RAIRO-Oper. Res. 54 (2020) 585–595
https://doi.org/10.1051/ro/2019013

RAIRO Operations Research
www.rairo-ro.org

MAKING HIGH SCHOOL STUDENTS AWARE OF OPTIMIZATION THROUGH GAMES AND PUZZLES

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Abstract. Optimization is among the oldest mathematical applications in history: human nature tends to obtain the best results with the minimum effort, in many sectors. Nevertheless, optimization and operations research as mathematical disciplines struggle to find their place in high school syllabi and it is usually very hard to explain what they are about to students and common people. This could be due to a lack of "optimization culture" that prevents one to see that many problems that we tackle everyday are actually optimization problems, or a lack of mathematical culture in a broader sense. With this paper we report on a small experience carried out in an Italian high school where we attempted to close this gap. Rather than using a conventional approach, we introduced a set of games, puzzles and challenges based on optimization problems that stimulated students' intuition and creativity. Indeed the aim of this experiment was not teaching specific optimization methods, but starting the construction of an "optimization awareness" in the students.

Mathematics Subject Classification. 97D80, 97D50, 90C27.

Received October 7, 2018. Accepted January 22, 2019.

1. Introduction

Operations research and optimization are not included in most of the Italian High School syllabi. It appears in some accounting schools at the end of the last year. It involves an introduction to linear programming and the application of the simplex method. Very rarely this part of the program is actually covered, for many reasons, including the lack of time, the concern of many teachers who feel like not having a sufficient knowledge of the field, and the (wrong) impression that it is a complicated and useless discipline.

We think, however, that optimization and operations research can be an effective opportunity to approach mathematics. The relatively simple background needed to understand the problems, the possibility of playing and proposing creative solutions, the easy way of "touching" the solutions, make operations research and optimization an invaluable chance to make mathematics appealing to most of the students [5] and to foster their creativity and intuition.

The main question that we faced was: What is the approach to follow to introduce optimization in a high school? Should we explore some specific solution methods, or rather open many windows on the applications?

Keywords. Creativity, intuition, applied mathematics, optimization, operations research, active learning, orienteering.

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Should we follow a formal approach or privilege the intuition and the creativity? The answers come from another question that we often encounter dealing with the interface between high school and university. Supposing that a student exposed to optimization and operations research joins a university course specialized in these topics, what do we expect from his/her previous knowledge? Mastering some (simple) solution techniques, or being curious and interested in exploring the discipline, being aware of some of its potentialities?

We strongly believe in this second type of expectation, and this guided our experiment in an early class of an Italian high school. In dealing with optimization we relied on the connection with other disciplines as history and literature, we followed the specific interests of the students and we encouraged them in proposing their own solution approaches. The idea was not to transmit them a specific mathematical method, but let the students touch with their hands and understand the importance of optimization for them. Also the use of the computer has been very limited, privileging a more physical and "handcraft" approach, putting the accent on the reasoning itself rather than in magically obtaining a solution out of a black box.

The experiment has been carried out in a second year class of the I.T. Bachelet of Ferrara, in a class that had been previously educated to a creative vision of mathematics [7]. The teaching-learning process followed in this experiment was to boost the "productive thinking" of the students [18], orienteering and guiding them into the independent solution of new and non trivial problems. The principal aim of the teacher has been to bring out the intuition, the so-called "insight" of Köhler [6] to improve original cognitive mechanisms, and to reach the soft skills and competences, described in the European Qualifications Framework (EQF 2008) [4, 13] using an active learning paradigm. The effect of the activation of the class was to experience the so called "flow" [14], that is the engagement of the students was motivated by the activity itself rather than by an incentive. Indeed, the students were often so involved in our puzzles and questions that they almost did not realize the time passing.

The experiment involved nine meetings of one hour where the Maths teacher of the class was helped by a university professor expert in operations research. The experiment has been concluded with an orienteering competition where some of the problems met in class could be useful to optimize the decisions to be made during the competition. The outcome has been extremely positive in terms of engagement and personal impression of both the students and the teachers. Moreover, also the objective evaluations showed the success of the experiment.

2. Preliminary activities and teaching method

2.1. The class

The class was composed by 19 students, with 3 students with special educational needs: one case of handicap, one case of dyslexia and one foreign student. In spite of the fact that the class had a good starting base in Maths, being above the average level of the school and of the region, the teacher experienced a very low interest in the discipline and a low level of engagement by most of the students. This scarce interest could be attributed to a set of misconceptions about Maths. Indeed the majority of the students associated Mathematics with the mere calculus, the memorization of formulas and involved procedures without having a clue of their practical use, ending up with the belief that Maths is useless in practice. This also resulted in a lack of motivation in finding strategies for solving problems ranging from the text book ones to the everyday life ones. More than 50% of the class also declared that, even if they got good grades in Maths, they did not love the discipline, finding it boring and far from real world.

2.2. Preliminary activity

The year preceding the introduction to mathematical optimization that we are going to describe further on, in the same class, we worked on the construction of a mathematical common ground. This work has not involved the introduction of new concepts or methods, rather it was intended to destroy the misbeliefs in mathematics maturated by the students in their career, and often preventing them from getting involved and being engaged in applying mathematics and logic in solving problems. In practice we wanted the students to understand that mathematics is creative as much as other disciplines such as music or literature, and that intuition is at the base of it. Certainly intuition is important, but it is not enough and must be subsequently supported by mathematical reasoning and tools in order to be confirmed. However, constructing the mathematical tools in the absence of the motivation may result sterile and only a matter of handling the semiotics without the background of the concepts, thus ending up in an extremely boring activity completely uncorrelated with reality. This work was intended mainly to empower the students with a dynamic set of characteristics like for example openness, tolerance for ambiguity, flexibility in thinking, perseverance, motivation for creativity, need for self-expression [11].

The activities, described in [7], have been carried out using drama, simple materials and small rewards, and most of the times implied physical activities. They have involved a variety of mathematical disciplines as logic, combinatorics, topology, probability, simulation, geometry and graphs, even though we did not deal with them in a traditional academic way.

The outcome of this preliminary project has been very positive from two viewpoints. On the one hand, from the qualitative point of view, the students have changed their initial opinion about mathematics and also their approach to the discipline has become more active and propositive. On the other hand, from the quantitative point of view, most of the students increased their proficiency and overall grade in Maths.

All students, at the end of the year, declared to be ready to continue the experience in the next year, and this is what we actually did. This time focusing the activities on optimization problems.

2.3. The teaching method

To recover a constructive and creative vision of mathematics we followed the *Puzzle based learning* method [15]. This teaching method has the aim of improving reasoning skills, perseverance, and motivation in tackling problems. That is, the basic skills of problem solving. The method does not consider real problems, but rather puzzles that do not require any particular background and that have the only feature of being challenging and appealing for the students. In certain cases even the students can propose their own puzzles. The main role of the teacher in this type of classes is to scaffold and to be a guide in the community of learners [3]. The student is the main actor in the discovery and learning process, becoming the constructor of his/her knowledge cooperating with the classmates according to the collaborative learning theory of Lev S. Vygotskij [16,17]. Thus the teacher does not teach how to solve problems, but keeps high the attention and the motivation by proposing stimulating problems, follows the student arguments, accompanies the students in reaching a solution. The teacher must also be able to accept methods to solve problems different from what he/she has initially thought. In the activities, the main scope is not reaching a solution, but the process that allows to reach it and the effort that must be put in. In this context the role of the teacher is similar to that of a sport coach, where the motivational skills must prevail with respect to the mere technical ones. In a certain sense, with this method, we reproduced in class the real work of a mathematician. Namely, to solve creatively new problems giving rise to different approaches and promoting a divergent thinking [9], rather than solving routine exercises and memorizing solution procedures.

3. Activities in class

The teaching approach used in these activities has been the same as in the previous year, and the learning process was centered on the constructivism principles. All the activities, apart from the shortest path one, have been organized in groups. The groups included 2–3 students and were formed by the class teacher considering the skills of every student and identifying a leader for each group. This was intended to maximize the involvement and the integration of all students. In each activity, the university teacher has first presented the problem statement, describing the application field in order to let the students accept the challenge and find the motivation in proposing their solution approaches. At the beginning of each activity, the groups have worked separately and independently, and the teachers assisted them helping in keeping the activities focused on the problem. In this phase, we tried to adopt the most intuitive and often “handicraft” approach to generate a solution. Finally the experiences were confronted in a debate phase coordinated by the teachers, where the various solution approaches were analyzed and discussed, summarizing all together a common strategy. This debate phase was then consolidated by a final contribution of the university teacher who gave a general framework to approach the problem, starting from what has been found by the students.

In the activities, we considered a variety of optimization problems arising in different fields, taking advantage of the ludic and challenging aspects. The optimization problems that we considered involve one *objective function* to be maximized (*e.g.*, profit, performance, area) or minimized (*e.g.*, cost, inconvenience, distance) subject to a set of *constraints*. The constraints may concern the available resources needed to reach a particular result, or the rules to be satisfied, and can be mathematically expressed by equalities or inequalities. The set of all solutions satisfying the constraints define the so called *feasible region*. In an optimization problem some decisions have to be made so as to find the *optimal solution* that is the solution that maximizes (minimizes) the objective function among those satisfying the constraints (*i.e.*, *feasible solutions*).

3.1. The isoperimetric problem

We started from the oldest documented optimization problem in history, finding a link with the literature and history classes [2]. The problem has been faced for the first time by queen Dido in the foundation of Carthage in 814 b.C when she asked a piece of land that could be contained in an oxhide to the local inhabitants of current Tunisia. Then she obtained a very long rope cutting the oxhide and she used used it to define the perimeter of the new city.

From the mathematical point of view, the problem consists in finding the shape of maximum area having a given perimeter. In this case the objective function is the area and the constraint is the perimeter. The decision to make is the geometric shape to be included in the perimeter. This problem is known as the isoperimetric problem.

We took a string loop about 40 cm long and started creating rectangles with different side lengths. We observed how the area did change depending on the length of the sides, finding out that the maximum area rectangle is the one with equal sides, that is the square.

We then tried to extend the analysis to regular polygons, that is the polygons with n sides of equal length and congruent angles. The polygon can be partitioned into n triangles having as base the side and as height the apothem. Considering a perimeter p, the side length is $s = p/n$, the rule to obtain the area A is:

$$A = \frac{1}{2} n \times s \times \text{apo} = \frac{p \times \text{apo}}{2},$$

where apo denotes the apothem.

We noticed that the area is actually growing with n. Letting n grow to infinity, the corresponding regular polygon approximates the circumference with increasing precision, suggesting that the optimal solution is actually the circumference. The students with a strong mathematical motivation have been invited to look at the formal proof of the isoperimetric problem and to its fascinating history.

3.2. Introduction to linear programming: the cell phone assembly game

We introduced a simple production mix problem. Given a fixed amount of resources of different types (electronic components) we wanted to assemble cell phones of two types, each yielding a known profit, so as to maximize the overall economic return. We divided the students into groups and we distributed to each group an envelope containing a certain number of cardboard rectangles of different colors simulating the electronic components, and the "recipes" of two possible cell phones (see Fig. 1), each yielding a given return (score). Each group had the same amount of components, and we opened a challenge among groups to find the best production mix, that is the number of cell phones that could be produced with the given components maximizing the overall return.

The problem objective function is the sum of the economic returns (scores) of the produced cell phones, the constraints concern the used components that cannot overcome the availabilities, and the decisions to be taken

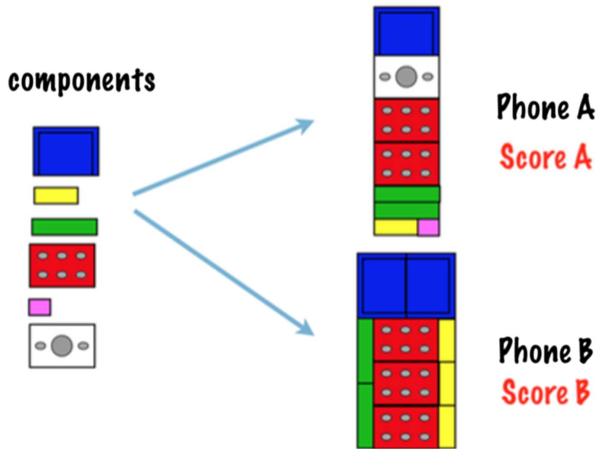

FIGURE 1. Cell phones assembly problem.

are how many cell phones to assemble for each type. If we neglect the requirement of constructing an integer number of cell phones, that is we accept also solutions where fractions of cell phones are allowed, the problem belongs to the class of Linear Programming problems.

After physically assembling the cell phones, we opened a discussion about the optimality of the solution. Some solutions yielded a better return than others, thus they were clearly better. After verifying that the best solution was actually feasible, we continued the discussion about proving its optimality: the solution of some groups was the best of the found ones, but was it the best one? We realized that the discussion had to be supported by something formal as a mathematical model. We introduced two variables x_A and x_B representing the number of cell phones of type A and of type B respectively, to be constructed and we represented the constraints on the used components as linear inequalities, for each component i:

$$\text{component}_{iA} x_A + \text{component}_{iB} x_B \leq \text{availability}_i$$

We observed that, from the geometric point of view, each inequality represents a half-space, and we could give the geometric representation of the whole set of constraints, whose intersection provides a polygon giving the region containing all feasible solutions (see Fig. 2). Any solution (x_A, x_B) has a value of the objective function given by:

$$\text{return}_A x_A + \text{return}_B x_B$$

and potentially all solutions inside the polygon should be evaluated to detect that maximizing the value of the objective function. However, someone argued that it is useless to evaluate the solutions that are strictly contained in the polygon since they do not exploit maximally the resources, thus they can be trivially improved. This implied that we could focus only on the vertices of the polygon and by evaluating only few solutions we could detect the best one in a short time and also prove its optimality.

3.3. An investment problem: continuous knapsack problem

In order to consider an optimization problem having more than two variables, we introduced a simple investment problem. We supposed to have a given amount of money to be invested by buying bonds selecting them from a given set. At the end of the investing horizon, each bond, besides the invested amount, returns a coupon, whose amount is proportional to the invested amount. For each bond there is a maximum amount that can be

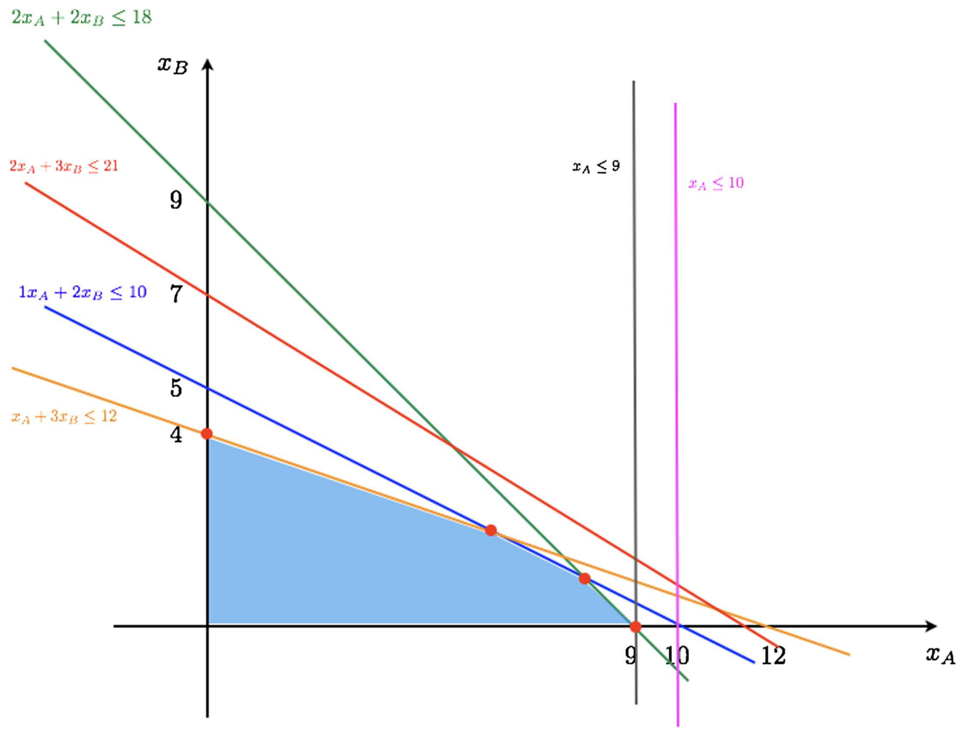

FIGURE 2. Geometric representation of the region of all feasible solutions.

TABLE 1. Data of the investment problem, total budget $b = 137$.

Bond	1	2	3	4	5
Max amount	40	12	130	5	400
Coupon	3.2	1.5	4.2	0.7	17

bought. The problem consists in determining the quantities to be invested for each bond so as to maximize the total economic return.

We used a small instance, summarized in Table 1.

The objective function is the total economic return, the constraints are about the maximum available quantity for every bond, and the total available budget. The decisions to be made are about the amount to be invested in each bond type.

Since the students were following also some business administration classes, the problem was quite familiar to them and they immediately proposed a simple strategy to decide about the investment. They proposed to sort the bonds $i = 1, \ldots, 5$ in decreasing order according to the interest rate computed as

$$\frac{\text{coupon}_i}{\text{max\_amount}_i},$$

that is the economic return per unit investment. The investment strategy that emerged from the debate considered the bonds according to the order, from the most profitable to the last one. The amount to buy was the maximum possible one, that is the minimum between the remaining budget and the available amount of the considered bond. This strategy turned out to be the *greedy algorithm*.

TABLE 2. Data of the fantasy sport problem, total budget $b = 8$.

Player	1	2	3	4	5	6
Projection	7	2	4	5	4	1
Cost	5	3	2	3	1	1

We opened a discussion whether the proposed strategy produced the optimal solution for the given instance, and it was quite easy to prove it attempting to change the solution and ending up with worse solutions. Having more time it could have been interesting to discuss also whether the strategy could be generalized to any instance of that type of problem showing (by induction or by contradiction) that the strategy is optimal.

3.4. A fantasy sport game: 0-1 knapsack problem

Some of the students of the class sometimes participated in a fantasy sport challenge, thus we used this context to introduce another optimization problem. We are given a fixed amount of fantasy money to hire players. For each player i in a given set we have a projection of the performance p_i and a cost c_i. The problem consists in selecting a subset of players to buy, maximizing the overall projected performance and complying with the budget limit. The small instance that we used to play with the problem is summarized in Table 2.

In this case the objective function is the sum of the projected performance over the selected players, the constraint is that the total expense must be less than or equal to the budget, the decisions are about the selection of players.

We observed that the problem is similar to the investment problem seen previously, with the only exception that we are not allowed to buy fractional amounts of players: a player can be either selected or discarded. This fact restricts the number of feasible solutions to be finite, while in the investment problem we had infinitely many solutions. This observation allowed us to adapt the greedy algorithm of the investment problem to the fantasy sport problem as follows. Consider players by decreasing ratio projection/cost: one player is selected if the remaining budget is sufficient to cover the cost, and the budget is updated, otherwise it is discarded. On the small example reported in the table above we noticed that, even though the number of possible feasible solutions is finite, the simple greedy algorithm does not guarantee to produce always the optimal solution. This outcome opened a discussion about the difficulty of problems related with the existence of efficient algorithms able to solve them. The curious students have been invited to look at the book [12] about knapsack problems and a fantasy sport application [1].

3.5. Shortest path on a map

We considered the problem of finding the shortest walking distance path in the city between a given origin and destination. We decided to start from the railway station and arrive to the school. We took advantage of a simplified map of the city[3] where some important sites of interest are pointed out and the distance in walking time between adjacent points is represented. The simplified map constitutes in practice a *graph*, that is an abstraction of the problem, where *nodes* represent sites and *arcs* represent the adjacency relation, having also the walking time as an attribute.

The problem consists in finding a sequence of consecutive arcs (a so called *path*) starting from the node representing the station (*origin*) and ending in the node representing the school (*destination*) having the minimum possible travel time. The objective is the total travel time, and the constraint is the definition of a path from a given origin to a given destination.

We wanted to start from a very "physical" intuition of the problem. For this reason, considering only a subset of nodes, thus neglecting nodes associated with sites too far from the origin and the destination, that would

[3]Metrominuto: http://servizi.comune.fe.it/7679/metrominuto-ferrara

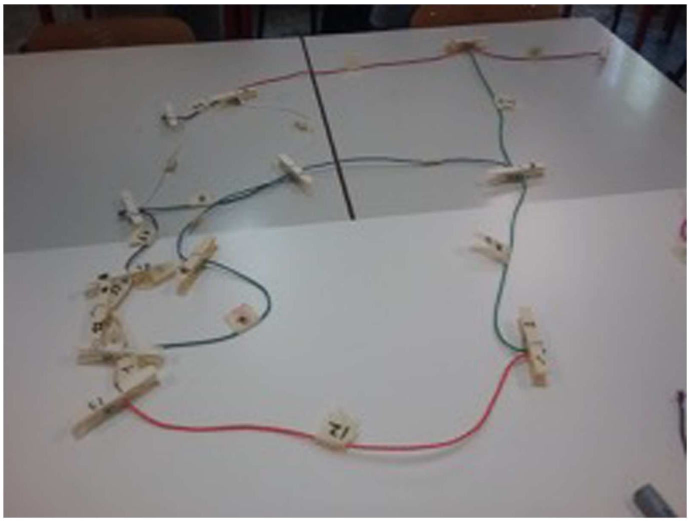

FIGURE 3. "Physical" graph for finding the path between the railway station and the school.

have been useless in the computation, we constructed a physical model of the graph. We used strings as arcs and clothespins as nodes. In this initial phase of the activity we used a more collaborative approach, and the whole class contributed to construct the graph by cutting strings and assembling them. We collaborated in cutting strings, one for each arc, having length proportional to the travel time. We used clothespins to connect strings and create our physical graph (see Fig. 3). We then discussed how to use our handicraft to find a solution. Someone observed that a way to find the shortest route is to rectify corners, thus we stretched the physical graph between the clothespin of the origin and that of the destination, avoiding to disconnect strings from the clothespins. The rectilinear route corresponds to the shortest path from the origin to the destination.

To conclude, we tried to skip the step requiring the construction of the physical graph, and we reproduced the "stretching" effect algorithmically. We associated to each node a *label* that eventually will assume a value equal to the distance from the origin to that node in the rectified trajectory. At the beginning the origin has label equal to 0 and the labels of the other nodes are set to infinity, meaning that no path has been found yet. In the physical graph, this is equivalent to cutting all strings pinned in the origin clothespin. We then fix the broken strings of the origin one at a time by suitably set the label of the nodes at the other end to the highest possible value. Note that setting the label equal to the highest possible value means that, in the physical graph the string is stretched. Doing this operation the strings pinned in the other extreme of the arc are broken. This operation is iterated from the node with the smallest label whose label has been updated. Eventually we obtained again the graph, without broken strings, and with the "rectified" path from the origin to the destination.

The students interested in this "strategy" have been suggested to navigate in the Internet looking at Dijkstra algorithm and to explore the world of shortest paths reading the novel [10].

3.6. Traveling Salesperson Problem

We continued to study shortest itinerary problems, where we considered an additional constraint: pass through each node of the graph once and return to the origin. This is the well known *Traveling Salesperson Problem* (TSP), one of the most studied and fascinating *combinatorial optimization* problems. The problem is described as follows: given a *complete graph*, that is a graph where each pair of nodes is connected by one arc, and a length associated with each arc, we have to find a *cycle*, that is a path starting and ending in a given node, that touches all nodes exactly once and minimizing the sum of the arc lengths. The history of this problem and other curiosities can be found in the TSP site[4].

The problem is more difficult than the shortest path problem and in principle all the sequences of nodes must be evaluated in order to find the best one. We observed that the number of sequences grows very quickly and its order of magnitude is the factorial of the node number, more specifically it is given by $(n-1)!/2$, where n is the number of nodes.

In order to understand the difficulty of the problem we went to the computer lab to use a web application. The application is available at http://vrp.upf.edu, and even though it has been designed to deal with the more general Vehicle Routing Problem, it can be helpful to generate also TSP instances and visualize the solutions on a map. We opened a challenge among groups: after having generated an instance with 13 cities, and having considered the trivial starting solution given by the sequence $1-2-\ldots-13-1$, we asked the groups to improve the solution by making simple exchanges in the sequence and evaluate them with the help of the web application that can compute the overall distance automatically.

Someone has been able to improve the initial solution, though, obviously, we could not prove the optimality of the obtained solutions.

3.7. Mathematical orienteering competition

To conclude our trip in the optimization world we organized a sport competition based on orienteering. The orienteering is a discipline that combines the run with the interpretation of a map to find some targets in the competition field.

In a public park of the city we have set up some targets to be found with the help of a map. The difference with the most common orienteering competitions was in the following aspects: i) each target has a score, ii) the sequence of visit is not fixed, iii) there is not the obligation of fetching all the targets, and iv) there is a time limit. Thus, in our competition, each team had to select a subset of the targets so as to maximize the total score and sequence them in the shortest possible way so as to finish within the time limit. The time limit has been set to a value such that even a well trained runner could not complete the shortest itinerary reaching all targets within the limit. A suitable penalty has been introduced for the teams exceeding the time limit.

From the mathematical optimization point of view the problem combines some aspects of the fantasy sport problem (select a subset of targets) with the TSP problem (sequence the targets in the shortest way). To the teams was given the map of the competition field with the targets, their score and an estimate of the distances between pairs of targets 10 min in advance with respect to their scheduled starting time. We gave a gap of 2 min between the starting of consecutive teams. In order to limit the differences among teams, so that the most athletic teams had not much competitive advantage, we decided to give a relatively short time limit (16 min), this made also challenging the target selection phase. The penalty for every minute exceeding the time limit was one point. We also asked the teams to punch their card according to the visit sequence of the targets so that we could reconstruct ex post their itinerary.

It was interesting to notice that teams made different choices both in the target selection and in the sequence to follow, and this gave us the opportunity to discuss the ranking of the competition in the final meeting. In the final meeting we first ranked the teams according to the overall score. Besides that, we made also a second ranking that awarded the most optimized choice. For each team we analyzed its sequence and computed the total run distance. With the help of an optimization model and a solver [8], we computed what could have been the optimal target choice and sequence having the computed distance as a limit. This allowed us to rank the teams with respect to their optimization skills rather than with respect to their athletic ability and actually we obtained a different ranking. The mathematical model also allowed us to compare the actual target selection and itinerary with the optimal one.

[4] http://www.math.uwaterloo.ca/tsp

Table 3. Outcome of the Maths school test: percentage values.

Class	Insufficient [0,4.5]	Basic [4.5, 6]	Intermediate [6, 7.5]	Excellent [7.5,10]
2A	0	6	70	24
2B	9	29	33	29
2C	8.7	66.2	21.7	4.4
2D	0	6	50	44
2L	15	47	32	6
2M	0	24	60	16

4. Feedback and conclusions

In this paper we reported on an unconventional teaching experience regarding mathematical optimization. The scope was not to teach optimization methods, but instead to make the students aware of this mathematical discipline. The feedback has been very positive, with a generalized involvement of the whole class. In order to assess the acquisition of the ideas seen in the teaching experience, we proposed a mathematical orienteering competition that has been appreciated by the students, including some students with special needs, and has created a general atmosphere of mutual aid and free exchange of ideas, very important to stimulate the motivation.

All the activities and the final orienteering competition had a beneficial effect on the students with inclusion problems. Both the foreign student and the handicapped one could actively participate and give a contribution. Thus they did not improve only their motivation and their mathematical skills, but they also improved their relational skills. The student with dislexia did not have any type of difficulty, and actually was one of those with the most brilliant ideas.

An anonymous questionnaire has been submitted to the class to evaluate the experience. All the students appreciated the experience and the active learning approach. Also the interaction with the external teacher has been positively evaluated by 100% of the students and they asked to repeat the experience in the following years. 80% of the students claimed that their motivation in approaching mathematics increased after the experience, and 70% of them thought to have broadened their mathematical skills. Among the free comments it emerged that the active learning approach made classes less tedious and allowed students to learn concepts in a more productive way. Moreover some students claimed that the contact with a university teacher has given them a different vision of Mathematics and, after the course, they looked at practical problems where Mathematics can be applied, from a different perspective.

From the objective point of view, in the national evaluation test (INVALSI[5]) the class has obtained a good overall score (62.2), much above the national average one (40.2). Moreover, considering the outcome of the internal test phase where the same test has been submitted to all the students of the school, a high percentage of students of our class (2D) reached an excellent level, and no one got an insufficient grade. In general the performance of the class has been significantly better than any other class of the school (Tab. 3).

In conclusion, we can state that this experience has been useful to improve the motivation towards Mathematics, to introduce the problem solving skills by cultivating the intuition and the creativity of the students, and to empower the excellent students.

Acknowledgements. This project has been partially financed by the Italian Instruction Ministry (Progetto CLIL). The authors are grateful to the editor and the reviewers for their constructive comments and helpful suggestions.

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