M)$,

$$\begin{aligned} &\int_{\gamma_{t_0,t_1}} f_\alpha(t,x(t),x_\gamma(t))\,\mathrm{d}t^\alpha - \int_{\gamma_{t_0,t_1}} f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))\,\mathrm{d}t^\alpha \\ &\quad \geq \int_{\gamma_{t_0,t_1}} \left\{ \left\langle x(t)-\bar{x}(t), \frac{\partial f_\alpha}{\partial x}(t,\bar{x}(t),\bar{x}_\gamma(t)) \right\rangle + \left\langle x_\gamma(t)-\bar{x}_\gamma(t), \frac{\partial f_\alpha}{\partial x_\gamma}(t,\bar{x}(t),\bar{x}_\gamma(t)) \right\rangle \right\} \mathrm{d}t^\alpha, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\int_{\gamma_{t_0,t_1}} \left\langle \bar{\mu}_{\alpha j}(t), g^j(t,x(t),x_\gamma(t)) \right\rangle \,\mathrm{d}t^\alpha - \int_{\gamma_{t_0,t_1}} \left\langle \bar{\mu}_{\alpha j}(t), g^j(t,\bar{x}(t),\bar{x}_\gamma(t)) \right\rangle \,\mathrm{d}t^\alpha \\ &\quad \geq \int_{\gamma_{t_0,t_1}} \left\{ \left\langle x(t)-\bar{x}(t), \left\langle \bar{\mu}_{\alpha j}(t), \frac{\partial g^j}{\partial x}(t,\bar{x}(t),\bar{x}_\gamma(t)) \right\rangle \right\rangle \right. \\ &\qquad \left. + \left\langle x_\gamma(t)-\bar{x}_\gamma(t), \left\langle \bar{\mu}_{\alpha j}(t), \frac{\partial g^j}{\partial x_\gamma}(t,\bar{x}(t),\bar{x}_\gamma(t)) \right\rangle \right\rangle \right\} \mathrm{d}t^\alpha,\ j=1,\ldots,m, \end{aligned} \tag{3.3}$$

$$\text{and} \int_{\gamma_{t_0,t_1}} \left\langle \bar{\nu}_{\alpha l}(t), h^l(t,x(t),x_\gamma(t)) \right\rangle \,\mathrm{d}t^\alpha - \int_{\gamma_{t_0,t_1}} \left\langle \bar{\nu}_{\alpha l}(t), h^l(t,\bar{x}(t),\bar{x}_\gamma(t)) \right\rangle \,\mathrm{d}t^\alpha$$

$$\geq \int_{\gamma_{t_0,t_1}} \left\{ \left\langle x(t) - \bar{x}(t), \left\langle \bar{\nu}_{\alpha l}(t), \frac{\partial h^l}{\partial x}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle \right\rangle + \left\langle x_\gamma(t) - \bar{x}_\gamma(t), \left\langle \bar{\nu}_{\alpha l}(t), \frac{\partial h^l}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle \right\rangle \right\} \mathrm{d}t^\alpha, \ l = 1, \ldots, k. \tag{3.4}$$

Taking summation over $j = 1, \ldots, m$ and $l = 1, \ldots, k$ in the inequalities (3.3) and (3.4) and then adding the obtained inequality with (3.2), we get

$$\begin{aligned}
&\int_{\gamma_{t_0,t_1}} \{ f_\alpha(t, x(t), x_\gamma(t)) - f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t)) + \langle \bar{\mu}_\alpha(t), g(t, x(t), x_\gamma(t)) - g(t, \bar{x}(t), \bar{x}_\gamma(t)) \rangle \\
&\quad + \langle \bar{\nu}_\alpha(t), h(t, x(t), x_\gamma(t)) - h(t, \bar{x}(t), \bar{x}_\gamma(t)) \rangle \} \, \mathrm{d}t^\alpha \geq \int_{\gamma_{t_0,t_1}} \left\{ \left\langle x(t) - \bar{x}(t), \frac{\partial f_\alpha}{\partial x}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right. \right. \\
&\quad \left. \left. + \left\langle \bar{\mu}_\alpha(t), \frac{\partial g}{\partial x}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle + \left\langle \bar{\nu}_\alpha(t), \frac{\partial h}{\partial x}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle \right\rangle \right. \\
&\quad + \left\langle x_\gamma(t) - \bar{x}_\gamma(t), \frac{\partial f_\alpha}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t)) + \left\langle \bar{\mu}_\alpha(t), \frac{\partial g}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle \right. \\
&\quad \left. \left. + \left\langle \bar{\nu}_\alpha(t), \frac{\partial h}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t)) \right\rangle \right\rangle \right\} \mathrm{d}t^\alpha.
\end{aligned} \tag{3.5}$$

We denote

$$\begin{aligned}
\chi_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t), \bar{\mu}_\alpha(t), \bar{\nu}_\alpha(t)) := {} & f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t)) + \langle \bar{\mu}_\alpha(t), g(t, \bar{x}(t), \bar{x}_\gamma(t)) \rangle \\
& + \langle \bar{\nu}_\alpha(t), h(t, \bar{x}(t), \bar{x}_\gamma(t)) \rangle.
\end{aligned}$$

Then, the inequality (3.5) can be written as

$$\begin{aligned}
&\int_{\gamma_{t_0,t_1}} \Big\{ f_\alpha(t, x(t), x_\gamma(t)) - f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t)) + \langle \bar{\mu}_\alpha(t), g(t, x(t), x_\gamma(t)) - g(t, \bar{x}(t), \bar{x}_\gamma(t)) \rangle \\
&\quad + \langle \bar{\nu}_\alpha(t), h(t, x(t), x_\gamma(t)) - h(t, \bar{x}(t), \bar{x}_\gamma(t)) \rangle \Big\} \, \mathrm{d}t^\alpha \\
&\quad \geq \int_{\gamma_{t_0,t_1}} \left\{ \left\langle x(t) - \bar{x}(t), \frac{\partial \chi_\alpha}{\partial x}(t, \bar{x}(t), \bar{x}_\gamma(t), \bar{\mu}_\alpha(t), \bar{\nu}_\alpha(t)) \right\rangle \right. \\
&\qquad \left. + \left\langle x_\gamma(t) - \bar{x}_\gamma(t), \frac{\partial \chi_\alpha}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t), \bar{\mu}_\alpha(t), \bar{\nu}_\alpha(t)) \right\rangle \right\} \mathrm{d}t^\alpha.
\end{aligned} \tag{3.6}$$

It is obvious that the following relation

$$\begin{aligned}
\left\langle x_\gamma(t) - \bar{x}_\gamma(t), \frac{\partial \chi_\alpha}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t), \bar{\mu}_\alpha(t), \bar{\nu}_\alpha(t)) \right\rangle &= D_\gamma \left\langle x(t) - \bar{x}(t), \frac{\partial \chi_\alpha}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t), \bar{\mu}_\alpha(t), \bar{\nu}_\alpha(t)) \right\rangle \\
&\quad - \left\langle x(t) - \bar{x}(t), D_\gamma \left(\frac{\partial \chi_\alpha}{\partial x_\gamma}(t, \bar{x}(t), \bar{x}_\gamma(t), \bar{\mu}_\alpha(t), \bar{\nu}_\alpha(t)) \right) \right\rangle
\end{aligned} \tag{3.7}$$

holds. Also, for $\alpha, \gamma = 1, \ldots, p$, we denote

$$Q_\alpha^\gamma(t) = \langle x(t) - \bar{x}(t), \frac{\partial \chi_\alpha}{\partial x_\gamma}(t, x(t), x_\gamma(t), \bar{\mu}_\alpha(t), \bar{\nu}_\alpha(t)) \rangle \quad \text{and } I = \int_{\gamma_{t_0,t_1}} D_\gamma Q_\alpha^\gamma(t) \, \mathrm{d}t^\alpha.$$

According to Udrişte *et al.* [21], a total divergence is equal to a total derivative. Therefore, there exists $Q(t)$, with $Q(t_0)=0$ and $Q(t_1)=0$ such that

$$D_\gamma Q_\alpha^\gamma(t) = D_\alpha Q(t) \text{ and } I = \int_{\gamma_{t_0,t_1}} D_\gamma Q_\alpha^\gamma(t)\,\mathrm{d}t^\alpha = Q(t_1) - Q(t_0) = 0. \tag{3.8}$$

On combining (3.6)–(3.8), we get

$$\begin{aligned}
&\int_{\gamma_{t_0,t_1}} \{f_\alpha(t,x(t),x_\gamma(t)) - f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t)) + \langle \bar{\mu}_\alpha(t), g(t,x(t),x_\gamma(t)) - g(t,\bar{x}(t),\bar{x}_\gamma(t))\rangle \\
&\quad + \langle \bar{\nu}_\alpha(t), h(t,x(t),x_\gamma(t)) - h(t,\bar{x}(t),\bar{x}_\gamma(t))\rangle\}\,\mathrm{d}t^\alpha \\
&\quad \geq \int_{\gamma_{t_0,t_1}} \left\{\left\langle x(t)-\bar{x}(t), \frac{\partial \chi_\alpha}{\partial x}(t,\bar{x}(t),\bar{x}_\gamma(t),\bar{\mu}_\alpha(t),\bar{\nu}_\alpha(t)) \right.\right. \\
&\qquad \left.\left. -\ D_\gamma\left(\frac{\partial \chi_\alpha}{\partial x_\gamma}(t,\bar{x}(t),\bar{x}_\gamma(t),\bar{\mu}_\alpha(t),\bar{\nu}_\alpha(t))\right)\right\rangle\right\}\,\mathrm{d}t^\alpha.
\end{aligned}$$

Since the necessary optimality conditions (2.3)–(2.5) are satisfied at $\bar{x}(\cdot)$, the above inequality together with equality (2.3) yields

$$\begin{aligned}
&\int_{\gamma_{t_0,t_1}} \Big\{f_\alpha(t,x(t),x_\gamma(t)) - f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t)) + \langle \bar{\mu}_\alpha(t), g(t,x(t),x_\gamma(t)) - g(t,\bar{x}(t),\bar{x}_\gamma(t))\rangle \\
&\quad + \langle \bar{\nu}_\alpha(t), h(t,x(t),x_\gamma(t)) - h(t,\bar{x}(t),\bar{x}_\gamma(t))\rangle\Big\}\,\mathrm{d}t^\alpha \geq 0.
\end{aligned}$$

Again, using the necessary optimality condition (2.4) together with the feasibility of $\bar{x}(\cdot)$ in the PDI and PDE constrained multitime variational problem (MVP), the above inequality implies

$$\begin{aligned}
&\int_{\gamma_{t_0,t_1}} \Big\{f_\alpha(t,x(t),x_\gamma(t)) + \langle \bar{\mu}_\alpha(t), g(t,x(t),x_\gamma(t))\rangle \\
&\qquad + \langle \bar{\nu}_\alpha(t), h(t,x(t),x_\gamma(t))\rangle\Big\}\,\mathrm{d}t^\alpha \geq \int_{\gamma_{t_0,t_1}} f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))\,\mathrm{d}t^\alpha,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\int_{\gamma_{t_0,t_1}} \left\{f_\alpha(t,x(t),x_\gamma(t)) + \sum_{a=1}^{s}\sum_{j=1}^{m} \bar{\mu}_{\alpha j}^a(t) g_a^j(t,x(t),x_\gamma(t)) \right. \\
&\qquad \left. + \sum_{a=1}^{s}\sum_{l=1}^{k} \bar{\nu}_{\alpha l}^a(t) h_a^l(t,x(t),x_\gamma(t))\right\}\,\mathrm{d}t^\alpha \geq \int_{\gamma_{t_0,t_1}} f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))\,\mathrm{d}t^\alpha.
\end{aligned}$$

On using the inequality (3.1) and the definition of absolute value function, it follows that

$$\begin{aligned}
&\int_{\gamma_{t_0,t_1}} \left\{f_\alpha(t,x(t),x_\gamma(t)) + \sum_{a=1}^{s}\sum_{j=1}^{m} \bar{\mu}_{\alpha j}^a(t) (g_a^j)^+(t,x(t),x_\gamma(t)) \right. \\
&\qquad \left. + \sum_{a=1}^{s}\sum_{l=1}^{k} |\bar{\nu}_{\alpha l}^a(t)||h_a^l(t,x(t),x_\gamma(t))|\right\}\,\mathrm{d}t^\alpha \geq \int_{\gamma_{t_0,t_1}} f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))\,\mathrm{d}t^\alpha.
\end{aligned} \tag{3.9}$$

Now, we consider two cases:

Case (i): Let $\rho(\cdot) = \max_{1\leq\alpha\leq p}\{\bar{\mu}_{\alpha j}^a(\cdot), |\bar{\nu}_{\alpha l}^a(\cdot)| : 1 \leq a \leq s,\ 1\leq j\leq m,\ 1\leq l\leq k\} > 0$ and $c'(\cdot) = (m+k)s\rho(\cdot)$. Then, $c'(\cdot) > 0$.

Therefore, from (3.9), it follows that

$$\int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t,x(t),x_\gamma(t)) + e_\alpha \sum_{a=1}^{s}\sum_{j=1}^{m} \rho(t) {(g_a^j)}^+ (t,x(t),x_\gamma(t)) + e_\alpha \sum_{a=1}^{s}\sum_{l=1}^{k} \rho(t) |h_a^l(t,x(t),x_\gamma(t))| \right\} \mathrm{d}t^\alpha \geq \int_{\gamma_{t_0,t_1}} f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))\, \mathrm{d}t^\alpha,$$

which can be rewritten as

$$\int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t,x(t),x_\gamma(t)) + c'(t)\frac{e_\alpha}{c'(t)} \sum_{a=1}^{s}\sum_{j=1}^{m} \rho(t) {(g_a^j)}^+ (t,x(t),x_\gamma(t)) + c'(t)\frac{e_\alpha}{c'(t)} \sum_{a=1}^{s}\sum_{l=1}^{k} \rho(t) |h_a^l(t,x(t),x_\gamma(t))| \right\} \mathrm{d}t^\alpha \geq \int_{\gamma_{t_0,t_1}} f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))\, \mathrm{d}t^\alpha. \tag{3.10}$$

We denote

$$\phi_r^q(t,x(t),x_\gamma(t)) := {(g_r^q)}^+ (t,x(t),x_\gamma(t)),\ t \in \Omega_{t_0,t_1},\ r=1,\dots,s,\ q=1,\dots,m, \tag{3.11}$$

$$\phi_r^{q+m}(t,x(t),x_\gamma(t)) := |h_r^q(t,x(t),x_\gamma(t))|,\ t \in \Omega_{t_0,t_1},\ r=1,\dots,s,\ q=1,\dots,k, \tag{3.12}$$

$$\bar{\xi}_q^r(\cdot) := \frac{\rho(\cdot)}{c'(\cdot)},\ r=1,\dots,s,\ q=1,\dots,m+k. \tag{3.13}$$

Thus, it follows that

$$\bar{\xi}_q^r(\cdot) \geq 0,\ r=1,\dots,s,\ q=1,\dots,m+k,\quad \sum_{r=1}^{s}\sum_{q=1}^{m+k} \bar{\xi}_q^r(\cdot) = 1.$$

Using (3.11)–(3.13), inequality (3.10) can be rewritten as

$$\int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t,x(t),x_\gamma(t)) + c'(t) e_\alpha \sum_{r=1}^{s}\sum_{q=1}^{m+k} \bar{\xi}_q^r(t) \phi_r^q(t,x(t),x_\gamma(t)) \right\} \mathrm{d}t^\alpha \geq \int_{\gamma_{t_0,t_1}} f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))\, \mathrm{d}t^\alpha. \tag{3.14}$$

Furthermore, by Lemma 3.1, we have

$$\max_{\substack{1\leq r\leq s\\ 1\leq q\leq m+k}} \phi_r^q(t,x(t),x_\gamma(t)) = \max_{\bar{\xi}\in Z} \sum_{r=1}^{s}\sum_{q=1}^{m+k} \bar{\xi}_q^r(t)\phi_r^q(t,x(t),x_\gamma(t)),$$

where $Z = \{\bar{\xi} \in \mathbb{R}^{s(m+k)} : \sum_{r=1}^{s}\sum_{q=1}^{m+k} \bar{\xi}_q^r = 1\}$. Thus, the inequality (3.14) implies

$$\int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t,x(t),x_\gamma(t)) + c'(t) e_\alpha \max_{\substack{1\leq r\leq s\\ 1\leq q\leq m+k}} \phi_r^q(t,x(t),x_\gamma(t)) \right\} \mathrm{d}t^\alpha \geq \int_{\gamma_{t_0,t_1}} f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))\, \mathrm{d}t^\alpha,$$

which on using (3.11) and (3.12), can be rewritten as

$$\int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, x(t), x_\gamma(t)) + c'(t) e_\alpha \max_{\substack{1 \le a \le s \\ 1 \le j \le m \\ 1 \le l \le k}} \left\{ \left(g_a^j\right)^+ (t, x(t), x_\gamma(t)), |h_a^l(t, x(t), x_\gamma(t))| \right\} \right\} \mathrm{d}t^\alpha$$
$$\ge \int_{\gamma_{t_0,t_1}} f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t)) \, \mathrm{d}t^\alpha.$$

Since, by hypothesis, $c(\cdot) \ge c'(\cdot)$, the above inequality yields

$$\int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, x(t), x_\gamma(t)) + e_\alpha c(t) \max_{\substack{1 \le a \le s \\ 1 \le j \le m \\ 1 \le l \le k}} \left\{ \left(g_a^j\right)^+ (t, x(t), x_\gamma(t)), |h_a^l(t, x(t), x_\gamma(t))| \right\} \right\} \mathrm{d}t^\alpha$$
$$\ge \int_{\gamma_{t_0,t_1}} f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t)) \, \mathrm{d}t^\alpha. \tag{3.15}$$

The feasibility of $\bar{x}(\cdot)$ in the considered multitime variational problem (MVP) yields

$$\max_{\substack{1 \le a \le s \\ 1 \le j \le m \\ 1 \le l \le k}} \left\{ \left(g_a^j\right)^+ (t, \bar{x}(t), \bar{x}_\gamma(t)), |h_a^l(t, \bar{x}(t), \bar{x}_\gamma(t))| \right\} = 0, \ t \in \Omega_{t_0,t_1}.$$

So, from the inequality (3.15), we have

$$\int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, x(t), x_\gamma(t)) + e_\alpha c(t) \max_{\substack{1 \le a \le s \\ 1 \le j \le m \\ 1 \le l \le k}} \left\{ \left(g_a^j\right)^+ (t, x(t), x_\gamma(t)), |h_a^l(t, x(t), x_\gamma(t))| \right\} \right\} \mathrm{d}t^\alpha$$
$$\ge \int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t)) + e_\alpha c(t) \max_{\substack{1 \le a \le s \\ 1 \le j \le m \\ 1 \le l \le k}} \left\{ \left(g_a^j\right)^+ (t, \bar{x}(t), \bar{x}_\gamma(t)), |h_a^l(t, \bar{x}(t), \bar{x}_\gamma(t))| \right\} \right\} \mathrm{d}t^\alpha.$$

Hence, by the definition of the penalized unconstrained multitime variational problem ($\mathrm{MVP}_\infty(c)$) with the exact minimax penalty function, we conclude that

$$F_\infty(x(\cdot), c(\cdot)) \ge F_\infty(\bar{x}(\cdot), c(\cdot))$$

holds for all $x(\cdot) \in C_\infty(\Omega_{t_0,t_1}, M)$, which proves that $\bar{x}(\cdot)$ is a minimizer of the penalized unconstrained multitime variational problem ($\mathrm{MVP}_\infty(c)$).

Case (ii): Let $\rho(\cdot) = \max_{1 \le \alpha \le p} \{\bar{\mu}_{\alpha j}^a(\cdot), |\bar{\nu}_{\alpha l}^a(\cdot)| : 1 \le a \le s, \ 1 \le j \le m, \ 1 \le l \le k\} = 0$. Then, $c'(\cdot) = (m+k)s\rho(\cdot) = 0$. Thus, from (3.9), it follows that

$$\int_{\gamma_{t_0,t_1}} f_\alpha(t, x(t), x_\gamma(t)) \, \mathrm{d}t^\alpha \ge \int_{\gamma_{t_0,t_1}} f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t)) \, \mathrm{d}t^\alpha.$$

Since, $c'(\cdot) = 0$, the above inequality can be rewritten as

$$\int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, x(t), x_\gamma(t)) + e_\alpha c'(t) \max_{\substack{1\le a\le s\\ 1\le j\le m\\ 1\le l\le k}} \left\{ \left(g_a^j\right)^+ (t, x(t), x_\gamma(t)), |h_a^l(t, x(t), x_\gamma(t))| \right\} \right\} \mathrm{d}t^\alpha$$
$$\geq \int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t)) + e_\alpha c'(t) \max_{\substack{1\le a\le s\\ 1\le j\le m\\ 1\le l\le k}} \left\{ \left(g_a^j\right)^+ (t, \bar{x}(t), \bar{x}_\gamma(t)), |h_a^l(t, \bar{x}(t), \bar{x}_\gamma(t))| \right\} \right\} \mathrm{d}t^\alpha.$$

Again, by hypothesis $c(\cdot) > c'(\cdot)$ and from the feasibility of $\bar{x}(\cdot)$, it follows that

$$\int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, x(t), x_\gamma(t)) + e_\alpha c(t) \max_{\substack{1\le a\le s\\ 1\le j\le m\\ 1\le l\le k}} \left\{ \left(g_a^j\right)^+ (t, x(t), x_\gamma(t)), |h_a^l(t, x(t), x_\gamma(t))| \right\} \right\} \mathrm{d}t^\alpha$$
$$\geq \int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t)) + e_\alpha c(t) \max_{\substack{1\le a\le s\\ 1\le j\le m\\ 1\le l\le k}} \left\{ \left(g_a^j\right)^+ (t, \bar{x}(t), \bar{x}_\gamma(t)), |h_a^l(t, \bar{x}(t), \bar{x}_\gamma(t))| \right\} \right\} \mathrm{d}t^\alpha,$$

which by the definition of the penalized unconstrained multitime variational problem (MVP$_\infty(c)$) with the exact minimax penalty function, yields

$$F_\infty(x(\cdot), c(\cdot)) \geq F_\infty(\bar{x}(\cdot), c(\cdot))$$

holds for all $x(\cdot) \in C_\infty(\Omega_{t_0,t_1}, M)$. Hence, it follows that $\bar{x}(\cdot)$ is a minimizer of the penalized unconstrained multitime variational problem (MVP$_\infty(c)$) with the exact minimax penalty function. □

Since every normal optimal solution to the multitime variational problem (MVP) satisfies the necessary optimality conditions (2.3)–(2.5), we reach to the following corollary:

Corollary 3.3. *Let $\bar{x}(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ be a normal optimal solution to the* PDI *and* PDE *constrained multitime variational problem (*MVP*). Assume that $F(x(\cdot))$, $\int_{\gamma_{t_0,t_1}} \langle \bar{\mu}_{\alpha j}(t), g^j(t, x(t), x_\gamma(t))\rangle \, \mathrm{d}t^\alpha$, $j = 1, \ldots, m$, $\int_{\gamma_{t_0,t_1}} \langle \bar{\nu}_{\alpha l}(t), h^l(t, x(t), x_\gamma(t))\rangle \, \mathrm{d}t^\alpha$, $l = 1, \ldots, k$ are convex at $\bar{x}(\cdot)$ on $C^\infty(\Omega_{t_0,t_1}, M)$. Furthermore, we assume that the penalty parameter $c(\cdot)$ is sufficiently large (it is sufficient to set $c(\cdot) \geq (m + k)s \max_{1\le\alpha\le p} \{\bar{\mu}^a_{\alpha j}(\cdot), |\bar{\nu}^a_{\alpha l}(\cdot)| :$ $1 \leq a \leq s,\ 1 \leq j \leq m,\ 1 \leq l \leq k\}$, where $\bar{\mu}(\cdot)$ and $\bar{\nu}(\cdot)$ are Lagrange matrix functions associated with the inequality and equality constraints, respectively), then $\bar{x}(\cdot)$ is a minimizer of the penalized unconstrained multitime variational problem (*MVP$_\infty$*(c)) involving the exact minimax penalty function.*

The following example is constructed to elucidate the results established in Theorem 3.2.

Example 3.4. Let us consider the following PDI and PDE constrained multitime variational problem:

$$\text{(MVP1)} \quad \min_{x(\cdot)} F(x(\cdot)) = \int_{\gamma_{t_0,t_1}} f_\alpha(t, x(t), x_\gamma(t))\, \mathrm{d}t^\alpha$$

subject to

$$x(t_0) = 0 = x(t_1)$$
$$g(t, x(t), x_\gamma(t)) \leqq 0, \; t \in \Omega_{t_0,t_1},$$
$$h(t, x(t), x_\gamma(t)) = 0, \; t \in \Omega_{t_0,t_1},$$

where $x(\cdot) = (x_1(\cdot), x_2(\cdot))$, $t_0 = (0,0)$, $t_1 = (1,1)$ and $f : J^1(T, M) \mapsto \mathbb{R}^2$, $g : J^1(T, M) \mapsto \mathbb{R}^3$ and $h : J^1(T, M) \mapsto \mathbb{R}^3$ are defined as

$$f = (f_\alpha) := (x_1^2(t) + e^{x_2(t)}, x_2(t) + 2x_2^2(t) - 9),$$
$$g = (g_a^j) := (x_1(t) - 1, -x_1(t), x_2^2(t) - x_2(t)),$$
$$h = (h_a^l) := (x_1(t) - x_2(t), x_1^2(t) - x_2^2(t), 2(x_1(t) - x_2(t))),$$
$$\alpha = 1, 2, \; a = 1, 2, 3, \; j = 1, \; l = 1.$$

Obviously,
$$\mathcal{F}(\Omega_{t_0,t_1}) = \{x(t) \in C^\infty(\Omega_{t_0,t_1}, \mathbb{R}^2) : x(t_0) = 0, \; x(t_1) = 0, \; 0 \leq x_1(t) \leq 1 \wedge 0 \leq x_2(t) \leq 1 \wedge x_1(t) = x_2(t)\}$$

is the set of all feasible solutions to (MVP1). Now, we construct the penalized unconstrained multitime variational problem (MVP1_∞(c)) involving the exact minimax penalty function as follows:

$$\begin{aligned}\min \quad F_\infty(x(\cdot), c(\cdot)) = \int_{\gamma_{t_0,t_1}} & (x_1^2(t) + e^{x_2(t)} + c(t) \max\{\max\{0, x_1(t) - 1\}, \max\{0, -x_1(t)\}, \\ & \max\{0, x_2^2(t) - x_2(t)\}, |x_1(t) - x_2(t)|, |x_1^2(t) - x_2^2(t)|, |2(x_1(t) - x_2(t))|\}, \\ & x_2(t) + 2x_2^2(t) - 9 + c(t) \max\{\max\{0, x_1(t) - 1\}, \max\{0, -x_1(t)\}, \max\{0, \\ & x_2^2(t) - x_2(t)\}, |x_1(t) - x_2(t)|, |x_1^2(t) - x_2^2(t)|, |2(x_1(t) - x_2(t))|\})(\, \mathrm{d}t^1, \, \mathrm{d}t^2).\end{aligned}$$

Clearly, $\bar{x}(\cdot) = (0,0)$ is a feasible solution to the considered PDI and PDE constrained multitime variational problem (MVP1) and the necessary optimality conditions (2.3)-(2.5) are satisfied at $\bar{x}(\cdot)$, with Lagrange multipliers $\bar{\lambda} = 1$, $\bar{\mu}_{11}(t) = \left(0, \frac{1}{2}, \frac{1}{2}\right)^T$, $\bar{\mu}_{21}(t) = \left(0, \frac{1}{4}, \frac{3}{4}\right)^T$, $\bar{\nu}_{11}(t) = \left(\frac{1}{2}, 0, 0\right)^T$ and $\bar{\nu}_{21}(t) = \left(\frac{1}{4}, 0, 0\right)^T$. It is easy to verify that $F(x(\cdot))$, $\int_{\gamma_{t_0,t_1}} \langle \bar{\mu}_{\alpha j}(t), g^j(t, x(t), x_\gamma(t)) \rangle \, \mathrm{d}t^\alpha$ and $\int_{\gamma_{t_0,t_1}} \langle \bar{\nu}_{\alpha l}(t), h^l(t, x(t), x_\gamma(t)) \rangle \, \mathrm{d}t^\alpha$ are convex at $\bar{x}(\cdot)$ on $C^\infty(\Omega_{t_0,t_1}, \mathbb{R}^2)$ for $j = 1$, $l = 1$. Hence, by Theorem 3.2, for $c(t) \geq 3$, $\bar{x}(\cdot)$ is a minimizer of the unconstrained penalized multitime variational problem (MVP1_∞(c)) involving the exact minimax penalty function.

Now, we shall prove the converse result in which we show that a minimizer of the penalized unconstrained multitime variational problem is also an optimal solution to the original PDI and PDE constrained multitime variational problem, when the penalty parameter exceeds a certain threshold value under the assumptions of convexity.

Theorem 3.5. *Let $\bar{x}(\cdot)$ be a minimizer of the penalized unconstrained multitime variational problem ($\text{MVP}_\infty(c)$) with the exact minimax penalty function. Assume that the penalty parameter is sufficiently large (it is sufficient to assume $c(\cdot) \geq (m + k)s \max_{1 \leq \alpha \leq p} \{\tilde{\mu}_{\alpha j}^a(\cdot), |\tilde{\nu}_{\alpha l}^a(\cdot)| : \; 1 \leq a \leq s, \; 1 \leq j \leq m, \; 1 \leq l \leq k\}$, where $\tilde{\mu}(\cdot)$ and $\tilde{\nu}(\cdot)$ are Lagrange multipliers satisfying the necessary optimality conditions (2.3)–(2.5) at any optimal solution $\tilde{x}(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$). Furthermore, we assume that $F(x(\cdot))$, $\int_{\gamma_{t_0,t_1}} \langle \tilde{\mu}_{\alpha j}(t), g^j(t, x(t), x_\gamma(t)) \rangle \, \mathrm{d}t^\alpha$,*

$j=1,\dots,m$, and $\int_{\gamma_{t_0,t_1}}\langle\tilde{\nu}_{\alpha l}(t),h^l(t,x(t),x_\gamma(t))\rangle\,\mathrm{d}t^\alpha$, $l=1,\dots,k$ are convex at $\tilde{x}(\cdot)$ on $C^\infty(\Omega_{t_0,t_1},M)$. Then, $\bar{x}(\cdot)$ is also an optimal solution to the PDI *and* PDE *constrained multitime variational problem (*MVP*).*

Proof. Since $\bar{x}(\cdot)$ is a minimizer of the penalized unconstrained multitime variational problem ($\mathrm{MVP}_\infty(c)$) with the exact minimax penalty function, the inequality

$$F_\infty(x(\cdot),c(\cdot))\geq F_\infty(\bar{x}(\cdot),c(\cdot))$$

holds for all $x(\cdot)\in C^\infty(\Omega_{t_0,t_1},M)$. By the definition of the penalized multitime variational problem, it follows that the following inequality

$$\int_{\gamma_{t_0,t_1}}\left\{f_\alpha(t,x(t),x_\gamma(t))+c(t)\max_{\substack{1\leq a\leq s\\1\leq j\leq m\\1\leq l\leq k}}\{(g_a^j)^+(t,x(t),x_\gamma(t)),|h_a^l(t,x(t),x_\gamma(t))|\}e_\alpha\right\}\mathrm{d}t^\alpha$$
$$\geq\int_{\gamma_{t_0,t_1}}\left\{f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))+c(t)\max_{\substack{1\leq a\leq s\\1\leq j\leq m\\1\leq l\leq k}}\{(g_a^j)^+(t,\bar{x}(t),\bar{x}_\gamma(t)),|h_a^l(t,\bar{x}(t),\bar{x}_\gamma(t))|\}e_\alpha\right\}\mathrm{d}t^\alpha \tag{3.16}$$

holds for all $x(\cdot)\in C^\infty(\Omega_{t_0,t_1},M)$. Thus, for all $x(\cdot)\in\mathcal{F}(\Omega_{t_0,t_1})$, we have

$$\int_{\gamma_{t_0,t_1}}f_\alpha(t,x(t),x_\gamma(t))\,\mathrm{d}t^\alpha\geq\int_{\gamma_{t_0,t_1}}\left\{f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))\right.$$
$$\left.+\,c(t)\max_{\substack{1\leq a\leq s\\1\leq j\leq m\\1\leq l\leq k}}\{(g_a^j)^+(t,\bar{x}(t),\bar{x}_\gamma(t)),|h_a^l(t,\bar{x}(t),\bar{x}_\gamma(t))|\}e_\alpha\right\}\mathrm{d}t^\alpha. \tag{3.17}$$

From (3.1) and the definition of absolute value function, it is clear that

$$\max_{\substack{1\leq a\leq s\\1\leq j\leq m\\1\leq l\leq k}}\{(g_a^j)^+(t,\bar{x}(t),\bar{x}_\gamma(t)),|h_a^l(t,\bar{x}(t),\bar{x}_\gamma(t))|\}\geq 0.$$

Thus, inequality (3.17) yields

$$\int_{\gamma_{t_0,t_1}}f_\alpha(t,x(t),x_\gamma(t))\,\mathrm{d}t^\alpha\geq\int_{\gamma_{t_0,t_1}}f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))\,\mathrm{d}t^\alpha, \tag{3.18}$$

for all $x(\cdot)\in\mathcal{F}(\Omega_{t_0,t_1})$. If $\bar{x}\in\mathcal{F}(\Omega_{t_0,t_1})$, then the result directly follows from (3.18).

Now, we consider the case when $\bar{x}\notin\mathcal{F}(\Omega_{t_0,t_1})$. By hypothesis, the necessary optimality conditions (2.3)–(2.5) are satisfied at $\tilde{x}(\cdot)$. Also, $F(x(\cdot))$, $\int_{\gamma_{t_0,t_1}}\langle\tilde{\mu}_{\alpha j}(t),g^j(t,x(t),x_\gamma(t))\rangle\,\mathrm{d}t^\alpha, j=1,\dots,m$ and $\int_{\gamma_{t_0,t_1}}\langle\tilde{\nu}_{\alpha l}(t),h^l(t,x(t),x_\gamma(t))\rangle\,\mathrm{d}t^\alpha$, $l=1,\dots,k$ are convex at $\tilde{x}(\cdot)$ on $C^\infty(\Omega_{t_0,t_1},M)$. Thus, for all $x(\cdot)\in C^\infty(\Omega_{t_0,t_1},M)$, we have

$$\begin{aligned}&\int_{\gamma_{t_0,t_1}} f_\alpha(t,x(t),x_\gamma(t))\,\mathrm{d}t^\alpha - \int_{\gamma_{t_0,t_1}} f_\alpha(t,\tilde{x}(t),\tilde{x}_\gamma(t))\,\mathrm{d}t^\alpha\\&\quad\geq \int_{\gamma_{t_0,t_1}} \left\{\left\langle x(t)-\tilde{x}(t), \frac{\partial f_\alpha}{\partial x}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\right\rangle + \left\langle x_\gamma(t)-\tilde{x}_\gamma(t), \frac{\partial f_\alpha}{\partial x_\gamma}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\right\rangle\right\}\mathrm{d}t^\alpha,\\&\int_{\gamma_{t_0,t_1}} \langle\tilde{\mu}_{\alpha j}(t), g^j(t,x(t),x_\gamma(t))\rangle\,\mathrm{d}t^\alpha - \int_{\gamma_{t_0,t_1}} \langle\tilde{\mu}_{\alpha j}(t), g^j(t,\tilde{x}(t),\tilde{x}_\gamma(t))\rangle\,\mathrm{d}t^\alpha\\&\quad\geq \int_{\gamma_{t_0,t_1}} \left\{\left\langle x(t)-\tilde{x}(t), \left\langle \bar{\mu}_{\alpha j}(t), \frac{\partial g^j}{\partial x}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\right\rangle\right\rangle\right.\\&\qquad\left.+ \left\langle x_\gamma(t)-\tilde{x}_\gamma(t), \left\langle \tilde{\mu}_{\alpha_j}(t), \frac{\partial g^j}{\partial x_\gamma}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\right\rangle\right\rangle\right\}\mathrm{d}t^\alpha,\ \ j=1,\ldots,m,\\\text{and}&\int_{\gamma_{t_0,t_1}} \langle\tilde{\nu}_{\alpha l}(t), h^l(t,x(t),x_\gamma(t))\rangle\,\mathrm{d}t^\alpha - \int_{\gamma_{t_0,t_1}} \langle\tilde{\nu}_{\alpha l}(t), h^l(t,\tilde{x}(t),\tilde{x}_\gamma(t))\rangle\,\mathrm{d}t^\alpha\\&\quad\geq \int_{\gamma_{t_0,t_1}} \left\{\left\langle x(t)-\tilde{x}(t), \left\langle \tilde{\nu}_{\alpha l}(t), \frac{\partial h^l}{\partial x}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\right\rangle\right\rangle\right.\\&\qquad\left.+ \left\langle x_\gamma(t)-\tilde{x}_\gamma(t), \left\langle \tilde{\nu}_{\alpha_l}(t), \frac{\partial h^l}{\partial x_\gamma}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\right\rangle\right\rangle\right\}\mathrm{d}t^\alpha,\ \ l=1,\ldots,k.\end{aligned}$$

Since, $\bar{x}(\cdot) \in C^\infty(\Omega_{t_0,t_1}, M)$, therefore, the above inequalities also hold for $x(\cdot) = \bar{x}(\cdot)$. Hence, it follows that

$$\begin{aligned}&\int_{\gamma_{t_0,t_1}} f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t))\,\mathrm{d}t^\alpha - \int_{\gamma_{t_0,t_1}} f_\alpha(t,\tilde{x}(t),\tilde{x}_\gamma(t))\,\mathrm{d}t^\alpha\\&\quad\geq \int_{\gamma_{t_0,t_1}} \left\{\left\langle \bar{x}(t)-\tilde{x}(t), \frac{\partial f_\alpha}{\partial x}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\right\rangle + \left\langle \bar{x}_\gamma(t)-\tilde{x}_\gamma(t), \frac{\partial f_\alpha}{\partial x_\gamma}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\right\rangle\right\}\mathrm{d}t^\alpha,\\&\int_{\gamma_{t_0,t_1}} \langle\tilde{\mu}_{\alpha j}(t), g^j(t,\bar{x}(t),\bar{x}_\gamma(t))\rangle\,\mathrm{d}t^\alpha - \int_{\gamma_{t_0,t_1}} \langle\tilde{\mu}_{\alpha j}(t), g^j(t,\tilde{x}(t),\tilde{x}_\gamma(t))\rangle\,\mathrm{d}t^\alpha\\&\quad\geq \int_{\gamma_{t_0,t_1}} \left\{\langle \bar{x}(t)-\tilde{x}(t), \langle \bar{\mu}_{\alpha j}(t), \frac{\partial g^j}{\partial x}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\rangle\rangle\right.\\&\qquad\left.+ \langle \bar{x}_\gamma(t)-\tilde{x}_\gamma(t), \langle \tilde{\mu}_{\alpha_j}(t), \frac{\partial g^j}{\partial x_\gamma}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\rangle\rangle\right\}\mathrm{d}t^\alpha,\ \ j=1,\ldots,m,\\\text{and}&\int_{\gamma_{t_0,t_1}} \langle\tilde{\nu}_{\alpha l}(t), h^l(t,\bar{x}(t),\bar{x}_\gamma(t))\rangle\,\mathrm{d}t^\alpha - \int_{\gamma_{t_0,t_1}} \langle\tilde{\nu}_{\alpha l}(t), h^l(t,\tilde{x}(t),\tilde{x}_\gamma(t))\rangle\,\mathrm{d}t^\alpha\\&\quad\geq \int_{\gamma_{t_0,t_1}} \left\{\left\langle \bar{x}(t)-\tilde{x}(t), \left\langle \tilde{\nu}_{\alpha l}(t), \frac{\partial h^l}{\partial x}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\right\rangle\right\rangle\right.\\&\qquad\left.+ \left\langle \bar{x}_\gamma(t)-\tilde{x}_\gamma(t), \left\langle \tilde{\nu}_{\alpha_l}(t), \frac{\partial h^l}{\partial x_\gamma}(t,\tilde{x}(t),\tilde{x}_\gamma(t))\right\rangle\right\rangle\right\}\mathrm{d}t^\alpha,\ \ l=1,\ldots,k.\end{aligned}$$

Proceeding along the lines of Theorem 3.2, we obtain

$$\begin{aligned}&\int_{\gamma_{t_0,t_1}} \left\{f_\alpha(t,\bar{x}(t),\bar{x}_\gamma(t)) + \sum_{a=1}^{s}\sum_{j=1}^{m} \tilde{\mu}^a_{\alpha j}(t)(g^j_a)^+(t,\bar{x}(t),\bar{x}_\gamma(t))\right.\\&\qquad\left.+ \sum_{a=1}^{s}\sum_{l=1}^{k} |\tilde{\nu}^a_{\alpha l}(t)||h^l_a(t,\bar{x}(t),\bar{x}_\gamma(t))|\right\}\mathrm{d}t^\alpha \geq \int_{\gamma_{t_0,t_1}} f_\alpha(t,\tilde{x}(t),\tilde{x}_\gamma(t))\,\mathrm{d}t^\alpha. \qquad (3.19)\end{aligned}$$

Now, we consider two cases:
Case (i): Let $\rho(\cdot) = \max_{1\leq\alpha\leq p}\{\tilde{\mu}^a_{\alpha j}(\cdot), |\tilde{\nu}^a_{\alpha l}(\cdot)| : 1 \leq a \leq s,\ 1 \leq j \leq m,\ 1 \leq l \leq k\} > 0$ and $c'(\cdot) = (m+k)s\rho(\cdot)$. Then, $c'(\cdot) > 0$.

Now, proceeding along the lines of Theorem 3.2, with the notations

$$\begin{gathered}\phi^q_r(t, \bar{x}(t), \bar{x}_\gamma(t)) := \left(g^q_r\right)^+(t, \bar{x}(t), \bar{x}_\gamma(t)),\ t \in \Omega_{t_0,t_1},\ r = 1,\dots,s,\ q = 1,\dots,m,\\ \phi^{q+m}_r(t, \bar{x}(t), \bar{x}_\gamma(t)) := |h^q_r(t, \bar{x}(t), \bar{x}_\gamma(t))|,\ t \in \Omega_{t_0,t_1},\ r = 1,\dots,s,\ q = 1,\dots,k,\\ \tilde{\xi}^r_q(\cdot) := \frac{\rho(\cdot)}{c'(\cdot)},\ r = 1,\dots,s,\ q = 1,\dots,m+k.\end{gathered}$$

It follows that

$$\begin{aligned}&\int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t)) + c(t) \max_{\substack{1\leq a\leq s\\1\leq j\leq m\\1\leq l\leq k}} \left\{ \left(g^j_a\right)^+(t, \bar{x}(t), \bar{x}_\gamma(t)), |h^l_a(t, \bar{x}(t), \bar{x}_\gamma(t))| \right\} e_\alpha \right\} \mathrm{d}t^\alpha\\ &\quad\geq \int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, \tilde{x}(t), \tilde{x}_\gamma(t)) + c(t) \max_{\substack{1\leq a\leq s\\1\leq j\leq m\\1\leq l\leq k}} \left\{ \left(g^j_a\right)^+(t, \tilde{x}(t), \tilde{x}_\gamma(t)), |h^l_a(t, \tilde{x}(t), \tilde{x}_\gamma(t))| \right\} e_\alpha \right\} \mathrm{d}t^\alpha,\end{aligned}$$

which contradicts (3.16). Hence, $\bar{x}(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ is an optimal solution to the PDI and PDE constrained multitime variational problem (MVP).
Case (ii): Let $\rho(\cdot) = \max_{1\leq\alpha\leq p}\{\tilde{\mu}^a_{\alpha j}(\cdot), |\tilde{\nu}^a_{\alpha l}(\cdot)| : 1 \leq a \leq s,\ 1 \leq j \leq m,\ 1 \leq l \leq k\} = 0$. Then, $c'(\cdot) = (m+k)s\rho(\cdot) = 0$. Thus, from the necessary optimality condition (2.5) and (3.19), it follows that

$$\int_{\gamma_{t_0,t_1}} f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t))\, \mathrm{d}t^\alpha \geq \int_{\gamma_{t_0,t_1}} f_\alpha(t, \tilde{x}(t), \tilde{x}_\gamma(t))\, \mathrm{d}t^\alpha.$$

Again proceeding along the lines of Theorem 3.2, we obtain

$$\begin{aligned}&\int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, \bar{x}(t), \bar{x}_\gamma(t)) + c(t) \max_{\substack{1\leq a\leq s\\1\leq j\leq m\\1\leq l\leq k}} \left\{ \left(g^j_a\right)^+(t, \bar{x}(t), \bar{x}_\gamma(t)), |h^l_a(t, \bar{x}(t), \bar{x}_\gamma(t))| \right\} e_\alpha \right\} \mathrm{d}t^\alpha\\ &\quad\geq \int_{\gamma_{t_0,t_1}} \left\{ f_\alpha(t, \tilde{x}(t), \tilde{x}_\gamma(t)) + c(t) \max_{\substack{1\leq a\leq s\\1\leq j\leq m\\1\leq l\leq k}} \left\{ \left(g^j_a\right)^+(t, \tilde{x}(t), \tilde{x}_\gamma(t)), |h^l_a(t, \tilde{x}(t), \tilde{x}_\gamma(t))| \right\} e_\alpha \right\} \mathrm{d}t^\alpha,\end{aligned}$$

which contradicts (3.16). Thus, $\bar{x}(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$. Hence, From (3.18), $\bar{x}(\cdot)$ is an optimal solution to the PDI and PDE constrained multitime variational problem (MVP). This completes the proof of the theorem. □

Example 3.6. Let us consider the following PDI and PDE constrained multitime variational problem:

$$\text{(MVP2)} \qquad \min_{x(\cdot)} F(x(\cdot)) = \int_{\gamma_{t_0,t_1}} f_\alpha(t, x(t), x_\gamma(t))\, \mathrm{d}t^\alpha$$

subject to

$$x(t_0) = 0 = x(t_1)$$
$$g(t, x(t), x_\gamma(t)) \leqq 0, \ t \in \Omega_{t_0,t_1},$$
$$h(t, x(t), x_\gamma(t)) = 0, \ t \in \Omega_{t_0,t_1},$$

where $x(\cdot) = (x_1(\cdot), x_2(\cdot))$, $t_0 = (0,0)$, $t_1 = (1,1)$ and $f : J^1(T,M) \mapsto \mathbb{R}^2$, $g : J^1(T,M) \mapsto \mathbb{R}^2$ and $h : J^1(T,M) \mapsto \mathbb{R}^2$ are defined as

$$f = (f_\alpha) := (x_1^2(t) + x_2^2(t), e^{\sin x_1^2(t) + \cos x_1(t)} + x_2^4(t)),$$
$$g = (g_a^j) := (x_1^2(t) - x_1(t), \sin^2 x_2(t) - \cos^2 x_2(t)),$$
$$h = (h_a^l) := (\cos x_1(t) - \cos x_2(t), x_1^2(t) - x_2^2(t)),$$
$$\alpha = 1,2, \ a = 1,2, \ j = 1, \ l = 1.$$

Obviously,
$$\mathcal{F}(\Omega_{t_0,t_1}) = \{x(t) \in C^\infty(\Omega_{t_0,t_1}, \mathbb{R}^2) : x(t_0) = 0, \ x(t_1) = 0, \ 0 \le x_1(t) \le 1 \wedge x_2(t) = \pm x_1(t) \wedge \ \sin^2 x_2(t) - \cos^2 x_2(t) \le 0\}$$

is the set of all feasible solutions to (MVP2). Now, we construct the penalized unconstrained multitime variational problem (MVP2$_\infty$(c)) involving the exact minimax penalty function as follows:

$$\begin{aligned}\min \quad F_\infty(x(\cdot), c(\cdot)) = \int_{\gamma_{t_0,t_1}} & (x_1^2(t) + x_2^2(t) + c(t) \max\{\max\{0, x_1^2(t) - x_1(t)\}, \max\{0, \\ & \sin^2 x_2(t) - \cos^2 x_2(t)\}, |\cos x_1(t) - \cos x_2(t)|, |x_1^2(t) - x_2^2(t)|\}, \\ & e^{\sin x_1^2(t) + \cos x_1(t)} + x_2^4(t) + c(t) \max\{\max\{0, x_1^2(t) - x_1(t)\}, \max\{0, \\ & \sin^2 x_2(t) - \cos^2 x_2(t)\}, |\cos x_1(t) - \cos x_2(t)|, |x_1^2(t) - x_2^2(t)|\})(\,\mathrm{d}t^1, \,\mathrm{d}t^2).\end{aligned}$$

It can be verified that $\bar{x}(\cdot) = (0,0)$ is a minimizer of the penalized unconstrained multitime variational problem (MVP2$_\infty$(c)). Also, the necessary optimality conditions (2.3)–(2.5) are satisfied at $\tilde{x}(\cdot) = (0,0)$, with Lagrange multipliers $\tilde{\lambda} = 1$, $\tilde{\mu}_{11}(t) = (0,k)^T$, $\bar{\mu}_{21}(t) = (0,k')^T$, $\bar{\nu}_{11}(t) = \bar{\nu}_{21}(t) = (0,0)^T$ for all $k, \ k' \in \mathbb{R}^+$. We take $k = 1, \ k' = \dfrac{1}{2}$. It is easy to verify that $F(x(\cdot))$, $\int_{\gamma_{t_0,t_1}} \langle \bar{\mu}_{\alpha j}(t), g^j(t, x(t), x_\gamma(t)) \rangle \,\mathrm{d}t^\alpha$ and $\int_{\gamma_{t_0,t_1}} \langle \bar{\nu}_{\alpha l}(t), h^l(t, x(t), x_\gamma(t)) \rangle \,\mathrm{d}t^\alpha$ are convex at $\tilde{x}$ on $C^\infty(\Omega_{t_0,t_1}, \mathbb{R}^2)$ for $j = 1, \ l = 1$. Hence, by Theorem 3.5, for $c(t) \ge 4$, $\bar{x}$ is an optimal solution to the multitime variational problem (MVP2).

Finally, based on the above established results, we can state the following result:

Corollary 3.7. *If all the hypotheses of Corollary 3.3 and Theorem 3.5 are satisfied, then the set of optimal solutions of the* PDI *and* PDE *constrained multitime variational problem (*MVP*) coincides with the set of minimizers of the penalized unconstrained multitime variational problem (*MVP$_\infty$*(c)) with the exact minimax penalty function.*

4. Conclusions and future perspectives

In this paper, we have investigated the applicability of the exact minimax penalty function method to find a minimizers of PDI and PDE constrained multitime variational problems under convexity assumptions of the functions involved. We find that the solutions of the penalized unconstrained multitime variational problem are equivalent to the solutions of the original constrained multitime variational problems, when the functionals under consideration are assumed to be convex and the penalty parameter exceeds a certain threshold value. Furthermore, we have also constructed examples to validate the results derived in this paper. Thus, we conclude that the exact minimax penalty function method also proves to be an effective way to solve PDI and PDE constrained multitime variational problems.

For future research work, it would be interesting to test the effectiveness of the exact minimax penalty function method to solve multiobjective multitime variational problems and related ones. Moreover, we could prepare a numerical experimentation using the approach proposed in this paper.

Acknowledgements. This research is financially supported by the Council of Scientific and Industrial Research, New Delhi, India through Grant no.: 25(0266)/17/EMR-II. Further, we would like to express our sincere thanks to referees for their helpful suggestions.

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