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OPTIMAL INVESTMENT-REINSURANCE PROBLEMS WITH COMMON SHOCK DEPENDENT RISKS UNDER TWO KINDS OF PREMIUM PRINCIPLES

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Abstract. This paper considers the optimal investment-reinsurance strategy in a risk model with two dependent classes of insurance business under two kinds of premium principles, where the two claim number processes are correlated through a common shock component. Under the criterion of maximizing the expected exponential utility with the expected value premium principle and the variance premium principle, we use the stochastic optimal control theory to derive the optimal strategy and the value function for the compound Poisson risk model as well as for the Brownian motion diffusion risk model. In particular, we find that the optimal investment strategy on the risky asset is independent to the reinsurance strategy and the reinsurance strategy for the compound Poisson risk model are very different from those for the diffusion model under both two kinds of premium principles, but the investment strategies are the same in this two risk models. Finally, numerical examples are presented to show the impact of model parameters in the optimal strategies.

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1. Introduction

In the past, people studied reinsurance and investment separately and both have achieved a lot of great conclusions. With the time going, more attention is paid to study the relation between reinsurance and investment, which will let the insurance company get more profit and make sure the operation of them. In fact, there are many techniques used to solve these problems, among which the stochastic control theory and HJB equations are most widely used, for example, [3, 13, 18, 20, 22, 23]. Other works about the optimal reinsurance and investment problems can be found in [2, 11, 12, 14, 25] and the references therein.

Some kinds of objective functions are commonly seen in the literature. Browne [3], Schmidli [24], Luo *et al.* [21], and Liang [15] consider the optimization problem of minimizing the ruin probability. Centeno [6, 7], Hald and Schmidli [9] and Liang and Guo [16, 17] use the objective function that maximizing the adjustment coefficient by the martingale approach. Moreover, Cai and Tan [4], Bernard and Tian [1] and Cai *et al.* [5] adopt the criteria of minimizing the tail risk measures. Besides, Liang and Yuen [19] and Yuen *et al.* [27] use different premium principle, one is the variance principle, the other is the expected value principle. In [10, 24], the insurer

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can invest in a risky asset to minimize the ruin probability.

Many literatures pay attention to just one class of insurance business or two independent classes of insurance business to gain the optimal proportional reinsurance. However, the insurance businesses are usually dependent in some way in practice. A typical example is that an earthquake (or hurricane, explosion, tsunami, and so on) often leads to various kinds of insurance claims such as medical claims, death claims and household claims. Common shock risk model is often used to describe the dependence between risks. This kind of model assumes that there is a common shock affecting the claim numbers of all classes in addition to their underlying risks. In reality, a common component can depict the effect of a natural disaster that causes various kinds of insurance claims. In this paper, we consider two dependent classes of insurance businesses which have a common shock in the claim number process.

Moreover, we study the optimal investment strategy in both risk-free asset and risky asset, and interest in whether there are some relations between investment and reinsurance. Under the criterion of maximizing the expected utility of terminal wealth, we compare the differences of the optimal strategy under two kinds of premium principles: expected value premium principle and variance premium principle. By the theory of stochastic optimal control, we get the closed-form expressions for the strategy and the value function for the compound Poisson risk model as well as the diffusion risk model.

In this paper, we pay attention to the investment-reinsurance problem with two dependent kinds of insurance contracts and investments under two premium principles. We get the expressions of the optimal strategies and the value functions in each situation and find that the investment strategy is separated to the reinsurance strategy under the criteria of maximizing the expected utility of terminal wealth. So the optimal investment strategy is independent to the optimal reinsurance strategy. The optimal investment strategy depends on the interest rate of the risk-free asset, the appreciation rate and the volatility coefficient of the risky asset. Besides, the forms of optimal reinsurance are very different under the expected value premium principle and the variance premium principle. The optimal reinsurance strategy of the compound Poisson risk model depends on the safety loading, the time, the interest rate of the risk-free asset, claim size distribution and the counting process, which is same under two different premium principles. While the optimal reinsurance strategy of the diffusion model under variance premium principle depends on the safety loading, time and interest rate of the risk-free asset. The optimal reinsurance strategy of the diffusion model under expected value premium principle depends on not only the safety loading, the time, the interest rate of the risk-free asset, but also the claim size distribution.

The rest of the paper is organized as follows. In Section 2, the models and assumptions are presented. In Sections 3 and 4, we discuss the optimal strategies in the compound Poisson model and diffusion model, respectively, under the expected value premium principle, and derive closed-form expressions for the optimal results. In Sections 5 and 6, we discuss the optimal problems in the compound Poisson model and diffusion model, respectively, under variance premium principle, and derive the closed-form expressions for the optimal results. In Section 7, numerical examples are carried out to assess the impact of some parameters. Finally, we conclude the paper in Section 8.

2. The model

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ containing all objects defined in the following. We suppose that the insurer has two classes of insurance business, where the claim arrivals are assumed to be dependent Poisson processes with common shocks. X_i is the size of the ith claim for the first class and $\{X_i, i \geq 1\}$ is assumed to be an independent and identically distributed (i.i.d.) sequence with common distribution $F_X(\cdot)$. Y_i is the size of the ith claim for the second class and $\{Y_i, i \geq 1\}$ is assumed to be an i.i.d. sequence with common distribution $F_Y(\cdot)$. Their means are denoted by $\mu_1 = \mathbb{E}(X_i)$ and $\mu_2 = \mathbb{E}(Y_i)$. Assume that $F_X(x) = 0$ for $x \leq 0$, $F_Y(y) = 0$ for $y \leq 0$, $0 \leq F_X(x) \leq 1$ for $x > 0$, $0 \leq F_Y(y) \leq 1$ for $y > 0$, and that their moment generating functions $M_X(z) = \mathbb{E}(\mathrm{e}^{zX})$ and $M_Y(z) = \mathbb{E}(\mathrm{e}^{zY})$ exist. Then, the aggregate claim processes for the two classes are given by

$$S_1(t) = \sum_{i=1}^{\widetilde{N}_1(t)} X_i,$$

and

$$S_2(t) = \sum_{i=1}^{\widetilde{N}_2(t)} Y_i,$$

where $\widetilde{N}_i(t)$ is the claim number process for class i $(i = 1, 2)$. It is assumed that X_i and Y_i are independent claim size random variables, and that they are independent of $\widetilde{N}_1(t)$ and $\widetilde{N}_2(t)$.

The two claim number processes are correlated in the way that

$$\widetilde{N}_1(t) = N_1(t) + N(t),$$

and

$$\widetilde{N}_2(t) = N_2(t) + N(t),$$

where $N_1(t)$, $N_2(t)$ and $N(t)$ are three independent Poisson processes with parameters λ_1, λ_2, and λ, respectively. In other words, the dependence of the two classes of the business is due to a common shock governed by the counting process $\{N(t)\}_{t\geq 0}$. Then the aggregate claims process generated from the two classes of business has the form

$$S_t = \sum_{i=1}^{N_1(t)+N(t)} X_i + \sum_{i=1}^{N_2(t)+N(t)} Y_i.$$

Assume that both $\mathbb{E}(X\mathrm{e}^{zX}) = M'_X(z)$ and $\mathbb{E}(Y\mathrm{e}^{zY}) = M'_Y(z)$ exist for $0 < z < \zeta$, and that both $\lim_{z\to\zeta} M'_X(z)$ and $\lim_{z\to\zeta} M'_Y(z)$ tend to ∞ for some $\zeta \in (0, +\infty]$.

We consider the financial market where the assets are traded continuously on a finite time horizon $[0, T]$. There are a risk-free asset (bond) and a risky asset (stock) in the financial market. The price of the risk-free asset is given by

$$\begin{cases} \mathrm{d}P_0(t) = rP_0(t)\mathrm{d}t, \quad t \in [0, T], \\ P_0(0) = 1, \end{cases}$$

where $r(> 0)$ is the interest rate of the bond.

The price of the risky asset (stock) is modeled by the following process

$$\begin{cases} \mathrm{d}P_1(t) = P_1(t)\left[b\mathrm{d}t + \sigma_1 \mathrm{d}W(t)\right], \quad t \in [0, T], \\ P_1(0) = P_1, \end{cases} \tag{2.1}$$

where $b(> r)$ is the appreciation rate and $\sigma_1 > 0$ is the volatility coefficient. We denote $a := b - r > 0$. $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion. We assume that r, b and σ_1 are deterministic.

The reserve process of the insurer is modeled by

$$R_t = u + ct - S_t,$$

where u is the amount of initial surplus and c is the rate of premium. The insurance company is allowed to continuously reinsure a fraction of its claim with the retention levels $q_1(t) \in [0, 1]$ and $q_2(t) \in [0, 1]$ for X_i and Y_i, respectively. It means that the insurer pays $q_1(t)X_i$ (or $q_2(t)Y_i$) of a claim occurring at time t and the reinsurer pays $(1 - q_1(t))X_i$ (or $(1 - q_2(t))Y_i$). Let the reinsurance premium rate at time t be $\delta(q_1(t), q_2(t))$. Furthermore, the company is allowed to invest $p_1(t)$ in the risky asset, and the rest is invested in the risk-free asset. $p_1(t) \geq 0$ means the prohibition of short-selling of the risky asset. Let $\{R_t^{p_1,q_1,q_2}\}_{t\geq 0}$ denote the associated surplus process, *i.e.*, $R_t^{p_1,q_1,q_2}$ is the wealth of the insurer at time t under the strategy $(p_1(t), q_1(t), q_2(t))$. This process then evolves as

$$\begin{aligned}\mathrm{d}R_t^{p_1,q_1,q_2} &= [rR_t^{p_1,q_1,q_2} + ap_1(t) + (c - \delta(q_1(t), q_2(t)))]\,\mathrm{d}t + p_1(t)\sigma_1 \mathrm{d}W(t) \\ &\quad - q_1(t)\mathrm{d}S_1(t) - q_2(t)\mathrm{d}S_2(t).\end{aligned} \tag{2.2}$$

As most studies on the mean-variance optimal investment-reinsurance problem (see, e.*g*., [3, 19]), we can consider the problem under the diffusion approximation of the reserve process. We know that the Brownian motion risk model given by

$$\widehat{S}_1(t) = a_1 t - \sigma_2 B_{1t},$$

with $a_1 = (\lambda_1 + \lambda)\mathbb{E}(X)$ and $\sigma_2^2 = (\lambda_1 + \lambda)\mathbb{E}(X^2)$ can be seen as a diffusion approximation to the compound Poisson process $S_1(t)$. Similarly,

$$\widehat{S}_2(t) = a_2 t - \sigma_3 B_{2t},$$

with $a_2 = (\lambda_2 + \lambda)\mathbb{E}(Y)$ and $\sigma_3^2 = (\lambda_2 + \lambda)\mathbb{E}(Y^2)$ can be seen as a diffusion approximation to the compound Poisson process $S_2(t)$. This diffusion approximation is widely used in the literature on the optimization problems for insurers. B_{1t} and B_{2t} are standard Brownian motions with the correlation coefficient

$$\rho = \frac{\lambda \mathbb{E}(X)\mathbb{E}(Y)}{\sqrt{(\lambda_1 + \lambda)\mathbb{E}(X^2)(\lambda_2 + \lambda)\mathbb{E}(Y^2)}}.$$

So, $\mathbb{E}(B_{1t}B_{2t}) = \rho t$. B_{1t} and B_{2t} are independent to the standard Brownian motion $W(t)$. Replacing $S_i(t)(i = 1, 2)$ of (2.2) by $\widehat{S}_i(t)(i = 1, 2)$, we obtain the following surplus process

$$\begin{aligned}\mathrm{d}\widehat{R_t}^{p_1,q_1,q_2} &= \Big[r\widehat{R_t}^{p_1,q_1,q_2} + ap_1(t) + (c - \delta(q_1(t), q_2(t))) - q_1(t)a_1 - q_2(t)a_2\Big]\,\mathrm{d}t \\ &\quad + p_1(t)\sigma_1 \mathrm{d}W(t) + q_1(t)\sigma_2 \mathrm{d}B_{1t} + q_2(t)\sigma_3 \mathrm{d}B_{2t},\end{aligned}$$

or equivalently,

$$\begin{aligned}\mathrm{d}\widehat{R_t}^{p_1,q_1,q_2} &= \Big[r\widehat{R_t}^{p_1,q_1,q_2} + ap_1(t) + (c - \delta(q_1(t), q_2(t))) - q_1(t)a_1 - q_2(t)a_2\Big]\,\mathrm{d}t \\ &\quad + p_1(t)\sigma_1 \mathrm{d}W(t) + \sqrt{\sigma_2^2(q_1(t))^2 + \sigma_3^2(q_2(t))^2 + 2q_1(t)q_2(t)\lambda\mu_1\mu_2}\,\mathrm{d}B_t,\end{aligned} \tag{2.3}$$

where B_t is a standard Brownian motion.

We assume now that the insurer's objective is maximizing the expected utility of terminal wealth at the terminal time T. The utility function is $u(x)$, which satisfies $u' > 0$ and $u'' < 0$. Then, the objective functions we will consider are

$$J^{p_1,q_1,q_2}(t, x) = \mathbb{E}[u(R_T^{p_1,q_1,q_2})|R_t^{p_1,q_1,q_2} = x], \tag{2.4}$$

and

$$J^{p_1,q_1,q_2}(t, x) = \mathbb{E}[u(\widehat{R}_T^{p_1,q_1,q_2})|\widehat{R}_t^{p_1,q_1,q_2} = x]. \tag{2.5}$$

Since (2.4) and (2.5) will be discussed separately in the following sections, the use of the same notation $J^{p_1,q_1,q_2}(t, x)$ will not cause any confusion. The corresponding value function is then given by

$$V(t, x) = \sup_{p_1,q_1,q_2} J^{p_1,q_1,q_2}(t, x). \tag{2.6}$$

We assume that the insurer has an exponential utility function

$$u(x) = -\frac{m}{\nu}\exp\{-\nu x\},$$

for $m > 0$ and $\nu > 0$. This utility has constant absolute risk aversion(CARA) parameter ν. Such a utility function plays an important role in insurance mathematics and actuarial practice, since this is the only function under which the principle of "zero utility" gives a fair premium that is independent of the level of reserve for an insurance company.

We use the theory of stochastic optimal control described in Fleming and Soner [8] to solve the problem defined above. Let $C^{1,2}$ denote the space of function $\phi(t,x)$ such that ϕ and its partial derivatives ϕ_t, ϕ_x, ϕ_{xx} are continuous on $[0,T]\times\mathbb{R}$. From the standard arguments, we see that if the value function $V\in C^{1,2}$, then V satisfies the following HJB equation

$$\sup_{p_1,q_1,q_2} \mathcal{A}^{p_1,q_1,q_2} V(t,x) = 0, \tag{2.7}$$

for $t < T$ with the boundary condition

$$V(T,x) = u(x) \tag{2.8}$$

where

$$\begin{aligned}\mathcal{A}^{p_1,q_1,q_2} V(t,x) = V_t &+ [rx + ap_1 + c - \delta(q_1,q_2)]\, V_x + \frac{1}{2}\sigma_1^2 p_1^2 V_{xx} \\ &+ \lambda_1 \mathbb{E}[V(t,x-q_1X) - V(t,x)] + \lambda_2 \mathbb{E}[V(t,x-q_2Y) - V(t,x)] \\ &+ \lambda \mathbb{E}[V(t,x-q_1X-q_2Y) - V(t,x)],\end{aligned} \tag{2.9}$$

for the risk process (2.2), and

$$\begin{aligned}\mathcal{A}^{p_1,q_1,q_2} V(t,x) = V_t &+ [rx + ap_1 + c - \delta(q_1,q_2) - q_1a_1 - q_2a_2]\, V_x + \frac{1}{2}\sigma_1^2 p_1^2 V_{xx} \\ &+ \frac{1}{2}(\sigma_2^2 q_1^2 + \sigma_3^2 q_2^2 + 2q_1q_2\lambda\mu_1\mu_2) V_{xx},\end{aligned} \tag{2.10}$$

for the risk process (2.3).

Applying the standard methods of Fleming and Soner [8] and Yang and Zhang [26], we have the following verification theorem:

Theorem 2.1. *Let $\widetilde{V}\in C^{1,2}$ be a classical solution to (2.7) and satisfies (2.8). Then, the value function V given by (2.6) coincides with $\widetilde{V}$. That is,*

$$\widetilde{V}(t,x) = V(t,x).$$

Furthermore, set (p_1^*, q_1^*, q_2^*) such that

$$\mathcal{A}^{p_1^*,q_1^*,q_2^*} V(t,x) = 0,$$

holds for all $(t,x)\in[0,T)\times R$. Then, $(p_1^*(t,R_t^*), q_1^*(t,R_t^*), q_2^*(t,R_t^*))$ is the optimal strategy. Here, R_t^* is the surplus process under the optimal strategy.

In this paper, we assume that continuous trading is allowed and that all assets are infinitely divisible. We work on a complete probability space $(\Omega,\mathcal{F},P)$ on which the process $R_t^{p_1,q_1,q_2}$ is well defined. The information at time t is given by the complete filtration $\mathcal{F}_t$ generated by $R_t^{p_1,q_1,q_2}$. We call $(p_1(t),q_1(t),q_2(t))$ an admissible strategy if $(p_1(t),q_1(t),q_2(t))$ is $\mathcal{F}_t$-predictable and satisfies $p_1(t)\ge 0$, $q_1(t)\ge 0$, $q_2(t)\ge 0$, $\mathbb{E}[\int_0^t p_1(s)^2 \mathrm{d}s] < \infty$ and $\mathbb{E}[\int_0^t q_i(s)^2 \mathrm{d}s] < \infty$, $i = 1,2$ for all $t\ge 0$.

Next we will study the optimal investment-reinsurance strategies under the expected value premium principle in the compound Poisson model and diffusion model in Sections 3 and 4, respectively. In Sections 5 and 6, we discuss the optimal investment-reinsurance problems under variance premium principle in the compound Poisson model and diffusion model, respectively.

3. Optimal results for the compound Poisson risk model by expected value principle

Throughout Sections 3 and 4, we assume that the reinsurance premium is calculated according to the expected value principle. That is,

$$\delta(q_1, q_2) = (1+\eta_1)(1-q_1)a_1 + (1+\eta_2)(1-q_2)a_2, \tag{3.1}$$

where η_1 and η_2 are the reinsurer's safety loading of the insurance business.

In this section, we consider the optimization problem for the compound Poisson risk model (2.2). The corresponding HJB equation is

$$\begin{aligned}\sup_{p_1,q_1,q_2} \Big\{ & V_t + [rx + ap_1 + c - \delta(q_1,q_2)]V_x + \frac{1}{2}\sigma_1^2 p_1^2 V_{xx} \\ & + \lambda_1 \mathbb{E}[V(t, x-q_1X) - V(t,x)] + \lambda_2 \mathbb{E}[V(t,x-q_2Y) - V(t,x)] \\ & + \lambda \mathbb{E}[V(t,x-q_1X-q_2Y) - V(t,x)] \Big\} = 0, \end{aligned} \tag{3.2}$$

with the boundary condition $V(T,x) = u(x)$. To solve this equation, we apply the method of Browne [3] to fit a solution of the form

$$V(t,x) = -\frac{m}{\nu}\exp\big\{ -\nu x \mathrm{e}^{r(T-t)} + h(T-t)\big\}, \tag{3.3}$$

where $m > 0$, $\nu > 0$ are the constant absolute risk aversion(CARA) parameters, and $h(\cdot)$ is a suitable function such that (3.3) is a solution to (3.2). The boundary condition $V(T,x) = u(x)$ implies $h(0) = 0$.

From (3.3), we get

$$\begin{cases} V_t = V(t,x)\big[\nu x r \mathrm{e}^{r(T-t)} - h'(T-t)\big], \\ V_x = V(t,x)[-\nu \mathrm{e}^{r(T-t)}], \\ V_{xx} = V(t,x)[\nu^2 \mathrm{e}^{2r(T-t)}], \\ \mathbb{E}[V(t,x-q_1X) - V(t,x)] = V(t,x)\big[M_X(\nu q_1 \mathrm{e}^{r(T-t)}) - 1\big], \\ \mathbb{E}[V(t,x-q_2Y) - V(t,x)] = V(t,x)\big[M_Y(\nu q_2 \mathrm{e}^{r(T-t)}) - 1\big], \\ \mathbb{E}[V(t,x-q_1X-q_2Y) - V(t,x)] = V(t,x)\big[M_X(\nu q_1 \mathrm{e}^{r(T-t)})M_Y(\nu q_2 \mathrm{e}^{r(T-t)}) - 1\big]. \end{cases} \tag{3.4}$$

Substituting (3.4) into (3.2), we obtain

$$\begin{aligned}\inf_{p_1,q_1,q_2} \Big\{ & -h'(T-t) - \lambda_1 - \lambda_2 - \lambda - c\nu \mathrm{e}^{r(T-t)} - \nu a p_1 \mathrm{e}^{r(T-t)} + \delta(q_1,q_2)\nu \mathrm{e}^{r(T-t)} \\ & + \frac{1}{2}\sigma_1^2 p_1^2 \nu^2 \mathrm{e}^{2r(T-t)} + \lambda_1 M_X(\nu q_1 \mathrm{e}^{r(T-t)}) \\ & + \lambda_2 M_Y(\nu q_2 \mathrm{e}^{r(T-t)}) + \lambda M_X(\nu q_1 \mathrm{e}^{r(T-t)}) M_Y(\nu q_2 \mathrm{e}^{r(T-t)}) \Big\} = 0, \end{aligned} \tag{3.5}$$

for $t < T$. Let

$$\begin{aligned}\widetilde{f}_1(p_1,q_1,q_2) = & -\nu a p_1 \mathrm{e}^{r(T-t)} + \delta(q_1,q_2)\nu \mathrm{e}^{r(T-t)} + \frac{1}{2}\sigma_1^2 p_1^2 \nu^2 \mathrm{e}^{2r(T-t)} + \lambda_1 M_X(\nu q_1 \mathrm{e}^{r(T-t)}) \\ & + \lambda_2 M_Y(\nu q_2 \mathrm{e}^{r(T-t)}) + \lambda M_X(\nu q_1 \mathrm{e}^{r(T-t)}) M_Y(\nu q_2 \mathrm{e}^{r(T-t)}). \end{aligned} \tag{3.6}$$

For any $t \in [0,T]$, we have

$$\begin{cases} \dfrac{\partial \widetilde{f}_1(p_1,q_1,q_2)}{\partial p_1} = -\nu a \mathrm{e}^{r(T-t)} + \sigma_1^2 p_1 \nu^2 \mathrm{e}^{2r(T-t)}, \\ \dfrac{\partial \widetilde{f}_1(p_1,q_1,q_2)}{\partial q_1} = -(1+\eta_1)a_1\nu \mathrm{e}^{r(T-t)} + M_X'(\nu q_1 \mathrm{e}^{r(T-t)}) \left[\lambda_1 + \lambda M_Y(\nu q_2 \mathrm{e}^{r(T-t)})\right] \nu \mathrm{e}^{r(T-t)}, \\ \dfrac{\partial \widetilde{f}_1(p_1,q_1,q_2)}{\partial q_2} = -(1+\eta_2)a_2\nu \mathrm{e}^{r(T-t)} + M_Y'(\nu q_2 \mathrm{e}^{r(T-t)}) \left[\lambda_2 + \lambda M_X(\nu q_1 \mathrm{e}^{r(T-t)})\right] \nu \mathrm{e}^{r(T-t)}, \\ \dfrac{\partial^2 \widetilde{f}_1(p_1,q_1,q_2)}{\partial q_1^2} = M_X''(\nu q_1 \mathrm{e}^{r(T-t)})\nu^2 \mathrm{e}^{2r(T-t)} \left[\lambda_1 + \lambda M_Y(\nu q_2 \mathrm{e}^{r(T-t)})\right] > 0, \\ \dfrac{\partial^2 \widetilde{f}_1(p_1,q_1,q_2)}{\partial q_2^2} = M_Y''(\nu q_2 \mathrm{e}^{r(T-t)})\nu^2 \mathrm{e}^{2r(T-t)} \left[\lambda_2 + \lambda M_X(\nu q_1 \mathrm{e}^{r(T-t)})\right] > 0, \\ \dfrac{\partial^2 \widetilde{f}_1(p_1,q_1,q_2)}{\partial q_1 q_2} = \lambda M_X'(\nu q_1 \mathrm{e}^{r(T-t)}) M_Y'(\nu q_2 \mathrm{e}^{r(T-t)})\nu^2 \mathrm{e}^{2r(T-t)}. \end{cases} \tag{3.7}$$

From (3.5) and (3.7), we firstly obtain the optimal p_1

$$p_1^* = \frac{a}{\sigma_1^2 \nu \mathrm{e}^{r(T-t)}}. \tag{3.8}$$

Next we calculate the optimal q_1 and q_2 according to (3.7). We know that $\widetilde{f}_1$ is a convex function with respect to q_1 and q_2. The minimizer (q_1,q_2) of $\widetilde{f}_1(p_1,q_1,q_2)$ satisfies the following equation

$$\begin{cases} -(1+\eta_1)a_1 + M_X'(n_1)(\lambda_1 + \lambda M_Y(n_2)) = 0, \\ -(1+\eta_2)a_2 + M_Y'(n_2)(\lambda_2 + \lambda M_X(n_1)) = 0, \end{cases} \tag{3.9}$$

where $n_1 = \nu q_1 \mathrm{e}^{r(T-t)}$ and $n_2 = \nu q_2 \mathrm{e}^{r(T-t)}$. The conclusion in Yuen *et al.* [27] can guarantee the existence and the uniqueness of the solution to (3.9). In fact, let M_i^{-1} and $(M_i^{'-1})$ be the inverse function of M_i and (M_i') for $i = X, Y$. If the following inequations

$$\begin{cases} M_Y^{-1}\left(1+\eta_1+\eta_1 \times \dfrac{\lambda_1}{\lambda}\right) > M_Y^{'-1}((1+\eta_2)\mu_2), \\ M_X^{-1}\left(1+\eta_2+\eta_2 \times \dfrac{\lambda_2}{\lambda}\right) > M_X^{'-1}((1+\eta_1)\mu_1), \end{cases}$$

or

$$\begin{cases} M_Y^{-1}\left(1+\eta_1+\eta_1 \times \dfrac{\lambda_1}{\lambda}\right) < M_Y^{'-1}((1+\eta_2)\mu_2), \\ M_X^{-1}\left(1+\eta_2+\eta_2 \times \dfrac{\lambda_2}{\lambda}\right) < M_X^{'-1}((1+\eta_1)\mu_1), \end{cases}$$

hold, equation (3.9) has a unique positive root $(\bar{n}_1, \bar{n}_2)$.

From the result above, we obtain $\bar{n}_1 = \nu q_1(T-t)\mathrm{e}^{r(T-t)}$, and $\bar{n}_2 = \nu q_2(T-t)\mathrm{e}^{r(T-t)}$, which in turn give

$$\begin{cases} q_1(T-t) = \dfrac{\bar{n}_1}{\nu} \mathrm{e}^{-r(T-t)}, \\ q_2(T-t) = \dfrac{\bar{n}_2}{\nu} \mathrm{e}^{-r(T-t)}. \end{cases} \tag{3.10}$$

Let

$$t_1 = T - \frac{1}{r} \ln\left(\frac{\bar{n}_1}{\nu}\right) \quad \text{for} \quad \nu < \bar{n}_1 < \nu \mathrm{e}^{rT};$$

$$t_2 = T - \frac{1}{r}\ln\left(\frac{\bar{n}_2}{\nu}\right) \quad \text{for} \quad \nu < \bar{n}_2 < \nu e^{rT}.$$

For $\bar{n}_1 \le \nu$ ($\bar{n}_2 \le \nu$), we set $t_1 = T$ ($t_2 = T$); and for $\bar{n}_1 \ge \nu e^{rT}$ ($\bar{n}_2 \ge \nu e^{rT}$), we set $t_1 = 0$ ($t_2 = 0$). To make sure that $q_1(t),\ q_2(t) \in [0,1]$, we need to discuss the optimal values in the following two cases: Case 1: $\bar{n}_1 \le \bar{n}_2$; and Case 2: $\bar{n}_1 > \bar{n}_2$.

Case 1. In this case, we have $t_1 \ge t_2$. When $t \in [0, t_2]$, we have $(q_1^*(t), q_2^*(t)) = (q_1(T-t), q_2(T-t))$. Denote the function h in (3.5) by h_1 for $t \in [0, t_2]$. Substituting $(q_1(T-t), q_2(T-t))$ into (3.5), we get

$$h_1(T-t) = \widetilde{h}_1(T-t) + C_1, \tag{3.11}$$

where

$$\begin{aligned}\widetilde{h}_1(T-t) = {} & \frac{1}{r}[(1+\eta_1)a_1 + (1+\eta_2)a_2 - c]\nu e^{r(T-t)} \\ & - [(\lambda_1 + \lambda_2 + \lambda + (1+\eta_1)a_1\bar{n}_1 + (1+\eta_2)a_2\bar{n}_2]\,(T-t) \\ & + \left[\lambda_1 M_X(\bar{n}_1) + \lambda_2 M_Y(\bar{n}_2) + \lambda M_X(\bar{n}_1)M_Y(\bar{n}_2) - \frac{a^2}{2\sigma_1^2}\right](T-t),\end{aligned} \tag{3.12}$$

and C_1 is a constant that will be determined later.

For $t \ge t_2$, $q_2(T-t) \ge 1$, and thus $q_2^* = 1$. Substituting $q_2^* = 1$ into (3.5), we get

$$\begin{aligned}\inf_{q_1}\Bigg\{ & -h'(T-t) - c\nu e^{r(T-t)} - \lambda_1 - \lambda_2 - \lambda - \nu a p_1^* e^{r(T-t)} + \frac{1}{2}\sigma_1^2 p_1^{*2}\nu^2 e^{2r(T-t)} \\ & + (1+\eta_1)(1-q_1)a_1\nu e^{r(T-t)} + \lambda_2 M_Y(\nu e^{r(T-t)}) \\ & + M_X(\nu q_1 e^{r(T-t)})\left[\lambda_1 + \lambda M_Y(\nu e^{r(T-t)})\right]\Bigg\} = 0,\end{aligned} \tag{3.13}$$

for $t < T$. Therefore, the minimum value of the inside of the curly brace in the left-hand side of (3.13) is attained at

$$\widehat{q}_1(T-t) = M_X^{'-1}\left[\frac{(1+\eta_1)a_1}{\lambda_1 + \lambda M_Y(\nu e^{r(T-t)})}\right] \times \frac{1}{r}e^{-r(T-t)}. \tag{3.14}$$

Since $M_X'(x)$ is an increasing function of x, it is not difficult to see that $\widehat{q}_1(T-t)$ is an increasing function of t.

Denote by t_{01} the solution to the equation $\widehat{q}_1(T-t) = 1$, and by h_2 the function h in (3.5) for $t \in [t_2, t_{01}]$. For $t \in [t_2, t_{01}]$, we have $(q_1^*(t), q_2^*(t)) = (\widehat{q}_1(T-t), 1)$. Substituting $(\widehat{q}_1(T-t), 1)$ into (3.5), we get

$$h_2(T-t) = \widetilde{h}_2(T-t) + C_2, \tag{3.15}$$

where

$$\begin{aligned}\widetilde{h}_2(T-t) = {} & \frac{1}{r}\left[(1+\eta_1)a_1 - c\right]\nu e^{r(T-t)} - \left(\lambda_1 + \lambda_2 + \lambda - \frac{a^2}{2\sigma_1^2}\right)(T-t) \\ & - \int_0^{T-t}(1+\eta_1)a_1\widehat{q}_1(s)\nu e^{rs}ds + \int_0^{T-t}\lambda_2 M_Y(\nu e^{rs}) \\ & + M_X(\widehat{q}_1(s)\nu e^{rs})\left[\lambda_1 + \lambda M_Y(\nu e^{rs})\right]ds,\end{aligned} \tag{3.16}$$

and C_2 is also a constant that will be determined later.

For $t \in [t_{01}, T]$, $(q_1^*(t), q_2^*(t)) = (1,1)$. Denote the function h in (3.5) by h_3. Then, putting $(1,1)$ into (3.5), we get

$$h_3(T-t) = -\frac{1}{r}c\nu(\mathrm{e}^{r(T-t)}-1) - \left(\lambda_1+\lambda_2+\lambda-\frac{a^2}{2\sigma_1^2}\right)(T-t) + \int_0^{T-t}[\lambda_1 M_X(\nu\mathrm{e}^{rs})+\lambda_2 M_Y(\nu\mathrm{e}^{rs})+\lambda M_X(\nu\mathrm{e}^{rs})M_Y(\nu\mathrm{e}^{rs})]\,\mathrm{d}s. \tag{3.17}$$

According to the continuity of the value function V and function h, we have $C_1 = h_2(T-t_2) - \widetilde{h}_1(T-t_2)$ and $C_2 = h_3(T-t_{01}) - \widetilde{h}_2(T-t_{01})$. Then

$$\begin{cases} h_2(T-t_2) = \widetilde{h}_1(T-t_2)+C_1 = h_1(T-t_2), \\ h_3(T-t_{01}) = \widetilde{h}_2(T-t_{01})+C_2 = h_2(T-t_{01}). \end{cases}$$

Case 2. In this case, we have $t_1 < t_2$. For $t \in [0,t_1]$, $(q_1^*(t), q_2^*(t)) = (q_1(T-t), q_2(T-t))$ from which the function h in (3.5) can be written as

$$h_4(T-t) = \widetilde{h}_1(T-t) + C_4. \tag{3.18}$$

For $t \ge t_1$, we have $q_1(T-t) \ge 1$, and thus $q_1^*(t) = 1$. Then, similar to the derivation of (3.14), we get the minimizer

$$\widehat{q}_2(T-t) = M_Y'^{-1}\left[\frac{(1+\eta_2)a_2}{\lambda_2+\lambda M_X(\nu\mathrm{e}^{r(T-t)})}\right]\times\frac{1}{r}\mathrm{e}^{-r(T-t)}, \tag{3.19}$$

which is an increasing function of t.

Denote by t_{02} the solution to the equation $\widehat{q}_2(T-t) = 1$, and by h_5 the function h in (3.5) for $t \in [t_1, t_{02}]$. We have $(q_1^*(t), q_2^*(t)) = (1, \widehat{q}_2(T-t))$ for $t \in [t_1, t_{02}]$. It follows from (3.5) and $(q_1^*(t), q_2^*(t)) = (1, \widehat{q}_2(T-t))$ that

$$h_5(T-t) = \widetilde{h}_5(T-t) + C_5, \tag{3.20}$$

where

$$\widetilde{h}_5(T-t) = \frac{1}{r}[(1+\eta_2)a_2 - c]\nu\mathrm{e}^{r(T-t)} - \left(\lambda_1+\lambda_2+\lambda-\frac{a^2}{2\sigma_1^2}\right)(T-t) - \int_0^{T-t}(1+\eta_2)a_2\widehat{q}_2(s)\nu\mathrm{e}^{rs}\mathrm{d}s + \int_0^{T-t}\lambda_1 M_X(\nu\mathrm{e}^{rs}) + M_Y(\widehat{q}_2(s)\nu\mathrm{e}^{rs})(\lambda_2+\lambda M_X(\nu\mathrm{e}^{rs}))\mathrm{d}s.$$

For $t \in [t_{02}, T]$, $(q_1^*(t), q_2^*(t)) = (1,1)$, and the corresponding function h in (3.5) is h_3 of (3.17).

According to the continuity of the value function V and function h, we have $C_4 = h_5(T-t_1) - \widetilde{h}_1(T-t_1)$ and $C_5 = h_3(T-t_{02}) - \widetilde{h}_5(T-t_{02})$, which yields

$$\begin{cases} h_5(T-t_1) = \widetilde{h}_1(T-t_1)+C_4 = h_4(T-t_1), \\ h_2(T-t_{02}) = \widetilde{h}_5(T-t_{02})+C_5 = h_5(T-t_{02}). \end{cases} \tag{3.21}$$

We summarize the above analysis in the following theorem.

Theorem 3.1. *Let p_1^* be given in (3.8), $(\bar{n}_1, \bar{n}_2)$ be the unique positive root of (3.9), and $(q_1(T-t), q_2(T-t))$, $\widehat{q}_1(T-t)$, and $\widehat{q}_2(T-t)$ be given in (3.10), (3.14), and (3.19) respectively. Recall the functions $h_1(T-t)$, $h_2(T-t)$, $h_3(T-t)$, $h_4(T-t)$ and $h_5(T-t)$ defined in (3.11), (3.15), (3.17), (3.18), and (3.20), respectively. Then, we have*

(i) *If Case (1) holds, i.e., $\bar{n}_1 \le \bar{n}_2$, for any $t \in [0,T]$, the optimal investment-reinsurance strategies $p_1^*(t)$, $q_1^*(t)$ and $q_2^*(t)$ under the model (2.2) are*

$$(p_1^*(t), q_1^*(t), q_2^*(t)) = \begin{cases} \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, q_1(T-t), q_2(T-t)\right), & 0 \le t \le t_2, \\ \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, \widehat{q}_1(T-t), 1\right), & t_2 \le t \le t_{01}, \\ \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, 1, 1\right), & t_{01} \le t \le T, \end{cases}$$

and the value function $V(t,x)$ is given by

$$V(t,x) = \begin{cases} -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_1(T-t)\}, & 0 \le t \le t_2, \\ -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_2(T-t)\}, & t_2 \le t \le t_{01}, \\ -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_3(T-t)\}, & t_{01} \le t \le T, \end{cases}$$

in which t_{01} is the solution to equation $\widehat{q}_1(T-t) = 1$ with $\widehat{q}_1(\cdot)$ given in (3.14).

(ii) *If Case (2) holds, i.e., $\bar{n}_1 > \bar{n}_2$, for any $t \in [0,T]$, the optimal investment-reinsurance strategies under the model (2.2) are*

$$(p_1^*(t), q_1^*(t), q_2^*(t)) = \begin{cases} \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, q_1(T-t), q_2(T-t)\right), & 0 \le t \le t_1, \\ \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, 1, \widehat{q}_2(T-t)\right), & t_1 \le t \le t_{02}, \\ \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, 1, 1\right), & t_{02} \le t \le T, \end{cases}$$

and the value function is given by

$$V(t,x) = \begin{cases} -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_4(T-t)\}, & 0 \le t \le t_1, \\ -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_5(T-t)\}, & t_1 \le t \le t_{02}, \\ -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_3(T-t)\}, & t_{02} \le t \le T, \end{cases}$$

where t_{02} is the solution to the equation $\widehat{q}_2(T-t) = 1$ with $\widehat{q}_2(\cdot)$ given in (3.19).

Remark 3.2. Since

$$\begin{cases} h_1(T-t_2) = h_2(T-t_2), \\ h_2(T-t_1) = h_3(T-t_1), \\ h_4(T-t_1) = h_5(T-t_1), \\ h_5(T-t_2) = h_3(T-t_2), \end{cases}$$

$V(t,x)$ is continuous function on $[0,T] \times R$. Furthermore, after calculation we have

$$\begin{cases} h_1'(T-t_2) = h_2'(T-t_2), \\ h_2'(T-t_{01}) = h_3'(T-t_{01}), \\ h_4'(T-t_1) = h_5'(T-t_1), \\ h_5'(T-t_{02}) = h_3'(T-t_{02}). \end{cases}$$

Therefore, we have $V(t,x) \in C^{1,2}$. That is, V(t,x) is a classical solution to the HJB equation (3.2).

4. Optimal results for the diffusion model by expected value principle

In this section, we discuss the optimization problem for the diffusion risk model (2.3) under the expected value premium principle. The corresponding HJB equation is

$$\begin{aligned}\sup_{p_1,q_1,q_2}\Big\{&V_t+[rx+ap_1+c-\delta(q_1,q_2)-q_1a_1-q_2a_2]V_x+\frac{1}{2}\sigma_1^2p_1^2V_{xx}\\&+\frac{1}{2}(\sigma_2^2q_1^2+\sigma_3^2q_2^2+2q_1q_2\lambda\mu_1\mu_2)V_{xx}\Big\}=0,\end{aligned}\tag{4.1}$$

for $t<T$, with the boundary condition $V(T,x)=u(x)$. Again, we consider a solution with the form of (3.3). After substituting (3.4) into (4.1) and some simple calculation, equation (4.1) becomes

$$\begin{aligned}\inf_{p_1,q_1,q_2}\Big\{&-h'(T-t)-[ap_1+c-\delta(q_1,q_2)-q_1a_1-q_2a_2]\,\nu\mathrm{e}^{r(T-t)}\\&+\frac{1}{2}(\sigma_1^2p_1^2+\sigma_2^2q_1^2+\sigma_3^2q_2^2+2q_1q_2\lambda\mu_1\mu_2)\nu^2\mathrm{e}^{2r(T-t)}\Big\}=0.\end{aligned}\tag{4.2}$$

Let

$$\begin{aligned}\widetilde{f}_2(p_1,q_1,q_2)=&\,[\delta(q_1,q_2)+q_1a_1+q_2a_2-ap_1]\,\nu\mathrm{e}^{r(T-t)}\\&+\frac{1}{2}\left(\sigma_1^2p_1^2+\sigma_2^2q_1^2+\sigma_3^2q_2^2+2q_1q_2\lambda\mu_1\mu_2\right)\nu^2\mathrm{e}^{2r(T-t)}.\end{aligned}\tag{4.3}$$

Then for any $t\in[0,T]$, we have

$$\begin{cases}\frac{\partial\widetilde{f}_2(p_1,q_1,q_2)}{\partial p_1}&=-\nu a\mathrm{e}^{r(T-t)}+\sigma_1^2p_1\nu^2\mathrm{e}^{2r(T-t)},\\ \frac{\partial\widetilde{f}_2(p_1,q_1,q_2)}{\partial q_1}&=-\eta_1a_1\nu\mathrm{e}^{r(T-t)}+(\sigma_2^2q_1+q_2\lambda\mu_1\mu_2)\nu^2\mathrm{e}^{2r(T-t)},\\ \frac{\partial\widetilde{f}_2(p_1,q_1,q_2)}{\partial q_2}&=-\eta_2a_2\nu\mathrm{e}^{r(T-t)}+(\sigma_3^2q_2+q_1\lambda\mu_1\mu_2)\nu^2\mathrm{e}^{2r(T-t)},\\ \frac{\partial^2\widetilde{f}_2(p_1,q_1,q_2)}{\partial q_1^2}&=\sigma_2^2\nu^2\mathrm{e}^{2r(T-t)}>0,\\ \frac{\partial^2\widetilde{f}_2(p_1,q_1,q_2)}{\partial q_2^2}&=\sigma_3^2\nu^2\mathrm{e}^{2r(T-t)}>0,\\ \frac{\partial^2\widetilde{f}_2(p_1,q_1,q_2)}{\partial q_1q_2}&=\lambda\mu_1\mu_2\nu^2\mathrm{e}^{2r(T-t)}>0.\end{cases}\tag{4.4}$$

From (4.2) and (4.4), we firstly obtain the optimal $p_1(t)$

$$p_1^*(t)=\frac{a}{\sigma_1^2\nu\mathrm{e}^{r(T-t)}}.\tag{4.5}$$

Next we calculate the optimal q_1 and q_2 according to (4.4). We know that $\widetilde{f}_2$ is a convex function with respect to q_1 and q_2. The minimizer (q_1,q_2) of $\widetilde{f}_2(p_1,q_1,q_2)$ satisfies the following equation

$$\begin{cases}q_1\sigma_2^2+q_2\lambda\mu_1\mu_2=\dfrac{\eta_1a_1}{\nu}\mathrm{e}^{-r(T-t)},\\ q_2\sigma_3^2+q_1\lambda\mu_1\mu_2=\dfrac{\eta_2a_2}{\nu}\mathrm{e}^{-r(T-t)}.\end{cases}\tag{4.6}$$

The conclusion in Yuen *et al.* [27] can guarantee the existence and the uniqueness of the solution to (4.6). Actually, it is easy to see that the solution to equation (4.6) is

$$\begin{cases}\bar{q}_1(T-t)=\dfrac{a_1\eta_1\sigma_3^2-\lambda\mu_1\mu_2a_2\eta_2}{\sigma_2^2\sigma_3^2-\lambda^2\mu_1^2\mu_2^2}\times\dfrac{1}{\nu}\mathrm{e}^{-r(T-t)},\\ \bar{q}_2(T-t)=\dfrac{a_2\eta_2\sigma_2^2-\lambda\mu_1\mu_2a_1\eta_1}{\sigma_2^2\sigma_3^2-\lambda^2\mu_1^2\mu_2^2}\times\dfrac{1}{\nu}\mathrm{e}^{-r(T-t)}.\end{cases}\tag{4.7}$$

Let

$$\begin{cases} A_1 = \dfrac{a_1\eta_1\sigma_3^2 - \lambda\mu_1\mu_2 a_2\eta_2}{\sigma_2^2\sigma_3^2 - \lambda^2\mu_1^2\mu_2^2}, \\ A_2 = \dfrac{a_2\eta_2\sigma_2^2 - \lambda\mu_1\mu_2 a_1\eta_1}{\sigma_2^2\sigma_3^2 - \lambda^2\mu_1^2\mu_2^2}, \end{cases} \tag{4.8}$$

and $t_3 = T - \frac{1}{r}\ln\frac{A_1}{\nu}$ for $\nu \le A_1 \le \nu e^{rT}$; $t_4 = T - \frac{1}{r}\ln\frac{A_2}{\nu}$ for $\nu \le A_2 \le \nu e^{rT}$.

For $A_1 \le \nu$ $(A_2 \le \nu)$, we set $t_3 = T$ $(t_4 = T)$; and for $A_1 \ge \nu e^{rT}$ $(A_2 \ge \nu e^{rT})$, we set $t_3 = 0$ $(t_4 = 0)$. To make sure that the optimal reinsurance strategies satisfy $q_1(t),\ q_2(t) \in [0,1]$, we need to investigate the optimal results in the following four cases:

Case 1: $\dfrac{\lambda\mu_1\mu_2 a_2}{\sigma_3^2 a_1}\eta_2 < \eta_1 \le \dfrac{\lambda\mu_1\mu_2 a_2 + \sigma_2^2 a_2}{\sigma_3^2 a_1 + \lambda\mu_1\mu_2 a_1}\eta_2$;

Case 2: $\dfrac{\lambda\mu_1\mu_2 a_2 + \sigma_2^2 a_2}{\sigma_3^2 a_1 + \lambda\mu_1\mu_2 a_1}\eta_2 < \eta_1 \le \dfrac{\sigma_2^2 a_2}{\lambda\mu_1\mu_2 a_1}\eta_2$;

Case 3: $\eta_1 \le \dfrac{\lambda\mu_1\mu_2 a_2}{\sigma_3^2 a_1}\eta_2$;

Case 4: $\eta_1 > \dfrac{\sigma_2^2 a_2}{\lambda\mu_1\mu_2 a_1}\eta_2$.

Case 1. In this case, we have $\bar{q}_1(T-t) > 0$, $\bar{q}_2(T-t) > 0$, $A_1 \le A_2$ and then $t_3 \ge t_4$. Denote by h_6 the function h in (4.2) for $t \in [0, t_4]$. For $t \in [0, t_4]$, we have $(q_1^*(t), q_2^*(t)) = (\bar{q}_1(T-t), \bar{q}_2(T-t))$. Then, substituting $(\bar{q}_1(T-t), \bar{q}_2(T-t))$ into (4.2), we obtain

$$h_6(T-t) = \widetilde{h}_6(T-t) + C_6, \tag{4.9}$$

where

$$\begin{aligned} \widetilde{h}_6(T-t) = {} & \frac{1}{r}[(1+\eta_1)a_1 + (1+\eta_2)a_2 - c]\nu e^{r(T-t)} \\ & + \left(\frac{1}{2}\sigma_2^2 A_1^2 + \frac{1}{2}\sigma_3^2 A_2^2 - \frac{a^2}{2\sigma_1^2} + A_1A_2\lambda\mu_1\mu_2 - a_1\eta_1 A_1 - a_2\eta_2 A_2\right)(T-t), \end{aligned}$$

and C_6 is a constant that will be determined later.

For $t \ge t_4$, we have $\bar{q}_2(T-t) \ge 1$, and thus $q_2^*(t) = 1$. Inserting $q_2^*(t) = 1$ into (4.2) yields

$$\begin{aligned} \inf_{q_1} \Big\{ & -h'(T-t) - ap_1^*\nu e^{r(T-t)} - c\nu e^{r(T-t)} - \lambda_1 - \lambda_2 - \lambda + \big[\delta(q_1,1) + q_1 a_1 \\ & + a_2\big]\nu e^{r(T-t)} + \frac{1}{2}(\sigma_1^1 p_1^{*2} + \sigma_2^2 q_1^2 + \sigma_3^2 + 2q_1\lambda\mu_1\mu_2)\nu^2 e^{2r(T-t)} \Big\} = 0, \end{aligned} \tag{4.10}$$

for $t < T$. Then the minimizer of (4.10) has the form

$$\widetilde{q}_1(T-t) = \frac{\eta_1 a_1 e^{-r(T-t)} - \lambda\mu_1\mu_2\nu}{\sigma_2^2\nu}. \tag{4.11}$$

Let $t_{03} = T - \frac{1}{r}\ln\big(\frac{\eta_1 a_1}{\sigma_2^2 + \lambda\mu_1\mu_2}\big)$. Denote by h_7 the function h in (4.2) for $t \in [t_4, t_{03}]$. For $t \in [t_4, t_{03}]$, it is easy to see that $(q_1^*(t), q_2^*(t)) = (\widetilde{q}_1(T-t), 1)$. Putting $(\widetilde{q}_1(T-t), 1)$ into (4.2) gives

$$h_7(T-t) = \widetilde{h}_7(T-t) + C_7, \tag{4.12}$$

where

$$\widetilde{h}_7(T-t) = \frac{1}{r}\left[(1+\eta_1)a_1 + a_2 - c\right]\nu \mathrm{e}^{r(T-t)} - \frac{a^2}{2\sigma_1^2}(T-t) + \frac{1}{4r}\sigma_3^2\nu^2\mathrm{e}^{2r(T-t)} - \int_0^{T-t} \eta_1\widetilde{q}_1(s)a_1\nu\mathrm{e}^{rs} - \frac{1}{2}\left(\sigma_2^2\widetilde{q}_1^2(s) + 2\widetilde{q}_1(s)\lambda\mu_1\mu_2\right)\nu^2\mathrm{e}^{2rs}\mathrm{d}s.$$

Denote by h_8 the function h in (4.2) for $t \in [t_{03}, T]$. For $t \in [t_{03}, T]$, it follows that $(q_1^*(t), q_2^*(t)) = (1, 1)$. Putting this into (4.2), we get

$$h_8(T-t) = \frac{1}{r}(a_1 + a_2 - c)\nu(\mathrm{e}^{r(T-t)} - 1) - \frac{a^2}{2\sigma_1^2}(T-t) + \frac{1}{4r}(\sigma_2^2 + \sigma_3^2 + 2\lambda\mu_1\mu_2)\nu^2(\mathrm{e}^{2r(T-t)} - 1). \tag{4.13}$$

According to the continuity of the value function V and function h, we know $C_6 = h_7(T-t_4) - \widetilde{h}_6(T-t_4)$ and $C_7 = h_8(T-t_{03}) - \widetilde{h}_7(T-t_{03})$. Then, we have

$$\begin{cases} h_7(T-t_4) &= \widetilde{h}_6(T-t_4) + C_6 = h_6(T-t_4), \\ h_8(T-t_{03}) &= \widetilde{h}_7(T-t_{03}) + C_7 = h_7(T-t_{03}). \end{cases} \tag{4.14}$$

Case 2. In this case, we still have $\bar{q}_1(T-t) > 0$ and $\bar{q}_2(T-t) \geq 0$, but $A_1 > A_2$. Thus, we obtain $t_3 < t_4$. Denote by h_9 the function h in (4.2) for $t \in [0, t_3]$. For $t \in [0, t_3]$, $(q_1^*(t), q_2^*(t)) = (\bar{q}_1(T-t), \bar{q}_2(T-t))$. This together with (4.2) gives

$$h_9(T-t) = \widetilde{h}_6(T-t) + C_9. \tag{4.15}$$

For $t \geq t_3$, we have $\bar{q}_1(T-t) \geq 1$, and thus $q_1^*(t) = 1$. By inserting $q_1^*(t) = 1$ into (4.2) and mimicking the derivation of (4.11), we obtain the minimizer

$$\widetilde{q}_2(T-t) = \frac{\eta_2 a_2 \mathrm{e}^{-r(T-t)} - \lambda\mu_1\mu_2\nu}{\sigma_3^2\nu}. \tag{4.16}$$

Denote

$$t_{04} = T - \frac{1}{r}\ln\left(\frac{\eta_2 a_2}{(\sigma_3^2 + \lambda\mu_1\mu_2)\nu}\right). \tag{4.17}$$

It is easy to see that $t_{04} \geq t_3$. Denote by h_{10} the function h in (4.2) for $t \in [t_3, t_{04}]$. For $t \in [t_3, t_{04}]$, we have $(q_1^*(t), q_2^*(t)) = (1, \widetilde{q}_2(T-t))$. Substituting $(1, \widetilde{q}_2(T-t))$ into (4.2), we obtain

$$h_{10}(T-t) = \widetilde{h}_{10}(T-t) + C_{10}, \tag{4.18}$$

where

$$\widetilde{h}_{10}(T-t) = \frac{1}{r}[(1+\eta_2)a_2 + a_1 - c]\nu\mathrm{e}^{r(T-t)} - \frac{a^2}{2\sigma_1^2}(T-t) + \frac{1}{4r}\sigma_2^2\nu^2\mathrm{e}^{2r(T-t)} - \int_0^{T-t}\left[\eta_2\widetilde{q}_2(s)a_2\nu\mathrm{e}^{rs} - \frac{1}{2}\sigma_3^2(\widetilde{q}_2(s))^2\lambda\mu_1\mu_2\nu^2\mathrm{e}^{2rs}\right]\mathrm{d}s.$$

For $t \in [t_{04}, T]$, $(q_1^*(t), q_2^*(t)) = (1, 1)$. Then, it can be shown that function $h(\cdot)$ in (4.2) is given by $h_8(\cdot)$ of (4.13).

According to the continuity of the value function V and function h, we have $C_9 = h_{10}(T-t_3) - \widetilde{h}_6(T-t_3)$ and $C_{10} = h_8(T-t_{04}) - \widetilde{h}_{10}(T-t_{04})$. Then, we get

$$\begin{cases} h_{10}(T-t_3) = \widetilde{h}_6(T-t_3) + C_9 = h_9(T-t_3), \\ h_8(T-t_{04}) = \widetilde{h}_{10}(T-t_{04}) + C_{10} = h_{10}(T-t_{04}). \end{cases}$$

Case 3. In this case, we have $\bar{q}_1(T-t) \le 0$ and $\bar{q}_2(T-t) > 0$, and thus $q_1^*(t) = 0$. Substituting $q_1^*(t) = 0$ into (4.2) yields the minimizer

$$\hat{q}_2(T-t) = \frac{\eta_2 a_2}{\sigma_3^2 \nu} \mathrm{e}^{-r(T-t)}. \tag{4.19}$$

Let $t_{05} = T - \frac{1}{r}\ln\left(\frac{\eta_2 a_2}{\sigma_3^2\nu}\right)$. Denote by h_{11} the function h in (4.2) for $t \in [0, t_{05}]$. For $t \in [0, t_{05}]$, it follows that $(q_1^*(t), q_2^*(t)) = (1, \hat{q}_2(T-t))$. Putting $(1, \hat{q}_2(T-t))$ into (4.2), we get

$$h_{11}(T-t) = \widetilde{h}_{11} + C_{11}, \tag{4.20}$$

where

$$\widetilde{h}_{11}(T-t) = \frac{1}{r}\left[(1+\eta_2)a_2 + (1+\eta_1)a_1 - c\right]\nu \mathrm{e}^{r(T-t)} - \frac{a^2}{2\sigma_1^2}(T-t) - \frac{1}{4r}\frac{\eta_2^2 a_2^2}{\sigma_3^2}(T-t).$$

Denote by h_{12} the function h in (4.2) for $t \in [t_{05}, T]$. For $t \in [t_{05}, T]$, $(q_1^*(t), q_2^*(t)) = (0, 1)$. Using these optimal values and (4.2), we obtain

$$h_{12}(T-t) = \frac{1}{r}[a_2 + (1+\eta_1)a_1 - c]\nu(\mathrm{e}^{r(T-t)} - 1) - \frac{a^2}{2\sigma_1^2}(T-t) + \frac{1}{4r}\sigma_3^2\nu^2(\mathrm{e}^{2r(T-t)} - 1). \tag{4.21}$$

According to the continuity of the value function V and function h, we have $C_{11} = h_{12}(T-t_{05}) - \widetilde{h}_{11}(T-t_{05})$. Then, we derive $h_{11}(T-t_{05}) = \widetilde{h}_{11}(T-t_{05}) + C_{11} = h_{12}(T-t_{05})$.

Case 4. In this case, we have $\bar{q}_1(T-t) > 0$ and $\bar{q}_2(T-t) < 0$, and thus $q_2^*(t) = 0$. Substituting $q_2^*(t) = 0$ into (4.2) yields the minimizer

$$\hat{q}_1(T-t) = \frac{\eta_1 a_1}{\sigma_2^2 \nu} \mathrm{e}^{-r(T-t)}. \tag{4.22}$$

Denote $t_{06} = T - \frac{1}{r}\ln\left(\frac{\eta_1 a_1}{\sigma_2^2\nu}\right)$. Denote by h_{13} the function h in (4.2) for $t \in [0, t_{06}]$. For $t \in [0, t_{06}]$, $(q_1^*(t), q_2^*(t)) = (\hat{q}_1(T-t), 0)$. This together with (4.2) yields

$$h_{13}(T-t) = \widetilde{h}_{13}(T-t) + C_{13}, \tag{4.23}$$

where

$$\widetilde{h}_{13}(T-t) = \frac{1}{r}\left[(1+\eta_2)a_2 + (1+\eta_1)a_1 - c\right]\nu \mathrm{e}^{r(T-t)} - \frac{a^2}{2\sigma_1^2}(T-t) - \frac{1}{4r}\frac{\eta_1^2 a_1^2}{\sigma_2^2}(T-t).$$

Denote by h_{14} the function h in (4.2) for $t \in [t_{06}, T]$. For $t \in [t_{06}, T]$, it follows that $(q_1^*(t), q_2^*(t)) = (1, 0)$. Putting this into (4.2), we get

$$h_{14}(T-t) = \frac{1}{r}\left[a_1 + (1+\eta_2)a_2 - c\right]\nu(\mathrm{e}^{r(T-t)} - 1) - \frac{a^2}{2\sigma_1^2}(T-t) + \frac{1}{4r}\sigma_2^2\nu^2(\mathrm{e}^{r(T-t)} - 1). \tag{4.24}$$

According to the continuity of the value function V and function h, we have $C_{13} = h_{14}(T-t_{05}) - \widetilde{h}_{13}(T-t_{06})$. Then, we obtain $h_{13}(T-t_{06}) = \widetilde{h}_{13}(T-t_{06}) + C_{13} = h_{14}(T-t_{05})$.

We summarize the results obtained above in the following theorem.

Theorem 4.1. *Let $p_1^*(t)$ be given in (4.5), $(\bar{q}_1(T-t), \bar{q}_2(T-t)), \widetilde{q}_1(T-t), \widetilde{q}_2(T-t), \hat{q}_2(T-t)$, and $\hat{q}_1(T-t)$ be given in (4.7), (4.11), (4.16), (4.19) and (4.22) respectively. Also, recall the functions $h_i(T-t)$ for $i = 6, 7, \ldots, 14$ given in (4.9), (4.12), (4.13), (4.15), (4.18), (4.20), (4.21), (4.23) and (4.24) respectively. Then, we have*

(i) *If Case 1 holds, the optimal investment-reinsurance strategies for model (2.3) are*

$$(p_1^*(t), q_1^*(t), q_2^*(t)) = \begin{cases} \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, \bar{q}_1(T-t), \bar{q}_2(T-t)\right), \ 0 \le t \le t_4, \\ \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, \widetilde{q}_1(T-t), 1\right), \ t_4 \le t \le t_{03}, \\ \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, 1, 1\right), \ t_{03} \le t \le T, \end{cases}$$

and the value function is given by

$$V(t,x) = \begin{cases} -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_6(T-t)\}, \ 0 \le t \le t_4, \\ -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_7(T-t)\}, \ t_4 \le t \le t_{03}, \\ -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_8(T-t)\}, \ t_{03} \le t \le T. \end{cases}$$

(ii) *If Case 2 holds, the optimal investment-reinsurance strategies for model (2.3) are*

$$(p_1^*(t), q_1^*(t), q_2^*(t)) = \begin{cases} \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, \bar{q}_1(T-t), \bar{q}_2(T-t)\right), \ 0 \le t \le t_3, \\ \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, 1, \widetilde{q}_2(T-t)\right), \ t_3 \le t \le t_{04}, \\ \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, 1, 1\right), \ t_{04} \le t \le T, \end{cases}$$

and the value function is given by

$$V(t,x) = \begin{cases} -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_9(T-t)\}, \ 0 \le t \le t_3, \\ -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_{10}(T-t)\}, \ t_3 \le t \le t_{04}, \\ -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_8(T-t)\}, \ t_{04} \le t \le T. \end{cases}$$

(iii) *If Case 3 holds, the optimal investment-reinsurance strategies for model (2.3) are*

$$(p_1^*(t), q_1^*(t), q_2^*(t)) = \begin{cases} \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, 0, \hat{q}_2(T-t)\right), \ 0 \le t \le t_{05}, \\ \left(\dfrac{a}{\sigma_1^2 \nu e^{r(T-t)}}, 0, 1\right), \ t_{05} \le t \le T, \end{cases}$$

and the value function is given by

$$V(t,x) = \begin{cases} -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_{11}(T-t)\}, \ 0 \le t \le t_{05}, \\ -\dfrac{m}{\nu} \exp\{-\nu x e^{r(T-t)} + h_{12}(T-t)\}, \ t_{05} \le t \le T. \end{cases}$$

(iv) *If Case 4 holds, the optimal investment-reinsurance strategies for model (2.3) are*

$$(p_1^*(t), q_1^*(t), q_2^*(t)) = \begin{cases} \left(\dfrac{a}{\sigma_1^2 \nu \mathrm{e}^{r(T-t)}}, \hat{q}_1(T-t), 0\right), \ 0 \le t \le t_{06}, \\ \left(\dfrac{a}{\sigma_1^2 \nu \mathrm{e}^{r(T-t)}}, 1, 0\right), \ t_{06} \le t \le T, \end{cases}$$

and the value function is given by

$$V(t,x) = \begin{cases} -\dfrac{m}{\nu}\exp\{-\nu x \mathrm{e}^{r(T-t)} + h_{13}(T-t)\}, \ 0 \le t \le t_{06}, \\ -\dfrac{m}{\nu}\exp\{-\nu x \mathrm{e}^{r(T-t)} + h_{14}(T-t)\}, \ t_{06} \le t \le T. \end{cases}$$

Remark 4.2. Since

$$\begin{cases} h_6(T-t_4) &= h_7(T-t_4), h_7(T-t_{03}) = h_8(T-t_{03}), \\ h_9(T-t_3) &= h_{10}(T-t_3), h_{10}(T-t_{04}) = h_8(T-t_{04}), \\ h_{11}(T-t_{05}) &= h_{12}(T-t_{05}), h_{13}(T-t_{06}) = h_{14}(T-t_{06}), \end{cases}$$

$V(t,x)$ is a continuous function for any $(t,x) \in [0,T] \times \mathbb{R}$. Furthermore,

$$\begin{cases} h_6'(T-t_4) &= h_7'(T-t_4), h_7'(T-t_{03}) = h_8'(T-t_{03}), \\ h_9'(T-t_3) &= h_{10}'(T-t_3), h_{10}'(T-t_{04}) = h_8'(T-t_{04}), \\ h_{11}'(T-t_{05}) &= h_{12}'(T-t_{05}), h_{13}'(T-t_{06}) = h_{14}'(T-t_{06}), \end{cases}$$

which includes that, $V(t,x)$ is a classical solution to the HJB equation (2.7).

5. Optimal results for the compound Poisson risk model by variance principle

Throughout Sections 5 and 6, we assume that the reinsurance premium is calculated according to the variance principle. That is,

$$\delta(q_1, q_2) = (1-q_1)a_1 + (1-q_2)a_2 + \Lambda \widetilde{h}(q_1, q_2), \tag{5.1}$$

here Λ is the reinsurer's safety loading of the insurer, and

$$\widetilde{h}(q_1, q_2) = (1-q_1)^2\sigma_2^2 + (1-q_2)^2\sigma_3^2 + 2(1-q_1)(1-q_2)\lambda\mu_1\mu_2.$$

In this section, we consider the optimization problem for the compound Poisson risk model (2.2). The corresponding HJB equation is

$$\begin{aligned} \sup_{p_1,q_1,q_2} \Big\{ & V_t + [rx + ap_1 + c - \delta(q_1,q_2)]V_x + \frac{1}{2}\sigma_1^2 p_1^2 V_{xx} \\ & + \lambda_1 \mathbb{E}[V(t, x - q_1 X) - V(t,x)] + \lambda_2 \mathbb{E}[V(t, x - q_2 Y) - V(t,x)] \\ & + \lambda \mathbb{E}[V(t, x - q_1 X - q_2 Y) - V(t,x)] \Big\} = 0, \end{aligned} \tag{5.2}$$

with the boundary condition $V(T,x) = u(x)$, we apply the method of Browne [3] to fit a solution of the form

$$V(t,x) = -\frac{m}{\nu}\exp[-\nu x \mathrm{e}^{r(T-t)} + h(T-t)], \tag{5.3}$$

where $m > 0$, $\nu > 0$ are constant absolute risk aversion (CARA) parameters. $h(\cdot)$ is a suitable function such that (5.3) is a solution to (5.2). The boundary condition $V(T,x) = u(x)$ implies $h(0) = 0$.

From (5.3), we obtain the following derivatives

$$\begin{cases} V_t = V(t,x)[\nu x r \mathrm{e}^{r(T-t)} - h'(T-t)], \\ V_x = V(t,x)[-\nu \mathrm{e}^{r(T-t)}], \\ V_{xx} = V(t,x)[\nu^2 \mathrm{e}^{2r(T-t)}], \\ \mathbb{E}[V(t,x-q_1X) - V(t,x)] = V(t,x)[M_X(\nu q_1 \mathrm{e}^{r(T-t)}) - 1], \\ \mathbb{E}[V(t,x-q_2Y) - V(t,x)] = V(t,x)[M_Y(\nu q_2 \mathrm{e}^{r(T-t)}) - 1], \\ \mathbb{E}[V(t,x-q_1X-q_2Y) - V(t,x)] = V(t,x)\left[M_X(\nu q_1 \mathrm{e}^{r(T-t)}) M_Y(\nu q_2 \mathrm{e}^{r(T-t)}) - 1\right]. \end{cases} \tag{5.4}$$

Then using (5.2) and (5.4), we obtain

$$\begin{aligned} \inf_{p_1,q_1,q_2} \Big\{ & - h'(T-t) - \lambda_1 - \lambda_2 - \lambda - c\nu \mathrm{e}^{r(T-t)} - \nu a p_1 \mathrm{e}^{r(T-t)} + \delta(q_1,q_2)\nu \mathrm{e}^{r(T-t)} \\ & + \frac{1}{2}\sigma_1^2 p_1^2 \nu^2 \mathrm{e}^{2r(T-t)} + \lambda_1 M_X(\nu q_1 \mathrm{e}^{r(T-t)}) + \lambda_2 M_Y(\nu q_2 \mathrm{e}^{r(T-t)}) \\ & + \lambda M_X(\nu q_1 \mathrm{e}^{r(T-t)}) M_Y(\nu q_2 \mathrm{e}^{r(T-t)}) \Big\} = 0, \ \ t < T. \end{aligned} \tag{5.5}$$

Let

$$\begin{aligned} \widetilde{f}_3(p_1,q_1,q_2) = & - \nu a p_1 \mathrm{e}^{r(T-t)} + \delta(q_1,q_2)\nu \mathrm{e}^{r(T-t)} + \frac{1}{2}\sigma_1^2 p_1^2 \nu^2 \mathrm{e}^{2r(T-t)} + \lambda_1 M_X(\nu q_1 \mathrm{e}^{r(T-t)}) \\ & + \lambda_2 M_Y(\nu q_2 \mathrm{e}^{r(T-t)}) + \lambda M_X(\nu q_1 \mathrm{e}^{r(T-t)}) M_Y(\nu q_2 \mathrm{e}^{r(T-t)}). \end{aligned} \tag{5.6}$$

For any $t \in [0,T]$, we have

$$\begin{cases} \dfrac{\partial \widetilde{f}_3(p_1,q_1,q_2)}{\partial p_1} = -\nu a \mathrm{e}^{r(T-t)} + \sigma_1^2 p_1 \nu^2 \mathrm{e}^{2r(T-t)}, \\ \dfrac{\partial \widetilde{f}_3(p_1,q_1,q_2)}{\partial q_1} = \Big\{ - a_1 - \Lambda\left[2(1-q_1)\sigma_2^2 + 2(1-q_2)\lambda\mu_1\mu_2\right] \\ \qquad\qquad + M_X'(\nu q_1 \mathrm{e}^{r(T-t)})\left[\lambda_1 + \lambda M_Y(\nu q_2 \mathrm{e}^{r(T-t)})\right] \Big\} \times \nu \mathrm{e}^{r(T-t)}, \\ \dfrac{\partial \widetilde{f}_3(p_1,q_1,q_2)}{\partial q_2} = \Big\{ - a_2 - \Lambda\left[2(1-q_2)\sigma_3^2 + 2(1-q_1)\lambda\mu_1\mu_2\right] \\ \qquad\qquad + M_Y'(\nu q_2 \mathrm{e}^{r(T-t)})\left[\lambda_2 + \lambda M_X(\nu q_1 \mathrm{e}^{r(T-t)})\right] \Big\} \times \nu \mathrm{e}^{r(T-t)}, \\ \dfrac{\partial^2 \widetilde{f}_3(p_1,q_1,q_2)}{\partial q_1^2} = M_X''(\nu q_1 \mathrm{e}^{r(T-t)})\nu^2 \mathrm{e}^{2r(T-t)}\left[\lambda_1 + \lambda M_Y(\nu q_2 \mathrm{e}^{r(T-t)})\right] + 2\Lambda\sigma_2^2 \times \nu \mathrm{e}^{r(T-t)} > 0, \\ \dfrac{\partial^2 \widetilde{f}_3(p_1,q_1,q_2)}{\partial q_2^2} = M_Y''(\nu q_2 \mathrm{e}^{r(T-t)})\nu^2 \mathrm{e}^{2r(T-t)}\left[\lambda_2 + \lambda M_X(\nu q_1 \mathrm{e}^{r(T-t)})\right] + 2\Lambda\sigma_3^2 \times \nu \mathrm{e}^{r(T-t)} > 0, \\ \dfrac{\partial^2 \widetilde{f}_3(p_1,q_1,q_2)}{\partial q_1 q_2} = 2\Lambda\lambda\mu_1\mu_2\nu \mathrm{e}^{r(T-t)} + \lambda M_X'(\nu q_1 \mathrm{e}^{r(T-t)}) \times M_Y'(\nu q_2 \mathrm{e}^{r(T-t)}) \times \nu^2 \mathrm{e}^{2r(T-t)}. \end{cases} \tag{5.7}$$

From (5.5) and (5.7), we firstly obtain the optimal $p_1(t)$

$$p_1^*(t) = \frac{a}{\sigma_1^2 \nu \mathrm{e}^{r(T-t)}}. \tag{5.8}$$

Next we calculate the optimal q_1 and q_2 according to (5.7). We know that $\widetilde{f}_3$ is a convex function with respect to q_1 and q_2. The minimizer (q_1, q_2) of $\widetilde{f}_3(p_1, q_1, q_2)$ satisfies the following equation

$$\begin{cases} a_1 + \Lambda\left[2(1-\frac{n}{\nu}\mathrm{e}^{-r(T-t)})\sigma_2^2 + 2(1-\frac{m}{\nu}\mathrm{e}^{-r(T-t)})\lambda\mu_1\mu_2\right] = M_X'(n)(\lambda_1+\lambda M_Y(m)), \\ a_2 + \Lambda\left[2(1-\frac{m}{\nu}\mathrm{e}^{-r(T-t)})\sigma_3^2 + 2(1-\frac{n}{\nu}\mathrm{e}^{-r(T-t)})\lambda\mu_1\mu_2\right] = M_Y'(m)(\lambda_2+\lambda M_X(n)), \end{cases} \tag{5.9}$$

where $n = \nu q_1 \mathrm{e}^{r(T-t)}$ and $m = \nu q_2 \mathrm{e}^{r(T-t)}$. Moreover, the conclusion in Liang and Yuen [19] can guarantee the existence and the uniqueness of the solution to (5.9).

The following three lemmas come from Liang and Yuen [19].

Lemma 5.1. *For any $t \in [0, T]$, there is a unique positive solution $m_1(t)$ $(n_1(t))$ to each of the following equation*

$$\lambda\mu_1 M_Y(m) = \lambda\mu_1 + 2\Lambda\sigma_2^2 + 2\Lambda\lambda\mu_1\mu_2\left(1-\frac{m}{\nu}\mathrm{e}^{-r(T-t)}\right), \tag{5.10}$$

and

$$(\lambda_1+\lambda)M_X'(n) = a_1 + 2\Lambda\lambda\mu_1\mu_2 + 2\Lambda\sigma_2^2\left(1-\frac{n}{\nu}\mathrm{e}^{-r(T-t)}\right). \tag{5.11}$$

Lemma 5.2. *For any $t \in [0, T]$, there is a unique positive solution $m_2(t)$ $(n_2(t))$ to each of the following equation*

$$\lambda\mu_2 M_X(n) = \lambda\mu_2 + 2\Lambda\sigma_3^2 + 2\Lambda\lambda\mu_1\mu_2\left(1-\frac{n}{\nu}\mathrm{e}^{-r(T-t)}\right), \tag{5.12}$$

and

$$(\lambda_2+\lambda)M_Y'(m) = a_2 + 2\Lambda\lambda\mu_1\mu_2 + 2\Lambda\sigma_3^2\left(1-\frac{m}{\nu}\mathrm{e}^{-r(T-t)}\right). \tag{5.13}$$

Lemma 5.3. *Let $m_1(t)$, $n_1(t)$, $m_2(t)$, and $n_2(t)$ be the unique positive roots of the equation (5.10)–(5.13), respectively. If*

$$\begin{cases} m_1(t) > m_2(t), \\ n_1(t) < n_2(t), \end{cases}$$

or

$$\begin{cases} m_1(t) < m_2(t), \\ n_1(t) > n_2(t), \end{cases}$$

hold for any $t \in [0, T]$, then the equation (5.9) has a unique positive root $(\bar{n}(T-t), \bar{m}(T-t))$.

From the Lemmas above, we obtain $\bar{n} = \nu q_1(T-t)\mathrm{e}^{r(T-t)}$, and $\bar{m} = \nu q_2(T-t)\mathrm{e}^{r(T-t)}$, which in turn give

$$\begin{cases} \bar{q}_1(T-t) = \dfrac{\bar{n}(T-t)}{\nu}\mathrm{e}^{-r(T-t)}, \\ \bar{q}_2(T-t) = \dfrac{\bar{m}(T-t)}{\nu}\mathrm{e}^{-r(T-t)}. \end{cases} \tag{5.14}$$

Assume that $\widehat{q}_1(T-t)$ and $\widehat{q}_2(T-t)$ are the unique positive solutions to the following equations:

$$a_1 + 2\Lambda(1-q_1)\sigma_2^2 = M_X'(\nu q_1 \mathrm{e}^{r(T-t)})\left[\lambda_1 + \lambda M_Y(\nu\mathrm{e}^{r(T-t)})\right], \tag{5.15}$$

and

$$a_2 + 2\Lambda(1-q_2)\sigma_3^2 = M_Y'(\nu q_2 \mathrm{e}^{r(T-t)})\left[\lambda_2 + \lambda M_X(\nu\mathrm{e}^{r(T-t)})\right], \tag{5.16}$$

respectively. Let t_1 (t_{01}) and t_2 (t_{02}) be the time points at which $\bar{q}_1(T-t)=1$ ($\widehat{q}_1(T-t)=1$) and $\bar{q}_2(T-t)=1$ ($\widehat{q}_2(T-t)=1$), respectively. By the convexity of the function $\widetilde{f}_3$, we obtain that, when $t_1>t_2$, the optimal investment-reinsurance strategy is

$$(p_1^*,q_1^*,q_2^*)=\begin{cases}\left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},\bar{q}_1(T-t),\bar{q}_2(T-t)\right),\ 0<t<t_2,\\ \left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},\widehat{q}_1(T-t),1\right),\ t_2\le t<t_{01},\\ \left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},1,1\right),\ t\ge t_{01}.\end{cases}$$

When $t_1\le t_2$, the optimal investment-reinsurance strategy is

$$(p_1^*,q_1^*,q_2^*)=\begin{cases}\left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},\bar{q}_1(T-t),\bar{q}_2(T-t)\right),\ 0<t<t_1,\\ \left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},1,\widehat{q}_2(T-t)\right),\ t_1\le t<t_{02},\\ \left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},1,1\right),\ t\ge t_{02}.\end{cases}$$

Denote by h_{15} the function h in (5.5) for this section. Putting the optimal strategy back into (5.5) yields

$$h_{15}(T-t)=-\frac{1}{r}c\nu(\mathrm{e}^{r(T-t)}-1)-(\lambda_1+\lambda_2+\lambda)(T-t)+\int_0^{T-t}K_1(s)\mathrm{d}s, \tag{5.17}$$

where

$$\begin{aligned}K_1(s)=&-\nu a p_1^*(s)\mathrm{e}^{rs}+\delta(q_1^*(s),q_2^*(s))\nu\mathrm{e}^{rs}+\frac{1}{2}\sigma_1^2p_1(s)^2\nu^2\mathrm{e}^{2rs}+\lambda_1M_X(\nu q_1^*(s)\mathrm{e}^s)\\&+\lambda_2M_Y(\nu q_2^*(s)\mathrm{e}^s)+\lambda M_X(\nu q_1^*(s)\mathrm{e}^s)M_Y(\nu q_2^*(s)\mathrm{e}^s).\end{aligned} \tag{5.18}$$

To summarize, we have

Theorem 5.4. *Let $p_1(T-t)$, $(\bar{q}_1(T-t),\bar{q}_2(T-t))$ be given in equations (5.8) and (5.14), and $(\widehat{q}_1(T-t),\widehat{q}_2(T-t))$ be the unique solution to the equations (5.15) and (5.16), respectively. Then, when $t_1>t_2$, the optimal strategy for the risk model (2.2) is*

$$(p_1^*(t),q_1^*(t),q_2^*(t))=\begin{cases}\left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},\bar{q}_1(T-t),\bar{q}_2(T-t)\right),\ 0<t<t_2,\\ \left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},\widehat{q}_1(T-t),1\right),\ t_2\le t<t_{01},\\ \left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},1,1\right),\ t\ge t_{01}.\end{cases}$$

When $t_1\le t_2$, the optimal strategy for the risk model (2.2) is

$$(p_1^*(t),q_1^*(t),q_2^*(t))=\begin{cases}\left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},\bar{q}_1(T-t),\bar{q}_2(T-t)\right),\ 0<t<t_1,\\ \left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},1,\widehat{q}_2(T-t)\right),\ t_1\le t<t_{02},\\ \left(\dfrac{a}{\sigma_1^2\nu \mathrm{e}^{r(T-t)}},1,1\right),\ t\ge t_{02}.\end{cases}$$

Moreover, the value function is given by

$$V(t,x) = -\frac{m}{\nu}\exp\left\{-\nu x \mathrm{e}^{r(T-t)} + h_{15}(T-t)\right\},$$

where $h_{15}(T-t)$ *is defined in (5.17).*

6. Optimal results for the diffusion model by variance principle

In this section, we discuss the optimization problem for the diffusion risk model (2.3). The corresponding HJB equation is

$$\begin{aligned}\sup_{p_1,q_1,q_2}\bigg\{&V_t + [rx + ap_1 + c - \delta(q_1,q_2) - q_1a_1 - q_2a_2]V_x + \frac{1}{2}\sigma_1^2 p_1^2 V_{xx}\\ &+ \frac{1}{2}(\sigma_2^2 q_1^2 + \sigma_3^2 q_2^2 + 2q_1q_2\lambda\mu_1\mu_2)V_{xx}\bigg\} = 0,\end{aligned} \tag{6.1}$$

for $t < T$, with the boundary condition $V(T,x) = u(x)$. Again, we consider a solution with the form of (5.3). After substituting (5.4) into (6.1) and some algebraic manipulation, the equation (6.1) becomes

$$\begin{aligned}\inf_{p_1,q_1,q_2}\bigg\{&-h'(T-t) - [ap_1 + c - \delta(q_1,q_2) - q_1a_1 - q_2a_2]\,\nu\mathrm{e}^{r(T-t)}\\ &+\frac{1}{2}(\sigma_1^2p_1^2 + \sigma_2^2q_1^2 + \sigma_3^2q_2^2 + 2q_1q_2\lambda\mu_1\mu_2)\nu^2\mathrm{e}^{2r(T-t)}\bigg\} = 0.\end{aligned} \tag{6.2}$$

Denote

$$\begin{aligned}\widetilde{f}_4(p_1,q_1,q_2) = {}& [\delta(q_1,q_2) + q_1a_1 + q_2a_2 - ap_1]\,\nu\mathrm{e}^{r(T-t)}\\ &+ \frac{1}{2}(\sigma_1^2p_1^2 + \sigma_2^2q_1^2 + \sigma_3^2q_2^2 + 2q_1q_2\lambda\mu_1\mu_2)\nu^2\mathrm{e}^{2r(T-t)}.\end{aligned} \tag{6.3}$$

Then for any $t \in [0,T]$, we have

$$\begin{cases}\frac{\partial\widetilde{f}_4(p_1,q_1,q_2)}{\partial p_1} &= -\nu a\mathrm{e}^{r(T-t)} + \sigma_1^2p_1\nu^2\mathrm{e}^{2r(T-t)},\\ \frac{\partial\widetilde{f}_4(p_1,q_1,q_2)}{\partial q_1} &= -2\Lambda\left[(1-q_1)\sigma_2^2 + (1-q_2)\lambda\mu_1\mu_2\right]\nu\mathrm{e}^{r(T-t)} + (q_1\sigma_2^2 + \lambda q_2\mu_1\mu_2)\nu^2\mathrm{e}^{2r(T-t)},\\ \frac{\partial\widetilde{f}_4(p_1,q_1,q_2)}{\partial q_2} &= -2\Lambda\left[(1-q_2)\sigma_3^2 + (1-q_1)\lambda\mu_1\mu_2\right]\nu\mathrm{e}^{r(T-t)} + (q_2\sigma_3^2 + \lambda q_1\mu_1\mu_2)\nu^2\mathrm{e}^{2r(T-t)},\\ \frac{\partial^2\widetilde{f}_4(p_1,q_1,q_2)}{\partial q_1^2} &= 2\Lambda\sigma_2^2\nu\mathrm{e}^{r(T-t)} + \sigma_2^2\nu^2\mathrm{e}^{2r(T-t)} > 0,\\ \frac{\partial^2\widetilde{f}_4(p_1,q_1,q_2)}{\partial q_2^2} &= 2\Lambda\sigma_3^2\nu\mathrm{e}^{r(T-t)} + \sigma_3^2\nu^2\mathrm{e}^{2r(T-t)} > 0,\\ \frac{\partial^2\widetilde{f}_4(p_1,q_1,q_2)}{\partial q_1q_2} &= 2\Lambda\lambda\mu_1\mu_2\nu\mathrm{e}^{r(T-t)} + \lambda\mu_1\mu_2\nu^2\mathrm{e}^{2r(T-t)} > 0.\end{cases} \tag{6.4}$$

From (6.1) and (6.4), we can firstly obtain the optimal $p_1(t)$

$$p_1^*(t) = \frac{a}{\sigma_1^2\nu\mathrm{e}^{r(T-t)}}. \tag{6.5}$$

Next we calculate the optimal q_1 and q_2 according to (6.4). We know that $\widetilde{f}_4$ is a convex function with respect to q_1 and q_2. The minimizer (q_1,q_2) of $\widetilde{f}_4(p_1,q_1,q_2)$ satisfies the following equation

$$\begin{cases}-2\Lambda\left[(1-q_1)\sigma_2^2 + (1-q_2)\lambda\mu_1\mu_2\right] + (q_1\sigma_2^2 + \lambda q_2\mu_1\mu_2)\nu\mathrm{e}^{r(T-t)} = 0,\\ -2\Lambda\left[(1-q_2)\sigma_3^2 + (1-q_1)\lambda\mu_1\mu_2\right] + (q_2\sigma_3^2 + \lambda q_1\mu_1\mu_2)\nu\mathrm{e}^{r(T-t)} = 0.\end{cases} \tag{6.6}$$

The conclusion in Liang and Yuen [19] can guarantee the existence and the uniqueness of the solution to (6.6). In fact, It is easy to show that the solution to equation (6.6) is

$$\begin{cases} q_1(T-t) = \dfrac{2\Lambda\sigma_2^2 + 2\Lambda\lambda\mu_1\mu_2}{2\Lambda + \nu \mathrm{e}^{r(T-t)}}, \\ q_2(T-t) = \dfrac{2\Lambda\sigma_2^2 + 2\Lambda\lambda\mu_1\mu_2}{2\Lambda + \nu \mathrm{e}^{r(T-t)}}. \end{cases} \tag{6.7}$$

Since

$$\frac{2\Lambda\sigma_2^2 + 2\Lambda\lambda\mu_1\mu_2}{2\Lambda + \nu \mathrm{e}^{r(T-t)}} \in (0,1),$$

we obtain the optimal strategy

$$\begin{cases} p_1^*(t) = \dfrac{a}{\sigma_1^2 \nu \mathrm{e}^{r(T-t)}}, \\ q_1^*(T-t) = q_2^*(T-t) = \dfrac{2\Lambda\sigma_2^2 + 2\Lambda\lambda\mu_1\mu_2}{2\Lambda + \nu \mathrm{e}^{r(T-t)}}. \end{cases}$$

Denote by h_{16} the function h in (6.2) for this section. Inserting the solution into (6.2) yields

$$h_{16}(T-t) = -\frac{1}{r} c\nu(\mathrm{e}^{r(T-t)} - 1) + \int_0^{T-t} K_2(s)\mathrm{d}s, \tag{6.8}$$

where

$$K_2(s) = -ap_1^*(s)^2 \nu \mathrm{e}^{rs} + \frac{1}{2}\left[\sigma_1^2 p_1^{*2} + \sigma_2^2 q_1^{*2} + \sigma_3^2 q_2^{*2} + 2q_1^*(s)q_2^*(s)\lambda\mu_1\mu_2\right]\nu^2 \mathrm{e}^{2rs}.$$

To summarize, we have

Theorem 6.1. *For any $t \in [0,T]$, the optimal strategy for the risk model (2.3) is*

$$\begin{cases} p_1^*(t) = \dfrac{a}{\sigma_1^2 \nu \mathrm{e}^{r(T-t)}}, \\ q_1^*(T-t) = q_2^*(T-t) = \dfrac{2\Lambda\sigma_2^2 + 2\Lambda\lambda\mu_1\mu_2}{2\Lambda + \nu \mathrm{e}^{r(T-t)}}, \end{cases}$$

and the value function is

$$V(t,x) = -\frac{m}{\nu}\exp\left\{-\nu x \mathrm{e}^{r(T-t)} + h_{16}(T-t)\right\},$$

where $h_{16}(T-t)$ is defined in (6.8).

Remark 6.2. From Sections 3 and 5, we can see that the optimal investment-reinsurance strategies in the compound Poisson risk model under two kinds of premium principles both depend not only on the safety loading, time, and interest rate, but also on the claim size distributions and the counting processes. But from Sections 4 and 6, we see that the optimal investment-reinsurance strategies in the diffusion model under the expected value premium principle and the variance principle depend on different parameters, the former depends on the safety loading, time, interest rate, the claim size distributions and the counting processes, the later depends on the safety loading, time, and interest rate only.

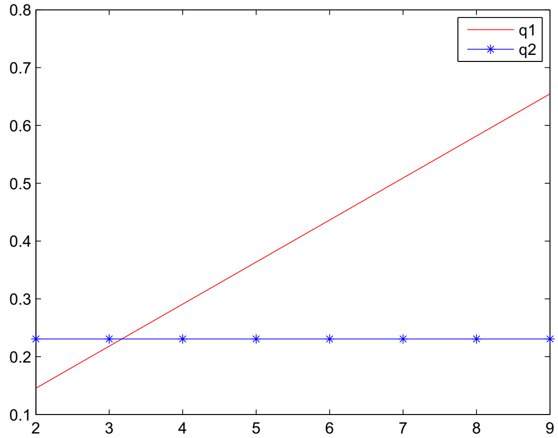

FIGURE 1. The effect of α_1 on the optimal reinsurance strategies in Section 3.

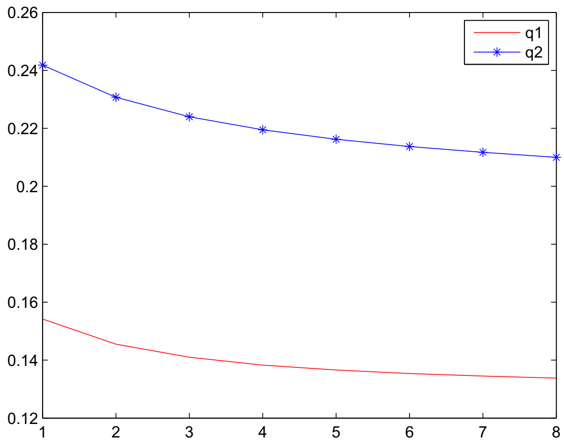

FIGURE 2. The effect of λ on the optimal reinsurance strategies in Section 3.

7. NUMERICAL EXAMPLES

Since we know that p_1 is independent to q_1 and q_2, and it depends on a and σ_1 only, we will not consider the effect of these two parameter to p_1 but pay more attention to q_1 and q_2.

We assume that the claim sizes X_i and Y_i are exponential distribution with parameters α_1 and α_2, respectively. Then, we have

$$\begin{cases} M_X(\nu q_1 \mathrm{e}^{r(T-t)}) = \dfrac{\alpha_1}{\alpha_1 - \nu q_1 \mathrm{e}^{r(T-t)}}, \\ M_Y(\nu q_2 \mathrm{e}^{r(T-t)}) = \dfrac{\alpha_2}{\alpha_2 - \nu q_2 \mathrm{e}^{r(T-t)}}. \end{cases}$$

Firstly, we consider the impact of parameters on the optimal reinsurance strategies in the compound Poisson risk model under expected value premium principle in Section 3. For computational convenience, we set $\eta_1 =$

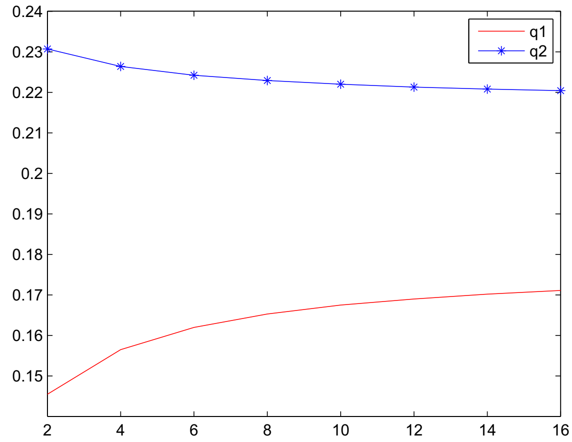

FIGURE 3. The effect of λ_1 on the optimal reinsurance strategies in Section 3.

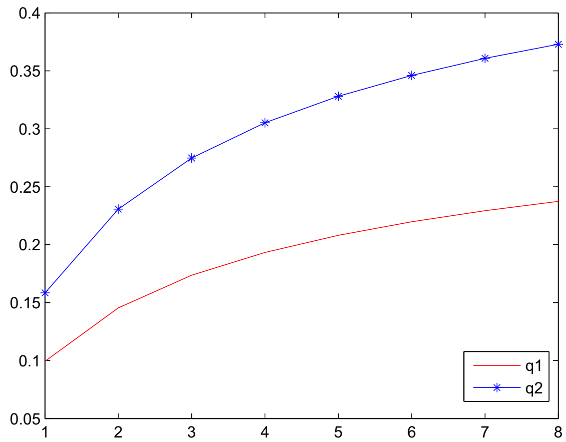

FIGURE 4. The effect of η on the optimal reinsurance strategies in Section 3.

$\eta_2 = \eta$. The minimizer $(q_1(T-t), q_2(T-t))$ of (3.5) satisfies the following equations

$$\begin{cases} -(1+\eta)(\lambda_1+\lambda)/\alpha_1 + \dfrac{\alpha_1}{(\alpha_1-\nu q_1 \mathrm{e}^{r(T-t)})^2}\left[\lambda_1 - \dfrac{\lambda\alpha_2}{\alpha_2-\nu q_2\mathrm{e}^{r(T-t)}}\right] = 0, \\ -(1+\eta)(\lambda_2+\lambda)/\alpha_2 + \dfrac{\alpha_2}{(\alpha_2-\nu q_2 \mathrm{e}^{r(T-t)})^2}\left[\lambda_2 - \dfrac{\lambda\alpha_1}{\alpha_1-\nu q_1\mathrm{e}^{r(T-t)}}\right] = 0. \end{cases}$$

Example 7.1. In this example, we set $\eta = 2$, $r = 0.3$, $T = 10$, $t = 2.5$, $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda = 2$, $\alpha_2 = 3$ and $\nu = 0.5$. Figure 1 shows the impact of α_1. From Figure 1, we see that q_1 increases as the α_1 increases, but α_1 has no impact on q_2. Along the same lines, q_2 increases as the α_2 increases, but α_2 has no impact on q_1. The optimal strategy is more sensitive to the claims size distribution than the counting process.

Example 7.2. In this example, we set $\eta = 2$, $r = 0.3$, $T = 10$, $t = 2.5$, $\alpha_1 = 2$, $\lambda_2 = 3$, $\alpha_2 = 3$ and $\nu = 0.5$. Figures 2 and 3 show the impact of λ and λ_1. From Figure 2 with $\lambda_1 = 2$, we see that q_1 and q_2 decrease as the

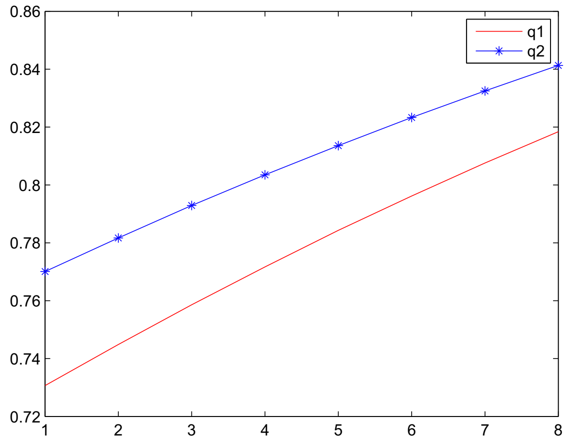

FIGURE 5. The effect of t on the optimal reinsurance strategies in Section 5.

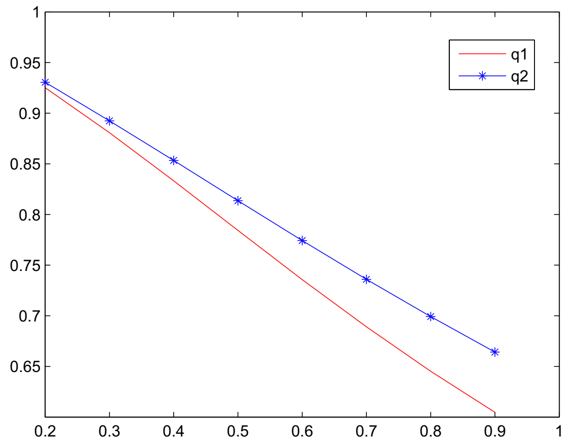

FIGURE 6. The effect of ν on the optimal reinsurance strategies in Section 5.

λ increases. From Figure 3 with $\lambda = 2$, we find that a greater value of λ_1 yields a greater value of the optimal strategy q_1 and a smaller value of q_2. But the changes of the optimal strategy are small, which shows that the optimal reinsurance strategy are insensitive to the change of the counting process.

Example 7.3. In this example, we set $\lambda = 2$, $r = 0.3$, $T = 10$, $t = 2.5$, $\alpha_1 = 2$, $\lambda_2 = 3$, $\alpha_2 = 3$, $\lambda_1 = 2$ and $\nu = 0.5$. Figure 4 shows the impact of η. From Figure 4, we see that q_1 and q_2 increase as η increases.

Secondly, we consider the impact of parameters in the compound Poisson risk model under the variance premium principle in Section 5. The minimizer $(q_1(T-t), q_2(T-t))$ of (5.5) satisfies the following equation

$$\begin{cases} a_1 + \Lambda\left[2(1-q_1)\sigma_2^2 + 2(1-q_2)\lambda\mu_1\mu_2\right] = \dfrac{\alpha_1}{(\alpha_1 - \nu q_1 \mathrm{e}^{r(T-t)})^2}\left[\lambda_1 + \dfrac{\lambda\alpha_2}{\alpha_2 - \nu q_2 \mathrm{e}^{r(T-t)}}\right], \\ a_2 + \Lambda\left[2(1-q_2)\sigma_3^2 + 2(1-q_1)\lambda\mu_1\mu_2\right] = \dfrac{\alpha_2}{(\alpha_2 - \nu q_2 \mathrm{e}^{r(T-t)})^2}\left[\lambda_2 + \dfrac{\lambda\alpha_1}{\alpha_1 - \nu q_1 \mathrm{e}^{r(T-t)}}\right], \end{cases}$$

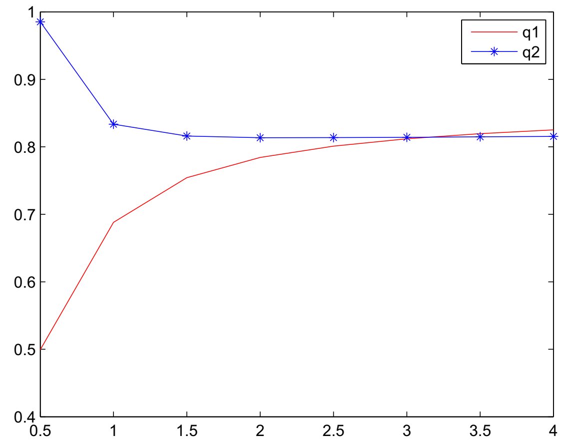

FIGURE 7. The effect of α_1 on the optimal reinsurance strategies in Section 5.

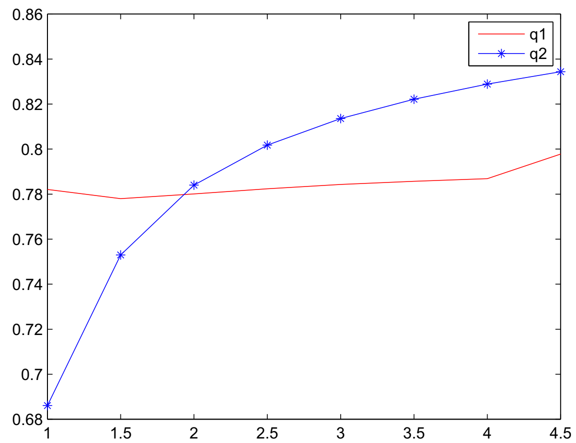

FIGURE 8. The effect of α_2 on the optimal reinsurance strategies in Section 5.

with $\mu_1 = 1/\alpha_1$, $\mu_2 = 1/\alpha_2$, $\sigma_2^2 = 2(\lambda_1 + \lambda)/\alpha_1^2$, and $\sigma_3^2 = 2(\lambda_2 + \lambda)/\alpha_2^2$.

Example 7.4. In this example, we set $\lambda_1 = 2$, $r = 0.05$, $T = 10$, $\lambda_2 = 3$, $\Lambda = 2$, $\lambda = 3$, $\alpha_1 = 2$ and $\alpha_2 = 3$. The results are showed in Figures 5 and 6. From Figure 5 with $\nu = 0.5$, we see that the optimal reinsurance strategies increase as t increase. Note that ν is the constant absolute risk aversion parameter of the utility function, a large value of ν means more risk averse. In Figure 6 with $t = 5$, we observe that the optimal reinsurance strategies decrease as ν increase.

Example 7.5. In this example, we set $\lambda_1 = 2$, $r = 0.05$, $T = 10$, $\lambda_2 = 3$, $\Lambda = 2$, $\lambda = 2$, $t = 5$ and $\nu = 0.5$. Figures 7 and 8 present the impact of α_1 and α_2. From Figure 7 with $\alpha_2 = 3$, we see that a greater value of α_1 yields a greater q_1 and smaller q_2. From Figure 8 with $\alpha_1 = 2$, we see that a greater value of (α_2) yields a greater q_2 and smaller q_1. Moreover, q_1 is always smaller than q_2 when $\alpha_1 < \alpha_2$ holds, and *vice versa*. This implies that the values of optimal strategies are sensitive to the change in the claim size distributions.

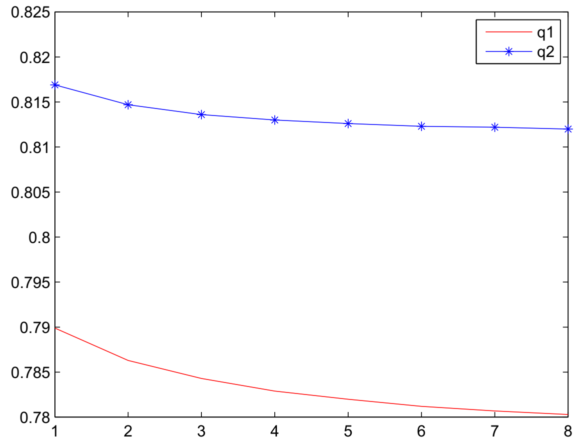

FIGURE 9. The effect of λ on the optimal reinsurance strategies in Section 5.

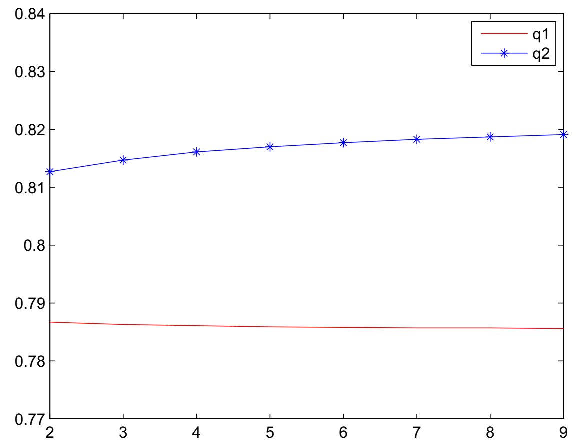

FIGURE 10. The effect of λ_2 on the optimal strategies.

Example 7.6. In this example, we set $\lambda_1 = 2$, $r = 0.05$, $T = 10$, $\alpha_1 = 2$, $\Lambda = 2$, $\alpha_2 = 3$, $t = 5$ and $\nu = 0.5$. Figures 9 and 10 present the impact of λ and λ_2. From Figure 9 with $\lambda_2 = 3$, we find that the optimal reinsurance strategies decrease while the value of λ increases. And in Figure 10 with $\lambda = 2$, a greater value of λ_2 yields a greater value of q_1 but a smaller value of q_2, along the same lines, a greater value of λ_1 yields a greater value of q_2 but a smaller value of q_1. Besides, we also find that the changes of the optimal strategies are small, which shows that the optimal strategies are kind of insensitive to the change in the counting process.

8. CONCLUSION

We first recap the main results of the paper. From an insurer's point of view, we consider the optimal investment and optimal proportional reinsurance in a risk model with two dependent classes of insurance businesses, where the two claim sizes are correlated. By the theory of stochastic optimal control and under the criterion of maximizing the expected exponential utility, we derive the closed-form expressions for the optimal strategies and the value functions under two kinds of premium principles both for the compound Poisson risk model and for the diffusion model. Furthermore, we find that the optimal investment strategy is independent to the optimal reinsurance strategy. Besides, the forms of optimal reinsurance are very different under the expected value premium principle and the variance premium principle. We also give some numerical examples to assess the impact of some parameters in the optimal strategies.

For a further extension, it is interesting to study the following problems: Firstly, we may consider the optimal investment-reinsurance problem with dependent risk under the criterion of minimizing the probability ruin or the mean-variance theory. Especially under the framework of mean-variance and the variance premium principle, we find that the value function is no longer the quadratic function. We need to find the proper form of the value function, which is a very challenging problem. Secondly, compared with the classical risk model or finance model, the Markovian regime-switching model seems to provide a better fit to the reality data of insurance and finance. In risk theory, the Markovian regime-switching risk model can capture the feature that insurance policies may need to change if the environment, such as weather condition, economical or political environment, *etc.*, changes. We may consider the optimal investment-reinsurance problems with common shock dependent risks in regime-switching financial market under the mean-variance criterion for insurers.

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