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A NEOTERIC APPROACH TO GEOMETRIC SHIPMENT POLICY IN AN INTEGRATED SUPPLY CHAIN WITH SETUP COST REDUCTION AND FREIGHT COST USING SERVICE LEVEL CONSTRAINT

Selvaraj Hemapriya* and Ramasamy Uthayakumar

Abstract. This paper explores a neoteric approach to geometric shipment policy and concerns the impact of controllable lead time, setup cost reduction, lost sales caused by stock-out and freight cost within an integrated vendor–buyer supply chain configuration using service-level constraint. In particular, the transportation cost is a function of shipping weight, distance and transportation modes. In other words, truckload (TL) and less-than-truckload (LTL) shipments. A heuristic model is developed to minimize the joint expected total cost (JETC), when the mode of transportation is limited to TL and LTL shipments. Numerical examples including the sensitivity analysis with some managerial insights of system parameters is implemented to endorse the outcome of the supply chain models.

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1. Introduction

To improve the competitive capacity of the business, firms tend to become integral part of a supply chain, rather than being single entities. According to this point of view, the development of Joint Economic Lot Size (JELS) models still represents one of the main research topic in the Supply Chain Management (SCM) field.

Goyal [12] studied a model in which the shipment size increases geometrically. Hill [16] developed the Goyal [12] model where the geometric growth rate in shipments size is a decision variable. Goyal and Nebebe [13] presented geometric-then-equal-shipments policy for the first time. In this policy, the size of shipments is small at first and then its size growths by multiplying a factor and finally the shipment size will be constant for some shipments. Hill [16] studied a problem where there is no pre-defined assumption on the shipment delivery policy. He showed that the optimal shipment delivery policy is geometric-then-equal-shipments policy. Goyal [11] presented a generalized model in which the lot-for-lot policy is relaxed and insisted that vendor's economic production quantity must be an integral multiple of buyer's order quantity. Later, several researchers [6,13,17,26,32,34] have developed integrated production-distribution inventory models by extending the idea of Goyal [11] and incorporating various realistic assumptions. But it is not always suitable in a realistic environment.

Keywords. Integrated inventory model, freight cost, geometric shipment policy, setup cost reduction, lead time.

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The controllable lead time becomes a prominent issue and its control leads to many benefits. The integrated inventory management system is a common practice in the global markets and provides economic advantages for both the vendor and the buyer. In recent years, most integrated inventory management systems have focused on the integration between vendor and buyer. Once they form a strategic alliance in order to minimize their own cost or maximize their own profit, then trading parties can collaborate and share information to achieve improved benefits. Therefore, several authors [1, 3–5, 17, 25, 36–39] have presented the integrated inventory management system. Porteus [27] developed an investing in setups in the EOQ model.

In lieu of multiple suppliers making independent decisions about carrier specification and modes of transportation, manufacturers should seize the opportunity to reduce overall freight costs by concentrating shipments among as few transportation carriers as possible and by specifying these carriers include lowered cost of partial shipments, more frequent delivery and shortened lead times. In previous studies, several scholars have taken the freight cost problem into account and proposed various deterministic JELS to gain more insight into the relationship between freight cost and production-inventory model. Ertogral *et al.* [9] were the early scholars who proposed production-inventory model in which the freight cost was explicitly considered in the model. They considered all-unit discount for freight cost structures with and without over declaration. Toptal [33] studied the replenishment decisions under a general cost structure which includes freight costs and all-unit quantity discounts.

Mendoza and Ventura [22] presented an algorithm based on a grossly simplified freight rate structure (using either a constant charge per truckload (TL) or a constant cost per unit for less-than-truckload (LTL) shipments). Their algorithm also included two types of purchase quantity discounts (all-units and incremental). Jauhari *et al.* [18] presented an integrated inventory model for single-vendor single-buyer system with freight rate discount and stochastic demand. Darma Wangsa and Wee [8] considered an integrated vendor–buyer inventory model with transportation cost.

The study of integrated inventory models in a single-vendor and a single-buyer can be divided into two broad groups: the full cost model and the service level approach model [19]. In the full cost model, the objective is to find the optimal inventory policy which minimizes the joint total relevant cost of the vendor–buyer integrated system including the cost of shortages. The service level approach introduces a Service Level Constraint (SLC) in place of the shortage cost which implies that the stock-out level per cycle is bounded and the availability of stock in a probabilistic or expected sense. Specifying a service level avoids the difficult practical issue of explicitly estimating the shortage cost. The Service Level Constraint (SLC) inventory problem has been studied in the literatures under single echelon (*e.g.* [2, 7, 23, 24]) with less attention being given to multi-echelon. Priyan and Uthayakumar [29] proposed permissible delay in payments in the two-echelon inventory system with controllable setup cost and lead time under service level constraint. In the above discussion, we look a detailed literature survey about integrated approach, inventory model of multi-item, inventory control under some investments and constraints, probabilistic environment. There are many literature available in the integrated inventory model for geometric shipment policy under the stochastic environment. Among the stochastic situations there is no literature discussed about the geometric shipment policy in an integrated supply chain with setup cost reduction and freight cost using service level constraint. The comparison of present stochastic model with some other existing literatures are tabulated in Table 14. So that in this paper our aim is to develop a geometric shipment policy in an integrated inventory model with setup cost reduction and freight cost using service level constraint. The main contribution of this paper is to minimize the total cost simultaneously by optimizing the order quantity, safety factor, investment and quality improvement. Therefore, this research paper intends to fill this remarkable gap in the inventory literature. There is a big research gap in this direction, which is fulfilled by this research.

2. Notations and assumptions

2.1. Notations

We need the following assumptions and notations, to develop the mathematical model of this model. The following terminology is used:

Q	Vendor's production batch size
q_1	Size of the first shipment from the vendor to the buyer
q_i	Size of the ith shipment
D	Average demand per year
P	Production rate of the buyer $P > D$
A	The ordering cost of the buyer per order
S_0	The initial setup cost of the vendor
S	Vendor's setup cost per setup
β	Geometric growth factor
h_b	The holding cost rate of the buyer per unit per unit time
h_v	The holding cost rate of the vendor per unit per unit time
θ	Probability of the vendor's production process that can go out-of-control
θ_0	Original probability of the vendor's production process that can go out-of-control
τ	Annual fractional cost of capital investment to reduce setup cost per year
s	Rework cost per unit defective item
$I(S,\theta)$	Total investment for setup cost reduction from S_0 to S and quality improvement from θ to θ_0.
n	Number of lots in which the item are delivered from the vendor to the buyer
L	Length of lead time
β^*	Fraction of the shortage that will be backordered at the buyer's end, $0 \leq \beta < 1$
π_0	Marginal profit per unit
π_x	Price discount offered on backorder by the vendor per unit, $0 \leq \pi_x \leq \pi_0$
R	Reorder point of the buyer
k	Safety factor
w	Weight of a unit part
d	Transportation distance
α	Discount factor for LTL shipments, $0 \leq \alpha < 1$
F_x	The freight rate in dollar per pound for a given per mile for full truckload (FTL)
F_y	The freight rate in dollar per pound for a given per mile for partial load
w_x	Full truckload (FTL) shipping weight
w_y	Actual shipping weight
X	The lead time demand with finite mean μL and standard deviation $\sigma\sqrt{L}$
$E(.)$	The mathematical expectation
x^+	The maximum value of x and 0, $x^+ = \max\{x,0\}$

2.2. Assumptions

The fundamental assumptions used in developing the model are as follows:

(1) The system deals with a single-vendor and a single-buyer.

(2) Replenishments are made when the on hand inventory reaches the reorder point R. (the inventory is reviewed continuously)

(3) Shortages are allowed at the buyer's point and they are partially backordered.

(4) The reorder point R equals the sum of the expected demand during lead time and safety stock (SS) and SS = $k\times$ standard deviation of lead time demand. *i.e.*, $R = \mu L + k\sigma\sqrt{L}$ where k is the safety factor. (see [15])

(5) The buyer orders a lot size of Q units and the vendor produces them with a finite production rate P ($P > D$) in units per unit time in one setup. Produced items are supplied to the buyer in n unequal sized shipments.

(6) If a shortened lead time is requested then the extra costs incurred by the vendor will be fully transferred to the buyer. Therefore, lead time crash cost is the buyer's cost component.
(7) For all products, the lead time L consists of n mutually independent components. The ith component has a normal duration b_i, minimum duration a_i, and crashing cost per unit time c_i such that $c_1 \leq c_2 \leq \ldots \leq c_n$. The components of lead time are crashed one at a time starting from the first component because it has the minimum unit crashing cost, and then the second component, and so on. Let $L_0 = \sum_{i=1}^{m} b_i$, and L_i be the length of the lead time with components $1, 2, \ldots, i$ crashed to their minimum duration, then L_i can be expressed as $L_i = L_0 - \sum_{j=1}^{i}(b_j - a_j)$, $i = 1, 2, \ldots, m$; and for all products, the lead time crashing cost per cycle $C(L)$ is given by $C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j)$, $L \in [L_i, L_{i-1}]$. (see [14])
(8) The buyer incurs all the cost for the transportation and items are purchased under the Free-On-Board (FOB).

3. Model development

3.1. Model definition

Production and delivery are structured as follows.

A single- vendor single-buyer system incorporating freight forwarding is considered in this study. The buyer sells items to the end-customers whose demand follows a normal distribution with a mean of D and standard deviation of σ. The buyer orders a lot size Q which is allocated to the vendor and the vendor produces Q units with a finite production rate ($P > D$). Consider a production batch of size Q which is made up of n unequal-sized shipments which are delivered to the buyer and the size of the ith shipment within a batch is $q_i = \beta q_{i-1} = \beta^{i-1} q_1, i = 2, 3, \ldots, n$ and represents the lot size of the first shipment. $\beta = P/D$ gives the policy described by Goyal [12]. The stock positions associated with this policy are illustrated in Figure 1. The intuitive attraction of this policy is that the time to consume a shipment exactly balances the time to manufacture the next shipment within that batch. Therefore, the total batch production lot size (the sum of the n shipments) is,

$$Q = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} \beta^{i-1} q_1 = \frac{q_1(\beta^n - 1)}{(\beta - 1)}$$

and the total cycle length of the inventory over finite horizon is (see [35])

$$T = \frac{\sum_{i=1}^{n} q_i}{D} = \frac{Q}{D} = \frac{q_1(\beta^n - 1)}{D(\beta - 1)}.$$

The vendor's transportation cost for delivering to the freight forwarding is F_0. When the first Q units have been produced, the vendor delivers to the freight forwarding who will make consolidation and delivery to the buyer with freight rate (F_x). The freight rates are functions of shipping weight (W_x), distance (d) and transportation modes. Rather than a discrete investment is used by the vendor to reduced the setup cost. The objective is to find the optimal inventory policy which minimizes the joint total relevant cost of the vendor–buyer integrated system including the cost of shortages.

3.2. Buyer's perspective

It is assumed that when the buyer's stock achieves the reorder level r. An order of length q_i is placed at the ith shipment. The vendor deliver these items after a constant lead time L in n unequal-sized shipments. The buyer receives a whole lot of q_i units in each shipments which will be applied for the duration of time period $\frac{q_i}{D}(\beta^{i-1} q_1)$.

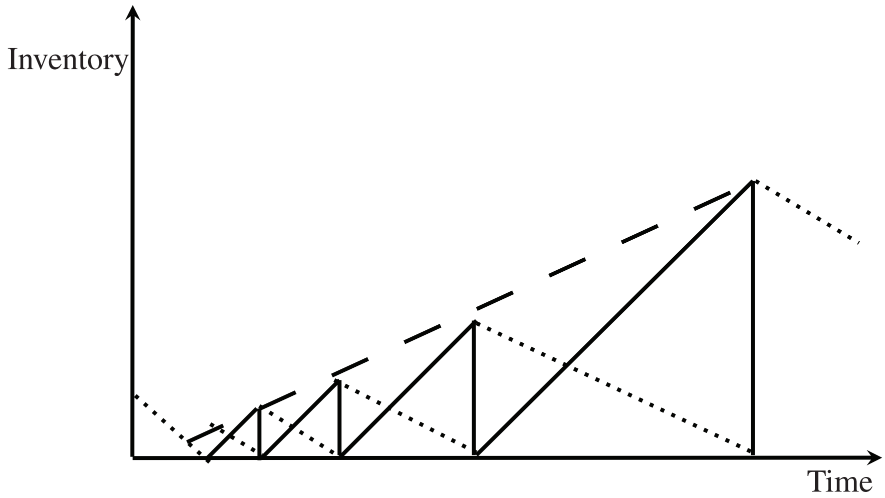

FIGURE 1. Graph of stock against time for the case $\beta = P/D$.

The average buyer stock level at some stage in the time period $\frac{q_i}{D}$ is $\frac{1}{2}\beta^{i-1}q_1$. The time-weighted stock-holding for the buyer throughout a complete production cycle is,

$$\sum_{i=1}^{n} \frac{q_i^2}{2D} = \sum_{i=1}^{n} \frac{(\beta^{i-1}q_1)^2}{2D} = \frac{q_1^2(\beta^{2n}-1)}{2D(\beta^2-1)}.$$

Therefore, the average buyer's inventory is the buyer's time-weighted stock holding throughout a cycle divided by the cycle length of the inventory as,

$$\frac{q_1^2(\beta^{2n}-1)}{2D(\beta^2-1)} / \frac{q_1(\beta^n-1)}{D(\beta-1)} = \frac{q_1(\beta^n+1)}{2(\beta+1)}.$$

For each shipment, the buyer sustain an ordering cost A and so the buyer's ordering cost per cycle is nA. Here, the lead time demand X follows a normal distribution with mean DL and the standard deviation $\sigma\sqrt{L}$. We assume that X has a cumulative distribution function F and reorder point $R = DL + k\sigma\sqrt{L}$, where k is the safety factor. If $X > R$, then shortages occur. Hence, the expected shortage at the end of the cycle is,

$$E(X-R)^+ = \int_R^\infty (X-R)\mathrm{d}F(x) = \sigma\sqrt{L}\Psi(k)$$

where $\Psi(k) = \phi(k) - k[1-\Phi(k)] > 0$ and ϕ, Φ are the standard normal probability density function and cumulative distribution function, respectively.

The expected number of back orders per cycle is $\beta^* E(X-R)^+$ and the expected number of lost sales is $(1-\beta^*)E(X-R)^+$ and the expected stock out cost per unit time is,

$$\frac{nD(\beta-1)}{q_1(\beta^n-1)}[\pi_x\beta^* + \pi_0(1-\beta^*)]E(X-R)^+.$$

Therefore, the expected net inventory level at the end of the ith batch cycle is $[R - DL + (1-\beta^*)E(X-R)^+]$. The buyer's expected holding cost per unit time is,

$$h_b\left[\frac{q_1(\beta^n+1)}{2(\beta+1)} + R - DL + (1-\beta^*)E(X-R)^+\right].$$

Darma Wangsa and Wee [8] determined the freight rate for partial load F_y primarily based adjusted inverse function given as;

$$F_y = F_x + \alpha F_x \left(\frac{w_x - w_y}{w_y}\right), 0 < \alpha \leq 1.$$

where α indicated as a discount factor for LTL shipments and the anticipated total freight cost per year which is the characteristic of shipping weight and distance with adapted inverse yields [21] is stated as,

$$F(D, q_1, w, d) = \frac{nD(\beta - 1)}{q_1(\beta^n - 1)}\alpha F_x w_x d + Ddw(1-\alpha)F_x.$$

The total cost per unit time of buyer can be computed by aggregating as the ordering cost, holding cost, stock out cost and the lead time crashing cost which is given by,

$$\begin{aligned} \text{EATC}_b(q_1, k, L, n) = & \left[A + (\pi_x\beta^* + \pi_0(1-\beta^*))\sigma\sqrt{L}\psi(k) + R(L)\right]\frac{nD(\beta-1)}{q_1(\beta^n - 1)} \\ & + h_b\left[\frac{q_1(\beta^n + 1)}{2(\beta+1)} + k\sigma\sqrt{L} + (1-\beta^*)\sigma\sqrt{L}\psi(k)\right] \\ & + \frac{nD(\beta-1)}{q_1(\beta^n - 1)}\alpha F_x w_x d + Ddw(1-\alpha)F_x. \end{aligned} \quad (3.1)$$

3.3. Vendor's perspective

The buyer order a lot size of Q units and the vendor produces them at the rate of $P > D$ and supplies them in n unequal-sized shipments. When the production process is about to start, the vendor's inventory level is zero and there are $\frac{Dq_1}{P}$ units in the buyer's inventory which is just first shipment arrives. Then the vendor's total inventory level increases at a rate of $P - D$ units and it reaches the maximum height of, $\frac{Dq_1}{P} + (P-D)\frac{q_1(\beta^n - 1)}{P(\beta-1)}$.

Therefore, the total average inventory for the vendor is given by,

$$\frac{Dq_1}{P} + \frac{q_1(P-D)(\beta^n - 1)}{2P(\beta - 1)}$$

and the vendor's average inventory = vendor's average total inventory − buyer's average inventory.

$$\frac{Dq_1}{P} + \frac{q_1(P-D)(\beta^n - 1)}{2P(\beta - 1)} - \frac{q_1(\beta^n + 1)}{2(\beta + 1)}.$$

Therefore, the vendor's expected holding cost per unit is,

$$h_v\left[\frac{Dq_1}{P} + \frac{q_1(P-D)(\beta^n - 1)}{2P(\beta - 1)} - \frac{q_1(\beta^n + 1)}{2(\beta + 1)}\right].$$

Further, the vendor sustain a setup cost S and the expected setup cost per unit time is $\frac{SD(\beta-1)}{q_1(\beta^n - 1)}$.

In this model, we consider two investments which is to reduce the total supply chain cost to make more profitable. First, we assumes that the relationship between setup cost reduction and capital investment. The capital investment $I_s(S)$ in reducing setup cost reduction is a logarithmic function of a setup cost S. That is, we consider to reduce the setup cost from the original level S_0 to a target level S and $I_s(S)$ is the one-time investment cost whose benefits will extend indefinitely into the future and it can be stated as,

$$I_S(S) = B \ln\left(\frac{S_0}{S}\right), \quad \text{for} \ \ 0 < S \leq S_0.$$

Considering an investment for quality improvement and setup cost reduction, the capital investment function is assumed as a logarithmic function as suggested by Porteus [27] $I_\theta(\theta)$, investment to reduce the "out-of-control" probability θ is given as,

$$I_\theta(\theta) = b\ln\left(\frac{\theta_0}{\theta}\right), \quad \text{for} \quad 0 < \theta \le \theta_0.$$

It is to be noted that lower value of the probability θ gives higher value of quality level, where θ_0 is the initial probability that the production process may go to "out-of-control" state. Hence, the total investment for setup cost reduction and quality improvement for the buyer is given by,

$$I(S,\theta) = I_\theta(\theta) + I_S(S)$$

where $G = b\ln(\theta_0) + B\ln(S_0)$.

Using the concept of defective items, the expected total cost for the vendor per unit time is given by,

$$\begin{aligned}\mathrm{EATC}_v(q_1,\theta,S,n) &= \frac{SD(\beta-1)}{q_1(\beta^n-1)} + h_v\left[\frac{Dq_1}{P} + \frac{q_1(P-D)(\beta^n-1)}{2P(\beta-1)} - \frac{q_1(\beta^n+1)}{2(\beta+1)}\right] \\ &\quad + \tau(G - b\ln\theta - B\ln S) + \frac{SDn\theta}{2}\left(\frac{q_1(\beta^n-1)}{(\beta-1)}\right) \end{aligned} \tag{3.2}$$

for $0 < \theta \le \theta_0$ and $0 < S \le S_0$ where τ is the annual fractional cost of capital investments (*e.g.*, interest rate).

4. Joint optimization

Considering vendor and buyer cooperation to each other, we framed the expected joint total cost per unit time which is given by,

$$\begin{aligned}\mathrm{JETC}(q_1,k,S,\theta,L,n) &= \mathrm{EATC}_b(q_1,k,L,n) + \mathrm{EATC}_v(q_1,\theta,S,n) \\ &= \left[\left(A + \frac{S}{n}\right) + (\pi_x\beta^* + (1-\beta^*)\pi_0)\sigma\sqrt{L}\psi(k) + R(L)\right] \\ &\quad \times \frac{nD(\beta-1)}{q_1(\beta^n-1)} + h_b\left[\frac{q_1(\beta^n+1)}{2(\beta+1)} + k\sigma\sqrt{L} + (1-\beta^*)E(X-R)^+\right] \\ &\quad + h_v\left[\frac{Dq_1}{P} + \frac{q_1(P-D)(\beta^n-1)}{2P(\beta-1)} - \frac{q_1(\beta^n+1)}{2(\beta+1)}\right] \\ &\quad + \tau(G - b\ln\theta - B\ln S) + \frac{SDn\theta}{2}\left(\frac{q_1(\beta^n-1)}{(\beta-1)}\right) \\ &\quad + \frac{nD(\beta-1)}{q_1(\beta^n-1)}\alpha F_x w_x d + Ddw(1-\alpha)F_x \end{aligned} \tag{4.1}$$

subject to $0 < \theta \le \theta_0$, $0 < S \le S_0$.

5. Buyer's service level constraint

To avoid the imprecision, inefficiency of estimation and the penalty costs associated with a shortage, several researchers use an SLC in their models. Service Level Constraint is a constraint on the system that defines at least an assumed proportion of demands should be met from on hand inventory. The proportion of demand which exceeds reorder point, at the end of each production cycle, should not exceed the desired value of γ. So the service-level constraint can be written as:

$$\frac{\text{The expected demand shortages at the end of cycle for a given safety factor}}{\text{Quantity available for satisfying the demand per cycle}} \le \gamma.$$

That is,

$$\frac{\sigma\sqrt{L}\Psi(k)}{Q} \leq \gamma, \qquad L \in [L_i, L_{i-1}] \tag{5.1}$$

Therefore, the complete model can be formulated as,

$$\begin{aligned}\text{Min JETC}(q_1,k,S,\theta,L,n) = & \left[\left(A+\frac{S}{n}\right) + (\pi_x\beta^* + (1-\beta^*)\pi_0)\sigma\sqrt{L}\psi(k) + R(L)\right] \\ & \times \frac{nD(\beta-1)}{q_1(\beta^n-1)} + h_b\left[\frac{q_1(\beta^n+1)}{2(\beta+1)} + k\sigma\sqrt{L} + (1-\beta^*)E(X-R)^+\right] \\ & + h_v\left[\frac{Dq_1}{P} + \frac{q_1(P-D)(\beta^n-1)}{2P(\beta-1)} - \frac{q_1(\beta^n+1)}{2(\beta+1)}\right] \\ & + \tau(G - b\ln\theta - B\ln S) + \frac{SDn\theta}{2}\left(\frac{q_1(\beta^n-1)}{(\beta-1)}\right) \\ & + \frac{nD(\beta-1)}{q_1(\beta^n-1)}\alpha F_x w_x d + Ddw(1-\alpha)F_x\end{aligned} \tag{5.2}$$

subject to $0 < \theta \leq \theta_0$, $0 < S \leq S_0$ and $\frac{\sigma\sqrt{L}\Psi(k)}{Q} \leq \gamma, \qquad L \in [L_i, L_{i-1}]$.

The problem expressed in the previous section occurs as constrained non-linear program. To solve this kind of non-linear program, we pursue the similar procedure of most of the literature dealing with non-linear program. That is, first we temporarily ignore the constraint SLC, and then we seek to define the properties which are satisfied by $\text{JETC}(q_1,k,S,\theta,L,n)$.

At first for fixed (q_1,k,S,θ,n), $\text{JETC}(q_1,k,S,\theta,L,n)$ is a strictly concave in L due to the fact that,

$$\begin{aligned}\frac{\partial}{\partial L}\text{JETC}(q_1,k,S,\theta,n) = & \left[(\pi_x\beta^* + \pi_0(1-\beta^*))\frac{\sigma\psi(k)}{2\sqrt{L}} - c_i\right]\frac{nD(\beta-1)}{(\beta^n-1)} \\ & + \frac{h_b}{2\sqrt{L}}\left[k\sigma + (1-\beta^*)\sigma\psi(k)\right] \\ \frac{\partial}{\partial L^2}\text{JETC}(q_1,k,S,\theta,n) = & -\frac{1}{2\sqrt{L}^{\frac{3}{2}}}\left[(\pi_x\beta^* + \pi_0(1-\beta^*))\sigma\psi(k)\right]\frac{nD(\beta-1)}{(\beta^n-1)} \\ & \times h_b\left[k\sigma + (1-\beta^*)\sigma\psi(k)\right] < 0.\end{aligned}$$

Therefore, for fixed (q_1,k,S,θ,n), the minimum value of $\text{JETC}(q_1,k,S,\theta,L,n)$ om L lies at the end point of the interval $[L_i, L_{i-1}]$.

Secondly, if we relax the integer constraint on n, it is possible to note that $\text{JETC}(q_1,k,S,\theta,L,n)$ is strictly convex in n, for fixed (q_1,k,S,θ,L), $L \in [L_i.L_{i-1}]$ and noticing that,

$$\begin{aligned}\frac{\partial}{\partial n}\text{JETC}(q_1,k,S,\theta,L,n) = & \left(\frac{\alpha F_x w_x dD(\beta-1)}{q_1} + \frac{YD(\beta-1)}{q_1}\right)\left[\frac{(\beta^n-1) - n\log\beta e^{n\log\beta}}{(\beta^n-1)^2}\right] \\ & - \frac{SD(\beta-1)}{q_1}\left[\frac{\log\beta e^{n\log\beta}}{(\beta^n-1)^2}\right] + (h_B - h_v)\left[\frac{q_1}{2(\beta+1)}\log\beta e^{n\log\beta}\right] \\ & + \frac{h_v q_1(P-D)\log\beta e^{n\log\beta}}{2P(\beta-1)} + \frac{q_1 SD\theta}{2(\beta-1)}\left[n\log\beta e^{n\log\beta} + (\beta^n-1)\right] \\ \frac{\partial}{\partial n^2}\text{JETC}(q_1,k,S,\theta,L,n) = & \frac{2M\log\beta\beta^n}{(\beta^n-1)^3}\{\beta^n + 1 + n\beta^n\log\beta\} + \frac{SD(\beta-1)(\log\beta)^2}{q_1(\beta^n-1)^3}(\beta^n+1) \\ & + \frac{(h_b-h_v)q_1(\log\beta)^2\beta^n}{2(\beta+1)} + \frac{h_v q_1(P-D)(\log\beta)^2\beta^n}{2P(\beta-1)} \\ & + \frac{q_1 SD\theta}{2(\beta-1)}\left[2\log\beta\beta^n + n(\log\beta)^2\beta^n\right] > 0\end{aligned} \tag{5.3}$$

where $Y = \left[\left(A+\frac{S}{n}\right) + (\pi_x\beta^* + (1-\beta^*)\pi_0)\sigma\sqrt{L}\psi(k) + R(L)\right]$ and $M = \left(\frac{\alpha F_x w_x d D(\beta-1)}{q_1} + \frac{YD(\beta-1)}{q_1}\right)$.

The equation (5.3) propose that the joint expected total cost is strictly convex in n for fixed (q_1, k, S, θ, L), $L \in [L_i, L_{i-1}]$.

Next for fixed (k, S, θ, L, n), $L \in [L_i, L_{i-1}]$ and with the aid of taking account of the equation (5.2), we may want to able to take the derivatives of JETC$(q_1, k, S, \theta, L, n)$ with respect to q_1 and to attain:

$$\begin{aligned}\frac{\partial}{\partial q_1}\mathrm{JETC}(q_1,k,S,\theta,L,n) = & -\frac{1}{q_1^2}\left[\left(A+\frac{S}{n}\right) + (\pi_x\beta^*+(1-\beta^*)\pi_0)\sigma\sqrt{L}\psi(k) + R(L)\right. \\ & \left. + \alpha F_x w_x d\right]\frac{nD(\beta-1)}{(\beta^n-1)} + h_b\left[\frac{(\beta^n+1)}{2(\beta+1)}\right] + \frac{SDn\theta}{2}\left(\frac{(\beta^n-1)}{(\beta-1)}\right) \\ & + h_v\left[\frac{D}{P} + \frac{(P-D)(\beta^n-1)}{2P(\beta-1)} - \frac{(\beta^n+1)}{2(\beta+1)}\right]. \end{aligned} \tag{5.4}$$

Solving this equation, we bear the value of q_1 as,

$$q_1 = \left[\frac{\left[\left(A+\frac{S}{n}\right) + (\pi_x\beta^*+(1-\beta^*)\pi_0)\sigma\sqrt{L}\psi(k) + R(L) + \alpha F_x w_x d\right]\frac{nD(\beta-1)}{(\beta^n-1)}}{(h_b-h_v)\left(\frac{(\beta^n+1)}{2(\beta+1)}\right) + h_v\left[\frac{D}{P} + \frac{(P-D)(\beta^n-1)}{2P(\beta-1)}\right] + \frac{SDn\theta}{2}\left(\frac{(\beta^n-1)}{(\beta-1)}\right)}\right]^{\frac{1}{2}}. \tag{5.5}$$

Likewise, for fixed n and $L \in [L_i, L_{i-1}]$, the values of the decision variables k, S and θ are obtained by equating the first order derivatives of JETC$(q_1, k, S, \theta, L, n)$ to zero.

$$\begin{aligned}\frac{\partial}{\partial k}\mathrm{JETC}(q_1,k,S,\theta,L,n) = \sigma\sqrt{L}\Bigg[&\left((\pi_x\beta+\pi_0(1-\beta))\frac{nD(\beta-1)}{q_1(\beta^n-1)} + (1-\beta)h_b\right) \\ & (\Phi(k)-1) + h_b\Bigg]\end{aligned} \tag{5.6}$$

$$\frac{\partial}{\partial S}\mathrm{JETC}(q_1,k,S,\theta,L,n) = \frac{D(\beta-1)}{nq_1(\beta^n-1)} - \frac{\tau B}{S} \tag{5.7}$$

$$\frac{\partial}{\partial \theta}\mathrm{JETC}(q_1,k,S,\theta,L,n) = \frac{SDnq_1(\beta^n-1)}{2(\beta-1)} - \frac{\tau b}{\theta}. \tag{5.8}$$

By setting the equations (5.6)–(5.8) to zero , we yield the values of k, S and θ as,

$$\Phi(k) = 1 - \frac{h_b q_1(\beta^n-1)}{(\pi_x\beta^*+\pi_0(1-\beta^*))nD(\beta-1) + (1-\beta^*)h_B q_1(\beta^n-1)} \tag{5.9}$$

$$S = \frac{\tau B q_1(\beta^n-1)}{(\beta-1)D} \tag{5.10}$$

$$\theta = \frac{2\tau b(\beta-1)}{SDnq_1(\beta^n-1)}. \tag{5.11}$$

Moreover, it can be shown that the SOSC are satisfied since the Hessian matrix is positive definite at point (q_1, k, S, θ) (see the Appendix for the proof).

Algorithm

Step 1: Set $n = 1$.
Step 2: Find Q, k, S and θ for fixed n.
Step 3: For every L_i, $i = 1, 2, \ldots, n$ perform steps (3.1)–(3.6).
3.1: Set $S_i^{(1)} = S_0$, $k_i^{(1)} = 0$, $\theta_i^{(1)} = \theta_0$. [implies $\psi(k_i)^{(1)} = 0.3989, \Phi(k_i)^{(1)} = 0.5$]

3.2: Substitute $S_i^{(1)}$, $k_i^{(1)}$ and $\theta_i^{(1)}$ in equation (5.5) and evaluate $q_{1i}^{(1)}$.
3.3: Calculate actual shipping weight $w_y = q_{1i}^{(1)}.w$. If $w_y \leq w_x$ is satisfied then go to step (3.5). Otherwise go to step (3.4), if truckload constraint is not satisfied. (*i.e.*, $w_y > w_x$)
3.4: Revised delivery lot size $q_{1i}^{(1)} = \frac{w_x}{w}$ and go to step (3.5).
3.5: Using $q_{1i}^{(1)}$ to determine the values of $k_i^{(2)}$, $\theta_i^{(2)}$ and $S_i^{(2)}$.
3.6: Repeat (3.1)–(3.4) until no change occurs in the values of q_{1i}, k_i, θ_i and $\S_i$. Denote the solutions by $(q_{1i}^*, k_i^*, \theta_i^*, S_i^*)$.

Step 4: Compare θ_i^* and S_i^* with θ_0 and S_0, respectively.
4.1: If $\theta_i^* < \theta_0$ and $S_i^* < S_0$, then the solution which can be found in step (3) is the optimal solution for the given L_i. Hence, we denote the optimal solution by $(\tilde{q}_{1i}, \tilde{k}_i, \tilde{S}_i, \tilde{\theta}_i,)$. *i.e.*, $(\tilde{q}_{1i}, \tilde{k}_i, \tilde{S}_i, \tilde{\theta}_i,) = (q_{1i}^*, k_i^*, S_i^*, \theta_i^*)$. Go to step 5.
4.2: If $\theta_i^* \geq \theta_0$ and $S_i^* < S_0$, then for this L_i, letting $\tilde{\theta}_i = \theta_0$, repeat step (3) to determine the new set (q_{1i}^*, k_i^*, S_i^*) and denote the optimal solutions by $(\tilde{q}_{1i}, \tilde{k}_i, \tilde{\theta}_i, \tilde{S}_i)$. *i.e.*, $(\tilde{q}_{1i}, \tilde{k}_i, \tilde{\theta}_i, \tilde{S}_i) = (q_{1i}^*, k_i^*, S_i^*, \theta_0)$ Go to step 5.
4.3: If $\theta_i^* < \theta_0$ and $S_i^* \geq S_0$, then for this L_i, letting $\tilde{S}_i = S_0$, repeat step (3) to determine the new set $(q_{1i}^*, k_i^*, \theta_i^*)$ and denote the optimal solutions by $(\tilde{q}_{1i}, \tilde{k}_i, \tilde{\theta}_i, \tilde{S}_i)$. *i.e.*, $(\tilde{q}_{1i}, \tilde{k}_i, \tilde{\theta}_i, \tilde{S}_i) = (q_{1i}^*, k_i^*, \theta_i^*, S_0)$ Go to step 5.
4.4: If $\theta_i^* \geq \theta_0$ and $S_i^* \geq S_0$, then for this L_i, letting $\tilde{\theta}_i = \theta_0$ and $\tilde{S}_i = S_0$ repeat step (2) to determine the new set (q_{1i}^*, k_i^*) and denote the optimal solutions by $(\tilde{q}_{1i}, \tilde{k}_i, \tilde{\theta}_i, \tilde{S}_i)$. *i.e.*, $(\tilde{q}_{1i}, \tilde{k}_i, \tilde{\theta}_i, \tilde{S}_i) = (q_{1i}^*, k_i^*, \theta_0, S_0)$ Go to step 5.

Step 5: Set q_{01i} =max$\{\tilde{q}_{1i}, (\sigma/\alpha)\sqrt{L}\psi(k)$, for every $L \in [L_i, L_{i-1}]$. Calculate the actual weight $w_y = q_{01i}.w$. If $w_y \leq w_x$ is satisfied, then go to step (6). Otherwise we can find $q_{01i} = \frac{w_x}{w}$ and go to step (6).
Step 6: By using q_{01i} to determine the values of k_{0i}, S_{0i} and θ_{0i}. The result is denoted by $(q_{01i}, k_{0i}, S_{0i}, \theta_{0i})$.
Step 7: Use equation (5.2) to evaluate the JETC$(q_{01i}, k_{0i}, S_{0i}, \theta_{0i}, L_i, n)$ for $i = 1, 2, \ldots, m$.
Step 8: Find $\min_{i=1,2,\ldots,m}$ JETC$(q_{01i}, k_{0i}, S_{0i}, \theta_{0i}, L_i, n)$ and denote it by JETC$(q_{01(n)}, k_{0(n)}, S_{0(n)}, \theta_{0(n)}, L_{(n)}, n) = \min_{i=1,2,\ldots,m}$ JETC$(q_{01i}, k_{0i}, S_{0i}, \theta_{0i}, L_i, n)$ and $(q_{01(n)}, k_{0(n)}, S_{0(n)}, \theta_{0(n)}, L_{(n)}, n)$ are the optimal solution for given n.
Step 9: Replace n by $n+1$ and repeat steps (3)–(8) to get JETC$(q_{01(n)}, k_{0(n)}, S_{0(n)}, \theta_{0(n)}, L_{(n)}, n)$.
9.1: If JETC$(q_{01(n)}, k_{0(n)}, S_{0(n)}, \theta_{0(n)}, L_{(n)}, n) \leq$ JETC$(q_{01(n-1)}, k_{0(n-1)}, S_{0(n-1)}, \theta_{0(n-1)}, L_{(n-1)}, n-1)$.
9.2: Set JETC$(q_{01}, k_0, S_0, \theta_0, L, n)$ = JETC$(q_{01(n-1)}, k_{0(n-1)}, S_{0(n-1)}, \theta_{0(n-1)}, L_{(n-1)}, n-1)$ and JETC$(q_{01}, k_0, S_0, \theta_0, L, n)$ is the minimum expected total cost and $(q_{01}, k_0, S_0, \theta_0, L, n)$ is a set of optimal solutions.

6. Numerical illustration

In order to illustrate the proposed model, we consider the flexible parameter similar to the Kim and Sarkar [20] and Darma Wangsa and Wee [8].

$D = 600$ units/year, $P = 1500$ units/year, $S_0 = \$1500$/setup, $h_b = \$25$/unit, $h_v = \$20$/unit, $\beta = 1.5$, $\sigma = 7$ units per week, $\pi_x = \$100$/unit, $\pi_0 = \$300$/unit, $\beta^* = 0.25$, $\tau = 0.5$ dollar/unit, $B = 5800$, $b = 400$, $\theta_0 = 0.0002$, $\alpha = 0.11246$, $w = 22\,lbs$/unit, $d = 600$ miles.

The lead time has three components with data shown in Table 1. The numerical example includes actual freight rate schedule and parameter data. The actual freight rate schedule is adopted from Darma Wangsa and Wee [8]. Table 2 represents the freight rate schedule as the function of shipping weight and distance.

Example 6.1. Assume that the lead time demand follows a normal distribution and the capital investment $I(S, \theta)$ for reducing the setup cost and quality improvement which is asserted in the form of logarithmic function. Letting $F_x = 0.000101343$ and $w_x = 9999\,lbs$, use the proposed algorithm to find the optimal solution of the model which is shown in Table 3.

TABLE 1. Lead time component with data.

Lead time component i	Normal duration b_i (days)	Minimum duration cost a_i (days)	Unit crashing c_i (\$/days)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

TABLE 2. Freight rate schedule.

Weight break	F_x/pound	F_x/pound/mile
1–227 lbs^a	\$40	\$0.000293685
228–420 lbs^a	\$0.176/lb	\$0.000293333
421–499 lbs^a	\$74	\$0.000247161
500–932 lbs^a	\$0.148/lb	\$0.000246666
933–999 lbs^a	\$138	\$0.000230230
1000–1855 lbs^a	\$0.138/lb	\$0.000230000
1856–1999 lbs^a	\$256	\$0.000213440
2000–4749 lbs^a	\$0.128/lb	\$0.000293333
4750–9999 lbs^a	\$0.608	\$0.000101343
10 000–18 256 lbs^a	\$0.0608/lb	\$0.000101333
18 527 lbs^a and $more^a$	\$1110	\$0.000040217

TABLE 3. The optimal solutions for descriptive Example 6.1 with capacity problem ($w_x = 9999$).

			Check actual weight			Check actual weight			With investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n=1$	8	169	3718	Satisfied	169	3718	Satisfied	1.40	254	0.00001578	5402
$n=2$	8	120	2640	Satisfied	120	2640	Satisfied	1.46	450	0.00004444	7324
$n=1$	6	117	2574	Satisfied	117	2574	Satisfied	1.55	176	0.000022792	4438
$n=2$	6	100	2200	Satisfied	100	2200	Satisfied	1.58	375	0.000053333	6782
$n=1$	4	85	1870	Satisfied	85	1870	Satisfied	1.73	128	0.000031373	4062
$n=2$	4	66	1452	Satisfied	66	1452	Satisfied	1.76	248	0.000080808	6044
$n=1$	3	76	1672	Satisfied	76	1672	Satisfied	1.78	114	0.000035088	4029
$n=2$	3	48	1056	Satisfied	48	1056	Satisfied	1.89	180	0.000081110	4577
			Check actual weight			Check actual weight			Without investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n=1$	8	396	8712	Satisfied	396	8712	Satisfied	0.94	1500	0.0002	8203
$n=2$	8	203	4466	Satisfied	203	4466	Satisfied	1.19	1500	0.0002	8934
$n=1$	6	279	6138	Satisfied	279	6138	Satisfied	1.14	1500	0.0002	6680
$n=2$	6	161	3542	Satisfied	161	3452	Satisfied	1.31	1500	0.0002	7624
$n=1$	4	186	4092	Satisfied	186	4092	Satisfied	1.36	1500	0.0002	5776
$n=2$	4	107	2354	Satisfied	107	2354	Satisfied	1.52	1500	0.0002	6203
$n=1$	3	155	3410	Satisfied	155	3410	Satisfied	1.45	1500	0.0002	5650
$n=2$	3	76	1672	Satisfied	76	1672	Satisfied	1.68	1500	0.0002	5786

TABLE 4. The study and differentiation of LTL and TL with incapacity problem ($w_x = 9999$).

	Unit	Actual decision (LTL)	Optimal decision (TL)
Data			
Weight capacity	*lbs*	1856–1999	4750–9999
Freight rate	$	256	608
Freight rate/lb^a	$/*lb*	0.1313	0.0608
Analysis			
Actual weight	*lbs*	1672	1672
Effect of freight cost	$	219.53	101.66
Effect of total transportation cost	$	1722.06	1258.12

Notes. Freight rate/*lb* = freight rate/weight capacity (weight capacity LTL = 1950 and TL = 9999). Effect of freight cost = freight rate/*lb* × actual weight. Effect of total transportation cost = formula of freight cost at this article.

To analyze the effects of setup cost reduction and quality improvement, optimal results of the no-investment and no quality improvement policy are provided in the Table 3. From this table, it may be clear that when no attempt is made to reduce the setup cost and if quality improvement is not allowed then the JETC is $5650. When $L = 3$ weeks and the ordered units could be supplied in first shipment with an initial lot size of 155 units. But if an additional cost $I(S, \theta)$ is spent, then the JETC is reduced to $4029 value when $L = 3$ weeks and the setup cost and quality improvement reduces to $S = \$114$ and $\theta = \$0.000035088$ which is shown in the same table. This expose that the system could earn an additional savings of $1621 which is cost savings of 28.69%.

The derived actual shipping weight of the buyer is $1672 *lbs*. The freight rate F_x is $608 or $0.0608/*lb* when TL is used ($w_x = 9999$). The outcome of freight rate to freight cost is based on its actual shipping weight with $101.66/shipment. Therefore, the total transportation cost is $1258.12/year. The freight rate is $256 or $0.1313/*lb* and the freight cost is based on actual shipping weight with $219.53/shipment when LTL is used. Thus, the transportation cost is $1722.06/year. The difference between TL and LTL is shown in Table 4.

Example 6.2. Consider the same data as in Example 6.1. Suppose that the freight rate in dollar for full truckload is $F_x = 0.000213440$ and the full truckload shipping weight $w_x = 1950$ and by applying a computational algorithm, the optimal solutions are expressed in Table 5.

To analyze the effects of setup cost reduction and quality improvement, optimal results of the no-investment and no quality improvement policy are provided in the Table 5. From this table, it may be clear that when no attempt is made to reduce the setup cost and if quality improvement is not allowed then the JETC is $5990. When $L = 3$ weeks and the ordered units could be supplied in first shipment with an initial lot size of 154 units. But if an additional cost $I(S, \theta)$ is spent, then the JETC is reduced to $4185 value when $L = 3$ weeks and the setup cost and quality improvement reduces to $S = \$107$ and $\theta = 0.00003756$ which is shown in the same table. This can expose that the system could earn an additional savings of $1805 Which is cost savings of 30.13% and the derived actual shipping of the buyer is $1562 *lbs*.

Example 6.3. Consider the same data as in Example 6.1. Suppose that the freight rate in dollar for full truckload is $F_x = 0.000213333$ and the full truckload shipping weight $w_x = 2500$ and by applying a computational algorithm, the optimal solutions are expressed in Table 6.

Similarly, to analyze the effects of setup cost reduction and quality improvement, optimal results of the no-investment and no quality improvement policy are provided in the Table 6. From this table, it may clear that when no attempt is made to reduce the setup cost and if quality improvement is not allowed then the JETC is $5573. When $L = 3$ weeks and the ordered units could be supplied in first shipment with an initial lot size of 154 units. But if an additional cost $I(S, \theta)$ is spent, then the JETC is reduced to $4197 value when $L = 3$ weeks and the setup cost and quality improvement reduces to $S = \$108$ and $\theta = 0.00003704$ which is shown in the

TABLE 5. The optimal solutions for descriptive Example 6.2 with capacity problem ($w_x = 1950$).

			Check actual weight			Check actual weight			With investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n=1$	8	157	3454	Not satisfied	88.6364	1950	Satisfied	1.71	133	0.000030085	4513
$n=2$	8	126	2774	Not satisfied	88.6364	1950	Satisfied	1.61	332	0.000006017	5536
$n=1$	6	313	6886	Not satisfied	88.6364	1950	Satisfied	1.71	133	0.000030085	4382
$n=2$	6	94	2068	Not satisfied	88.6364	1950	Satisfied	1.61	332	0.000006017	5410
$n=1$	4	108	2376	Not satisfied	88.6364	1950	Satisfied	1.71	133	0.000030085	4382
$n=2$	4	63	1386	Satisfied	63	1386	Satisfied	1.77	236	0.000008466	4843
$n=1$	3	71	1562	Satisfied	71	1562	Satisfied	1.81	107	0.00003756	4180
$n=2$	3	50	1100	Satisfied	50	1100	Satisfied	1.87	188	0.00001067	4715
			Check actual weight			Check actual weight			Without investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n=1$	8	379	8338	Not satisfied	88.6364	1950	Satisfied	1.71	1500	0.0002	6362
$n=2$	8	201	4620	Not satisfied	88.6364	1950	Satisfied	1.61	1500	0.0002	6885
$n=1$	6	290	6380	Not satisfied	88.6364	1950	Satisfied	1.71	1500	0.0002	6236
$n=2$	6	151	3322	Not satisfied	88.6364	1950	Satisfied	1.61	1500	0.0002	6755
$n=1$	4	194	4268	Not satisfied	88.6364	1950	Satisfied	1.71	1500	0.0002	6105
$n=2$	4	107	2354	Satisfied	107	2354	Satisfied	1.70	1500	0.0002	6623
$n=1$	3	154	3388	Satisfied	154	3388	Satisfied	1.71	1500	0.0002	5990
$n=2$	3	80	1760	Satisfied	80	1760	Satisfied	1.71	1500	0.0002	6601

TABLE 6. The optimal solutions for descriptive Example 6.3 with capacity problem ($w_x = 2500$).

			Check actual weight			Check actual weight			With investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n=1$	8	157	3454	Not satisfied	114	2500	Satisfied	1.60	171	0.00002347	4691
$n=2$	8	129	2838	Not satisfied	114	2500	Satisfied	1.49	426	0.000004693	6062
$n=1$	6	118	2596	Not satisfied	114	2500	Satisfied	1.60	171	0.000023467	4566
$n=2$	6	97	2134	Satisfied	97	2134	Satisfied	1.57	364	0.0000054983	5410
$n=1$	4	75	1650	Satisfied	75	1650	Satisfied	1.79	113	0.00003556	4210
$n=2$	4	65	1430	Satisfied	65	1430	Satisfied	1.75	244	0.00000205	4897
$n=1$	3	72	1584	Satisfied	72	1584	Satisfied	1.81	108	0.00003704	4197
$n=2$	3	46	1012	Satisfied	46	1012	Satisfied	1.91	173	0.00001159	4726
			Check actual weight			Check actual weight			Without investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n=1$	8	379	8338	Not satisfied	114	2500	Satisfied	1.60	1500	0.0002	6381
$n=2$	8	201	4422	Not satisfied	114	2500	Satisfied	1.49	1500	0.0002	6803
$n=1$	6	284	6248	Not satisfied	114	2500	Satisfied	1.60	1500	0.0002	6256
$n=2$	6	151	3322	Not satisfied	114	2500	Satisfied	1.49	1500	0.0002	6681
$n=1$	4	55	3410	Not satisfied	114	2500	Satisfied	1.60	1500	0.0002	6125
$n=2$	4	101	2222	Satisfied	101	3410	Satisfied	1.55	1500	0.0002	6304
$n=1$	3	154	3388	Not satisfied	114	2500	Satisfied	1.60	1500	0.0002	5573
$n=2$	3	80	1760	Satisfied	80	1760	Satisfied	1.66	1500	0.0002	5999

Table 7. The change in the value of JETC when the reduction parameters are represented by the values as $\beta = 2.00$ for $w_x = (9999)$.

			Check actual weight			Check actual weight			With investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n = 1$	8	169	3718	Satisfied	169	3718	Satisfied	1.40	254	0.00001578	5019
$n = 2$	8	161	5019	Satisfied	161	5019	Satisfied	1.22	725	0.00000276	7600
$n = 1$	6	117	2574	Satisfied	117	2574	Satisfied	1.58	176	0.000022792	4396
$n = 2$	6	93	2046	Satisfied	93	2200	Satisfied	1.50	419	0.000004779	5628
$n = 1$	4	85	1870	Satisfied	85	1870	Satisfied	1.73	128	0.000031373	4082
$n = 2$	4	63	1386	Satisfied	63	1386	Satisfied	1.68	248	0.000007055	4834
$n = 1$	3	76	1672	Satisfied	76	1672	Satisfied	1.78	114	0.000035088	4039
$n = 2$	3	48	1056	Satisfied	48	1056	Satisfied	1.81	216	0.000009259	4591
			Check actual weight			Check actual weight			Without investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n = 1$	8	396	8712	satisfied	396	8712	Satisfied	0.94	1500	0.0002	8203
$n = 2$	8	203	4466	satisfied	203	4466	Satisfied	1.09	1500	0.0002	10061
$n = 1$	6	279	6138	satisfied	279	6138	Satisfied	1.14	1500	0.0002	6580
$n = 2$	6	152	3344	satisfied	152	3344	Satisfied	1.26	1500	0.0002	8138
$n = 1$	4	190	4180	satisfied	190	4180	Satisfied	1.34	1500	0.0002	5804
$n = 2$	4	101	2222	Satisfied	101	2222	Satisfied	1.46	1500	0.0002	6440
$n = 1$	3	155	3410	satisfied	155	3410	Satisfied	1.45	1500	0.0002	5650
$n = 2$	3	79	1738	Satisfied	79	1738	Satisfied	1.58	1500	0.0002	5917

same table. This can expose that the system could earn an additional savings of \$1376 which is cost of savings of 24.69% and the derived actual shipping of the buyer is \$1584.

7. Sensitivity analysis

The sensitivity analysis has been conducted based on the variation of the following parameters:

(i) Geometric growth factor.
(ii) Price discount offered on backorder by the vendor per unit.
(iii) Transportation distance (miles).

Based on our numerical results, we accomplish the following managerial circumstances:

(1) From Tables 7–9, it shows that the setup cost (S) and joint expected total cost (JETC) decreases when the geometric growth factor (β) increases without affecting the lead time L. This result is expected because this fact occurs in the real life. Since, if the geometric growth function increases then naturally the total cost function decreases.
(2) From Tables 10 and 11, it is amusing to observe that if there is an increase in price discount then there is a gradual decrease in the joint expected total cost (JETC). This result is expected because this situation arises in a real life.
(3) From Tables 12 and 13, it shows that if there is an increase in transportation distance (d) then there is a moderate decrease in the joint expected total cost (JETC), setup cost (S) and safety factor (k) increases.

Table 8. The change in the value of JETC when the reduction parameters are represented by the values as $\beta = 2.50$ for $w_x = (9999)$.

			Check actual weight			Check actual weight			With investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n=1$	8	185	4070	Satisfied	185	4070	Satisfied	1.36	278	0.000014414	5187
$n=2$	8	133	2926	Satisfied	133	2926	Satisfied	1.24	698	0.00000286	7386
$n=1$	6	117	2574	Satisfied	117	2574	Satisfied	1.58	176	0.000022792	4456
$n=2$	6	99	2178	Satisfied	99	2178	Satisfied	1.39	520	0.00003848	6182
$n=1$	4	85	1870	Satisfied	85	1870	Satisfied	1.73	128	0.000031373	4062
$n=2$	4	67	1474	Satisfied	67	1474	Satisfied	1.58	352	0.00000568	5124
$n=1$	3	76	1672	Satisfied	76	1672	Satisfied	1.78	114	0.000035088	4049
$n=2$	3	49	1078	Satisfied	49	1078	Satisfied	1.73	257	0.00000777	4689
			Check actual weight			Check actual weight			Without investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n=1$	8	396	8712	satisfied	396	8712	Satisfied	0.94	1500	0.0002	8203
$n=2$	8	215	4730	satisfied	215	4730	Satisfied	0.97	1500	0.0002	11778
$n=1$	6	279	6138	satisfied	279	6138	Satisfied	1.14	1500	0.0002	6680
$n=2$	6	161	3542	satisfied	161	3542	Satisfied	1.13	1500	0.0002	9350
$n=1$	4	190	4180	satisfied	190	4180	Satisfied	1.34	1500	0.0002	5804
$n=2$	4	108	2276	Satisfied	108	2276	Satisfied	1.35	1500	0.0002	7125
$n=1$	3	154	3385	satisfied	154	3385	Satisfied	1.45	1500	0.0002	5680
$n=2$	3	78	1716	Satisfied	78	1716	Satisfied	1.51	1500	0.0002	6077

Table 9. The change in the value of JETC when the reduction parameters are represented by the values as $\beta = 3.00$ for $w_x = (9999)$.

			Check actual weight			Check actual weight			With investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n=1$	8	169	3718	Satisfied	169	3718	Satisfied	1.40	254	0.000015779	5019
$n=2$	8	130	2860	Satisfied	130	2860	Satisfied	1.18	780	0.000002564	7874
$n=1$	6	127	2794	Satisfied	127	2794	Satisfied	1.55	191	0.00002099	4469
$n=2$	6	98	2156	Satisfied	98	2156	Satisfied	1.33	588	0.000003401	6566
$n=1$	4	85	1870	Satisfied	85	1870	Satisfied	1.73	128	0.000031373	4062
$n=2$	4	65	1430	Satisfied	65	1430	Satisfied	1.53	390	0.000005128	5306
$n=1$	3	76	1672	Satisfied	76	1672	Satisfied	1.78	114	0.000035088	4029
$n=2$	3	53	1166	Satisfied	53	1166	Satisfied	1.63	318	0.000006289	4916
			Check actual weight			Check actual weight			Without investment		
Iterations	L	q_1	w_y	Conclusion	q_1^* revised	w_y	Conclusion	k	S	θ	JETC
$n=1$	8	396	8712	satisfied	396	8712	Satisfied	0.94	1500	0.0002	8203
$n=2$	8	203	4466	satisfied	203	4466	Satisfied	0.89	1500	0.0002	12506
$n=1$	6	152	3344	satisfied	152	3344	Satisfied	1.09	1500	0.0002	6849
$n=2$	6	279	6138	satisfied	279	6138	Satisfied	1.14	1500	0.0002	6680
$n=1$	4	190	4180	satisfied	190	4180	Satisfied	1.34	1500	0.0002	5804
$n=2$	4	101	2222	Satisfied	101	2222	Satisfied	1.31	1500	0.0002	7378
$n=1$	3	155	3410	satisfied	155	3410	Satisfied	1.44	1500	0.0002	5650
$n=2$	3	77	1694	Satisfied	77	1694	Satisfied	1.45	1500	0.0002	6392

TABLE 10. Sensitivity of Price discount $\pi_x = 110$ when $\beta = 1.5$, $F_x = 0.000101343$, $w_x = 9999$.

Iterations	L	q_1	Check actual weight w_y	Conclusion	q_1^* revised	Check actual weight w_y	Conclusion	k	With investment S	θ	JETC
$n=1$	8	175	3850	Satisfied	175	3850	Satisfied	1.39	263	0.00001524	5083
$n=1$	6	123	2706	Satisfied	123	2706	Satisfied	1.57	185	0.00002168	4436
$n=1$	4	88	1936	Satisfied	88	1936	Satisfied	1.72	132	0.00003030	4074
$n=1$	3	75	1650	Satisfied	75	1650	Satisfied	1.79	113	0.00003556	4031
Iterations	L	q_1	Check actual weight w_y	Conclusion	q_1^* revised	Check actual weight w_y	Conclusion	k	Without investment S	θ	JETC
$n=1$	8	376	8272	satisfied	376	8272	Satisfied	0.98	1500	0.0002	7942
$n=1$	6	287	6314	satisfied	287	6314	Satisfied	1.13	1500	0.0002	6769
$n=1$	4	191	4202	satisfied	191	4202	Satisfied	1.35	1500	0.0002	5804
$n=1$	3	155	3410	satisfied	155	3410	Satisfied	1.45	1500	0.0002	5655

TABLE 11. Sensitivity of Price discount $\pi_x = 90$ when $\beta = 1.5$, $F_x = 0.000101343$, $w_x = 9999$.

Iterations	L	q_1	Check actual weight w_y	Conclusion	q_1^* revised	Check actual weight w_y	Conclusion	k	With investment S	θ	JETC
$n=1$	8	171	3762	Satisfied	171	3762	Satisfied	1.39	257	0.00001524	5083
$n=1$	6	128	2816	Satisfied	128	2816	Satisfied	1.54	192	0.00002083	4477
$n=1$	4	86	1892	Satisfied	86	1892	Satisfied	1.72	129	0.00003101	4065
$n=1$	3	75	1650	Satisfied	75	1650	Satisfied	1.79	113	0.00003556	4039
Iterations	L	q_1	Check actual weight w_y	Conclusion	q_1^* revised	Check actual weight w_y	Conclusion	k	Without investment S	θ	JETC
$n=1$	8	385	8470	satisfied	385	8470	Satisfied	0.95	1500	0.0002	8057
$n=1$	6	295	6490	satisfied	295	6490	Satisfied	1.10	1500	0.0002	6857
$n=1$	4	195	4290	satisfied	195	4290	Satisfied	1.33	1500	0.0002	5823
$n=1$	3	155	3410	satisfied	155	3410	Satisfied	1.44	1500	0.0002	5654

TABLE 12. Sensitivity of Transportation distance $d = 660$ when $\beta = 1.5$, $F_x = 0.000101343$, $w_x = 9999$.

Iterations	L	q_1	Check actual weight w_y	Conclusion	q_1^* revised	Check actual weight w_y	Conclusion	k	With investment S	θ	JETC
$n=1$	8	179	3938	Satisfied	179	3938	Satisfied	1.37	269	0.00001490	5160
$n=1$	6	127	2794	Satisfied	127	2794	Satisfied	1.55	191	0.00002099	4503
$n=1$	4	76	1672	Satisfied	76	1672	Satisfied	1.68	285	0.00000701	4909
$n=1$	3	73	1606	Satisfied	73	1606	Satisfied	1.80	110	0.00003653	4261
Iterations	L	q_1	Check actual weight w_y	Conclusion	q_1^* revised	Check actual weight w_y	Conclusion	k	Without investment S	θ	JETC
$n=1$	8	396	8712	satisfied	396	8712	Satisfied	0.94	1500	0.0002	8230
$n=1$	6	279	6138	satisfied	279	6138	Satisfied	1.14	1500	0.0002	6709
$n=1$	4	186	4092	satisfied	186	4092	Satisfied	1.36	1500	0.0002	5807
$n=1$	3	156	3432	satisfied	156	3432	Satisfied	1.44	1500	0.0002	5689

TABLE 13. Sensitivity of Transportation distance $d = 540$ when $\beta = 1.5$, $F_x = 0.000101343$, $w_x = 9999$.

Iterations	L	q_1	Check actual weight w_y	Conclusion	q_1^* revised	Check actual weight w_y	Conclusion	k	With investment S	θ	JETC
$n = 1$	8	167	3674	Satisfied	167	3674	Satisfied	1.14	251	0.00001597	4960
$n = 1$	6	125	2750	Satisfied	125	2750	Satisfied	1.55	188	0.00002133	4426
$n = 1$	4	83	1826	Satisfied	83	1826	Satisfied	1.74	125	0.00003213	4018
$n = 1$	3	75	1650	Satisfied	75	1650	Satisfied	1.79	113	0.00003556	3982
Iterations	L	q_1	Check actual weight w_y	Conclusion	q_1^* revised	Check actual weight w_y	Conclusion	k	Without investment S	θ	JETC
$n = 1$	8	373	8206	satisfied	373	8206	Satisfied	0.97	1500	0.0002	7880
$n = 1$	6	279	6138	satisfied	279	6138	Satisfied	1.14	1500	0.0002	6652
$n = 1$	4	190	4180	satisfied	190	4180	Satisfied	1.34	1500	0.0002	5773
$n = 1$	3	155	3410	satisfied	155	3410	Satisfied	1.45	1500	0.0002	5618

TABLE 14. A comparison of the present model with related existing models.

Model Model	Variable lead time	Investment	Unequal shipments	Quality improvement	Constraints	Freight cost
Porteus [28]		✓		✓		
Giri & sharma [10]			✓			
Moon & choi [23]	✓					
Sarkar [30]	✓	✓		✓		
Sarkar *et al.* [31]		✓		✓		
Hemapriya [15]	✓	✓			✓	
Rameswari [35]	✓		✓		✓	
Kim and Sarkar [20]	✓	✓		✓	✓	
Hoque [17]	✓		✓	✓		
Darma Wangsa and Wee [8]						✓
Present paper	✓	✓	✓	✓	✓	✓

8. Conclusion

In this paper, we presented a neoteric approach to geometric shipment policy and concerning the impact of lead time reduction, setup cost reduction, lost sales and freight cost which is a function of shipping weight, distance and transportation modes within an integrated vendor–buyer supply chain system using service-level constraint. We then developed an exact algorithm that authorizes the optimization of ordering quantity, safety stock and lead time. Numerical application conferred that this optimization approach achieves a high level of efficiency, which may offer promising application in practice. One of the reverberation of this work is that if the setup cost per setup could be reduced effectively, then the joint expected total cost per unit time could be automatically minimized. The results of the sensitivity analysis indicate that our models can achieve a significant savings. The model can be extended to embody price discount, various reduction factors etc.

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Appendix

For a given value of $L \in [L_i, L_{i-1}]$, we first obtain the Hessian Matrix $\mathbf{H}$ as follows:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1^2} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1 \partial k} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1 \partial S} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1 \partial \theta} \\ \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k \partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k^2} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k \partial S} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k \partial \theta} \\ \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S \partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S \partial k} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S^2} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S \partial \theta} \\ \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta \partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta \partial k} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta \partial S} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta^2} \end{bmatrix}$$

$$\begin{aligned}
\frac{\partial^2}{\partial q_1^2}\mathrm{JETC}(q_1,k,S,\theta,L,n) &= \frac{2}{q_1^3}\left[\left(A+\frac{S}{n}\right)+(\pi_x\beta^*+(1-\beta^*)\pi_0)\sigma\sqrt{L}\psi(k)+R(L)\right.\\
&\qquad\left.+\,\alpha F_x w_x d\right]\frac{nD(\beta-1)}{(\beta^n-1)}\\
\frac{\partial^2}{\partial k^2}\mathrm{JETC}(q_1,k,S,\theta,L,n) &= \sigma\sqrt{L}\left[\left((\pi_x\beta+\pi_0(1-\beta))\frac{nD(\beta-1)}{q_1(\beta^n-1)}+(1-\beta)h_b\right)\right]\phi(k)\\
\frac{\partial^2}{\partial S^2}\mathrm{JETC}(q_1,k,S,\theta,L,n) &= \frac{\tau B}{S^2}\\
\frac{\partial^2}{\partial \theta^2}\mathrm{JETC}(q_1,k,S,\theta,L,n) &= \frac{\tau b}{\theta^2}\\
\frac{\partial^2}{\partial q_1\partial S}\mathrm{JETC}(q_1,k,S,\theta,L,n) &= \frac{\partial^2}{\partial S\partial q_1}\mathrm{JETC}(q_1,k,S,\theta,L,n) = -\frac{D(\beta-1)}{q_1^2(\beta^n-1)}\\
\frac{\partial^2}{\partial q_1\partial \theta}\mathrm{JETC}(q_1,k,S,\theta,L,n) &= \frac{\partial^2}{\partial \theta\partial q_1}\mathrm{JETC}(q_1,k,S,\theta,L,n) = \frac{SDn(\beta^n-1)}{2(\beta-1)}\\
\frac{\partial^2}{\partial k\partial S}\mathrm{JETC}(q_1,k,S,\theta,L,n) &= \frac{\partial^2}{\partial S\partial k}\mathrm{JETC}(q_1,k,S,\theta,L,n) = 0\\
\frac{\partial^2}{\partial k\partial \theta}\mathrm{JETC}(q_1,k,S,\theta,L,n) &= \frac{\partial^2}{\partial \theta\partial k}\mathrm{JETC}(q_1,k,S,\theta,L,n) = 0\\
\frac{\partial^2}{\partial S\partial \theta}\mathrm{JETC}(q_1,k,S,\theta,L,n) &= \frac{\partial^2}{\partial \theta\partial S}\mathrm{JETC}(q_1,k,S,\theta,L,n) = 0\\
\frac{\partial^2}{\partial q_1\partial k}\mathrm{JETC}(q_1,k,S,\theta,L,n) &= \frac{\partial^2}{\partial k\partial q_1}\mathrm{JETC}(q_1,k,S,\theta,L,n)\\
&= -\frac{nD(\beta-1)}{q^2(\beta^n-1)}\left[(\pi_x\beta^*+(1-\beta^*)\pi_0)\sigma\sqrt{L}(\Phi(k)-1)\right].
\end{aligned}$$

For the first minor, one can easily obtain as,

$$\begin{aligned}
|H_{11}| = \det\left[\frac{\partial^2}{\partial q_1^2}\mathrm{JETC}(q_1,k,S,\theta,L,n)\right] &= \frac{2}{q_1^3}\left[\left(A+\frac{S}{n}\right)+(\pi_x\beta^*+(1-\beta^*)\pi_0)\sigma\sqrt{L}\psi(k)\right.\\
&\qquad\left.+\,R(L)+\alpha F_x w_x d\right]\frac{nD(\beta-1)}{(\beta^n-1)}\cdot
\end{aligned}$$

For the second minor, one can easily obtain easily as,

$$\begin{aligned}
|H_{22}| &= \det\begin{bmatrix} \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1^2} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1 \partial k} \\ \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k \partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k^2} \end{bmatrix}\\
&= A^*C^* - {B^*}^2
\end{aligned}$$

where,

$$
\begin{aligned}
A^* &= \frac{2}{q_1^3}\left[\left(A+\frac{S}{n}\right)+(\pi_x\beta^*+(1-\beta^*)\pi_0)\sigma\sqrt{L}\psi(k)+R(L)+\alpha F_x w_x d\right]\frac{nD(\beta-1)}{(\beta^n-1)}\\
B^* &= -\frac{nD(\beta-1)}{q^2(\beta^n-1)}\left[(\pi_x\beta^*+(1-\beta^*)\pi_0)\sigma\sqrt{L}(\Phi(k)-1)\right]\\
C^* &= \sigma\sqrt{L}\left[\left((\pi_x\beta+\pi_0(1-\beta))\frac{nD(\beta-1)}{q_1(\beta^n-1)}+(1-\beta)h_b\right)\right]\phi(k)\\
&= \left(\frac{2}{q_1^3}\left[\left(A+\frac{S}{n}\right)+(\pi_x\beta^*+(1-\beta^*)\pi_0)\sigma\sqrt{L}\psi(k)+R(L)+\alpha F_x w_x d\right]\frac{nD(\beta-1)}{(\beta^n-1)}\right)\\
&\quad\times\left(\sigma\sqrt{L}\left[\left((\pi_x\beta+\pi_0(1-\beta))\frac{nD(\beta-1)}{q_1(\beta^n-1)}+(1-\beta)h_b\right)\right]\phi(k)\right)\\
&\quad+\left(\frac{nD(\beta-1)}{q^2(\beta^n-1)}\left[(\pi_x\beta^*+(1-\beta^*)\pi_0)\sigma\sqrt{L}(\Phi(k)-1)\right]\right)^2>0.
\end{aligned}
$$

Since $\phi(k)>0$, $\psi(k)>0$ and $2\phi(k)\psi(k)-[\Phi(k)-1]^2>0$ for all $k>0$.
Therefore, $|H_{22}|>0$.

$$
\begin{aligned}
|H_{33}| &= \det\begin{bmatrix}
\frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1^2} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1\partial k} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1\partial S}\\
\frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k\partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k^2} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k\partial S}\\
\frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S\partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S\partial k} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S^2}
\end{bmatrix}\\
&= \det\begin{bmatrix}
A^* & B^* & -\frac{D(\beta-1)}{q_1^2(\beta^n-1)}\\
B^* & C^* & 0\\
-\frac{D(\beta-1)}{q_1^2(\beta^n-1)} & 0 & \frac{\tau B}{S^2}
\end{bmatrix}+\frac{\tau B}{S^2}|H_{22}|\\
&= \left(-\frac{D(\beta-1)}{q_1^2(\beta^n-1)}\right)\left(-\frac{D(\beta-1)C^*}{q_1^2(\beta^n-1)}\right)+\frac{\tau B}{S^2}|H_{22}|\\
&= C^*\left(\frac{\tau B}{S^2}A^*-\frac{D^2(\beta-1)^2}{q_1^4(\beta^n-1)^2}\right)-\frac{\tau B}{S^2}B^{*2}.
\end{aligned}
$$

It is enough to prove that, $C^*\left(\frac{\tau B}{S^2}A^*-\frac{D^2(\beta-1)^2}{q_1^4(\beta^n-1)^2}\right)>\frac{\tau B}{S^2}B^{*2}$

$$
\begin{aligned}
&\Rightarrow \frac{\tau B}{S^2}(A^*C^*-B^{*2})>\frac{C^*D^2(\beta-1)^2}{q_1^4(\beta^n-1)^2}\\
&\Rightarrow (A^*C^*-B^{*2})>\frac{C^*S^2D^2(\beta-1)^2}{q_1^4(\beta^n-1)^2\tau B}\\
&\Rightarrow (A^*C^*-B^{*2})-\frac{C^*S^2D^2(\beta-1)^2}{q_1^4(\beta^n-1)^2\tau B}>0. \qquad (.1)
\end{aligned}
$$

Thus, $|H_{33}|>0$.

Finally, for the 4th minor, the optimum value can be obtained as,

$$\begin{aligned}
|H_{44}| &= \begin{bmatrix} \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1^2} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1 \partial k} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1 \partial S} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1 \partial \theta} \\ \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k \partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k^2} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k \partial S} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k \partial \theta} \\ \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S \partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S \partial k} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S^2} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S \partial \theta} \\ \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta \partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta \partial k} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta \partial S} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta^2} \end{bmatrix} \\
&= -\frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial q_1 \partial \theta} \begin{bmatrix} \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k \partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k^2} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k \partial S} \\ \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S \partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial S \partial k} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial k \partial S} \\ \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta \partial q_1} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta \partial k} & \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta \partial S} \end{bmatrix} \\
&\quad + \frac{\partial^2 \mathrm{JETC}(q_1,k,S,\theta,L,n)}{\partial \theta^2} |H_{33}| \\
&= -\frac{SDn}{2}\left(\frac{(\beta^n-1)}{\beta-1}\right) \begin{bmatrix} B^* & C^* & 0 \\ -\frac{D(\beta-1)}{q_1^2(\beta^n-1)} & 0 & \frac{\tau B}{S^2} \\ \frac{SDn}{2}\left(\frac{\beta^n-1}{\beta-1}\right) & 0 & 0 \end{bmatrix} + \frac{\tau b}{\theta^2}|H_{33}| \\
&= \frac{C^* \tau B D n}{2S}\left(\frac{\beta^n-1}{\beta-1}\right) + \frac{\tau b}{\theta^2}|H_{33}|.
\end{aligned}$$

Here, the first part is positive and already we have proved that $|H_{33}|$ is positive.

Therefore, $|H_{44}| > 0$.

From the above derivations, all the principal minors of the Hessian matrix is positive. Hence, the given Hessian matrix H is positive definite at (q_1, k, S, θ).

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